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SOME mTH-ORDER DIFFERENCE SEQUENCE SPACES OF GENERALIZED MEANS AND COMPACT OPERATORS

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Abstract. In this paper, new sequence spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$ defined by using generalized means and difference operator of order m are introduced. It is shown that these spaces are complete normed linear spaces and the spaces $c_0(r,s,t;\Delta^{(m)})$, $c(r,s,t;\Delta^{(m)})$ have Schauder basis. Furthermore, the α-, β-, γ- duals of these spaces are computed and also obtained necessary and sufficient conditions for some matrix transformations from $X(r,s,t;\Delta^{(m)})$ to X. Finally, some classes of compact operators on the spaces $c_0(r,s,t;\Delta^{(m)})$ and $l_\infty(r,s,t;\Delta^{(m)})$ are characterized by using the Hausdorff measure of .

1. Introduction

The study of sequence spaces has importance in the several branches of analysis, namely, the structural theory of topological vector spaces, summability theory, Schauder basis theory etc. In addition, the theory of sequence spaces is a powerful tool for obtaining some topological and geometrical results using Schauder basis.

Let w be the space of all real or complex sequences $x = (x_n)$, where $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. For an infinite matrix A and a sequence space λ, the matrix domain of A, which is denoted by λ_A and defined as $\lambda_A = \{x \in w : Ax \in \lambda\}$ [22]. Basic methods, which are used to determine the topologies, matrix transformations and inclusion relations on sequence spaces can also be applied to study the matrix domain λ_A. In recent times, there is an approach of forming new sequence spaces by using matrix domain of a suitable matrix and characterize the matrix mappings between these sequence spaces.

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Kizmaz[12] first introduced and studied the difference sequence space. Later on, several authors including Ahmad and Mursaleen [1], Çolak and Et [7], Başar and Altay [2], Polat and Başar [19], Aydin and Başar [4] and others have introduced and studied new sequence spaces defined by using difference operator.

On the other hand, sequence spaces are also defined by using generalized weighted mean. Some of them can be viewed in Malkowsky and Savaş [14], Altay and Başar [3]. Mursaleen and Noman [18] also introduced a sequence space of generalized means, which includes most of the earlier known sequence spaces. But till 2011, there was no such literature available in which a sequence space is generated by combining both the weighted mean and the difference operator. This was first initiated by Polat et al. [20]. Later on, Başarir and Kara [5] generalized the sequence spaces of Polat et al. [20] to an mth-order difference sequence spaces $X(u, v; \Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$ which are defined as

$$X(u, v; \Delta^{(m)}) = \Big\{x = (x_n) \in w : \big((G(u, v).\Delta^{(m)}x)_n\big) \in X\Big\},$$

where $u = (u_n), v = (v_n) \in w$ such that $u_n, v_n \neq 0$ for all n, $\Delta^{(m)} = \Delta^{(m-1)} \circ \Delta^{(1)}$ for $m \in \mathbb{N} = \{1, 2, \dots\}$ and the matrices $G(u, v) = (g_{nk})$, $\Delta^{(1)} = (\delta_{nk})$ are defined by

$$g_{nk} = \begin{cases} u_n v_k, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \qquad \delta_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

for all $n, k \in \mathbb{N}_0$, respectively. For some results related to generalized weighted mean and difference operator one can see [6] and [11].

The aim of this present paper is to introduce new sequence spaces defined by using both the generalized means and the difference operator of order m. We investigate some topological properties as well as the α-, β-, γ- duals and bases of the new sequence spaces are obtained. We also characterize some matrix mappings between these new sequence spaces. Moreover, we give the characterization of some classes of compact operators on the spaces $c_0(r, s, t; \Delta^{(m)})$ and $l_\infty(r, s, t; \Delta^{(m)})$ by using the Hausdorff measure of .

2. Preliminaries

Let l_∞, c and c_0 be the spaces of all bounded, convergent and null sequences $x = (x_n)$ respectively with the norm $\|x\|_\infty = \sup_n |x_n|$. Let bs and cs be the sequence spaces of all bounded and convergent series respectively. We denote by $e = (1, 1, \cdots)$ and e_n for the sequence whose n-th term is 1 and others are zero and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. A sequence (b_n) in a normed linear space $(X, \|.\|)$ is called a Schauder basis for X if for every $x \in X$ there is a unique sequence of scalars (μ_n) such that

$$\Big\|x - \sum_{n=0}^{k} \mu_n b_n\Big\| \to 0 \text{ as } k \to \infty,$$

i.e., $x = \sum_{n=0}^{\infty} \mu_n b_n$ [22].
For any subsets U and V of w, the multiplier space $M(U,V)$ of U and V is defined as

$$M(U,V) = \{a = (a_n) \in w : au = (a_n u_n) \in V \text{ for all } u = (u_n) \in U\}.$$

In particular,

$$U^\alpha = M(U, l_1),\ U^\beta = M(U, cs) \text{ and } U^\gamma = M(U, bs)$$

are called the α-, β- and γ- duals of U respectively [15].

Let $A = (a_{nk})_{n,k}$ be an infinite matrix with real or complex entries a_{nk}. We write A_n as the sequence of the n-th row of A, i.e., $A_n = (a_{nk})_k$ for every n. For $x = (x_n) \in w$, the A-transform of x is defined as the sequence $Ax = ((Ax)_n)$, where

$$A_n(x) = (Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,$$

provided the series on the right side converges for each n. For any two sequence spaces U and V, we denote by (U,V), the class of all infinite matrices A that map from U into V. Therefore $A \in (U,V)$ if and only if $Ax = ((Ax)_n) \in V$ for all $x \in U$. In other words, $A \in (U,V)$ if and only if $A_n \in U^\beta$ for all n [22].

The theory of BK spaces is the most powerful tool in the characterization of matrix transformations between sequence spaces. A sequence space X is called a BK space if it is a Banach space with continuous coordinates $p_n : X \to \mathbb{K}$, where $\mathbb{K}$ denotes the real or complex field and $p_n(x) = x_n$ for all $x = (x_n) \in X$ and each $n \in \mathbb{N}_0$. The space l_1 is a BK space with the usual norm defined by $\|x\|_1 = \sum_{k=0}^{\infty} |x_k|$. An infinite matrix $T = (t_{nk})_{n,k}$ is called a triangle if $t_{nn} \neq 0$ and $t_{nk} = 0$ for all $k > n$. Let T be a triangle and X be a BK space. Then X_T is also a BK space with the norm given by $\|x\|_{X_T} = \|Tx\|_X$ for all $x \in X_T$ [22].

3. Sequence spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$

In this section, we first begin with the notion of generalized means given by Mursaleen et al. [18].
We denote the sets $\mathcal{U}$ and $\mathcal{U}_0$ as

$$\mathcal{U} = \Big\{u = (u_n) \in w : u_n \neq 0 \text{ for all } n\Big\} \text{ and } \mathcal{U}_0 = \Big\{u = (u_n) \in w : u_0 \neq 0\Big\}.$$

Let $r = (r_n), t = (t_n) \in \mathcal{U}$ and $s = (s_n) \in \mathcal{U}_0$. The sequence $y = (y_n)$ of generalized means of a sequence $x = (x_n)$ is defined by

$$y_n = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k x_k \qquad (n \in \mathbb{N}_0).$$

The infinite matrix $A(r,s,t)$ of generalized means is defined by

$$(A(r,s,t))_{nk} = \begin{cases} \frac{s_{n-k}t_k}{r_n}, & 0 \le k \le n \\ 0, & k > n. \end{cases}$$

Since $A(r,s,t)$ is a triangle, it has a unique inverse and the inverse is also a triangle [10]. Take $D_0^{(s)} = \frac{1}{s_0}$ and

$$D_n^{(s)} = \frac{1}{s_0^{n+1}} \begin{vmatrix} s_1 & s_0 & 0 & 0\cdots & 0 \\ s_2 & s_1 & s_0 & 0\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4}\cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3}\cdots & s_1 \end{vmatrix} \qquad \text{for } n = 1,2,3,\cdots$$

Then the inverse of $A(r,s,t)$ is the triangle $B = (b_{nk})_{n,k}$, which is defined as

$$b_{nk} = \begin{cases} (-1)^{n-k}\, \frac{D_{n-k}^{(s)}}{t_n} r_k, & 0 \le k \le n \\ 0, & k > n. \end{cases}$$

We now introduce the sequence spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$ as

$$X(r,s,t;\Delta^{(m)}) = \Big\{ x = (x_n) \in w : (((A(r,s,t).\Delta^{(m)})x)_n) \in X \Big\},$$

which are the combination of the generalized means and the difference operator of order m. By using matrix domain, we can write $X(r,s,t;\Delta^{(m)}) = X_{A(r,s,t;\Delta^{(m)})} = \{x \in w : A(r,s,t;\Delta^{(m)})x \in X\}$, where $A(r,s,t;\Delta^{(m)}) = A(r,s,t).\Delta^{(m)}$, product of two triangles $A(r,s,t)$ and $\Delta^{(m)}$. The sequence $y = (y_n)$ is $A(r,s,t).\Delta^{(m)}$-transform of a sequence $x = (x_n)$, i.e.,

$$y_n = \frac{1}{r_n} \sum_{j=0}^{n} \left[\sum_{i=j}^{n} (-1)^{i-j} \binom{m}{i-j} s_{n-i} t_i \right] x_j.$$

These sequence spaces include many known sequence spaces studied by several authors. For examples,

(1) if $r_n = \frac{1}{u_n}$, $t_n = v_n$, $s_n = 1$ for all n, then the sequence spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$ reduce to $X(u,v;\Delta^{(m)})$ studied by Başarir and Kara [5] and in particular for $m = 1$, the sequence space $X(u,v;\Delta)$ introduced by Polat et al. [20].

(2) if $r_n = \frac{1}{n!}$, $t_n = \frac{\alpha^n}{n!}$, $s_n = \frac{(1-\alpha)^n}{n!}$, where $0 < \alpha < 1$, then the sequence spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$ reduce to $e_\infty^\alpha(\Delta^{(m)})$, $e^\alpha(\Delta^{(m)})$ and $e_0^\alpha(\Delta^{(m)})$ respectively studied by Polat and Başar [19].

(3) if $r_n = n+1$, $t_n = 1 + \alpha^n$, where $0 < \alpha < 1$ and $s_n = 1$ for all n, then the sequence spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{c, c_0\}$ reduce to the spaces of sequences $a_c^\alpha(\Delta)$ and $a_0^\alpha(\Delta)$ studied by Aydin and Başar [4]. For $X = l_\infty$, the sequence space $X(r,s,t;\Delta^{(m)})$ reduces to $a_\infty^\alpha(\Delta)$ studied by Djolović [8].

(4) if $r_n = \lambda_n\, t_n = \lambda_n - \lambda_{n-1}$, $s_n = 1$ and $m = 1$ then the spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{c, c_0\}$ reduce to $c_0^\lambda(\Delta)$ and $c^\lambda(\Delta)$ respectively studied by Mursaleen and Noman [16].

4. Main results

In this section, we begin with some topological results of the newly defined sequence spaces.

Theorem 4.1. *The sequence spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$ are complete normed linear spaces under the norm defined by*

$$\|x\|_{X(r,s,t;\Delta^{(m)})} = \sup_n \left| \frac{1}{r_n} \sum_{j=0}^{n} \left[\sum_{i=j}^{n} (-1)^{i-j} \binom{m}{i-j} s_{n-i} t_i \right] x_j \right|$$
$$= \sup_n |(A(r,s,t;\Delta^{(m)})x)_n|.$$

Proof. Since $\Delta^{(m)}$ is a linear operator, it is easy to show that $X(r,s,t;\Delta^{(m)})$ is a linear space and the functional $\|.\|_{X(r,s,t;\Delta^{(m)})}$ defined above gives a norm on the linear space $X(r,s,t;\Delta^{(m)})$.

To show completeness, let (x^i) be a Cauchy sequence in $X(r,s,t;\Delta^{(m)})$, where $x^i = (x^i_k) = (x^i_0, x^i_1, x^i_2, \ldots) \in X(r,s,t;\Delta^{(m)})$ for each $i \in \mathbb{N}_0$. Then for every $\epsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that

$$\|x^i - x^j\|_{X(r,s,t;\Delta^{(m)})} < \epsilon \text{ for } i, j \geq i_0.$$

The above implies that for each $k \in \mathbb{N}_0$,

$$|(A(r,s,t;\Delta^{(m)})x^i)_k - (A(r,s,t;\Delta^{(m)})x^j)_k| < \epsilon \text{ for all } i, j \geq i_0, \tag{4.1}$$

Therefore the sequence $((A(r,s,t;\Delta^{(m)})x^i)_k)_i$ is a Cauchy sequence of scalars for each $k \in \mathbb{N}_0$ and hence $((A(r,s,t;\Delta^{(m)})x^i)_k)_i$ converges for each k. We write

$$\lim_{i\to\infty} (A(r,s,t;\Delta^{(m)})x^i)_k = (A(r,s,t;\Delta^{(m)})x)_k \quad \text{ for each } k \in \mathbb{N}_0.$$

Letting $j \to \infty$ in (4.1), we obtain

$$\left|(A(r,s,t;\Delta^{(m)})x^i)_k - (A(r,s,t;\Delta^{(m)})x)_k\right| < \epsilon \text{ for all } i \geq i_0 \text{ and each } k \in \mathbb{N}_0.$$

Hence by definition, $\|x^i - x\|_{X(r,s,t;\Delta^{(m)})} < \epsilon$ for all $i \geq i_0$. Next we show that $x \in X(r,s,t;\Delta^{(m)})$. Since $(x^i) \subset X(r,s,t;\Delta^{(m)})$, we have

$$\|x\|_{X(r,s,t;\Delta^m)} \leq \|x^i\|_{X(r,s,t;\Delta^{(m)})} + \|x^i - x\|_{X(r,s,t;\Delta^{(m)})},$$

which is finite for $i \geq i_0$. So $x \in X(r,s,t;\Delta^{(m)})$. This completes the proof. $\square$

Theorem 4.2. *The sequence spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$ are linearly isomorphic to the spaces $X \in \{l_\infty, c, c_0\}$ respectively, i.e., $l_\infty(r,s,t;\Delta^{(m)}) \cong l_\infty$, $c(r,s,t;\Delta^{(m)}) \cong c$ and $c_0(r,s,t;\Delta^{(m)}) \cong c_0$.*

Proof. We prove the theorem only for the case $X = c_0$. For this, we need to show that there exists a bijective linear map from $c_0(r,s,t;\Delta^{(m)})$ to c_0.

We define a map $T : c_0(r,s,t;\Delta^{(m)}) \to c_0$ by $x \longmapsto Tx = y = (y_n)$, where

$$y_n = \frac{1}{r_n} \sum_{j=0}^{n} \left[\sum_{i=j}^{n} (-1)^{i-j} \binom{m}{i-j} s_{n-i} t_i \right] x_j.$$

Since $\Delta^{(m)}$ is a linear operator, so the linearity of T is trivial. It is clear from the definition that $Tx = 0$ implies $x = 0$. Thus T is injective. To prove T is surjective, let $y = (y_n) \in c_0$. Since $y = (A(r,s,t).\Delta^{(m)})x$, i.e.,

$$x = (A(r,s,t).\Delta^{(m)})^{-1}y = (\Delta^{(m)})^{-1}.A(r,s,t)^{-1}y.$$

So we can get a sequence $x = (x_n)$ as

$$x_n = \sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}\binom{m+n-k-1}{n-k}\frac{D_{k-j}^{(s)}}{t_k}r_j y_j, \qquad n \in \mathbb{N}_0. \tag{4.2}$$

Then

$$\|x\|_{c_0(r,s,t;\Delta^{(m)})} = \sup_n \left| \frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right]x_j\right| = \sup_n |y_n| = \|y\|_\infty < \infty.$$

Thus $x \in c_0(r,s,t;\Delta^{(m)})$ and this shows that T is surjective. Hence T is a linear bijection from $c_0(r,s,t;\Delta^{(m)})$ to c_0. Also T is norm preserving. So $c_0(r,s,t;\Delta^{(m)}) \cong c_0$.
Similarly, we can prove that $l_\infty(r,s,t;\Delta^{(m)}) \cong l_\infty$, $c(r,s,t;\Delta^{(m)}) \cong c$. This completes the proof. □

Since $X(r,s,t;\Delta^{(m)}) \cong X$ for $X \in \{c_0, c\}$, the Schauder bases of the sequence spaces $X(r,s,t;\Delta^{(m)})$ are the inverse image of the bases of X for $X \in \{c_0, c\}$. So, we have the following theorem without proof.

Theorem 4.3. *Let $\mu_k = (A(r,s,t;\Delta^{(m)})x)_k$, $k \in \mathbb{N}_0$. For each $j \in \mathbb{N}_0$, define the sequence $b^{(j)} = (b_n^{(j)})_n$ of the elements of the space $c_0(r,s,t;\Delta^{(m)})$ as*

$$b_n^{(j)} = \begin{cases} \displaystyle\sum_{k=j}^{n}(-1)^{k-j}\binom{m+n-k-1}{n-k}\frac{D_{k-j}^{(s)}}{t_k}r_j, & 0 \le j \le n \\ 0, & j > n \end{cases}$$

and

$$b_n^{(-1)} = \sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}\binom{m+n-k-1}{n-k}\frac{D_{k-j}^{(s)}}{t_k}r_j.$$

Then the followings are true:
(i) The sequence $(b^{(j)})_{j=0}^{\infty}$ is a basis for the space $c_0(r,s,t;\Delta^{(m)})$ and any $x \in c_0(r,s,t;\Delta^{(m)})$ has a unique representation of the form

$$x = \sum_{j=0}^{\infty}\mu_j b^{(j)}.$$

(ii) The set $(b^{(j)})_{j=-1}^{\infty}$ is a basis for $c(r,s,t;\Delta^{(m)})$ and any $x \in c(r,s,t;\Delta^{(m)})$ has a unique representation of the form

$$x = \ell b^{(-1)} + \sum_{j=0}^{\infty}(\mu_j - \ell)b^{(j)},$$

where $\ell = \lim_{n\to\infty}(A(r,s,t;\Delta^{(m)})x)_n$.

Remark 4.4. In particular, if we choose $r_n = \frac{1}{u_n}$, $t_n = v_n$, $s_n = 1$ for all n, then the sequence spaces $X(r,s,t;\Delta^{(m)})$ reduce to $X(u,v;\Delta^{(m)})$ for $X \in \{c_0, c\}$. With this choice of s_n, we have $D_0^{(s)} = D_1^{(s)} = 1$ and $D_n^{(s)} = 0$ for $n \geq 2$. Then the sequences $b^{(j)} = (b_n^{(j)})$ for $j = -1, 0, 1, \dots$ reduce to

$$b_n^{(j)} = \begin{cases} \sum\limits_{k=j}^{j+1} (-1)^{k-j} \binom{m+n-k-1}{n-k} \frac{1}{u_j v_k}, & 0 \leq j \leq n \\ 0, & j > n. \end{cases}$$

and

$$b_n^{(-1)} = \sum_{j=0}^{n} \sum_{k=j}^{j+1} (-1)^{k-j} \binom{m+n-k-1}{n-k} \frac{1}{u_j v_k}.$$

The sequences $(b^{(j)})_{j=0}^{\infty}$ and $(b^{(j)})_{j=-1}^{\infty}$ are the bases for the spaces $c_0(u,v;\Delta^{(m)})$ and $c(u,v;\Delta^{(m)})$ respectively [5].

Let $\mathcal{F}$ be the collection of all finite nonempty subsets of the set of all natural numbers. Let $A = (a_{nk})_{n,k}$ be an infinite matrix and consider the following conditions:

$$\sup_{K \in \mathcal{F}} \sum_{n=0}^{\infty} \Big| \sum_{k \in K} a_{nk} \Big| < \infty \tag{4.3}$$

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty \tag{4.4}$$

$$\lim_n \sum_{k=0}^{\infty} |a_{nk}| = 0 \tag{4.5}$$

$$\lim_n a_{nk} = 0 \text{ for all } k \tag{4.6}$$

$$\lim_n \sum_{k=0}^{\infty} a_{nk} = 0 \tag{4.7}$$

$$\lim_n a_{nk} \text{ exists for all } k \tag{4.8}$$

$$\lim_n \sum_{k=0}^{\infty} |a_{nk} - \lim_n a_{nk}| = 0 \tag{4.9}$$

$$\lim_n \sum_{k=0}^{\infty} a_{nk} \text{ exists} \tag{4.10}$$

We now state some results given by Stieglitz and Tietz [21] which are required to obtain the duals and matrix transformations.

Theorem 4.5. [21]
(a) $A \in (c_0, l_1), A \in (c, l_1), A \in (l_\infty, l_1)$ *if and only if* (4.3) *holds.*
(b) $A \in (c_0, l_\infty), A \in (c, l_\infty), A \in (l_\infty, l_\infty)$ *if and only if* (4.4) *holds.*
(c) $A \in (c_0, c_0)$ *if and only if* (4.4) *and* (4.6) *hold.*

(d) $A \in (l_\infty, c_0)$ *if and only if* (4.5) *holds.*
(e) $A \in (c, c_0)$ *if and only if* (4.4), (4.6) *and* (4.7) *hold.*
(f) $A \in (c_0, c)$ *if and only if* (4.4) *and* (4.8) *hold.*
(g) $A \in (l_\infty, c)$ *if and only if* (4.4), (4.8) *and* (4.9) *hold.*
(h) $A \in (c, c)$ *if and only if* (4.4), (4.8) *and* (4.10) *hold.*

4.1. **The α-, γ-duals of $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$.** Now we compute the α-, γ-duals of $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$.

Theorem 4.6. *The α-dual of the space $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$ is the set*

$$\Lambda = \left\{a = (a_n) \in w : \sup_{K \in \mathcal{F}} \sum_{n=0}^{\infty} \Big| \sum_{j \in K} \sum_{k=j}^{n} (-1)^{k-j} \binom{m+n-k-1}{n-k} \frac{D_{k-j}^{(s)}}{t_k} r_j a_n \Big| < \infty \right\}.$$

Proof. Let $a = (a_n) \in w$, $x \in X(r,s,t;\Delta^{(m)})$ and $y \in X$ for $X \in \{l_\infty, c, c_0\}$. Then for each $n \in \mathbb{N}_0$, we have

$$a_n x_n = \sum_{j=0}^{n} \sum_{k=j}^{n} (-1)^{k-j} \binom{m+n-k-1}{n-k} \frac{D_{k-j}^{(s)}}{t_k} r_j a_n y_j = (Cy)_n,$$

where the matrix $C = (c_{nj})_{n,j}$ is defined as

$$c_{nj} = \begin{cases} \displaystyle\sum_{k=j}^{n} (-1)^{k-j} \binom{m+n-k-1}{n-k} \frac{D_{k-j}^{(s)}}{t_k} r_j a_n, & 0 \le j \le n \\ 0, & j > n \end{cases}$$

and x_n is given by (4.2). Thus for each $x \in X(r,s,t;\Delta^{(m)})$, $(a_n x_n)_n \in l_1$ if and only if $Cy \in l_1$, where $y \in X$ for $X \in \{l_\infty, c, c_0\}$. Therefore $a = (a_n) \in [X(r,s,t;\Delta^{(m)})]^\alpha$ if and only if $C \in (X, l_1)$. By using Theorem 4.5(a), we have

$$[X(r,s,t;\Delta^{(m)})]^\alpha = \Lambda.$$

□

Theorem 4.7. *The γ-dual of the space $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$ is the set*

$$\Gamma = \left\{a = (a_n) \in w : \ \sup_{l} \sum_{n=0}^{\infty} |e_{ln}| < \infty \right\},$$

where the matrix $E = (e_{ln})$ is defined by

$$e_{ln} = \begin{cases} r_n \Bigg[\dfrac{a_n}{s_0 t_n} + \displaystyle\sum_{k=n}^{n+1} (-1)^{k-n} \frac{D_{k-n}^{(s)}}{t_k} \sum_{j=n+1}^{l} \binom{m+j-k-1}{j-k} a_j \\ \qquad + \displaystyle\sum_{k=n+2}^{l} (-1)^{k-n} \frac{D_{k-n}^{(s)}}{t_k} \sum_{j=k}^{l} \binom{m+j-k-1}{j-k} a_j \Bigg], & 0 \le n \le l \\ 0, & n > l. \end{cases} \tag{4.11}$$

Note: We mean $\sum_{j=n}^{l} = 0$ *if* $n > l$.

Proof. Let $a = (a_n) \in w$, $x \in X(r,s,t;\Delta^{(m)})$ and $y \in X$ for $X \in \{l_\infty, c, c_0\}$, which are connected by the relation (4.2). Then, we have

$$\begin{aligned}
\sum_{n=0}^{l} a_n x_n &= \sum_{n=0}^{l}\sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}\binom{m+n-k-1}{n-k}\frac{D_{k-j}^{(s)}}{t_k} r_j a_n y_j \\
&= \sum_{n=0}^{l-1}\sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}\binom{m+n-k-1}{n-k}\frac{D_{k-j}^{(s)}}{t_k} r_j y_j a_n \\
&\quad + \sum_{j=0}^{l}\sum_{k=j}^{l}(-1)^{k-j}\binom{m+l-k-1}{l-k}\frac{D_{k-j}^{(s)}}{t_k} r_j y_j a_l \\
&= \Bigg[\frac{D_0^{(s)}}{t_0} a_0 + \sum_{k=0}^{1}(-1)^k\frac{D_k^{(s)}}{t_k}\sum_{j=1}^{l}\binom{m+j-k-1}{j-k} a_j \\
&\quad + \sum_{k=2}^{l}(-1)^k\frac{D_k^{(s)}}{t_k}\sum_{j=k}^{l}\binom{m+j-k-1}{j-k} a_j\Bigg] r_0 y_0 \\
&\quad + \Bigg[\frac{D_0^{(s)}}{t_1} a_1 + \sum_{k=1}^{2}(-1)^{k-1}\frac{D_{k-1}^{(s)}}{t_k}\sum_{j=2}^{l}\binom{m+j-k-1}{j-k} a_j \\
&\quad + \sum_{k=3}^{l}(-1)^{k-1}\frac{D_{k-1}^{(s)}}{t_k}\sum_{j=k}^{l}\binom{m+j-k-1}{j-k} a_j\Bigg] r_1 y_1 + \cdots + \frac{D_0^{(s)}}{t_l} a_l r_l y_l \\
&= \sum_{n=0}^{l} r_n \Bigg[\frac{a_n}{s_0 t_n} + \sum_{k=n}^{n+1}(-1)^{k-n}\frac{D_{k-n}^{(s)}}{t_k}\sum_{j=n+1}^{l}\binom{m+j-k-1}{j-k} a_j \\
&\quad + \sum_{k=n+2}^{l}(-1)^{k-n}\frac{D_{k-n}^{(s)}}{t_k}\sum_{j=k}^{l}\binom{m+j-k-1}{j-k} a_j\Bigg] y_n \\
&= (Ey)_l,
\end{aligned}$$

where E is the matrix defined in (4.11).
Thus $a \in \left[X(r,s,t;\Delta^{(m)})\right]^\gamma$ if and only if $ax = (a_n x_n) \in bs$ for $x \in X(r,s,t;\Delta^{(m)})$ if and only if $\left(\sum_{n=0}^{l} a_n x_n\right) \in l_\infty$, i.e., $Ey \in l_\infty$ for $y \in X$. Hence by Theorem 4.5(b), we have

$$\left[X(r,s,t;\Delta^{(m)})\right]^\gamma = \Gamma.$$

□

Remark 4.8. In particular, if we choose $r_n = \frac{1}{u_n}$, $t_n = v_n$, $s_n = 1$ for all n, then the sequence spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$ reduce to $X(u,v;\Delta^{(m)})$

[5]. With this choice of s_n, we have $D_0^{(s)} = D_1^{(s)} = 1$ and $D_n^{(s)} = 0$ for $n \geq 2$. Therefore the γ-dual of the space $X(u, v; \Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$ is the set

$$\left\{ a = (a_n) \in w : \sup_l \sum_{n=0}^{\infty} \left| \frac{1}{u_n} \left[\frac{a_n}{v_n} + \sum_{k=n}^{n+1} \frac{(-1)^{k-n}}{v_k} \sum_{j=n+1}^{l} \binom{m+j-k-1}{j-k} a_j \right] \right| < \infty \right\}.$$

4.2. **β-dual and Matrix transformations.** Here we first discuss about the β-dual and then characterize the matrix transformations. Let T be a triangle and X_T be the matrix domain of T in X.

Theorem 4.9. *([10], Theorem 2.6) Let X be a BK space with AK property and $R = S^t$, the transpose of S, where $S = (s_{jk})$ is the inverse of the matrix T. Then $a \in (X_T)^\beta$ if and only if $a \in (X^\beta)_R$ and $W \in (X, c_0)$, where the triangle $W = (w_{pk})$ is defined by $w_{pk} = \sum_{j=p}^{\infty} a_j s_{jk}$. Moreover if $a \in (X_T)^\beta$, then*

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty} R_k(a) T_k(z) \qquad \text{for all } z \in X_T.$$

Remark 4.10. ([10], Remark 2.7) The conclusion of Theorem 4.9 is also true for $X = l_\infty$.

Remark 4.11. ([10], [15]) We have $a \in (c_T)^\beta$ if and only if $R(a) \in l_1$ and $W \in (c, c)$. Moreover, if $a \in (c_T)^\beta$ then we have for all $z \in c_T$

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty} R_k(a) T_k(z) - \eta\gamma,$$

where $\eta = \lim_{k \to \infty} T_k(z)$ and $\gamma = \lim_{p \to \infty} \sum_{k=0}^{p} w_{pk}$.

To find the β-duals of the sequence spaces $X(r, s, t; \Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$, we define the following sets:

$$B_1 = \Big\{a \in w : \sum_{k=0}^{\infty} |R_k(a)| < \infty\Big\}$$
$$B_2 = \Big\{a \in w : \lim_{p\to\infty} w_{pk} = 0 \text{ for all } k\Big\}$$
$$B_3 = \Big\{a \in w : \sup_p \sum_{k=0}^{\infty} |w_{pk}| < \infty\Big\}$$
$$B_4 = \Big\{a \in w : \lim_{p\to\infty} \sum_{k=0}^{p} |w_{pk}| = 0\Big\}$$
$$B_5 = \Big\{a \in w : \lim_{p\to\infty} w_{pk} \ \text{ exists for all } k\Big\}$$
$$B_6 = \Big\{a \in w : \lim_{p\to\infty} \sum_{k=0}^{p} w_{pk} \text{ exists}\Big\},$$

where

$$R_k(a) = r_k\Bigg[\frac{a_k}{s_0 t_k} + \sum_{i=k}^{k+1}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\sum_{j=k+1}^{\infty}\binom{m+j-i-1}{j-i}a_j + \sum_{l=2}^{\infty}(-1)^{l}\frac{D_{l}^{(s)}}{t_{l+k}}\sum_{j=k+l}^{\infty}\binom{m+j-k-l-1}{j-k-l}a_j\Bigg]$$

and
$$w_{pk} = r_k\Bigg[\sum_{i=k}^{p}(-1)^{i-k}\tfrac{D_{i-k}^{(s)}}{t_i}\sum_{j=p}^{\infty}\tbinom{m+j-i-1}{j-i}a_j + \sum_{i=p+1}^{\infty}(-1)^{i-k}\tfrac{D_{i-k}^{(s)}}{t_i}\sum_{j=i}^{\infty}\tbinom{m+j-i-1}{j-i}a_j\Bigg].$$

Theorem 4.12. *We have* $[c_0(r,s,t;\Delta^{(m)})]^\beta = B_1 \bigcap B_2 \bigcap B_3$, $[l_\infty(r,s,t;\Delta^{(m)})]^\beta = B_1 \bigcap B_4$ *and* $[c(r,s,t;\Delta^{(m)})]^\beta = B_1 \bigcap B_3 \bigcap B_5 \bigcap B_6$.

Proof. Here the triangle $T = A(r,s,t).\Delta^{(m)}$. So $T^{-1} = (A(r,s,t).\Delta^{(m)})^{-1} = (\Delta^{(m)})^{-1}.A(r,s,t)^{-1}$. Let $S = (s_{jk})$ be the inverse of T. Then we have

$$s_{jk} = \begin{cases} \sum\limits_{i=k}^{j}(-1)^{i-k}\binom{m+j-i-1}{j-i}\frac{D_{i-k}^{(s)}}{t_i}r_k, & 0 \le k \le j \\ 0, & k > j. \end{cases}$$

To find the β-dual of $X(r,s,t;\Delta^{(m)})$ for $X \in \{l_\infty, c, c_0\}$, we need to show $R(a) = (R_k(a)) \in l_1$, where $R = S^t$ and characterize the classes $W \in (c_0, c_0), W \in (l_\infty, c_0)$

and $W \in (c, c)$. Now

$$\begin{aligned}
R_k(a) &= \sum_{j=k}^{\infty} a_j s_{jk} \\
&= \sum_{j=k}^{\infty} \sum_{i=k}^{j} (-1)^{i-k} \binom{m+j-i-1}{j-i} \frac{D_{i-k}^{(s)}}{t_i} r_k a_j \\
&= \frac{D_0^{(s)}}{t_k} r_k a_k + \sum_{j=k+1}^{\infty} \sum_{i=k}^{j} (-1)^{i-k} \binom{m+j-i-1}{j-i} \frac{D_{i-k}^{(s)}}{t_i} r_k a_j \\
&= r_k \Bigg[\frac{a_k}{s_0 t_k} + \sum_{i=k}^{k+1} (-1)^{i-k} \frac{D_{i-k}^{(s)}}{t_i} \sum_{j=k+1}^{\infty} \binom{m+j-i-1}{j-i} a_j \\
&\quad + \sum_{l=2}^{\infty} (-1)^{l} \frac{D_l^{(s)}}{t_{l+k}} \sum_{j=k+l}^{\infty} \binom{m+j-k-l-1}{j-k-l} a_j \Bigg]
\end{aligned}$$

and

$$\begin{aligned}
w_{pk} &= \sum_{j=p}^{\infty} a_j s_{jk} \\
&= \sum_{j=p}^{\infty} \sum_{i=k}^{j} (-1)^{i-k} \binom{m+j-i-1}{j-i} \frac{D_{i-k}^{(s)}}{t_i} r_k a_j \\
&= r_k \Bigg[\sum_{i=k}^{p} (-1)^{i-k} \frac{D_{i-k}^{(s)}}{t_i} \sum_{j=p}^{\infty} \binom{m+j-i-1}{j-i} a_j \\
&\quad + \sum_{i=p+1}^{\infty} (-1)^{i-k} \frac{D_{i-k}^{(s)}}{t_i} \sum_{j=i}^{\infty} \binom{m+j-i-1}{j-i} a_j \Bigg].
\end{aligned}$$

Using Theorem 4.9 and Remark 4.10 & 4.11, we have $[c_0(r, s, t; \Delta^{(m)})]^\beta = B_1 \bigcap B_2 \bigcap B_3$, $[l_\infty(r, s, t; \Delta^{(m)})]^\beta = B_1 \bigcap B_4$ and $[c(r, s, t; \Delta^{(m)})]^\beta = B_1 \bigcap B_3 \bigcap B_5 \bigcap B_6$. □

Theorem 4.13. *([10], Theorem 2.13) Let X be a BK space with AK property, Y be an arbitrary subset of w and $R = S^t$, where $S = (s_{jk})$ is the inverse of the matrix T. Then $A \in (X_T, Y)$ if and only if $B^A \in (X, Y)$ and $W^{A_n} \in (X, c_0)$ for all $n = 0, 1, 2, \cdots$, where B^A is the matrix with rows $B_n^A = R(A_n)$, A_n are the rows of A and the triangles W^{A_n} for $n \in \mathbb{N}_0$ are defined by*

$$w_{pk}^{A_n} = \begin{cases} \sum\limits_{j=p}^{\infty} a_{nj} s_{jk}, & 0 \le k \le p \\ 0, & k > p. \end{cases}$$

Theorem 4.14. *([10]) Let Y be any linear subspace of w. Then $A \in (c_T, Y)$ if and only if $R_k(A_n) \in (c_0, Y)$ and $W^{A_n} \in (c, c)$ for all n and $R_k(A_n)e - (\gamma_n) \in Y$,* *where* $\gamma_n = \lim_{p\to\infty} \sum_{k=0}^{p} w_{pk}^{A_n}$ *for* $n = 0, 1, 2 \cdots$.

Moreover, if $A \in (c_T, Y)$ *then we have*

$$Az = R_k(A_n)(T(z)) - \eta(\gamma_n) \quad \text{for all } z \in c_T, \text{ where } \eta = \lim_{k\to\infty} T_k(z).$$

To characterize the matrix transformations $A \in (X(r, s, t; \Delta^{(m)}), Y)$ for $X, Y \in \{l_\infty, c, c_0\}$, we list the following conditions:

$$\sup_n \sum_{k=0}^{\infty} |R_k(A_n)| < \infty \tag{4.12}$$

$$\lim_{n\to\infty} R_k(A_n) = 0 \quad \text{for all } k \tag{4.13}$$

$$\sup_p \sum_{k=0}^{p} |w_{pk}^{A_n}| < \infty \quad \text{for all } n \tag{4.14}$$

$$\lim_{p\to\infty} w_{pk}^{A_n} = 0 \text{ for all } n \tag{4.15}$$

$$\lim_{n\to\infty} R_k(A_n) \text{ exists for all } k \tag{4.16}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |R_k(A_n)| = 0 \tag{4.17}$$

$$\lim_{p\to\infty} \sum_{k=0}^{p} |w_{pk}^{A_n}| = 0 \quad \text{for all } n \tag{4.18}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \Big| R_k(A_n) - \lim_{n\to\infty} R_k(A_n) \Big| = 0 \tag{4.19}$$

$$\lim_{p\to\infty} w_{pk}^{A_n} \text{ exists for all } k, n \tag{4.20}$$

$$\lim_{p\to\infty} \sum_{k=0}^{p} w_{pk}^{A_n} \text{ exists for all } n \tag{4.21}$$

$$R_k(A_n)e - (\gamma_n) \in c_0 \quad \text{for all } \gamma_n,\ n = 0, 1, 2, \cdots \tag{4.22}$$

$$R_k(A_n)e - (\gamma_n) \in l_\infty \quad \text{for all } \gamma_n,\ n = 0, 1, 2, \cdots \tag{4.23}$$

$$R_k(A_n)e - (\gamma_n) \in c \quad \text{for all } \gamma_n,\ n = 0, 1, 2, \cdots, \tag{4.24}$$

where $\gamma_n = \lim_{p\to\infty} \sum_{k=0}^{p} w_{pk}^{A_n}$,

$$R_k(A_n) = r_k \Bigg[\frac{a_{nk}}{s_0 t_k} + \sum_{i=k}^{k+1} (-1)^{i-k} \frac{D_{i-k}^{(s)}}{t_i} \sum_{j=k+1}^{\infty} \binom{m+j-i-1}{j-i} a_{nj} + \sum_{l=2}^{\infty} (-1)^l \frac{D_l^{(s)}}{t_{l+k}} \sum_{j=k+l}^{\infty} \binom{m+j-k-l-1}{j-k-l} a_{nj} \Bigg]$$

and
$$w_{pk}^{A_n}=r_k\Big[\sum_{i=k}^{p}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\sum_{j=p}^{\infty}\binom{m+j-i-1}{j-i}a_{nj}+\sum_{i=p+1}^{\infty}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\sum_{j=i}^{\infty}\binom{m+j-i-1}{j-i}a_{nj}\Big].$$

Theorem 4.15. *(a) $A\in(c_0(r,s,t;\Delta^{(m)}),c_0)$ if and only if (4.12), (4.13), (4.14) and (4.15) hold.*
(b) $A\in(c_0(r,s,t;\Delta^{(m)}),c)$ if and only if (4.12), (4.14), (4.15) and (4.16)hold.
(c) $A\in(c_0(r,s,t;\Delta^{(m)}),l_\infty)$ if and only if (4.12), (4.14) and (4.15) hold.

Proof. We only prove the part (a) of this theorem. The other parts follow in a similar way. We first compute the matrices $B^A=(R_k(A_n))$ and $W^{A_n}=(w_{pk}^{A_n})$ for $n=0,1,2,\cdots$ of Theorem 4.13 to determine the conditions $B^A\in(c_0,c_0)$ and $W^{A_n}\in(c_0,c_0)$. Using the same lines of proof as used in Theorem 4.12, we have

$$\begin{aligned}
R_k(A_n)&=\sum_{j=k}^{\infty}s_{jk}a_{nj}\\
&=\frac{D_0^{(s)}}{t_k}r_ka_{nk}+\sum_{j=k+1}^{\infty}\sum_{i=k}^{j}(-1)^{i-k}\binom{m+j-i-1}{j-i}\frac{D_{i-k}^{(s)}}{t_i}r_ka_{nj}\\
&=r_k\Big[\frac{a_{nk}}{s_0t_k}+\sum_{i=k}^{k+1}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\sum_{j=k+1}^{\infty}\binom{m+j-i-1}{j-i}a_{nj}\\
&\qquad+\sum_{l=2}^{\infty}(-1)^{l}\frac{D_l^{(s)}}{t_{l+k}}\sum_{j=k+l}^{\infty}\binom{m+j-k-l-1}{j-k-l}a_{nj}\Big]
\end{aligned}$$

and

$$\begin{aligned}
w_{pk}^{A_n}&=\sum_{j=p}^{\infty}s_{jk}a_{nj}\\
&=r_k\Big[\sum_{i=k}^{p}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\sum_{j=p}^{\infty}\binom{m+j-i-1}{j-i}a_{nj}\\
&\qquad+\sum_{i=p+1}^{\infty}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\sum_{j=i}^{\infty}\binom{m+j-i-1}{j-i}a_{nj}\Big].
\end{aligned}$$

Using Theorem 4.13, we have $A\in(c_0(r,s,t;\Delta^{(m)}),c_0)$ if and only if the conditions (4.12), (4.13), (4.14) and (4.15) hold. □

We can also obtain the following results.

Corollary 4.16. *(a) $A\in(l_\infty(r,s,t;\Delta^{(m)}),c_0)$ if and only if the conditions (4.17) and (4.18) hold.*
(b) $A\in(l_\infty(r,s,t;\Delta^{(m)}),c)$ if and only if the conditions (4.12), (4.16), (4.18) and (4.19) hold.
(c) $A\in(l_\infty(r,s,t;\Delta^{(m)}),l_\infty)$ if and only if the conditions (4.12) and (4.18) hold.

Corollary 4.17. *(a) $A \in (c(r,s,t;\Delta^{(m)}), c_0)$ if and only if the conditions* (4.12), (4.13), (4.14), (4.20), (4.21) *and* (4.22) *hold.*
(b) $A \in (c(r,s,t;\Delta^{(m)}), c)$ if and only if the conditions (4.12), (4.14), (4.16), (4.20), (4.21) *and* (4.24) *hold.*
(c) $A \in (c(r,s,t;\Delta^{(m)}), l_\infty)$ if and only if the conditions (4.12), (4.14), (4.20), (4.21) *and* (4.23) *hold.*

5. Compact operators on the spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{c_0, l_\infty\}$

In this section, we apply the Hausdorff measure of to establish necessary and sufficient conditions for an infinite matrix to be a compact operator from the space $X(r,s,t;\Delta^{(m)})$ to X for $X \in \{c_0, l_\infty\}$.

As the matrix transformations between BK spaces are continuous, it is quite natural to find necessary and sufficient conditions for a matrix mapping between BK spaces to be a compact operator. This can be achieved with the help of Hausdorff measure of . Recently several authors, namely, Malkowsky and Rakočević [13], Djolović et al. [9], Djolović [8], Mursaleen and Noman [17], Başarir and Kara [5] and others have established some identities or estimates for the operator norms and the Hausdorff measure of of matrix operators from an arbitrary BK space to arbitrary BK space. Let us recall some definitions and well-known results.

Let X, Y be two Banach spaces and S_X denotes the unit sphere in X, i.e., $S_X = \{x \in X : \|x\| = 1\}$. We denote by $\mathcal{B}(X,Y)$, the set of all bounded (continuous) linear operators $L : X \to Y$, which is a Banach space with the operator norm $\|L\| = \sup_{x \in S_X} \|L(x)\|_Y$ for all $L \in \mathcal{B}(X,Y)$. A linear operator $L : X \to Y$ is said to be compact if the domain of L is all of X and for every bounded sequence $(x_n) \in X$, the sequence $(L(x_n))$ has a subsequence which is convergent in Y and we denote by $\mathcal{C}(X,Y)$, the class of all compact operators in $\mathcal{B}(X,Y)$. An operator $L \in \mathcal{B}(X,Y)$ is said to be finite rank if $\dim R(L) < \infty$, where $R(L)$ is the range space of L. If X is a BK space and $a = (a_k) \in w$, then we consider

$$\|a\|_X^* = \sup_{x \in S_X} \Big| \sum_{k=0}^{\infty} a_k x_k \Big|, \tag{5.1}$$

provided the expression on the right side exists and is finite which is the case whenever $a \in X^\beta$ [17].
Let (X,d) be a metric space and $\mathcal{M}_X$ be the class of all bounded subsets of X. Let $B(x,r) = \{y \in X : d(x,y) < r\}$ denotes the open ball of radius $r > 0$ with centre at x. The Hausdorff measure of of a set $Q \in \mathcal{M}_X$, denoted by $\chi(Q)$, is defined as

$$\chi(Q) = \inf \Big\{ \epsilon > 0 : Q \subset \bigcup_{i=0}^{n} B(x_i, r_i), x_i \in X, r_i < \epsilon, n \in \mathbb{N}_0 \Big\}.$$

The function $\chi : \mathcal{M}_X \to [0,\infty)$ is called the Hausdorff measure of . The basic properties of the Hausdorff measure of can be found in ([9], [13]). For example, if Q, Q_1 and Q_2 are bounded subsets of a metric space (X, d) then

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is totally bounded and}$$
$$\text{if } Q_1 \subset Q_2 \text{ then } \chi(Q_1) \leq \chi(Q_2).$$

Also if X is a normed space, the function χ has some additional properties due to linear structure, namely,

$$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),$$
$$\chi(\alpha Q) = |\alpha|\chi(Q) \text{ for all } \alpha \in \mathbb{K}.$$

Let ϕ denotes the set of all finite sequences, i.e., of sequences that terminate in zeros. Throughout we denote p' as the conjugate of p for $1 \leq p < \infty$, i.e., $p' = \frac{p}{p-1}$ for $p > 1$ and $p' = \infty$ for $p = 1$. The following known results are fundamental for our investigation.

Lemma 5.1. [17] *Let X denote any of the sequence spaces c_0 or l_∞. If $A \in (X, c)$, then*

$$(i)\ \alpha_k = \lim_{n\to\infty} a_{nk} \text{ exists for all } k \in \mathbb{N}_0,$$
$$(ii)\ \alpha = (\alpha_k) \in l_1,$$
$$(iii)\ \sup_n \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| < \infty,$$
$$(iv)\ \lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k \text{ for all } x = (x_k) \in X.$$

Lemma 5.2. ([13], Theorem 1.29) *Let X denote any of the spaces c_0, c or l_∞. Then, $X^\beta = l_1$ and $\|a\|_X^* = \|a\|_1$ for all $a \in l_1$.*

Lemma 5.3. [17] *Let $X \supset \phi$ and Y be BK spaces. Then $(X, Y) \subset \mathcal{B}(X, Y)$, i.e., every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$, where $L_A(x) = Ax$ for all $x \in X$.*

Lemma 5.4. [8] *Let $X \supset \phi$ be a BK space and Y be any of the spaces c_0, c or l_∞. If $A \in (X, Y)$, then*

$$\|L_A\| = \|A\|_{(X,l_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

Lemma 5.5. [13] *Let $Q \in \mathcal{M}_{c_0}$ and $P_l : c_0 \to c_0$ $(l \in \mathbb{N}_0)$ be the operator defined by $P_l(x) = (x_0, x_1, \cdots, x_l, 0, 0, \cdots)$ for all $x = (x_k) \in c_0$. Then*

$$\chi(Q) = \lim_{l\to\infty} \left(\sup_{x\in Q} \|(I - P_l)(x)\|_\infty \right),$$

where I is the identity operator on c_0.

Let $z = (z_n) \in c$. Then z has a unique representation $z = \hat{\ell}e + \sum_{n=0}^{\infty}(z_n - \hat{\ell})e_n$, where $\hat{\ell} = \lim_{n\to\infty} z_n$. We now define the operators P_l $(l \in \mathbb{N}_0)$ from c onto the linear span of $\{e, e_0, e_1, \cdots, e_l\}$ as

$$P_l(z) = \hat{\ell}e + \sum_{n=0}^{l} (z_n - \hat{\ell})e_n,$$

for all $z \in c$ and $\hat{\ell} = \lim_{n\to\infty} z_n$.
Then the following result gives an estimate for the Hausdorff measure of in the BK space c.

Lemma 5.6. [13] *Let $Q \in \mathcal{M}_c$ and $P_l : c \to c$ be the operator from c onto the linear span of $\{e, e_0, e_1, \ldots e_l\}$. Then*

$$\frac{1}{2}\lim_{l\to\infty}\Big(\sup_{x\in Q}\|(I-P_l)(x)\|_\infty\Big) \leq \chi(Q) \leq \lim_{l\to\infty}\Big(\sup_{x\in Q}\|(I-P_l)(x)\|_\infty\Big),$$

where I is the identity operator on c.

Lemma 5.7. [13] *Let X, Y be two Banach spaces and $L \in \mathcal{B}(X, Y)$. Then*

$$\|L\|_\chi = \chi(L(S_X))$$

and

$$L \in \mathcal{C}(X, Y) \text{ if and only if } \|L\|_\chi = 0.$$

We establish the following lemmas which are required to characterize the classes of compact operators with the help of Hausdorff measure of .

Lemma 5.8. *Let $X(r, s, t; \Delta^{(m)})$ be any sequence spaces for $X \in \{c_0, l_\infty\}$. If $a = (a_k) \in [X(r, s, t; \Delta^{(m)})]^\beta$ then $\tilde{a} = (\tilde{a}_k) \in X^\beta = l_1$ and the equality*

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \tilde{a}_k y_k$$

holds for every $x = (x_k) \in X(r, s, t; \Delta^{(m)})$ and $y = (y_k) \in X$, where $y = (A(r, s, t).\Delta^{(m)})x$. In addition

$$\begin{aligned} \tilde{a}_k = r_k\Bigg[\frac{a_k}{s_0 t_k} &+ \sum_{i=k}^{k+1}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\sum_{j=k+1}^{\infty}\binom{m+j-i-1}{j-i}a_j \\ &+ \sum_{l=2}^{\infty}(-1)^l\frac{D_l^{(s)}}{t_{l+k}}\sum_{j=k+l}^{\infty}\binom{m+j-k-l-1}{j-k-l}a_j\Bigg]. \end{aligned} \tag{5.2}$$

Proof. Let $a = (a_k) \in [X(r, s, t; \Delta^{(m)})]^\beta$. Then by Theorem 4.9 and Remark 4.10, we have $R(a) = (R_k(a)) \in X^\beta = l_1$ and also

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} R_k(a)T_k(x) \quad \text{for all } x \in X(r, s, t; \Delta^{(m)}),$$

where

$$R_k(a) = r_k\Bigg[\frac{a_k}{s_0t_k} + \sum_{i=k}^{k+1}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\sum_{j=k+1}^{\infty}\binom{m+j-i-1}{j-i}a_j + \sum_{l=2}^{\infty}(-1)^l\frac{D_l^{(s)}}{t_{l+k}}\sum_{j=k+l}^{\infty}\binom{m+j-k-l-1}{j-k-l}a_j\Bigg] = \tilde{a}_k,$$

and $y = T(x) = (A(r,s,t).\Delta^{(m)})x$. This completes the proof. □

Lemma 5.9. *Let $X(r,s,t;\Delta^{(m)})$ be any sequence spaces for $X \in \{c_0, l_\infty\}$. Then we have*

$$\|a\|^*_{X(r,s,t;\Delta^{(m)})} = \|\tilde{a}\|_1 = \sum_{k=0}^{\infty}|\tilde{a}_k| < \infty$$

for all $a = (a_k) \in [X(r,s,t;\Delta^{(m)})]^\beta$, where $\tilde{a} = (\tilde{a}_k)$ is defined in (5.2).

Proof. Let $a = (a_k) \in [X(r,s,t;\Delta^{(m)})]^\beta$. Then from Lemma 5.8, we have $\tilde{a} = (\tilde{a}_k) \in l_1$. Also $x \in S_{X(r,s,t;\Delta^{(m)})}$ if and only if $y = T(x) \in S_X$ as $\|x\|_{X(r,s,t;\Delta^{(m)})} = \|y\|_\infty$. From (5.1), we have

$$\|a\|^*_{X(r,s,t;\Delta^{(m)})} = \sup_{x\in S_{X(r,s,t;\Delta^{(m)})}}\Big|\sum_{k=0}^{\infty}a_kx_k\Big| = \sup_{y\in S_X}\Big|\sum_{k=0}^{\infty}\tilde{a}_ky_k\Big| = \|\tilde{a}\|^*_X.$$

Using by Lemma 5.2, we have $\|a\|^*_{X(r,s,t;\Delta^{(m)})} = \|\tilde{a}\|^*_X = \|\tilde{a}\|_1$, which is finite as $\tilde{a} \in l_1$. This completes the proof. □

Lemma 5.10. *Let $X(r,s,t;\Delta^{(m)})$ be any sequence space for $X \in \{c_0, l_\infty\}$, Y any sequence space and $A = (a_{nk})_{n,k}$ an infinite matrix. If $A \in (X(r,s,t;\Delta^{(m)}), Y)$, then $\tilde{A} \in (X,Y)$ such that $Ax = \tilde{A}y$ for all $x \in X(r,s,t;\Delta^{(m)})$ and $y \in X$, which are connected by the relation $y = (A(r,s,t).\Delta^{(m)})x$ and*
$\tilde{A} = (\tilde{a}_{nk})_{n,k}$ is given by

$$\tilde{a}_{nk} = r_k\Bigg[\frac{a_{nk}}{s_0t_k} + \sum_{i=k}^{k+1}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\sum_{j=k+1}^{\infty}\binom{m+j-i-1}{j-i}a_{nj} + \sum_{l=2}^{\infty}(-1)^l\frac{D_l^{(s)}}{t_{l+k}}\sum_{j=k+l}^{\infty}\binom{m+j-k-l-1}{j-k-l}a_{nj}\Bigg], \tag{5.3}$$

provided the series on the right side converges for all n,k.

Proof. We assume that $A \in (X(r,s,t;\Delta^{(m)}), Y)$, then $A_n \in [X(r,s,t;\Delta^{(m)})]^\beta$ for all n. Thus it follows from Lemma 5.8, we have $\tilde{A}_n \in X^\beta = l_1$ for all n and $Ax = \tilde{A}y$ holds for every $x \in X(r,s,t;\Delta^{(m)})$, $y \in X$, which are connected by the relation $y = (A(r,s,t).\Delta^{(m)})x$. Hence $\tilde{A}y \in Y$. Since $x = (\Delta^{(m)})^{-1}(A(r,s,t))^{-1}y$, for every $y \in X$, we get some $x \in X(r,s,t;\Delta^{(m)})$ and hence $\tilde{A} \in (X,Y)$. This completes the proof. □

Lemma 5.11. *Let $X(r,s,t;\Delta^{(m)})$ be any sequence spaces for $X \in \{c_0, l_\infty\}$, $A = (a_{nk})_{n,k}$ be an infinite matrix and $\tilde{A} = (\tilde{a}_{nk})_{n,k}$ be the associate matrix defined in (5.3). If $A \in (X(r,s,t;\Delta^{(m)}), Y)$, where $Y \in \{c_0, c, l_\infty\}$, then*

$$\|L_A\| = \|A\|_{(X,l_\infty)} = \sup_n \sum_{k=0}^{\infty} |\tilde{a}_{nk}| < \infty.$$

Proof. Since the spaces $X(r,s,t;\Delta^{(m)})$ for $X \in \{c_0, l_\infty\}$ are BK spaces, using Lemma 5.4 we have

$$\|L_A\| = \|A\|_{(X,l_\infty)} = \sup_n \|A_n\|^*_{X(r,s,t;\Delta^{(m)})}.$$

Now from Lemma 5.9, we have

$$\|A_n\|^*_{X(r,s,t;\Delta^{(m)})} = \|\tilde{A}_n\|_1 = \sum_{k=0}^{\infty} |\tilde{a}_{nk}|,$$

which is finite as $(\tilde{A}_n) \in l_1$. This completes the proof. $\square$

Now we give the main results.

Theorem 5.12. *Let $X(r,s,t;\Delta^{(m)})$ be any sequence spaces, where $X \in \{c_0, l_\infty\}$.*
(a) If $A \in (X(r,s,t;\Delta^{(m)}), c_0)$ then

$$\|L_A\|_\chi = \limsup_{n\to\infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk}| \tag{5.4}$$

(b) If $A \in (X(r,s,t;\Delta^{(m)}), c)$ then

$$\frac{1}{2} \limsup_{n\to\infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{\alpha}_k| \le \|L_A\|_\chi \le \limsup_{n\to\infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{\alpha}_k|, \tag{5.5}$$

where $\tilde{\alpha}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ for all k.
(c) If $A \in (X(r,s,t;\Delta^{(m)}), l_\infty)$ then

$$0 \le \|L_A\|_\chi \le \limsup_{n\to\infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk}|. \tag{5.6}$$

Proof. (a) Let us first observe that the expressions in (5.4) and in (5.6) exist by Lemma 5.11. Also by using the Lemma 5.1 & 5.10, we can deduce that the expressions in (5.5) exists.
We write $S = S_{X(r,s,t;\Delta^{(m)})}$ in short. Then by Lemma 5.7, we have $\|L_A\|_\chi = \chi(AS)$. Since $X(r,s,t;\Delta^{(m)})$ and c_0 are BK spaces, A induces a continuous map L_A from $X(r,s,t;\Delta^{(m)})$ to c_0 by Lemma 5.3. Thus AS is bounded in c_0, i.e., $AS \in \mathcal{M}_{c_0}$. Now by Lemma 5.5,

$$\chi(AS) = \lim_{l\to\infty} \left(\sup_{x\in S} \|(I - P_l)(Ax)\|_\infty \right),$$

where $P_l : c_0 \to c_0$ is defined by $P_l(x) = (x_0, x_1, \cdots, x_l, 0, 0, \cdots)$ for all $x = (x_k) \in c_0$ and $l \in \mathbb{N}_0$. Therefore $\|(I - P_l)(Ax)\|_\infty = \sup_{n>l} |A_n(x)|$ for all $x \in X(r, s, t; \Delta^{(m)})$. Using (5.1) and Lemma 5.9, we have

$$\begin{aligned}\sup_{x\in S} \|(I - P_l)(Ax)\|_\infty &= \sup_{n>l} \|A_n\|^*_{X(r,s,t;\Delta^{(m)})} \\ &= \sup_{n>l} \|\tilde{A}_n\|_1\end{aligned}$$

Therefore $\chi(AS) = \lim_{l\to\infty} \left(\sup_{n>l} \|\tilde{A}_n\|_1\right) = \limsup_{n\to\infty} \|\tilde{A}_n\|_1 = \limsup_{n\to\infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk}|$. This completes the proof.

(b) We have $AS \in \mathcal{M}_c$. Let $P_l : c \to c$ be the operator from c onto the span of $\{e, e_0, e_1, \cdots, e_l\}$ defined as

$$P_l(z) = \hat{\ell}e + \sum_{k=0}^{r} (z_k - \hat{\ell})e_k,$$

where $\hat{\ell} = \lim_{k\to\infty} z_k$. Thus for every $l \in \mathbb{N}_0$, we have

$$(I - P_l)(z) = \sum_{k=l+1}^{\infty} (z_k - \hat{\ell})e_k.$$

Therefore $\|(I - P_l)(z)\|_\infty = \sup_{k>l} |z_k - \hat{\ell}|$ for all $z = (z_k) \in c$. Applying Lemma 5.6, we have

$$\frac{1}{2} \lim_{l\to\infty} \left(\sup_{x\in S} \|(I - P_l)(Ax)\|_\infty\right) \le \|L_A\|_\chi \le \lim_{l\to\infty} \left(\sup_{x\in S} \|(I - P_l)(Ax)\|_\infty\right). \quad (5.7)$$

Since $A \in (X(r, s, t; \Delta^{(m)}), c)$, we have by Lemma 5.10, $\tilde{A} \in (X, c)$ and $Ax = \tilde{A}y$ for every $x \in X(r, s, t; \Delta^{(m)})$ and $y \in X$, which are connected by the relation $y = (A(r, s, t).\Delta^{(m)})x$. Using Lemma 5.1, we have $\tilde{\alpha}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ exists for all k, $\tilde{\alpha} = (\tilde{\alpha}_k) \in X^\beta = l_1$ and $\lim_{n\to\infty} \tilde{A}_n(y) = \sum_{k=0}^{\infty} \tilde{\alpha}_k y_k$. Since $\|(I - P_l)(z)\|_\infty = \sup_{k>l} |z_k - \hat{\ell}|$, we have

$$\begin{aligned}\|(I - P_l)(Ax)\|_\infty &= \|(I - P_l)(\tilde{A}y)\|_\infty \\ &= \sup_{n>l} \left|\tilde{A}_n(y) - \sum_{k=0}^{\infty} \tilde{\alpha}_k y_k\right| \\ &= \sup_{n>l} \left|\sum_{k=0}^{\infty} (\tilde{a}_{nk} - \tilde{\alpha}_k) y_k\right|.\end{aligned}$$

Also we know that $x \in S = S_{X(r,s,t;\Delta^{(m)})}$ if and only if $y \in S_X$. From (5.1) and Lemma 5.2, we deduce

$$\begin{aligned}\sup_{x\in S}\|(I-P_l)(Ax)\|_\infty &= \sup_{n>l}\Big(\sup_{y\in S_X}\Big|\sum_{k=0}^{\infty}(\tilde{a}_{nk}-\tilde{\alpha}_k)y_k\Big|\Big)\\ &= \sup_{n>l}\|\tilde{A}_n-\tilde{\alpha}\|_X^* = \sup_{n>l}\|\tilde{A}_n-\tilde{\alpha}\|_1.\end{aligned}$$

Hence from (5.7), we have

$$\tfrac{1}{2}\limsup_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{\alpha}_k| \le \|L_A\|_\chi \le \limsup_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{\alpha}_k|.$$

(c) We first define an operator $P_l : l_\infty \to l_\infty$, as $P_l(x) = (x_0, x_1, \cdots, x_l, 0, 0, \cdots)$ for all $x = (x_k) \in l_\infty$, $l \in \mathbb{N}_0$. We have

$$AS \subset P_l(AS) + (I-P_l)(AS).$$

By the property of χ, we have

$$\begin{aligned}0 \le \chi(AS) &\le \chi(P_l(AS)) + \chi((I-P_l)(AS))\\ &= \chi((I-P_l)(AS))\\ &\le \sup_{x\in S}\|(I-P_l)(Ax)\|_\infty\\ &= \sup_{n>l}\|\tilde{A}_n\|_1.\end{aligned}$$

Hence

$$0 \le \chi(AS) \le \limsup_{n\to\infty}\|\tilde{A}_n\|_1 = \limsup_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}|.$$

This completes the proof. $\square$

Corollary 5.13. *Let $X(r,s,t;\Delta^{(m)})$ be any sequence spaces for $X \in \{c_0, l_\infty\}$.*

(a) If $A \in (X(r,s,t;\Delta^{(m)}), c_0)$, then L_A is compact if and only if $\lim_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}| = 0$

(b) If $A \in (X(r,s,t,\Delta^{(m)}), c)$ then L_A is compact if and only if

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{\alpha}_k| = 0, \text{ where } \tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk} \text{ for all } k.$$

(c) If $A \in (X(r,s,t,\Delta^{(m)}), l_\infty)$ then L_A is compact if $\lim_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}| = 0.$

Proof. The proof is immediate from Theorem 5.12. $\square$

Corollary 5.14. *For every matrix $A \in (l_\infty(r,s,t;\Delta^{(m)}), c_0)$ or $A \in (l_\infty(r,s,t;\Delta^{(m)}), c)$ the operator L_A induced by matrix A is compact.*

Proof. Let $A \in (l_\infty(r,s,t;\Delta^{(m)}), c_0)$ then $\tilde{A} \in (l_\infty, c_0)$, where $Ax = \tilde{A}y$ holds for every $x \in l_\infty(r,s,t;\Delta^{(m)})$ and $y \in l_\infty$, which are connected by the relation $y = (A(r,s,t).\Delta^{(m)})x$. Since $\tilde{A} \in (l_\infty, c_0)$, by Theorem 4.5(d), we have $\lim_{n\to\infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk}| = 0$. Hence by Corollary 5.13(a) the operator L_A is compact.
Similarly if $A \in (l_\infty(r,s,t;\Delta^{(m)}), c)$ then $\tilde{A} \in (l_\infty, c)$. From Theorem 4.5(g), we have $\lim_{n\to\infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{\alpha}_k| = 0$, where $\tilde{\alpha}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ for all k. Thus by Corollary 5.13(b), we have L_A is compact. □

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