Annals of Functional Analysis

Ann. Funct. Anal. 6 (2015), no. 1, 221–226
http://doi.org/10.15352/afa/06-1-16
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

dist-FORMULAS AND TOEPLITZ OPERATORS

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Communicated by J. Esterle

ABSTRACT. The distance from the nonconstant function φ in $L^\infty(\mathbb{T})$ to the set $\mathcal{F}_{\text{const}}$ of all constant functions is estimated in terms of Hankel operators on the Hardy space $H^2(\mathbb{D})$ over the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We give a sufficient condition ensuring the equality $dist(\varphi, \mathcal{F}_{\text{const}}) = \|\varphi\|_{L^\infty}$. Some other *dist*-formulas are also discussed.

1. Introduction and preliminaries

Let $L^\infty = L^\infty(\mathbb{T})$ denote the Lebesgue-Banach space of all essentially bounded functions f on the unit circle $\mathbb{T} := \{\xi \in \mathbb{C} : |\xi| = 1\}$ with the finite norm $\|f\|_\infty := ess-\sup_{\xi\in\mathbb{T}} |f(\xi)| < +\infty$. Recall also that the norm of a bounded linear operator A on a Banach space X is defined as $\|A\| := \sup_{x\neq 0} \frac{\|A(x)\|}{\|x\|} < \infty$.

In the present article we estimate in terms of Hankel operators the distance from any nonconstant essentially bounded function φ on the unit circle $\mathbb{T}$ to the set $\mathcal{F}_{\text{const}}$ of all constant functions (see Section 2). We also investigate the equality $dist(\varphi, \mathcal{F}_{\text{const}}) = \|\varphi\|_{L^\infty}$. Some other results related with Toeplitz operators are also obtained.

Recall that a derivation on a Banach algebra $\mathcal{B}$ is a linear transformation $\mathcal{D} : \mathcal{B} \to \mathcal{B}$ which satisfies

$$\mathcal{D}(ab) = a\mathcal{D}(b) + \mathcal{D}(a)\, b$$

for all $a,\, b \in \mathcal{B}$. If for a fixed a, $\mathcal{D}_a : b \to ab - ba$, then $\mathcal{D}_a$ is called an inner derivation. It is well known that every derivation on a von Neumann algebra or on a simple C^*-algebra is inner (see Kadison [4], Sakai [8, 9]). Obviously,

Date: Received: Mar. 15, 2013; Revised: Jul. 24, 2013; Accepted: Dec. 17, 2013.

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2010 *Mathematics Subject Classification.* Primary 47B35; Secondary 47B10.

Key words and phrases. Hardy space, Hankel operator, Toeplitz operator, Maximal numerical range, *dist*-formula.

$\|\mathcal{D}_a\| \leq 2\|a\|$. Stampfli proved that ([10], Theorem 4) if $\mathcal{D}_T$ is an inner derivation on $\mathcal{B}(H)$ (the Banach algebra of all bounded linear operators on a Hilbert space H), then $\|\mathcal{D}_T\| = 2dist(T, \mathbb{C}I)$, where $\mathbb{C}I$ denotes the set of all scalar operators λI ($\lambda \in \mathbb{C}$) on H. Stampfli also proved in terms of so-called "maximal numerical range" of T that $\|\mathcal{D}_T\| = 2\|T\|$ if and only if $0 \in W_0(T)$ see ([10], Theorem 4); here

$$W_0(T) := \{\lambda : \langle Tx_n, x_n\rangle \to \lambda \text{ where } \|x_n\| = 1 \text{ and } \|Tx_n\| \to \|T\|\}$$

is the maximal numerical range of operator T.

Before giving our results, let us introduce some necessary definitions and notations. The Hardy space $H^2 = H^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$\|f\|_2^2 := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{it})\right|^2 dt < +\infty.$$

The symbol $H^\infty = H^\infty(\mathbb{D})$ denotes the Banach algebra of functions bounded and analytic on the unit disc $\mathbb{D}$ equipped with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. A function $\theta \in H^\infty$ such that $|\theta(\xi)| = 1$ almost everywhere in the unit circle $\mathbb{T}$ is called an inner function. It is convenient to establish a natural embedding of the space H^2 in the space $L^2 = L^2(\mathbb{T})$ by associating to each function $f \in H^2$ its radial boundary values $(bf)(\xi) := \lim_{r\to 1^-} f(r\xi)$, which (by the Fatou Theorem [3]) exist for m-almost all $\xi \in \mathbb{T}$; where m is the normalized Lebesgue measure on $\mathbb{T}$. Then we have

$$H^2 = \left\{f \in L^2 : \hat{f}(n) = 0, n < 0\right\},$$

where $\hat{f}(n) := \int_{\mathbb{T}} \bar{\xi}^n f(\xi) dm(\xi)$ is the Fourier coefficient of the function f. We denote

$$H_-^2 = \left\{f \in L^2 : \hat{f}(n) = 0, n > 0\right\}.$$

For $\varphi \in L^\infty = L^\infty(\mathbb{T})$, the Toeplitz operator T_φ with symbol φ is the operator on H^2 defined by $T_\varphi f = P_+(\varphi f)$; here P_+ is the orthogonal projection from $L^2(\mathbb{T})$ onto H^2. The Hankel operator H_φ is defined by $H_\varphi f = P_-(\varphi f)$, $f \in H^2$, where $P_- := I - P_+$.

Clearly, when $T = T_\varphi$, the Toeplitz operator defined on $H^2(\mathbb{D})$ (Hardy space) by $T_\varphi f = P_+\varphi f$, Stampfli's result mentioned above "$\left\|\mathcal{D}_{T_\varphi}\right\| = 2\|T_\varphi\| \Leftrightarrow 0 \in W_0(T_\varphi)$" is equivalent to "$dist(\varphi, \mathcal{F}_{\text{const}}) = \|\varphi\|_{L^\infty} \Leftrightarrow 0 \in W_0(T_\varphi)$" (because it is easy to see that $dist(T_\varphi, \mathbb{C}I) = dist(\varphi, \mathcal{F}_{\text{const}})$). In this article we give another sufficient condition under which $dist(\varphi, \mathcal{F}_{\text{const}}) = \|\varphi\|_{L^\infty}$; namely, we prove that if $\varphi \in L^\infty$ and

$$\max\left\{\|H_\varphi\|, \sup_{\theta\in(\Sigma)} \left\|H_{\overline{\varphi}}^* H_{\overline{\theta}}\right\|\right\} = \|\varphi\|_{L^\infty},$$

then $dist(\varphi, \mathcal{F}_{\text{const}}) = \|\varphi\|_{L^\infty}$, where (Σ) denotes the set of all scalar inner functions.

2. Distance estimates from H^∞-functions and operators

Our main result is the following.

Theorem 2.1. *Let $\varphi \in L^\infty$ be any nonconstant function. Then we have:*

$$\max\left\{\sup_{\theta\in(\Sigma)}\left\|H^*_{\overline{\varphi}}H_{\overline{\theta}}\right\|, \left\|H_\varphi\right\|\right\} \leq dist\left(\varphi, \mathcal{F}_{const}\right) \leq \left\|\varphi\right\|_{L^\infty}.$$

Therefore, if the function φ satisfies $\max\left\{\sup_{\theta\in(\Sigma)}\left\|H^*_{\overline{\varphi}}H_{\overline{\theta}}\right\|, \left\|H_\varphi\right\|\right\} = \left\|\varphi\right\|_{L^\infty}$, *then*

$$dist\left(\varphi, \mathcal{F}_{const}\right) = \left\|\varphi\right\|_{L^\infty}.$$

Proof. By the well known Nehari formula (see [7])

$$\left\|H_\varphi\right\| = dist\left(\varphi, H^\infty\right). \tag{2.1}$$

Then by using formula (2.1) we have:

$$\begin{aligned}\left\|H_\varphi\right\| = dist\left(\varphi, H^\infty\right) &= \inf_{h\in H^\infty}\left\|\varphi - h\right\| \\ &\leq \inf_{\lambda\in\mathbb{C}}\left\|\varphi-\lambda\right\|_{L^\infty} = dist\left(\varphi, \mathcal{F}_{\text{const}}\right),\end{aligned}$$

thus

$$\left\|H_\varphi\right\| \leq dist\left(\varphi, \mathcal{F}_{\text{const}}\right) \leq \left\|\varphi\right\|_{L^\infty}. \tag{2.2}$$

(Note that the first inequality in (2.2) can be also be proved without using of formula (2.1) as follows: for any complex number λ, since $H_\lambda = 0$,

$$\left\|\varphi-\lambda\right\|_{L^\infty} \geq \left\|H_{\varphi-\lambda}\right\| = \left\|H_\varphi - H_\lambda\right\| = \left\|H_\varphi\right\|.$$

Taking infimum with respect to λ, we obtain the desired inequality.)

On the other hand, for any inner function θ, since $\left\|H_{\overline{\theta}}\right\| \leq \left\|\overline{\theta}\right\|_{L^\infty} = 1$, by using the first inequality in (2.2), we have

$$\left\|H^*_{\overline{\varphi}}H_{\overline{\theta}}\right\| \leq \left\|H^*_{\overline{\varphi}}\right\|\left\|H_{\overline{\theta}}\right\| \leq \left\|H_{\overline{\varphi}}\right\| \leq dist\left(\overline{\varphi}, \mathcal{F}_{\text{const}}\right) = dist\left(\varphi, \mathcal{F}_{\text{const}}\right)$$

and thus

$$\sup_{\theta\in(\Sigma)}\left\|H^*_{\overline{\varphi}}H_{\overline{\theta}}\right\| \leq dist\left(\varphi, \mathcal{F}_{\text{const}}\right). \tag{2.3}$$

Now the desired result is immediate from (2.2) and (2.2).

Now it is clear from (2.2) and (2.2) that if φ is a function such that

$$\max\left\{\sup_{\theta\in(\Sigma)}\left\|H^*_{\overline{\varphi}}H_{\overline{\theta}}\right\|, \left\|H_\varphi\right\|\right\} = \left\|\varphi\right\|_{L^\infty},$$

then

$$dist\left(\varphi, \mathcal{F}_{\text{const}}\right) = \left\|\varphi\right\|_{L^\infty}.$$

Thus, the theorem has been proved. □

For the proof of the following lemma we are indebted to Mustafaev.

Lemma 2.2. *Let H be a Hilbert space, $\mathcal{N}$ the set of all nilpotent operators on H, and $\{\mathcal{N}\}'$ be the commutant of the set $\mathcal{N}$. Then $\{\mathcal{N}\}' = \{\lambda I : \lambda \in \mathbb{C}\} := \mathbb{C}I$, i.e., the set $\{\mathcal{N}\}'$ consist from the scalar operators.*

Proof. Let $x \in H$, and $y \in H$ be a nonzero vector which is orthogonal to x, that is $< x, y >= 0$. Let us consider the one-dimensional operator $x \circ y$ on H defined by $(x \circ y)(z) :=< z, y > x$. Then $x \circ y$ is a nilpotent operator with nilpotency degree 2. Let $T \in \{\mathcal{N}\}^{'}$ be an arbitrary operator. Then, in particular, T commutes with $x \circ y$. Then we can write $T(x \circ y) = (x \circ y)T$ which is equivalent to $Tx \circ y = x \circ T^{*}y$. This implies that $< z, y > Tx =< z, T^{*}y > x$ for all z in H. By taking in this identity $z = y$, we see that $< y, y > Tx =< Ty, y > x$. It follows that $Tx = (\frac{<Ty,y>}{<y,y>})\ x$, which means that T is a scalar operator. This proves the lemma. $\square$

Corollary 2.3. *Let $\mathcal{N}$ be the set of all nilpotent operators on H^2, and $\{\mathcal{N}\}^{'} := \{A \in \mathcal{B}(H^2) : AN = NA \text{ for all } N \in \mathcal{N}\}$ be the commutant of the set $\mathcal{N}$. Then*

$$dist(T_{\varphi}, \{\mathcal{N}\}^{'}) \geq \max \left\{ \sup_{\theta \in (\Sigma)} \left\| H_{\overline{\varphi}}^{*} H_{\overline{\theta}} \right\|, \left\| H_{\varphi} \right\| \right\}.$$

The proof of this corollary is immediate from Theorem 2.1 and Lemma 2.2.

Now, we will separately consider the particular cases $\varphi \in H^{\infty}$ and $\Phi \in H^{\infty}(E \to E)$, and will demonstrate the roles of another *dist*-formulas (belonging to Davidson [1], Mustafaev [5] and Mustafaev and Shulman [6]) in estimating $dist\,(\varphi, \mathcal{F}_{\text{const}})$ and $dist(T_{\Phi}, \{\mathcal{N}\}^{'})$ for some suitable algebra $\mathcal{N}$ of operators on $H^2(E)$.

Let E and E_* be separable Hilbert spaces, $H^2(E)$ the vector-valued Hardy space with values in $E, L^{\infty}(E \to E_*)$ the class of bounded functions on the unit circle $\mathbb{T}$ whose values are bounded operators from E to E_*, and let $H^{\infty}(E \to E_*)$ be an operator Hardy class of bounded analytic functions whose values are bounded operators F from E to E_* with

$$\|F\|_{\infty} := \sup_{z \in \mathbb{D}} \|F(z)\| = ess \sup_{\xi \in \mathbb{T}} \|F(\xi)\| < +\infty.$$

The Toeplitz operator T_{Φ} with symbol $\Phi \in L^{\infty}(E \to E)$ is defined as

$$T_{\Phi} f := P_{+}(\Phi f),$$

where P_{+} is an orthogonal projection of $L^2(E)$ onto $H^2(E)$. The function Θ, $\Theta \in H^{\infty}(E \to E)$, is called inner operator-function (or two sided inner function in sense of Sz.-Nagy and Foias [11]) if its angular limiting values $\Theta(e^{it})$ are unitary operators in E for almost all $t \in [0, 2\pi]$.

The Hankel operator $H_{\Phi}, H_{\Phi} : H^2(E) \to H^2_{-}(E)$ is defined by

$$H_{\Phi} f := P_{-}(\Phi f),$$

where $P_{-} := I - P_{+}$.

Any inner operator-function $\Theta \in H^{\infty}(E \to E)$ determines the following square-zero operator on the vectorial Hardy space $H^2(E)$:

$$N_{\Theta} := T_{\Theta} P_{\Theta},$$

where $P_{\Theta} := I - T_{\Theta} T_{\Theta}^{*} : H^2(E) \to H^2(E) \ominus \Theta H^2(E)$ is the orthogonal projection of $H^2(E)$ onto $\mathcal{K}_{\Theta} := H^2(E) \ominus \Theta H^2(E)$. Clearly, $\|N_{\Theta}\| = 1$, because T_{Θ} is an isometry.

Theorem 2.4. *Let $\varphi \in H^\infty$ be any nonconstant function. Then*

$$\sup_{\theta\in(\Sigma)} \left\|H^*_{\overline{\varphi}} H_{\overline{\theta}}\right\| \le dist\,(\varphi, \mathcal{F}_{const}) \le \|\varphi\|_{L^\infty}\,. \tag{2.4}$$

Proof. Since the zero function is in $\mathcal{F}_{\text{const}}$, the right inequality is obvious. Let us prove the left inequality. For this purpose, for any inner function θ let us consider the orthogonal projection $P_{\theta H^2}$ onto subspace θH^2. By considering that the analytic Toeplitz operator T_θ is isometry, $T_\varphi T_\theta = T_\theta T_\varphi$ and $T_{\varphi\overline{\theta}} - T_\varphi T_{\overline{\theta}} = H^*_{\overline{\varphi}} H_{\overline{\theta}}$ (see, for instance, Douglas [2] and Nikolski [7]), we have:

$$\begin{aligned}
\|T_\varphi P_{\theta H^2} - P_{\theta H^2} T_\varphi\| &= \|T_\varphi T_\theta T^*_\theta - T_\theta T^*_\theta T_\varphi\| \\
&= \|T_\theta (T_\varphi T_{\overline{\theta}} - T_{\overline{\theta}} T_\varphi)\| \\
&= \left\|T_\varphi T_{\overline{\theta}} - T_{\overline{\theta}\varphi}\right\| \\
&= \left\|T_{\varphi\overline{\theta}} - T_\varphi T_{\overline{\theta}}\right\| \\
&= \left\|H^*_{\overline{\varphi}} H_{\overline{\theta}}\right\|
\end{aligned}$$

From this, by using the formula (which is due to Mustafaev [5])

$$dist(T_\varphi, \mathbb{C}I) = \sup_{P\in\mathcal{P}} \|T_\varphi P - PT_\varphi\|\,,$$

where $\mathcal{P}$ denotes the set of all orthogonal projections of the space H^2, we obtain that

$$\begin{aligned}
\sup_{\theta\in(\Sigma)} \left\|H^*_{\overline{\varphi}} H_{\overline{\theta}}\right\| &= \sup_{\theta\in(\Sigma)} \|T_\varphi P_{\theta H^2} - P_{\theta H^2} T_\varphi\| \\
&\le \sup_{p\in\mathcal{P}} \|T_\varphi P - PT_\varphi\| \\
&= dist(T_\varphi, \mathbb{C}I) = dist(\varphi, \mathcal{F}_{\text{const}}),
\end{aligned}$$

which proves the theorem. □

Our next result estimates the distance from any Toeplitz operator T_Φ with $\Phi \in H^\infty(E \to E)$ to some algebra $\mathcal{N}$ of operators on the space $H^2(E)$, which apparently is new even in the case $\dim E = 1$. In what follows the symbol (Σ_{inn}) will denote the set of all inner operator-functions.

Theorem 2.5. *Let $\Phi \in H^\infty(E \to E)$ be an operator function, and let $\mathcal{N}$ be an algebra of operators on the space $H^2(E)$ such that $T_\Phi \notin \overline{\mathcal{N}}^u$ (where $\overline{\mathcal{N}}^u$ denotes the uniform closure of the operator algebra $\mathcal{N}$) and $N_\Theta \in \{\mathcal{N}\}'$ for some inner operator function Θ. If $\Phi\Theta = \Theta\Phi$, then*

$$dist(T_\Phi, \mathcal{N}) \ge \frac{1}{2} \|H^*_{\Phi^*} H_{\Theta^*}\|\,.$$

Proof. We will use the following known estimate, which is due to Davidson ([1], Lemma 3) and Mustafaev and Shulman [6]:

$$\sup_{B\in\mathcal{T}',\|B\|\le 1} \|AB - BA\| \le 2dist(A, \mathcal{T}), \tag{2.5}$$

where $\mathcal{T}$ is an algebra of operators, $\mathcal{T}'$ is its commutant and $A \in \mathcal{B}(H)$ is any operator.

Indeed, since $N_\Theta \in \{\mathcal{N}\}'$ for some $\Theta \in (\Sigma_{inn})$, by considering inequality (2.5) and the condition $\Phi\Theta = \Theta\Phi$, we obtain:

$$\begin{aligned}
2dist(T_\Phi, \mathcal{N}) &\geq \sup_{N\in\{\mathcal{N}\}'} \|T_\Phi N - NT_\Phi\| \\
&\geq \|T_\Phi N_\Theta - N_\Theta T_\Phi\| \\
&= \|T_\Phi T_\Theta P_\Theta - T_\Theta P_\Theta T_\Phi\| \\
&= \|T_\Phi T_\Theta (I - T_\Theta T_\Theta^*) - T_\Theta (I - T_\Theta T_\Theta^*) T_\Phi\| \\
&= \left\|T_\Phi T_\Theta - T_\Theta T_\Phi + T_\Theta^2 (T_\Theta^* T_\Phi - T_\Phi T_\Theta^*\right\| \\
&= \left\|T_{\Phi\Theta} - T_{\Theta\Phi} + T_\Theta^2 (T_\Theta^* T_\Phi - T_\Phi T_\Theta^*\right\| \\
&= \left\|T_{\Phi\Theta - \Theta\Phi} + T_\Theta^2 (T_\Theta^* T_\Phi - T_\Phi T_\Theta^*\right\| \\
&= \|T_{\Theta^*\Phi} - T_\Phi T_{\Theta^*}\| \\
&= \|H_{\Phi^*}^* H_{\Theta^*}\| .
\end{aligned}$$

Here we have used the known formula $T_{\Theta^*\Phi} - T_\Phi T_{\Theta^*} = H_{\Phi^*}^* H_{\Theta^*}$.The theorem is proved. □

Acknowledgement. The authors would like to thank the referees for giving useful comments and suggestions for the improvement of this paper.

This work supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.

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