Annals of Functional Analysis
Ann. Funct. Anal. 6 (2015), no. 2, 11–25
http://doi.org/10.15352/afa/06-2-2
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

APPLICATIONS TO THE CAMERON–STORVICK TYPE THEOREM WITH RESPECT TO THE GAUSSIAN PROCESS

IL YONG LEE, HYUN SOO CHUNG AND SEUNG JUN CHANG*

Communicated by S. Barza

Abstract. In this paper, we establish a Cameron–Storvick type theorem with respect to the Gaussian process. We then use this theorem to obtain various integration formulas involving the transform, the $\diamond$-product and the first variation.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$.

In [2], the authors introduced the Cameron–Storvick theorem; that is to say, the Wiener integral of the first variation of a functional F is expressed in terms of the Wiener integral of F multiplied by a linear factor. In [14, 17], the authors used the Cameron–Storvick theorem to obtained the various integration formulas involving the Fourier-Feynman transform, the convolution product and the first variation.

In [7], the authors established a Cameron–Storvick type theorem on $C_{a,b}[0,T]$ which is general function space rather than the Wiener space $C_0[0,T]$. Also, they obtained various integration formulas [3, 7]. Recently, in [15], the authors introduced the concept of the transform with respect to the Gaussian process on function space $C_{a,b}[0,T]$. And then they used the Gaussian process to defined the $\diamond$-product and the first variation. Also, they established all possible relationships involving all three of these concepts.

Date: Received: Oct. 17, 2014; Revised: Jan. 31, 2014; Accepted: Feb. 21, 2014.
* Corresponding author.
2000 *Mathematics Subject Classification.* Primary 60J25; Secondary 46E30, 28C20.
Key words and phrases. Generalized Brownian motion process, Cameron–Storvick type theorem, translation theorem, $\diamond$-product, first variation.

In this paper, we show that the function space integral, involving two or three concepts, can be expressed in terms of the function space integral of one concept. First, a Cameron–Storvick relationship and translation theorem was established with respect to the Gaussian process on function space. Simply by allowing $h = s = 1$, we demonstrate that equation (3.4) in Section 3 is a Cameron–Storvick type theorem, similar to that introduced by Chang and Skoug [7]. Using this, we obtain various results involving the transform, the $\diamond$-product, and the first variation with respect to the Gaussian process. Finally, we demonstrate several applications of the proposed method to explain the usefulness of our results.

2. Definitions and preliminaries

In this section, we list some definitions and properties from [4, 5, 15].

Let $D = [0, T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0, \omega)$=0 almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1, \omega), \cdots, Y(t_n, \omega))$ is normally distributed with the density function

$$K(\vec{t}, \vec{\eta}) = \Big((2\pi)^n \prod_{j=1}^{n} \big(b(t_j) - b(t_{j-1})\big)\Big)^{-1/2}$$
$$\cdot \exp\Big\{-\frac{1}{2}\sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\Big\}$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \cdots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0, T]$ with $a(0) = 0$, $a'(t) \in L^2[0, T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0, T]$.

As explained in [18, pp.18-20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real-valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

In [18], Yeh showed that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s, t) = \min\{b(s), b(t)\}$, and that the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0, T]$ (which is equivalent to the Banach space of continuous functions x on $[0, T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0, T], \mathcal{B}(C_{a,b}[0, T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0, T])$ is the Borel σ-algebra of $C_{a,b}[0, T]$. We then complete this function space to obtain $(C_{a,b}[0, T], \mathcal{W}(C_{a,b}[0, T]), \mu)$ where $\mathcal{W}(C_{a,b}[0, T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0, T]$.

Given two $\mathbb{C}$-valued measurable functions F and G on $C_{a,b}[0,T]$, F is said to be equal to G scale almost everywhere(s-a.e.) if for each $\rho > 0$, $\mu(\{x \in C_{a,b}[0,T] : F(\rho x) \neq G(\rho x)\}) = 0$ [8, 12].

Let $L^2_{a,b}[0,T]$ be the set of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue–Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T |v^2(s)| db(s) < \infty \text{ and } \int_0^T |v^2(s)| d|a|(s) < \infty \right\}$$

where $|a|(t)$ denotes the total variation of the function $a(\cdot)$ on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)\overline{v(t)} d[b(t) + |a|(t)].$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular, note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore, $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space. Note that all functions of bounded variation on $[0,T]$ are elements of $L^2_{a,b}[0,T]$. Also note that if $a(t) \equiv 0$ and $b(t) = t$, then $L^2_{a,b}[0,T] = L^2[0,T]$. In fact,

$$(L^2_{a,b}[0,T], \|\cdot\|_{a,b}) \subset (L^2_{0,b}[0,T], \|\cdot\|_{0,b}) = (L^2[0,T], \|\cdot\|_2)$$

since the two norms $\|\cdot\|_{0,b}$ and $\|\cdot\|_2$ are equivalent. For $u \in L^2_{a,b}[0,T]$, let

$$(u,a') = \int_0^T u(t)a'(t)dt = \int_0^T u(t)da(t)$$

and

$$(u^2,b') = \int_0^T u^2(t)b'(t)dt = \int_0^T u^2(t)db(t).$$

It is well-known that for each $v \in L^2_{a,b}[0,T]$, the Paley–Wiener–Zygmund (PWZ) stochastic integral $\langle v, x\rangle$, see [4, 5, 7], exists for μ-a.e. $x \in C_{a,b}[0,T]$.

Next we state several definitions and introduce various notations which are used throughout the remainder of this paper.

For $h \in L^2_{a,b}[0,T]$, we define the Gaussian process Z_h by

$$Z_h(x,t) = \int_0^t h(s)\tilde{d}x(t)$$

where $\int_0^t h(s)\tilde{d}x(t)$ denotes the PWZ integral. For each $v \in L^2_{a,b}[0,T]$, let $\langle v, x\rangle = \int_0^T v(t)\tilde{d}x(t)$. From [9], we note that

$$\langle v, Z_h(x,\cdot)\rangle = \langle vh, x\rangle$$

for $h \in L_\infty[0,T]$ and s-a.e. $x \in C_{a,b}[0,T]$. Thus, throughout this paper, we require h to be in $L_\infty[0,T]$ rather than simply in $L^2_{a,b}[0,T]$.

Let $K_{a,b}[0,T]$ be the set of all complex-valued continuous functions $x(t)$ defined on $[0,T]$ which vanish at $t=0$ and whose real and imaginary parts are elements of $C_{a,b}[0,T]$; namely,

$$K_{a,b}[0,T] = \{x : [0,T] \to \mathbb{C} | x(0) = 0, \ \mathrm{Re}(x) \in C_{a,b}[0,T] \quad \text{and} \\ \mathrm{Im}(x) \in C_{a,b}[0,T]\}.$$

Thus $C_{a,b}[0,T]$ is the subspace of all real-valued functions in $K_{a,b}[0,T]$.

Now, we state the definitions of the transform with respect to the Gaussian process, the $\diamond$-product and the first variation.

Definition 2.1. Let F and G be a functionals on $K_{a,b}[0,T]$ and let ρ, β, γ and τ be non-zero complex numbers. Then the transform with respect to the Gaussian process, the $\diamond$-product and the first variation are defined by formulas

$$(T^{h_1,h_2}_{\gamma,\beta}(F))(y) = \int_{C_{a,b}[0,T]} F\Big(\gamma Z_{h_1}(x,\cdot) + \beta Z_{h_2}(y,\cdot)\Big) d\mu(x), \tag{2.1}$$

$$\begin{aligned}((F\diamond G)^{s_1,s_2}_{\rho,\tau})(y) = \int_{C_{a,b}[0,T]} & F\Big(\tau Z_{s_2}(y,\cdot) + \rho Z_{s_1}(x,\cdot)\Big) \\ & \cdot G\Big(\tau Z_{s_2}(y,\cdot) - \rho Z_{s_1}(x,\cdot)\Big) d\mu(x),\end{aligned} \tag{2.2}$$

$$\delta F\Big(Z_h(x,\cdot) | Z_s(z,\cdot)\Big) = \frac{\partial}{\partial k} F\Big(Z_h(x,\cdot) + kZ_s(z,\cdot)\Big)\Big|_{k=0} \tag{2.3}$$

if they exist.

Remark 2.2. (1) When $h_1(t) = h_2(t) = 1$ on $[0,T]$, $\gamma = \sqrt{2}$ and $\beta = i$, $T^{1,1}_{\sqrt{2},i}(F)$ is the generalized Fourier-Wiener function space transform introduced by Chang and Chung [4]. Also, $T^{1,1}_{\gamma,\beta}(F)$ is the generalized integral transform used by Chang, Chung and Skoug [5]. In particular, if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then $T^{1,1}_{\sqrt{2},i}(F)$ is the Fourier-Wiener transform used by Cameron and Martin [1] and the $T^{1,1}_{\gamma,\beta}(F)$ is the integral transform used by Kim and Skoug [13].

(2) If $s_1(t) = s_2(t) = 1$ on $[0,T]$, $\tau = \frac{1}{\sqrt{2}}$ and $\rho = \frac{1}{\sqrt{2\lambda}}$ for $\lambda \in \tilde{\mathbb{C}}_+$, then the $\diamond$-product $(F\diamond G)^{s_1,s_2}_{\rho,\tau}$ coincides with the convolution product $(F*G)_\lambda$ [6, 10, 11]; that is to say, $(F\diamond G)^{s_1,s_2}_{\rho,\tau} = (F*G)_\lambda$ for $\lambda \in \tilde{\mathbb{C}}_+$. Hence many results for convolution products are corollaries of the results for $\diamond$-product.

(3) If $h(t) = s(t) = 1$ on $[0,T]$, then the first variation of F with respect to Gaussian process coincides with the first variation of δF [7, 14].

3. Integration by parts formulas on function space

In [7], Chang and Skoug established a translation theorem and a Cameron–Storvick type theorem on function space. In this section, we will derive a generalized translation theorem and a Cameron–Storvick type theorem with respect to a Gaussian process. It is clear (by assigning $h = s = 1$) that the results in [7] are corollaries of the results obtained in this section. Finally, we establish several integration formulas involving a transform with respect to a Gaussian process.

The following lemma was established in [7, Theorem 3.1].

Lemma 3.1. (Translation theorem) *Let $x_0(t) = \int_0^t z(s)db(s)$ for some $z \in L^2_{a,b}[0,T]$. Let F be a μ-integrable functional on $C_{a,b}[0,T]$. Then*

$$\begin{aligned} &\int_{C_{a,b}[0,T]} F(x+x_0)d\mu(x) \\ &= \exp\Big\{-\frac{1}{2}(z^2,b') - (z,a')\Big\}\int_{C_{a,b}[0,T]} F(x)\exp\{\langle z,x\rangle\}d\mu(x). \end{aligned} \tag{3.1}$$

In our next theorem, we obtain a translation theorem for the transform with respect to the Gaussian process.

Theorem 3.2. (Generalized translation theorem) *Let $w(t) = \int_0^t z(u)db(u)$ for some $z \in L^2_{a,b}[0,T]$ and let $\frac{1}{h} \in L_\infty[0,T]$. Let $F\Big(Z_h(x,\cdot)\Big)$ be a μ-integrable functional on $K_{a,b}[0,T]$. Then*

$$\begin{aligned} &\int_{C_{a,b}[0,T]} F\Big(Z_h(x,\cdot)+Z_s(w,\cdot)\Big)d\mu(x) \\ &= \exp\Big\{-\frac{1}{2}\Big(\Big(\frac{sz}{h}\Big)^2,b'\Big) - \Big(\frac{sz}{h},a'\Big)\Big\} \\ &\qquad\qquad \cdot\int_{C_{a,b}[0,T]} F\Big(Z_h(x,\cdot)\Big)\exp\Big\{\Big\langle\frac{sz}{h},x\Big\rangle\Big\}d\mu(x). \end{aligned} \tag{3.2}$$

Proof. Let $G(x) = F\Big(Z_h(x,\cdot)\Big)$. Then

$$G(x+x_0) = F\Big(Z_h(x,\cdot)+Z_s(w,\cdot)\Big) \tag{3.3}$$

where $x_0(t) = \int_0^t \frac{s(u)}{h(u)}dw(u)$. Hence using equations (3.1) and (3.3), it follows that

$$\begin{aligned} &\int_{C_{a,b}[0,T]} F\Big(Z_h(x,\cdot)+Z_s(w,\cdot)\Big)d\mu(x) = \int_{C_{a,b}[0,T]} G(x+x_0)d\mu(x) \\ &= \exp\Big\{-\frac{1}{2}\Big(\Big(\frac{sz}{h}\Big)^2,b'\Big) - \Big(\frac{sz}{h},a'\Big)\Big\}\int_{C_{a,b}[0,T]} G(x)\exp\Big\{\Big\langle\frac{sz}{h},x\Big\rangle\Big\}d\mu(x) \\ &= \exp\Big\{-\frac{1}{2}\Big(\Big(\frac{sz}{h}\Big)^2,b'\Big) - \Big(\frac{sz}{h},a'\Big)\Big\}\int_{C_{a,b}[0,T]} F\Big(Z_h(x,\cdot)\Big)\exp\Big\{\Big\langle\frac{sz}{h},x\Big\rangle\Big\}d\mu(x). \end{aligned}$$

Thus we have the desired result. □

Remark 3.3. (1) The result established by Chang and Skoug in [7] follows immediately from Theorem 3.2 above by choosing $h(t) = s(t) = 1$ on $[0,T]$.

(2) In the setting of one parameter Wiener space $C_0[0,T]$ (i.e., in the case where $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$ in our research), the function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$. Thus the result of in [16, Theorem 3.2] follows immediately from Theorem 3.2 above.

The following theorem is one of main results in this paper and is called a Cameron–Storvick type theorem with respect to the Gaussian process on function space.

Theorem 3.4. (Cameron–Storvick type theorem) *Let F, w and h be as in Theorem 3.2. Assume that*

$$\int_{C_{a,b}[0,T]} \Big|\delta F\Big(Z_h(x,\cdot)|Z_s(w,\cdot)\Big)\Big| d\mu(x) < \infty.$$

Then

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} \delta F\Big(Z_h(x,\cdot)|Z_s(w,\cdot)\Big) d\mu(x) \\
&= \int_{C_{a,b}[0,T]} \Big\langle \frac{sz}{h}, x\Big\rangle F\Big(Z_h(x,\cdot)\Big) d\mu(x) - \Big(\frac{sz}{h}, a'\Big) \int_{C_{a,b}[0,T]} F\Big(Z_h(x,\cdot)\Big) d\mu(x).
\end{aligned} \tag{3.4}$$

Proof. By using equations (2.2) and (3.2), we have

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} \delta F\Big(Z_h(x,\cdot)|Z_s(w,\cdot)\Big) d\mu(x) \\
&= \frac{\partial}{\partial k}\bigg[\int_{C_{a,b}[0,T]} F\Big(Z_h(x,\cdot) + kZ_s(w,\cdot)\Big) d\mu(x)\bigg]\bigg|_{k=0} \\
&= \frac{\partial}{\partial k}\bigg[\exp\bigg\{-\frac{k^2}{2}\Big(\Big(\frac{sz}{h}\Big)^2, b'\Big) - k\Big(\frac{sz}{h}, a'\Big)\bigg\} \\
&\qquad\qquad \cdot \int_{C_{a,b}[0,T]} F\Big(Z_h(x,\cdot)\Big) \exp\Big\{k\Big\langle \frac{sz}{h}, x\Big\rangle\Big\} d\mu(x)\bigg]\bigg|_{k=0} \\
&= \int_{C_{a,b}[0,T]} \Big\langle \frac{sz}{h}, x\Big\rangle F\Big(Z_h(x,\cdot)\Big) d\mu(x) - \Big(\frac{sz}{h}, a'\Big) \int_{C_{a,b}[0,T]} F\Big(Z_h(x,\cdot)\Big) d\mu(x),
\end{aligned}$$

which completes the proof of Theorem 3.4. □

Remark 3.5. (1)The main result [7] follows immediately from Theorem 3.4 above by choosing $h(t) = s(t) = 1$ on $[0,T]$.

(2) If $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then Theorem 3.4 in [16] follows immediately from Theorem 3.4 above.

We establish an integration by parts formula in our next theorem.

Theorem 3.6. *Let F and G be complex-valued Borel measurable functionals on $K_{a,b}[0,T]$. Let h and w be as in Theorem 3.2. Assume that $F(Z_h(x,\cdot))G(Z_h(x,\cdot))$*

and $\delta F(Z_h(x,\cdot)|Z_s(w,\cdot))G(Z_h(x,\cdot)|Z_s(w,\cdot))$ *are* μ*-integrable on* $K_{a,b}[0,T]$*. Then*

$$\begin{aligned}\int_{C_{a,b}[0,T]} &F\Big(Z_h(x,\cdot)\Big)\delta G\Big(Z_h(x,\cdot)|Z_s(w,\cdot)\Big)\\ &\qquad\qquad + G\Big(Z_h(x,\cdot)\Big)\delta F\Big(Z_h(x,\cdot)|Z_s(w,\cdot)\Big)d\mu(x)\\ &= \int_{C_{a,b}[0,T]}\Big\langle\frac{sz}{h},x\Big\rangle F\Big(Z_h(x,\cdot)\Big)G\Big(Z_h(x,\cdot)\Big)d\mu(x)\\ &\qquad - \Big(\frac{sz}{h},a'\Big)\int_{C_{a,b}[0,T]} F\Big(Z_h(x,\cdot)\Big)G\Big(Z_h(x,\cdot)\Big)d\mu(x).\end{aligned}$$

Proof. Let $H\Big(Z_h(x,\cdot)\Big) = F\Big(Z_h(x,\cdot)\Big)G\Big(Z_h(x,\cdot)\Big)$. Then

$$\begin{aligned}\delta H\Big(Z_h(x,\cdot)|Z_s(w,\cdot)\Big) &= F\Big(Z_h(x,\cdot)\Big)\delta G\Big(Z_h(x,\cdot)|Z_s(w,\cdot)\Big)\\ &\qquad + G\Big(Z_h(x,\cdot)\Big)\delta F\Big(Z_h(x,\cdot)|Z_s(w,\cdot)\Big).\end{aligned}\tag{3.5}$$

Thus, applying Theorem 3.4 to equation (3.5) above, we have

$$\begin{aligned}\int_{C_{a,b}[0,T]}&\delta H\Big(Z_h(x,\cdot)|Z_s(w,\cdot)\Big)d\mu(x)\\ &= \int_{C_{a,b}[0,T]}\Big\langle\frac{sz}{h},x\Big\rangle H\Big(Z_h(x,\cdot)\Big)d\mu(x) - \Big(\frac{sz}{h},a'\Big)\int_{C_{a,b}[0,T]} H\Big(Z_h(x,\cdot)\Big)d\mu(x),\end{aligned}$$

which completes the proof of Theorem 3.6. □

Remark 3.7. If $a(t)\equiv 0$ and $b(t)=t$ on $[0,T]$, then the result in [16, Theorem 3.6] follows immediately from Theorem 3.6 above.

The following corollary is a special case of Theorem 3.6.

Corollary 3.8. *Under the hypotheses of Theorem 3.6.*

$$\begin{aligned}\int_{C_{a,b}[0,T]}&F\Big(Z_h(x,\cdot)\Big)\delta F\Big(Z_h(x,\cdot)|Z_s(w,\cdot)\Big)d\mu(x)\\ &= \frac{1}{2}\int_{C_{a,b}[0,T]}\Big\langle\frac{sz}{h},x\Big\rangle\Big[F\Big(Z_h(x,\cdot)\Big)\Big]^2 d\mu(x)\\ &\qquad - \frac{1}{2}\Big(\frac{sz}{h},a'\Big)\int_{C_{a,b}[0,T]}\Big[F\Big(Z_h(x,\cdot)\Big)\Big]^2 d\mu(x).\end{aligned}$$

In our next theorem, we establish an integration by parts formula involving the transforms with respect to the Gaussian process.

Theorem 3.9. *Let* h *and* w *be as in Theorem 3.2. Let* F *and* G *be complex-valued Borel measurable functionals on* $K_{a,b}[0,T]$*. Assume that* $T^{h_1,h_2}_{\gamma,\beta}(F)T^{h_1,h_2}_{\gamma,\beta}(G)$ *and*

$\delta(T_{\gamma,\beta}^{h_1,h_2}(F)T_{\gamma,\beta}^{h_1,h_2}(G))$ *are* μ*-integrable on* $K_{a,b}[0,T]$*. Then*

$$\begin{aligned}\int_{C_{a,b}[0,T]} & T_{\gamma,\beta}^{h_1,h_2}(F)\Big(Z_h(y,\cdot)\Big)\delta T_{\gamma,\beta}^{h_1,h_2}(G)\Big(Z_h(y,\cdot)|Z_s(w,\cdot)\Big) \\ &\qquad + \delta T_{\gamma,\beta}^{h_1,h_2}(F)\Big(Z_h(y,\cdot)|Z_s(w,\cdot)\Big)T_{\gamma,\beta}^{h_1,h_2}(G)\Big(Z_h(y,\cdot)\Big)d\mu(y) \\ &= \int_{C_{a,b}[0,T]}\Big\langle\frac{sz}{h},y\Big\rangle T_{\gamma,\beta}^{h_1,h_2}(F)\Big(Z_h(y,\cdot)\Big)T_{\gamma,\beta}^{h_1,h_2}(G)\Big(Z_h(y,\cdot)\Big)d\mu(y) \\ &\qquad - \Big(\frac{sz}{h},a'\Big)\int_{C_{a,b}[0,T]}T_{\gamma,\beta}^{h_1,h_2}(F)\Big(Z_h(y,\cdot)\Big)T_{\gamma,\beta}^{h_1,h_2}(G)\Big(Z_h(y,\cdot)\Big)d\mu(y)\end{aligned} \tag{3.6}$$

for $y\in K_{a,b}[0,T]$*.*

Proof. Equation (3.6) follows from Theorem 3.5 with F and G replaced with $T_{\gamma,\beta}^{h_1,h_2}(F)$ and $T_{\gamma,\beta}^{h_1,h_2}(G)$, respectively. □

The following corollary is a special case of Theorem 3.9.

Corollary 3.10. *Under the hypotheses of Theorem 3.9.*

$$\begin{aligned}\int_{C_{a,b}[0,T]} & T_{\gamma,\beta}^{h_1,h_2}(F)\Big(Z_h(y,\cdot)\Big)\delta T_{\gamma,\beta}^{h_1,h_2}(F)\Big(Z_h(y,\cdot)|Z_s(w,\cdot)\Big)d\mu(y) \\ &= \frac{1}{2}\int_{C_{a,b}[0,T]}\Big\langle\frac{sz}{h},y\Big\rangle\Big[T_{\gamma,\beta}^{h_1,h_2}(F)\Big(Z_h(y,\cdot)\Big)\Big]^2 d\mu(y) \\ &\qquad - \frac{1}{2}\Big(\frac{sz}{h},a'\Big)\int_{C_{a,b}[0,T]}\Big[T_{\gamma,\beta}^{h_1,h_2}(F)\Big(Z_h(y,\cdot)\Big)\Big]^2 d\mu(y).\end{aligned}$$

Our next lemma plays a key role in obtaining various integration formulas. In Lemma 3.11 below, we establish that the transform of the first variation equals the first variation of the transform with respect to the Gaussian process.

Lemma 3.11. *Let* F *be as in Theorem 3.4. Let* h *and* w *be as in Theorem 3.2. Let* h*,* s*,* $h_j(j=1,2,3,4)$*,* l *and* m *satisfy the following conditions;*

(1) $h_3(t)=h(t)h_1(t)$
(2) $l(t)h_4(t)=h(t)h_2(t)$
(3) $m(t)h_4(t)=s(t)$ *on* $[0,T]$*.*

Then for all non-zero complex numbers γ *and* β

$$T_{\gamma,\beta}^{h_1,h_2}\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w,\cdot)\Big)\Big)(y) = \delta T_{\gamma,\beta}^{h_3,h_4}(F)\Big(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(w,\cdot)\Big)$$

for $y\in K_{a,b}[0,T]$*.*

Proof. By using equations (2.1) and (2.3), we have

$$
\begin{aligned}
&T_{\gamma,\beta}^{h_1,h_2}\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w,\cdot)\Big)\Big)(y)\\
&=\frac{\partial}{\partial k}\bigg[\int_{C_{a,b}[0,T]}F\Big(\gamma Z_{hh_1}(x,\cdot)+\beta Z_{hh_2}(y,\cdot)+kZ_s(w,\cdot)\Big)d\mu(x)\bigg]\bigg|_{k=0}\\
&=\frac{\partial}{\partial k}\bigg[\int_{C_{a,b}[0,T]}F\Big(\gamma Z_{h_3}(x,\cdot)+\beta Z_{lh_4}(y,\cdot)+\frac{\beta k}{\beta}Z_{mh_4}(w,\cdot)\Big)d\mu(x)\bigg]\bigg|_{k=0}\\
&=\frac{\partial}{\partial k}T_{\gamma,\beta}^{h_3,h_4}(F)\Big(Z_l(y,\cdot)+\frac{k}{\beta}Z_m(w,\cdot)\Big)\bigg|_{k=0}\\
&=\delta T_{\gamma,\beta}^{h_3,h_4}(F)\Big(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(w,\cdot)\Big).
\end{aligned}
$$

Thus we have the desired results. □

The following theorem follows immediately from Theorem 3.4 and Lemma 3.11.

Theorem 3.12. *Let F, h, s, l, m and $h_j (j = 1, 2, 3, 4)$ be as in Lemma 3.11. Let $\frac{1}{l}$ be in $L_\infty[0,T]$. Assume that $T_{\gamma,\beta}^{h_1,h_2}(F)$ and $T_{\gamma,\beta}^{h_1,h_2}(\delta F)$ are μ-integrable on $K_{a,b}[0,T]$. Then for all non-zero complex numbers γ and β*

$$
\begin{aligned}
&\int_{C_{a,b}[0,T]}T_{\gamma,\beta}^{h_1,h_2}\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w,\cdot)\Big)\Big)(y)d\mu(y)\\
&=\frac{1}{\beta}\int_{C_{a,b}[0,T]}\Big\langle\frac{mz}{l},y\Big\rangle T_{\gamma,\beta}^{h_3,h_4}(F)\Big(Z_l(y,\cdot)\Big)d\mu(y)\\
&\qquad\qquad-\frac{1}{\beta}\Big(\frac{mz}{l},a'\Big)\int_{C_{a,b}[0,T]}T_{\gamma,\beta}^{h_3,h_4}(F)\Big(Z_l(y,\cdot)\Big)d\mu(y)
\end{aligned}
$$

for $y \in K_{a,b}[0,T]$.

4. Various integration by parts formulas

In Section 3, we established a Cameron–Storvick type theorem and the translation theorem, with respect to the Gaussian process. We then obtained integration by parts formulas, using the transform with respect to the Gaussian process. In this section, we establish the relationships between the transform, the $\diamond$-product, and the first variation with respect to the Gaussian process.

First we introduce some notation which will be used throughout this section.

(1) For each pair of non-zero complex numbers γ and β, let

$$\nu(t)\equiv\nu_{\gamma,\beta}^{h_1,h_2}(t)=\sum_{j=1}^{\infty}\Big(\gamma^2(h_1,\phi_j)_{a,b}^2+\beta^2(h_2,\phi_j)_{a,b}^2\Big)^{\frac{1}{2}}\phi_j(t)$$

$$\Phi_{\phi_j}^{\pm}=\gamma(h_1,\phi_j)_{a,b}\pm\beta(h_2,\phi_j)_{a,b}-\Big(\gamma^2(h_1,\phi_j)_{a,b}^2+\beta^2(h_2,\phi_j)_{a,b}^2\Big)^{\frac{1}{2}},$$

$$W^{h_1,h_2}_{\gamma,\beta}(\cdot)=\sum_{j=1}^{\infty}\Phi^{+}_{\phi_j}Z_{\phi_j}(a,\cdot)\quad\text{and}\quad B^{h_1,h_2}_{\gamma,\beta}(\cdot)=\sum_{j=1}^{\infty}\Phi^{-}_{\phi_j}Z_{\phi_j}(a,\cdot).$$

Since $|\gamma^2+\beta^2|^{\frac12}\le|\gamma^2|^{\frac12}+|\beta^2|^{\frac12}$ for all complex numbers γ and β, we have

$$\begin{aligned}|\nu(t)|&\le\sum_{j=1}^{\infty}\Big|\gamma^2(h_1,\phi_j)^2_{a,b}+\beta^2(h_2,\phi_j)^2_{a,b}\Big|^{\frac12}|\phi_j(t)|\\&=\sum_{j=1}^{\infty}\Big|\gamma^2(h_1,\phi_j)^2_{a,b}\phi_j^2(t)+\beta^2(h_2,\phi_j)^2_{a,b}\phi_j^2(t)\Big|^{\frac12}\\&\le\sum_{j=1}^{\infty}\Big|\gamma^2(h_1,\phi_j)^2_{a,b}\phi_j^2(t)\Big|^{\frac12}+\sum_{j=1}^{\infty}\Big|\beta^2(h_2,\phi_j)^2_{a,b}\phi_j^2(t)\Big|^{\frac12}\\&=|\gamma^2|^{\frac12}h_1(t)+|\beta^2|^{\frac12}h_2(t)\le|\gamma^2|^{\frac12}\|h_1\|_\infty+|\beta^2|^{\frac12}\|h_2\|_\infty.\end{aligned}$$

Hence ν is an element of $L_\infty[0,T]$.

(2) Let F be a functional defined on $K_{a,b}[0,T]$ and let

$$F_y(x)=F(x+y)\quad\text{for}\quad x,y\in K_{a,b}[0,T].$$

The following theorem was established in [15].

Theorem 4.1. (1) *Let F be a complex-valued Borel measurable functional on $K_{a,b}[0,T]$ such that*

$$\int_{C^2_{a,b}[0,T]}\Big|F\Big(\gamma Z_{h_1}(x,\cdot)+\beta Z_{h_2}(y,\cdot)\Big)\Big|d(\mu\times\mu)(x,y)<\infty.$$

Then for the non-zero complex numbers γ and β,

$$\begin{aligned}&\int_{C^2_{a,b}[0,T]}F\Big(\gamma Z_{h_1}(x,\cdot)+\beta Z_{h_2}(y,\cdot)\Big)d(\mu\times\mu)(x,y)\\&=\int_{C_{a,b}[0,T]}F\Big(Z_\nu(w,\cdot)+W^{h_1,h_2}_{\gamma,\beta}(\cdot)\Big)d\mu(w)\end{aligned}$$

where $\nu\equiv\nu^{h_1,h_2}_{\gamma,\beta}$ and $W^{h_1,h_2}_{\gamma,\beta}$.

(2) *Let F be as in (1) above. Then for non-zero complex numbers γ_j and $\beta_j(j=1,2,3)$*

$$T^{h_1,h_2}_{\gamma_1,\beta_1}(T^{h_3,h_4}_{\gamma_2,\beta_2}(F))(y)=T^{\nu,h_2h_4}_{1,\beta_1\beta_2}(F_W)(y)$$

where $\nu\equiv\nu^{h_3,h_1h_4}_{\gamma_2,\gamma_1\beta_2}$ and $W\equiv W^{h_3,h_1h_4}_{\gamma_2,\gamma_1\beta_2}$.

(3) Let F be as in (1) above and assume that $G:K_{a,b}[0,T]\to\mathbb{C}$ satisfies the same condition as F. Assume that $\tau\gamma h_1(t)s_2(t)=\rho s_1(t)$ on $[0,T]$. Then for each non-zero complex numbers γ, β, ρ and τ

$$(T^{h_1,h_2}_{\gamma,\beta}((F\diamond G)^{s_1,s_2}_{\rho,\tau}))(y)=T^{s_1,h_2s_2}_{\sqrt2\rho,\tau\beta}(F_W)(y)T^{s_1,h_2s_2}_{\sqrt2\rho,\tau\beta}(G_B)(y)$$

where $W=(2-\sqrt2\rho)Z_{s_1}(a,\cdot)$ and $B=-\sqrt2\rho Z_{s_1}(a,\cdot)$.

Now, we consider the various integration formulas involving the transform, the first variation and $\diamond$-product with respect to the Gaussian process. We will only state the formulas without proofs.

The following formula 1 tells us that the function space integral of the double transform for the first variation of F is expressed in terms of the function space integral of the transform of F.

Formula 1. Let F be as in Theorem 3.4. Let h, s, $h_j (j=1,2,3,4)$, l and m satisfy the following conditions;

(1) $h_3(t) = h(t)\nu$
(2) $l(t)h_4(t) = h(t)h_2^2(t)$
(3) $m(t)h_4(t) = s(t)$ on $[0,T]$.

Assume that $T_{\gamma,\beta}^{h_1,h_2}(T_{\gamma,\beta}^{h_1,h_2}(\delta F))$ is μ-integrable on $K_{a,b}[0,T]$. Then for all non-zero complex numbers γ and β

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} T_{\gamma,\beta}^{h_1,h_2}\Big(T_{\gamma,\beta}^{h_1,h_2}\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w,\cdot)\Big)\Big)(y)d\mu(y)\\
&= \frac{1}{\beta^2}\int_{C_{a,b}[0,T]} \Big\langle \frac{mz}{l}, y\Big\rangle T_{1,\beta^2}^{h_3,h_4}(F_W)\Big(Z_l(y,\cdot)\Big)d\mu(y)\\
&\qquad\qquad - \frac{1}{\beta^2}\Big(\frac{mz}{l}, a'\Big)\int_{C_{a,b}[0,T]} T_{1,\beta^2}^{h_3,h_4}(F_W)\Big(Z_l(y,\cdot)\Big)d\mu(y)
\end{aligned}$$

for $y \in K_{a,b}[0,T]$ where $\nu \equiv \nu_{\gamma,\gamma\beta}^{h_2,h_1h_2}$ and $W \equiv W_{\gamma,\gamma\beta}^{h_2,h_1h_2}$.

Formula 2 below shows that the function space integral of the transform of the $\diamond$-product of two functionals can be expressed in terms of the function space integral of the product of their transforms.

Formula 2. Let F and G be as in Theorem 3.6. Let h, s, $h_j (j=1,2,3,4)$, l and m satisfy the following conditions;

(1) $h_3(t) = h(t)h_1(t)$
(2) $l(t)h_4(t) = h(t)h_2(t)$
(3) $m(t)h_4(t) = s(t)$
(4) $\tau\gamma h_3(t)s_2(t) = \rho s_1(t)$ on $[0,T]$.

Assume that $T_{\gamma,\beta}^{h_1,h_2}(\delta(F\diamond G)_{\rho,\tau}^{s_1,s_2})$ is μ-integrable on $K_{a,b}[0,T]$. Then for all non-zero complex numbers γ, β, ρ and τ

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} T_{\gamma,\beta}^{h_1,h_2}\Big(\delta(F\diamond G)_{\rho,\tau}^{s_1,s_2}\Big(Z_h(\cdot,\cdot)|Z_s(w,\cdot)\Big)\Big)(y)d\mu(y)\\
&= \frac{1}{\beta}\int_{C_{a,b}[0,T]} \Big\langle \frac{mz}{l}, y\Big\rangle T_{\sqrt{2}\rho,\tau\beta}^{s_1,h_4s_2}(F_W)\Big(Z_l(y,\cdot)\Big)T_{\sqrt{2}\rho,\tau\beta}^{s_1,h_4s_2}(G_B)\Big(Z_l(y,\cdot)\Big)d\mu(y)\\
&\quad - \frac{1}{\beta}\Big(\frac{mz}{l}, a'\Big)\int_{C_{a,b}[0,T]} T_{\sqrt{2}\rho,\tau\beta}^{s_1,h_4s_2}(F_W)\Big(Z_l(y,\cdot)\Big)T_{\sqrt{2}\rho,\tau\beta}^{s_1,h_4s_2}(G_B)\Big(Z_l(y,\cdot)\Big)d\mu(y)
\end{aligned}$$

where $W = (2-\sqrt{2})\rho Z_{s_1}(a,\cdot)$ and $B = -\sqrt{2}\rho Z_{s_1}(a,\cdot)$.

Formula 3 below shows that the function space integral of the transform of the first variation of the $\diamond$-product of their transforms can be expressed in terms of the function space integral of the product of their transforms.

Formula 3. Let F and G be as in Theorem 3.6. Let h, s, $h_j(j=1,2,3,4)$, l and m be satisfying the following conditions;

(1) $h_3(t)=h(t)h_1(t)$
(2) $l(t)h_4(t)=h(t)h_2(t)$
(3) $m(t)h_4(t)=s(t)$
(4) $\tau\gamma h_1(t)s_2(t)=\rho s_1(t)$ on $[0,T]$.

Assume that $T^{h_1,h_2}_{\gamma,\beta}(\delta(T^{h_1,h_2}_{\gamma,\beta}(F)\diamond T^{h_1,h_2}_{\gamma,\beta}(G))^{s_1,s_2}_{\rho,\tau})$ is μ-integrable on $K_{a,b}[0,T]$. Then for all non-zero complex numbers γ, β, ρ and τ

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} T^{h_1,h_2}_{\gamma,\beta}\Big(\delta\Big(T^{h_1,h_2}_{\gamma,\beta}(F)\diamond T^{h_1,h_2}_{\gamma,\beta}(G)\Big)^{s_1,s_2}_{\rho,\tau}\Big(Z_h(\cdot,\cdot)|Z_s(w,\cdot)\Big)\Big)(y)d\mu(y)\\
&=\frac{1}{\beta}\int_{C_{a,b}[0,T]}\Big\langle\frac{mz}{l},y\Big\rangle T^{\nu,h_2^2s_2}_{1,\tau\beta^2}(F_{W+\tilde{W}})\Big(Z_l(y,\cdot)\Big)T^{\nu,h_2^2s_2}_{1,\tau\beta^2}(G_{B+\tilde{W}})\Big(Z_l(y,\cdot)\Big)d\mu(y)\\
&\quad-\frac{1}{\beta}\Big(\frac{mz}{l},a'\Big)\int_{C_{a,b}[0,T]}T^{\nu,h_2^2s_2}_{1,\tau\beta^2}(F_{W+\tilde{W}})\Big(Z_l(y,\cdot)\Big)T^{\nu,h_2^2s_2}_{1,\tau\beta^2}(G_{B+\tilde{W}})\Big(Z_l(y,\cdot)\Big)d\mu(y)
\end{aligned}$$

where $\nu\equiv\nu^{h_1,s_1h_2}_{\gamma,\sqrt{2}\rho\beta}$, $W=(2-\sqrt{2})\rho Z_{s_1}(a,\cdot)$, $B=-\sqrt{2}\rho Z_{s_1}(a,\cdot)$ and $\tilde{W}\equiv\tilde{W}^{h_1,s_1h_2}_{\gamma,\sqrt{2}\rho\beta}$.

The following simple example illustrates the results in Section 4.

Let $F,G:K_{a,b}[0,T]\to\mathbb{R}$ be defined by the formula

$$F(x)=\langle u,x\rangle, G(x)=\langle v,x\rangle,\quad \text{for}\quad u,v\in L^2_{a,b}[0,T]. \tag{4.1}$$

Then for all non-zero complex numbers γ and β, direct calculations show that

$$T^{h_1,h_2}_{\gamma,\beta}(F_W)\Big(Z_l(y,\cdot)\Big)=\beta\langle uh_2l,y\rangle+2\gamma(uh_1,a')+\beta(uh_2,a')-(u\nu,a')$$

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}\langle z,y\rangle T^{h_1,h_2}_{\gamma,\beta}(F_W)\Big(Z_l(y,\cdot)\Big)d\mu(y)\\
&\qquad-(z,a')\int_{C_{a,b}[0,T]}T^{h_1,h_2}_{\gamma,\beta}(F_W)\Big(Z_l(y,\cdot)\Big)d\mu(y)=\beta(uzh_2l,b')
\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}\langle z,y\rangle T^{h_1,h_2}_{\gamma,\beta}(F_W)\Big(Z_l(y,\cdot)\Big)T^{h_1,h_2}_{\gamma,\beta}(G_B)\Big(Z_l(y,\cdot)\Big)d\mu(y)\\
&\qquad-(z,a')\int_{C_{a,b}[0,T]}T^{h_1,h_2}_{\gamma,\beta}(F_W)\Big(Z_l(y,\cdot)\Big)T^{h_1,h_2}_{\gamma,\beta}(G_B)\Big(Z_l(y,\cdot)\Big)d\mu(y)\\
&=\beta^2[(uh_2lz,b')(vh_2l,a')+(vh_2lz,b')(uh_2l,a')]\\
&\qquad\qquad+2\gamma\beta[(vh_2lz,b')(uh_1,a')+(uh_2lz,b')(vh_1,a')]\\
&\qquad\qquad+\beta(vh_2lz,b')[\beta(uh_2,a')-(u\nu,a')]\\
&\qquad\qquad-\beta(uh_2lz,b')[\beta(uh_2,a')+(u\nu,a')]
\end{aligned} \tag{4.3}$$

where $\nu\equiv\nu^{h_1,h_2}_{\gamma,\beta}$, $W\equiv W^{h_1,h_2}_{\gamma,\beta}$ and $B\equiv B^{h_1,h_2}_{\gamma,\beta}$. Equations (4.2) and (4.3) follow from the following well-known integration formulas

$$\int_{C_{a,b}[0,T]}\langle u,y\rangle\langle v,y\rangle d\mu(y)=(uv,b')+(u,a')(v,a')$$

and

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}\langle z,y\rangle\langle u,y\rangle\langle v,y\rangle d\mu(y)\\
&=(uz,b')(v,a')+(vz,b')(u,a')+(z,a')(uv,b')+(z,a')(u,a')(v,a')
\end{aligned}$$

for all $z,u,v\in L^2_{a,b}[0,T]$.

In the following example the function space integral of the double transform of the first variation of F can be calculated from the function space integral of the transform of F, without using the first.

Example using the Formula 1. Let F be given by equation (4.1). Then by using equation (4.2), we obtain that

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}\Big\langle\frac{mz}{l},y\Big\rangle T^{h_3,h_4}_{1,\beta^2}(F_W)\Big(Z_l(y,\cdot)\Big)d\mu(y)\\
&\qquad\qquad-\Big(\frac{mz}{l},a'\Big)\int_{C_{a,b}[0,T]}T^{h_3,h_4}_{1,\beta^2}(F_W)\Big(Z_l(y,\cdot)\Big)d\mu(y)=\beta^2(usz,b')
\end{aligned}$$

and hence from Formula 1,

$$\int_{C_{a,b}[0,T]}T^{h_1,h_2}_{\gamma,\beta}\Big(T^{h_1,h_2}_{\gamma,\beta}\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w,\cdot)\Big)\Big)(y)d\mu(y)=(usz,b').$$

The function space integral of the transform for the $\diamond$-product of two functionals can be calculated from the function space integral of the product of their transforms, without the concepts of the $\diamond$-product or the first variation. An example of this approach is given below.

Example using the Formula 2. Let F and G be given by equation (4.1). Then by using equation (4.3), we obtain that

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} \Big\langle \frac{mz}{l}, y\Big\rangle T^{s_1,h_4s_2}_{\sqrt{2}\rho,\tau\beta}(F_W)\Big(Z_l(y,\cdot)\Big) T^{s_1,h_4s_2}_{\sqrt{2}\rho,\tau\beta}(G_B)\Big(Z_l(y,\cdot)\Big) d\mu(y) \\
&\qquad - \Big(\frac{mz}{l}, a'\Big) \int_{C_{a,b}[0,T]} T^{s_1,h_4s_2}_{\sqrt{2}\rho,\tau\beta}(F_W)\Big(Z_l(y,\cdot)\Big) T^{s_1,h_4s_2}_{\sqrt{2}\rho,\tau\beta}(G_B)\Big(Z_l(y,\cdot)\Big) d\mu(y) \\
&= \tau^2\beta^2[(uss_2z, b')(vhh_2s_2, a') + (vss_2z, b')(uhh_2s_2, a')] + 2\tau\beta(vss_2z, b')(us_1, a')
\end{aligned}$$

and hence from Formula 2,

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} T^{h_1,h_2}_{\gamma,\beta}\Big(\delta(F \diamond G)^{s_1,s_2}_{\rho,\tau}\Big(Z_h(\cdot,\cdot)|Z_s(w,\cdot)\Big)\Big)(y) d\mu(y) \\
&= \tau^2\beta[(uss_2z, b')(vhh_2s_2, a') + (vss_2z, b')(uhh_2s_2, a')] + 2\tau(vss_2z, b')(us_1, a').
\end{aligned}$$

Additionally, in the following example we show that the function space integral of the transform for the first variation of the $\diamond$-product of their transforms can be used to calculate the function space integral of the product of their transforms, without having to use the concepts of the $\diamond$-product or the first variation

Example using the Formula 3. Let F and G be given by equation (4.1). Then by using equation (4.3), we obtain that

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} \Big\langle \frac{mz}{l}, y\Big\rangle T^{\nu,h_2^2s_2}_{1,\tau\beta^2}(F_{W+\tilde{W}})\Big(Z_l(y,\cdot)\Big) T^{\nu,h_2^2s_2}_{1,\tau\beta^2}(G_{B+\tilde{W}})\Big(Z_l(y,\cdot)\Big) d\mu(y) \\
&\qquad - \Big(\frac{mz}{l}, a'\Big) \int_{C_{a,b}[0,T]} T^{\nu,h_2^2s_2}_{1,\tau\beta^2}(F_{W+\tilde{W}})\Big(Z_l(y,\cdot)\Big) T^{\nu,h_2^2s_2}_{1,\tau\beta^2}(G_{B+\tilde{W}})\Big(Z_l(y,\cdot)\Big) d\mu(y) \\
&= \tau^2\beta^4[(uh_2^2s_2mz, b')(vh_2^2s_2l, a') + (vh_2^2s_2mz, b')(uh_2^2s_2l, a')] \\
&\qquad + \tau\beta^2(vh_2^2s_2mz, b')[(2-\sqrt{2})\rho(us_1, a') + \gamma(uh_1, a') + \sqrt{2}\rho\beta(us_1h_2, a')] \\
&\qquad + \tau\beta^2(uh_2^2s_2mz, b')[-\sqrt{2}\rho(vs_1, a') + \gamma(vh_1, a') + \sqrt{2}\rho\beta(vs_1h_2, a')]
\end{aligned}$$

and hence from Formula 3,

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} T^{h_1,h_2}_{\gamma,\beta}\Big(\delta\Big(T^{h_1,h_2}_{\gamma,\beta}(F) \diamond T^{h_1,h_2}_{\gamma,\beta}(G)\Big)^{s_1,s_2}_{\rho,\tau}\Big(Z_h(\cdot,\cdot)|Z_s(w,\cdot)\Big)\Big)(y) d\mu(y) \\
&= \tau^2\beta^3[(uh_2^2s_2mz, b')(vh_2^2s_2l, a') + (vh_2^2s_2mz, b')(uh_2^2s_2l, a')] \\
&\qquad + \tau\beta(vh_2^2s_2mz, b')[(2-\sqrt{2})\rho(us_1, a') + \gamma(uh_1, a') + \sqrt{2}\rho\beta(us_1h_2, a')] \\
&\qquad + \tau\beta(uh_2^2s_2mz, b')[-\sqrt{2}\rho(vs_1, a') + \gamma(vh_1, a') + \sqrt{2}\rho\beta(vs_1h_2, a')]
\end{aligned}$$

where $\nu \equiv \nu^{h_1,s_1h_2}_{\gamma,\sqrt{2}\rho\beta}$.

Acknowledgement. The authors thank the referees for their helpful suggestions which led to the present version of this paper. This re-search was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Science, ICT & Future Planning (2012R1A1A1004774).

References

1. R.H. Cameron and W.T. Martin, *Fourier-Wiener transforms of functionals belonging to L_2 over the space C*, Duke Math. J. **14** (1947), 99–107.
2. R.H. Cameron and D.A. Storvick, *Feynman integral of variations of functionals*, in: Gaussian Random Fields, World Scientific, Singapore (1991), 144-157.
3. S.J. Chang, J.G. Choi and D. Skoug, *Integration by parts formulas involving generalized Fourier-Feynman transforms on function space*, Trans. Amer. Math. Soc. **355** (2003), 2925–2948.
4. S.J. Chang and H.S. Chung, *Generalized Fourier-Wiener function space transforms*, J. Korean Math. Soc. **46** (2009), 327-345.
5. S.J. Chang, H.S. Chung and D. Skoug, *Integral transforms of functionals in $L^2(C_{a,b}[0,T])$*, J. Fourier Anal. Appl. **15** (2009), 441-462.
6. S.J. Chang, H.S. Chung and D. Skoug, *Some basic relationships among transforms, convolution products, first variations and inverse transforms*, Cent. Eur. J. Math. **11** (2013), 538-551.
7. S.J. Chang and D. Skoug, *Generalized Fourier-Feynman transforms and a first variation on function space*, Integral Transforms Spec. Funct. **14** (2003), 375-393.
8. D.M. Chung, *Scale-invariant measurability in abstract Wiener spaces*, Pacific J. Math. **130** (1987), 27–40.
9. D.M. Chung, C. Park and D. Skoug, *Generalized Feynman integrals via conditional Feynamn integrals*, Michigan Math. J. **40** (1993), 337-391.
10. H.S. Chung, D. Skoug and S.J. Chang, *Relationships involving transform and convolutions via the translation theorem*, Stoch. Anal. Appl. **32** (2014), 348–363.
11. T. Huffman, D. Skoug and D. Storvick, *Integration formulas involving Fourier-Feynman transforms via a Fubini theorem*, J. Korean Math. Soc. **38** (2001), 421–435.
12. G.W. Johnson and D.L. Skoug, *Scale-invariant measurability in Wiener space*, Pacific J. Math. **83** (1979), 157–176.
13. B.S. Kim and D. Skoug, *Integral transforms of functionals in $L_2(C_0[0,T])$*, Rocky Mountain J. Math. **33** (2003), 1379-1393.
14. B.S. Kim and B.S. Kim, *Parts formulas involving integral transforms on function space*, Commun. Korean Math. Soc. **22** (2007), 553–564.
15. I.Y. Lee, H.S. Chung and S.J. Chang, *Relationships among the transform with respect to the Gaussian process, the $\diamond$-product and the first variation, of functionals on function space*, preprint.
16. I.Y. Lee, H.S. Chung and S.J. Chang, *Integration formulas involving the transform with respect to the Gaussian process on Wiener space*, preprint.
17. C. Park, D. Skoug and D.A. Storvick, *Relationships among the first variation, the convolution product and the Fourier-Feynman transform*, Rocky Mountain J. Math. **27** (1997), no. 3, 827–841.
18. J. Yeh, *Stochastic Processes and the Wiener Integral*, Marcel Dekker, Inc., New York, 1973.

Department of Mathematics, Dankook University, Cheonan 330-714, Korea.
E-mail address: iylee@dankook.ac.kr
E-mail address: hschung@dankook.ac.kr
E-mail address: sejchang@dankook.ac.kr