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p-QUASIPOSINORMAL COMPOSITION AND WEIGHTED COMPOSITION OPERATORS ON $L^2(\mu)$

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ABSTRACT. An operator T on a Hilbert space H is called p-quasiposinormal operator if $c^2T^*(T^*T)^pT \geq T^*(TT^*)^pT$ where $0 < p \leq 1$ and for some $c > 0$. In this paper, we have obtained conditions for composition and weighted composition operators to be p-quasiposinormal operators.

INTRODUCTION AND PRELIMINARIES

Let H be an infinite dimensional complex Hilbert space and $B(H)$ be the algebra of all bounded operators on H. An operator T is called p-quasiposinormal [6] if for some $c > 0$ and $0 < p \leq 1$, it satisfies the inequality

$$c^2T^*(T^*T)^pT \geq T^*(TT^*)^pT.$$

Let T be a measurable transformation on X. The composition operator C_T on the space $L^2(\mu)$ is given by

$$C_Tf = f \circ T \quad \text{for } f \in L^2(\mu)$$

Let ϕ be a complex-valued measurable function then the weighted composition operator $W_{\phi,T}$ on the space $L^2(\mu)$ induced by ϕ and T is given by

$$W_{\phi,T}f = \phi \cdot f \circ T \quad \text{for } f \in L^2(\mu)$$

In [1], G.Datt has described the conditions for the composition and weighted composition operators to be k-quasiposinormal operators. The aim of this paper is to study the p-quasiposinormal composition and p-quasiposinormal weighted composition operators and their corresponding adjoints in terms of Radon–Nikodym

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derivative and conditional expectation operators. The *Radon–Nikodym Theorem* and the *conditional expectation operators* defined on $L^2(\mu)$ and its properties play an important role. In the second section we have proved the conditions for composition operators to be p-quasiposinormal. In the third section we prove the same results for weighted composition operators.

1. Radon–Nikodym Theorem and Conditional expectation operator

Let $L^2(\mu) = L^2(X, \Sigma, \mu)$ be the space where (X, Σ, μ) is a σ-finite measure space. A transformation T is said to be measurable if $T^{-1}(A) \in \Sigma$ for $A \in \Sigma$. A measurable transformation T is said to be non-singular if

$$\mu(T^{-1}(A)) = 0 \quad \text{whenever } \mu(A) = 0 \quad \text{for every } A \in \Sigma\,.$$

If T is non-singular, then we say that μT^{-1} is absolutely continuous with respect to μ. Hence, by Radon–Nikodym theorem there exists a unique non-negative measurable function h such that

$$(\mu T^{-1})(A) = \int_A h d\mu \quad \text{for } A \in \Sigma.$$

The non-negative measure function h is called the Radon–Nikodym derivative and is denoted by $\dfrac{d\mu T^{-1}}{d\mu}$. We always assume that h is almost everywhere finite-valued or equivalently $T^{-1}(\Sigma) \subset \Sigma$ is a sub-sigma finite algebra.

The conditional expectation operator $E(\,\cdot \mid T^{-1}(\Sigma)) = E(f)$ is defined for each non-negative function f in L^p $(1 \le p < \infty)$ and is uniquely determined by the following set of conditions:

(1) E(f) is $T^{-1}(\Sigma)$ measurable.
(2) If A is any $T^{-1}(\Sigma)$ measurable set for which $\int_A f\,d\mu$ converges then we have

$$\int_A f\,d\mu = \int_A E(f)\,d\mu.$$

The conditional expectation operator E has the following properties:

(1) $E(f \cdot g \circ T) = (E(f))(g \circ T)$.
(2) E is monotonically increasing, i.e., if $f \le g$ a.e. then

$$E(f) \le E(g) \ \text{ a.e.}$$

(3) $E(1) = 1$.
(4) $E(f)$ has the form $E(f) = g \circ T$.

for exactly one Σ-measurable function g provided that the support of g lies in the support of h which is given by

$$\sigma(h) = \{x : h(x) \ne 0\}.$$

As an operator on L^p, E is the projection operator onto the closure of the range of the composition operator C_T. This operator plays an important role in the study of composition and weighted composition operators on various Banach function spaces [4, 5, 7].

2. Composition Operators

Let (X, Σ, μ) be a σ-finite measure space and C_T be the composition operator induced by the measurable transformation T on $L^2(\mu)$.

The adjoint C_T^* is given by $C_T^* f = hE(f) \circ T^{-1}$ for f in $L^2(\mu)$.

The following lemma [2, 7] is intrumental in proving the subsequent result.

Lemma 2.1. *Let P be the projection of $L^2(X, \Sigma, \mu)$ onto $\overline{R(C_T)}$. Then*

(1) $C_T^* C_T f = hf$ *and* $C_T C_T^* f = (h \circ T)Pf \ \forall \ f \in L^2(\mu)$.
(2) $\overline{R(C_T)} = \{f \in L^2(\mu) : f$ *is* $T^{-1}(\Sigma)$ *measurable*$\}$.
(3) *If f is $T^{-1}(\Sigma)$ measurable and g and fg belong to $L^2(\mu)$, then $P(fg) = fP(g)$, (f need not be in $L^2(\mu)$).*

Proposition 2.2. *For $0 < p \leq 1$,*

(1) $(C_T^* C_T)^p f = h^p f$.
(2) $(C_T C_T^*)^p f = (h \circ T)^p P(f)$.
(3) *E is the identity operator on $L^2(\mu)$ if and only if $T^{-1}(\Sigma) = \Sigma$.*

The following theorem characterizes the p-quasiposinormal composition operators.

Theorem 2.3. *If C_T be the composition operator induced by T on $L^2(\mu)$. Then the following statements are equivalent*:

(1) *C_T is p-quasiposinormal.*
(2) *$c^2 hE(h^p) \geq hE((h \circ T)^p)$ where $0 < p \leq 1$ and for some $c > 0$.*

Proof. For $f \in L^2(\mu)$,

$$\begin{aligned} C_T^*(C_T^* C_T)^p C_T f &= C_T^*(C_T^* C_T)^p f \circ T \\ &= C_T^*(h^p \cdot f \circ T) \\ &= hE(h^p \cdot f \circ T) \circ T^{-1}. \end{aligned}$$

Also,

$$\begin{aligned} C_T^*(C_T C_T^*)^p C_T f &= C_T^*(C_T C_T^*)^p f \circ T \\ &= C_T^*((h \circ T)^p E(f \circ T)) \\ &= hE((h \circ T)^p E(f \circ T)) \circ T^{-1}. \end{aligned}$$

If C_T is a p-quasiposinormal, then

$$\begin{aligned}
& c^2 C_T^*(C_T^* C_T)^p C_T \geq C_T^*(C_T C_T^*)^p C_T \quad \text{for some } c > 0 \\
\Leftrightarrow\ & c^2 hE(h^p \cdot f \circ T) \circ T^{-1} \geq hE((h \circ T)^p E(f \circ T)) \circ T^{-1} \\
\Leftrightarrow\ & c^2 hE(h^p) \cdot f \circ T \geq hE(h \circ T)^p f \circ T \\
\Leftrightarrow\ & c^2 hE(h^p) g \geq hE((h \circ T)^p) g \quad \text{where } g = f \circ T \in L^2 \\
\Leftrightarrow\ & c^2 hE(h^p) \geq hE(h \circ T)^p.
\end{aligned}$$

□

Corollary 2.4. *If $T^{-1}(\Sigma) = \Sigma$. Then the following statements are equivalent:*

(1) *C_T is p-quasiposinormal.*
(2) *$c^2 h^{p+1} \geq h(h \circ T)^p$ where $0 < p \leq 1$ and for some $c > 0$.*

Proof. Result follows from the Theorem 2.3 and the fact that E is the identity operator. □

The following theorem gives us an equivalent condition for the adjoint of composition operator to be p-quasiposinormal.

Theorem 2.5. *If C_T be a composition operator on $L^2(\mu)$. Then the following statements are equivalent:*

(1) *C_T^* is p-quasiposinormal.*
(2) *$h^{p+1} \leq c^2 h^p \circ TE(h)$ where $0 < p \leq 1$ and for some $c > 0$.*

Proof. For every $f \in L^2(\mu)$,

$$\begin{aligned}
C_T(C_T^* C_T)^p C_T^* f &= C_T(C_T^* C_T)^p (hE(f) \circ T^{-1}) \\
&= C_T(h^p \cdot hE(f) \circ T^{-1}) \\
&= (h^{p+1} E(f) \circ T^{-1}) \circ T
\end{aligned}$$

and

$$\begin{aligned}
C_T(C_T C_T^*)^p C_T^* f &= C_T(C_T C_T^*)^p (hE(f) \circ T^{-1}) \\
&= C_T((h \circ T)^p \cdot E(hE(f) \circ T^{-1})) \\
&= ((h \circ T)^p \cdot E(hE(f) \circ T^{-1})) \circ T.
\end{aligned}$$

Thus, if C_T^* is p-quasiposinormal then

$$\langle (C_T(C_T^* C_T)^p C_T^* - c^2 C_T(C_T C_T^*)^p C_T^*) f, f \rangle \leq 0$$

Let $f = \chi_{T^{-1}(A)}$ with $\mu(T^{-1}(A)) < \infty$ and $E(\chi_{T^{-1}(A)}) \circ T^{-1} = E(\chi_A \circ T) \circ T^{-1} = \chi_A$ on $\sigma(h)$, therefore

$$\int_{T^{-1}(A)} (h^{p+1} \circ TE(\chi_{T^{-1}(A)}) - c^2 h^p \circ T^2 \cdot (E(hE(\chi_{T^{-1}A}) \circ T^{-1}) \circ T)) d\mu \leq 0$$

$$\Leftrightarrow \int ((h^{p+1}E(\chi_{T^{-1}(A)}) \circ T^{-1} - c^2h^p \circ T \cdot E(hE(\chi_{T^{-1}A}) \circ T^{-1}) \circ T \circ T^{-1})d\mu T^{-1} \leq 0$$

$$\Leftrightarrow \int (h^{p+1}\chi_A - c^2h^p \circ T \cdot E(h\chi_A))hd\mu \leq 0$$

$$\Leftrightarrow \int (h^{p+1} - c^2h^p \circ T \cdot E(h))hd\mu \leq 0$$

$$\Leftrightarrow h^{p+1} \leq c^2h^p \circ T \cdot E(h).$$

□

Corollary 2.6. *If $T^{-1}(\Sigma) = \Sigma$. Then the following statements are equivalent:*

(1) *C_T^* is p-quasiposinormal.*
(2) *$h^{p+1} \leq c^2h^p \circ T \cdot h$ where $0 < p \leq 1$ and for some $c > 0$.*

Proof. Since $T^{-1}(\Sigma) = \Sigma$ then $E = I$ and hence the result follows. □

3. Weighted Composition Operators

Let (X, Σ, μ) be a σ-finite measure space and $W \equiv W_{\phi,T}$ be the weighted composition operator on $L^2(\mu)$ induced by the complex valued function ϕ and a measurable transformation T. Define

$$J = hE(|\phi|^2) \circ T^{-1}.$$

In [2, 7], it has been shown that W is bounded on $L^p(\mu)$ for $1 \leq p < \infty$ if and only if J $\in L^\infty(\mu)$.

The adjoint W^* is given by $W^*f = h \cdot E(\phi f) \circ T^{-1}$ for f in $L^2(\mu)$.

Also,

$$\begin{aligned}(W^*W)f = W^*(Wf) &= W^*(\phi \cdot f \circ T)\\ &= h \cdot E(\phi \cdot f \circ T) \circ T^{-1}\\ &= h \cdot E(\phi^2) \circ T^{-1}f.\\ (W^*W)^pf &= h^p \cdot [E(\phi^2)]^p \circ T^{-1}f\\ &= J^pf.\end{aligned}$$

The following lemma [3] is instrumental in proving the next theorem.

Lemma 3.1. *Let $f \in L^2(\mu)$ and $(WW^*)f = \phi(h \circ T)E(\phi f)$. Then for all $p \in (0, \infty)$,*

$$(WW^*)^pf = \phi(h^p \circ T)[E(\phi^2)]^{p-1}E(\phi f).$$

In the following theorem, an equivalent condition for the weighted composition operator to be p-quasiposinormal has been obtained in terms of Radon–Nikodym derivative h and the function J.

Theorem 3.2. *If W be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:*

(1) *W is p-quasiposinormal.*
(2) *$c^2hE(\phi^2J^p) \geq h^{p+1}[E(\phi^2)]^{p+1}$ where $0 < p \leq 1$ and for some $c > 0$.*

Proof. Using the properties of conditional expectation operator E and for every $f \in L^2(\mu)$,

$$\begin{aligned} W^*(WW^*)^p Wf &= W^*(WW^*)^p(\phi \cdot f \circ T) \\ &= W^*(\phi(h^p \circ T)[E(\phi^2)]^{p-1} E(\phi\phi \cdot f \circ T)) \\ &= W^*(\phi(h^p \circ T)[E(\phi^2)]^{p-1} E(\phi^2) \cdot f \circ T) \\ &= W^*(\phi(h^p \circ T)[E(\phi^2)]^{p} \cdot f \circ T) \\ &= h \cdot E(\phi^2(h^p \circ T)[E(\phi^2)]^{p} \cdot f \circ T) \circ T^{-1} \\ &= h \cdot h^p \circ T[E(\phi^2)]^{p+1} f \circ T \circ T^{-1} \\ &= h \cdot h^p[E(\phi^2)]^{p+1} \circ T^{-1} f \\ &= h^{p+1}[E(\phi^2)]^{p+1} \circ T^{-1} f \end{aligned}$$

and

$$\begin{aligned} W^*(W^*W)^p Wf &= W^*(W^*W)^p(\phi \cdot f \circ T) \\ &= W^*(J^p\phi \cdot f \circ T) \\ &= h \cdot E(\phi^2 J^p \cdot f \circ T) \circ T^{-1} \\ &= h \cdot E(\phi^2 J^p) \circ T^{-1} f. \end{aligned}$$

Now, W is p-quasiposinormal if and only if

$$\begin{aligned} & c^2 W^*(W^*W)^p W \geq W^*(WW^*)^p W \\ \Leftrightarrow \quad & c^2 h[E(\phi^2 J^p)] \circ T^{-1} \geq h^{p+1} \circ T[E(\phi^2)]^{p+1} \circ T^{-1} \\ \Leftrightarrow \quad & c^2 h[E(\phi^2 J^p)] \geq h^{p+1}[E(\phi^2)]^{p+1}. \end{aligned}$$

□

An equivalent condition for the adjoint of weighted composition operator to become p-quasiposinormal has been derived in the following theorem:

Theorem 3.3. *If W be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:*

(1) *W^* is p-quasiposinormal.*
(2) $J^p h E(\phi f) \leq c^2 \phi(h^p \circ T)[E(\phi^2)]^{p-1} E(\phi h E(\phi f) \circ T^{-1}) \circ T.$

Proof. The proof is along the similar lines as in the preceding theorem. □

Example 3.4. Let w $= < w_n >_{n=1}^{\infty}$ be a sequence of positive real numbers. Consider the weighted Banach space $l^2(w)$ with X $= \mathbb{N}$ and μ is a measure given by $\mu(n) = w_{n+r}$ for a fixed natural number r. Let T be a measurable transformation given by $T(n) = n + r$ for all $n \in \mathbb{N}$. We note that $\mu \circ T$ is absolutely continuous with respect to μ. Also, Let $< \phi(n) >$ be a sequence of non-negative real numbers given by

$$\phi(n) = \begin{cases} \frac{1}{2^n}, & \text{if n is even} \\ 0, & \text{otherwise} \end{cases}$$

Direct computations shows that

$$h(k) = \frac{\sum_{j \in T^{-1}(k)} m_j}{m_{k+r}}$$

$$E(f)(k) = \frac{\sum_{j \in T^{-1}(T(k))} f_j m_j}{\sum_{j \in T^{-1}(T(k))} m_j}$$

for all non-negative sequence $f =< f_n >_{n=1}^{\infty}$ and $k \in \mathbb{N}$.

By Theorem 2.3, C_T is p-quasiposinormal if and only if

$$c^2 \sum_{j \in T^{-1}(T(k))} (h(j))^p m_j \geq \sum_{j \in T^{-1}(T(k))} (h(T(j)))^p m_j.$$

By Theorem 3.2, W is p-quasiposinormal if and only if

$$c^2 \sum_{j \in T^{-1}(T(k))} ((\frac{1}{2^n})^2 (J(j)))^p m_j \geq \{\frac{\sum_{j \in T^{-1}(T(k))} (\frac{1}{2^n})^2 m_j}{m_{T(k)}}\}^{p+1} \frac{1}{m_{T(k)}^{p-2}} (\sum_{j \in T^{-1}(T(k))} m_j)^{p-1}.$$

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