Various generalized Ulam-Hyers stabilities of a nonic functional equations

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Abstract

In this paper, we have established the general solution and generalized Ulam - Hyers stability of the following nonic functional equation

$$\begin{aligned} f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) + 126f(x+y) - 126f(x) \\ + 84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) = 9!f(y) \end{aligned}$$

where $9! = 362880$ in a Banach Space (**BS**), Felbin's type Fuzzy Normed Space (**FFNS**) and Intuitionistic Fuzzy Normed Space (**IFNS**) using the standard direct and fixed point method.

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1 Introduction

One of the most interesting questions in the theory of functional equations concerning the famous Ulam stability problem is, as follows: when is it true that a mapping satisfying a functional equation approximately, must be close to an exact solution of the given functional equation?

The first stability problem was raised by S.M. Ulam [49] during his talk at the University of Wisconsin in 1940. In fact we are given a group $(G_1, \cdot)$ and let $(G_2, *)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

D.H. Hyers [16] gave the first affirmative partial answer to the question of Ulam for Banach spaces. It was further generalized via excellent results obtained by a number of authors [2, 12, 34, 42, 44].

One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of A.L. Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost

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every field of natural and social sciences. Every solution of the additive functional equation (1.1) is called an additive function.

The second famous functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.2}$$

is said to be **quadratic functional equation** because the quadratic function $f(x)=ax^2$ is a solution of the functional equation (1.2).

J.M. Rassias [36] introduced the following **cubic functional equation**

$$g(x+2y)+3g(x)=3g(x+y)+g(x-y)+6g(y) \tag{1.3}$$

and investigated its Ulam stability problem. The **quartic functional equation**

$$F(x+2y)+F(x-2y)+6F(x)=4[F(x+y)+F(x-y)+6F(y)] \tag{1.4}$$

was first introduced by J.M. Rassias [35], who solved its Ulam stability problem.

The general solution and the generalized Hyers-Ulam-Rassias stability of the generalized mixed type of functional equation

$$\begin{aligned} f(x+ay)+f(x-ay) &= a^2\left[f(x+y)+f(x-y)\right]+2\left(1-a^2\right)f(x) \\ &\quad+\frac{\left(a^4-a^2\right)}{12}\left[f(2y)+f(-2y)-4f(y)-4f(-y)\right] \end{aligned}$$

for fixed integers a with $a\neq 0,\pm 1$ having solution **additive, quadratic, cubic and quartic** was discussed by K. Ravi et. al., [45]. Its generalized Ulam-Hyers stability in multi-Banach spaces and non-Archimedean normed spaces via fixed point approach was respectively investigated by T.Z. Xu et. al., [52, 54].

Recently, C. Park and J.R. Lee [31] proved the Hyers - Ulam stability of the following **additive - quadratic - cubic - quartic functional equation**

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \tag{1.5}$$

in paranormed spaces.

The general solution of **Quintic and Sextic functional equations**

$$f(x+3y)-5f(x+2y)+10f(x+y)-10f(x)+5f(x-y)-f(x-2y)=120f(y) \tag{1.6}$$

and

$$\begin{aligned} &f(x+3y)-6f(x+2y)+15f(x+y)-20f(x)+15f(x-y) \\ &\qquad -6f(x-2y)+f(x-3y)=720f(y) \end{aligned} \tag{1.7}$$

was introduced and investigated on the generalized Ulam - Hyers stability in quasi $\beta-$normed spaces via fixed point method by T.Z. Xu et. al., [53]. Also, T.Z. Xu et. al.,[55] introduce and discuss the general solution and generalized Ulam-Hyers stability of **Septic and Octic functional equations**

$$\begin{aligned} &f(x+4y)-7f(x+3y)+21f(x+2y)-35f(x+y)+35f(x) \\ &\qquad -21f(x-y)+7f(x-2y)-f(x-3y)=5040f(y) \end{aligned} \tag{1.8}$$

and

$$f(x+4y)-8f(x+3y)+28f(x+2y)-56f(x+y)+70f(x)$$
$$-56f(x-y)+28f(x-2y)-8f(x-3y)+f(x-4y)=40320f(y) \tag{1.9}$$

in quasi $\beta-$normed spaces, respectively.

In this paper, we present the general solution and generalized Ulam - Hyers stability of the following **nonic functional equation**

$$f(x+5y)-9f(x+4y)+36f(x+3y)-84f(x+2y)+126f(x+y)$$
$$-126f(x)+84f(x-y)-36f(x-2y)+9f(x-3y)-f(x-4y)=9!f(y) \tag{1.10}$$

where $9! = 362880$ in a Banach Space (BS), Felbin's type Fuzzy Normed Space (FFNS) and Intuitionistic Fuzzy Normed Space (IFNS) by using the standard direct and fixed point method.

Now, we present the following theorem due to B. Margolis and J.B. Diaz [26] for the fixed point theory.

Theorem 1.1. [26] Suppose that for a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either

$$d(T^n x, T^{n+1}x) = \infty, \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 such that the properties hold:

(FP1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T;

(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;

(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

In Section 2, the general solution of (1.10) is provided.

In Section 3, 4 and 5 the generalized Ulam - Hyers stability of (1.10) is discussed in BS, FFNS and IFNS, respectively using both direct and fixed point methods.

2 General Solution of (1.10)

In this section, the general solution of the nonic functional equation (1.10) is given. For this, let us consider A and B be real vector spaces.

Theorem 2.1. If $f : A \to B$ be a mapping satisfying (1.10) for all $x, y \in A$ then f is Nonic.

Proof. Letting (x, y) by $(0, 0)$ in (1.10), one finds that

$$f(0) = 0. \tag{2.1}$$

Replacing (x, y) by $(0, x)$ in (1.10), we get

$$f(5x)-9f(4x)+36f(3x)-84f(2x)+126f(x)-126f(0)$$
$$+84f(-x)-36f(-2x)+9f(-3x)-f(-4x)=9!f(x) \tag{2.2}$$

for all $x \in A$. Again replacing (x,y) by $(x,-x)$ in (1.10), we obtain

$$\begin{aligned} f(-4x) - 9f(-3x) + 36f(-2x) - 84f(-x) + 126f(0) \\ - 126f(x) + 84f(2x) - 36f(3x) + 9f(4x) - f(5x) = 9!f(-x) \end{aligned} \tag{2.3}$$

for all $x \in A$. Adding (2.2) and (2.3), we reach

$$9!f(x) + 9!f(-x) = 0 \tag{2.4}$$

for all $x \in A$. It follows from (2.4), we arrive

$$f(-x) = -f(x) \tag{2.5}$$

for all $x \in A$. Hence f is an odd function. Setting (x,y) by $(0,2x)$ in (1.10), we get

$$\begin{aligned} & f(10x) - 9f(8x) + 36f(6x) - 84f(4x) + 126f(2x) \\ & \qquad - 126f(0) - 84f(2x) + 36f(4x) - 9f(6x) + f(8x) = 362880f(2x) \\ \text{i.e .,} \quad & f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - 362838f(2x) = 0 \end{aligned} \tag{2.6}$$

for all $x \in A$. Again setting (x,y) by $(5x,x)$ in (1.10), we obtain

$$\begin{aligned} & f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) \\ & \qquad - 126f(5x) + 84f(4x) - 36f(3x) + 9f(2x) - f(x) = 362880f(x) \\ \text{i.e .,} \quad & f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) \\ & \qquad - 126f(5x) + 84f(4x) - 36f(3x) + 9f(2x) - 362881f(x) = 0 \end{aligned} \tag{2.7}$$

for all $x \in A$. Subtracting (2.6) and (2.7), we arrive

$$\begin{aligned} 9f(9x) - 44f(8x) + 84f(7x) - 99f(6x) + 126f(5x) - 134f(4x) \\ + 36f(3x) - 362847f(2x) + 362881f(x) = 0 \end{aligned} \tag{2.8}$$

for all $x \in A$. Replacing (x,y) by $(4x,x)$ in (1.10), we get

$$\begin{aligned} f(9x) - 9f(8x) + 36f(7x) - 84f(6x) + 126f(5x) \\ - 126f(4x) + 84f(3x) - 36f(2x) + 9f(x) - f(0) = 9!f(x) \end{aligned} \tag{2.9}$$

for all $x \in A$. Multiplying by 9 on both sides of (2.9), one obtains

$$\begin{aligned} 9f(9x) - 81f(8x) + 324f(7x) - 756f(6x) + 1134f(5x) \\ - 1134f(4x) + 756f(3x) - 324f(2x) - 3265839f(x) = 0 \end{aligned} \tag{2.10}$$

for all $x \in A$. Subtracting (2.10) from (2.8), we arrive

$$\begin{aligned} 37f(8x) - 240f(7x) + 657f(6x) - 1008f(5x) \\ + 1002f(4x) - 720f(3x) - 362523f(2x) + 3628720f(x) = 0 \end{aligned} \tag{2.11}$$

for all $x \in A$. Letting (x, y) by $(3x, x)$ in (1.10), we have

$$\begin{aligned} f(8x) - 9f(7x) + 36f(6x) - 84f(5x) + 126f(4x) \\ - 126f(3x) + 84f(2x) - 36f(x) + 9f(0) - f(-x) = 9!f(x) \end{aligned} \tag{2.12}$$

for all $x \in A$. Multiplying by 37 on both sides of (2.12) and using oddness of f, one obtains

$$\begin{aligned} 37f(8x) - 333f(7x) + 1332f(6x) - 3108f(5x) + 46621f(4x) \\ - 46621f(3x) + 3108f(2x) - 13427855f(x) = 0 \end{aligned} \tag{2.13}$$

for all $x \in A$. Subtracting equations (2.11) and (2.13), we arrive at

$$\begin{aligned} 93(7x) - 675f(6x) + 2100f(5x) - 3660(4x) \\ + 3942f(3x) - 365631f(2x) + 17056575f(x) = 0 \end{aligned} \tag{2.14}$$

for all $x \in A$. Replacing (x, y) by $(2x, x)$ in (1.10), we obtain

$$\begin{aligned} f(7x) - 9f(6x) + 36f(5x) - 84f(4x) + 126f(3x) \\ - 126f(2x) + 84f(x) - 36f(0) + 9f(-x) - f(-2x) = 9!f(x) \end{aligned} \tag{2.15}$$

for all $x \in A$. Multiplying by 93 on both sides of (2.15) and using oddness of f, one finds

$$\begin{aligned} 93f(7x) - 837f(6x) + 3348f(5x) - 7812f(4x) \\ + 11718f(3x) - 11625f(2x) - 33740865f(x) = 0 \end{aligned} \tag{2.16}$$

for all $x \in A$. Subtracting equations (2.14) and (2.16), we reach

$$162(6x) - 1248f(5x) + 4152(4x) - 7776f(3x) - 354006f(2x) + 50797440f(x) = 0 \tag{2.17}$$

for all $x \in A$. Dividing (2.17) by 2, we arrive at

$$81(6x) - 624f(5x) + 2076(4x) - 3888f(3x) - 177003f(2x) + 25398720f(x) = 0 \tag{2.18}$$

for all $x \in A$. Replacing (x, y) by (x, x) in (1.10), we get

$$\begin{aligned} f(6x) - 9f(5x) + 36f(4x) - 84f(3x) + 126f(2x) \\ - 126f(x) + 84f(0) - 36f(-x) + 9f(-2x) - f(-3x) = 9!f(x) \end{aligned} \tag{2.19}$$

for all $x \in A$. Multiplying (2.19) by 81 and using oddness of f, one finds

$$81f(6x) - 729f(5x) + 2916f(4x) - 6723f(3x) + 9477f(2x) - 29400570f(x) = 0 \tag{2.20}$$

for all $x \in A$. Subtracting equations (2.18) and (2.20), we arrive at

$$105f(5x) - 840f(4x) + 2835f(3x) - 186480f(2x) + 54799290f(x) = 0 \tag{2.21}$$

for all $x \in A$. Replacing (x, y) by $(0, x)$ in (1.10), we obtain

$$\begin{aligned} f(5x) - 9f(4x) + 36f(3x) - 84f(2x) + 126f(x) \\ - 126f(0) + 84f(-x) - 36f(-2x) + 9f(-3x) - f(-4x) = 9!f(x) \end{aligned} \tag{2.22}$$

for all $x \in A$. Multiplying (2.22) by 105 and using oddness of f, one finds that

$$105f(5x) - 840f(4x) + 2835f(3x) - 5040f(2x) - 38097990f(x) = 0 \tag{2.23}$$

for all $x \in A$. Subtracting equations (2.21) and (2.23), we arrive at

$$-181440f(2x) + 92897280f(x) = 0 \tag{2.24}$$

for all $x \in A$. It follows from (2.24), we reach

$$f(2x) = 512f(x) \qquad \text{or} \qquad f(2x) = 2^9 f(x) \tag{2.25}$$

for all $x \in A$. Hence f is a nonic function. This completes the proof of the theorem. Q.E.D.

3 Stability Results In Banach Space

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.10) in Banach space using direct and fixed point methods.

Throughout this section, let we consider $\mathcal{G}$ be a normed space and $\mathcal{H}$ be a Banach space. Define a mapping $Df_9 : \mathcal{G} \to \mathcal{H}$ by

$$\begin{aligned} Df_9(x,y) =& f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) \\ &+ 126f(x+y) - 126f(x) + 84f(x-y) - 36f(x-2y) \\ &+ 9f(x-3y) - f(x-4y) - 9!f(y), \end{aligned}$$

where $9! = 362880$ for all $x, y \in \mathcal{G}$.

3.1 Banach Space: Direct Method

Theorem 3.1. Let $b = \pm 1$ and $\zeta, Z : \mathcal{G}^2 \to [0,\infty)$ be a function such that

$$\lim_{a\to\infty} \frac{\zeta\left(2^{ab}x, 2^{ab}y\right)}{2^{9ab}} = 0 \tag{3.1}$$

for all $x, y \in \mathcal{G}$. Let $Df_9 : \mathcal{G} \to \mathcal{H}$ be a function satisfying the inequality

$$\|Df_9(x,y)\| \leq \zeta(x,y) \tag{3.2}$$

for all $x, y \in \mathcal{G}$. Then there exists a unique Nonic function $\mathcal{N} : \mathcal{G} \to \mathcal{H}$ which satisfies (1.10) and

$$\|f(x) - \mathcal{N}(x)\| \leq \frac{1}{2^9} \sum_{c=\frac{1-b}{2}}^{\infty} \frac{Z\left(2^{cb}x, 2^{cb}x\right)}{2^{9cb}} \tag{3.3}$$

where $Z\left(2^{cb}x, 2^{cb}x\right)$ and $\mathcal{N}(x)$ are defined by

$$\begin{aligned} Z\left(2^{cb}x, 2^{cb}x\right) = \frac{1}{181440}\Big\{ &\frac{1}{2}\Big[\zeta(0, 2\cdot 2^{cb}x) + \zeta(5\cdot 2^{cb}x, 2^{cb}x) + 9\zeta(4\cdot 2^{cb}x, 2^{cb}x) \\ &+ 37\zeta(3\cdot 2^{cb}x, 2^{cb}x) + 93\zeta(2\cdot 2^{cb}x, 2^{cb}x)\Big] \\ &+ 81\zeta(2^{cb}x, 2^{cb}x) + 105\zeta(0, 2^{cb}x)\Big\} \end{aligned} \tag{3.4}$$

and

$$\mathcal{N}(x) = \lim_{c\to\infty} \frac{f(2^{cb}x)}{2^{cb}} \tag{3.5}$$

for all $x \in \mathcal{G}$, respectively.

Proof. Setting (x, y) by $(0, 2x)$ in (3.2), we get

$$\Big\| f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - 362838f(2x) \Big\| \leq \zeta(0, 2x) \tag{3.6}$$

for all $x \in \mathcal{G}$. Again setting (x, y) by $(5x, x)$ in (3.2), we obtain

$$\begin{aligned} &\Big\| f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) \\ &\quad - 126f(5x) + 84f(4x) - 36f(3x) + 9f(2x) - 362881f(x) \Big\| \leq \zeta(5x, x) \end{aligned} \tag{3.7}$$

for all $x \in \mathcal{G}$. Combining (3.6) and (3.7), we arrive

$$\begin{aligned} &\Big\| 9f(9x) - 44f(8x) + 84f(7x) - 99f(6x) + 126f(5x) - 134f(4x) \\ &\qquad + 36f(3x) - 362847f(2x) + 362881f(x) \Big\| \\ &= \Big\| f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - 362838f(2x) \\ &\qquad - f(10x) + 9f(9x) - 36f(8x) + 84f(7x) - 126f(6x) \\ &\qquad + 126f(5x) - 84f(4x) + 36f(3x) - 9f(2x) + 362881f(x) \Big\| \\ &\leq \Big\| f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - 362838f(2x) \Big\| \\ &\qquad + \Big\| f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) \\ &\qquad - 126f(5x) + 84f(4x) - 36f(3x) + 9f(2x) - 362881f(x) \Big\| \\ &\leq \zeta(0, 2x) + \zeta(5x, x) \end{aligned} \tag{3.8}$$

for all $x \in \mathcal{G}$. Replacing (x, y) by $(4x, x)$ in (3.2), we get

$$\begin{aligned} &\Big\| f(9x) - 9f(8x) + 36f(7x) - 84f(6x) + 126f(5x) \\ &\quad - 126f(4x) + 84f(3x) - 36f(2x) + 9f(x) - f(0) - 9!f(x) \Big\| \leq \zeta(4x, x) \end{aligned} \tag{3.9}$$

for all $x \in \mathcal{G}$. Multiplying by 9 on both sides of (3.9), one obtains

$$\begin{aligned} &\Big\| 9f(9x) - 81f(8x) + 324f(7x) - 756f(6x) + 1134f(5x) \\ &\quad - 1134f(4x) + 756f(3x) - 324f(2x) - 3265839f(x) \Big\| \leq 9\zeta(4x, x) \end{aligned} \tag{3.10}$$

for all $x \in \mathcal{G}$. It follows from (3.8) and (3.10), we arrive

$$\Big\| 37f(8x) - 240f(7x) + 657f(6x) - 1008f(5x) + 1002f(4x) - 720f(3x) \\ - 362523f(2x) + 3628720f(x) \Big\| \leq \zeta(0,2x) + \zeta(5x,x) + 9\zeta(4x,x) \tag{3.11}$$

for all $x \in \mathcal{G}$. Letting (x, y) by $(3x, x)$ in (3.2), we have

$$\Big\| f(8x) - 9f(7x) + 36f(6x) - 84f(5x) + 126f(4x) - 126f(3x) \\ + 84f(2x) - 36f(x) + 9f(0) - f(-x) - 9!f(x) \Big\| \leq \zeta(3x,x) \tag{3.12}$$

for all $x \in \mathcal{G}$. Multiplying by 37 on both sides of (3.12) and using oddness of f, one obtains

$$\Big\| 37f(8x) - 333f(7x) + 1332f(6x) - 3108f(5x) + 46621f(4x) \\ - 46621f(3x) + 3108f(2x) - 13427855f(x) \Big\| \leq 37\zeta(3x,x) \tag{3.13}$$

for all $x \in \mathcal{G}$. It follows form (3.11) and (3.13), we arrive at

$$\Big\| 93(7x) - 675f(6x) + 2100f(5x) - 3660(4x) + 3942f(3x) - 365631f(2x) \\ + 17056575f(x) \Big\| \leq \zeta(0,2x) + \zeta(5x,x) + 9\zeta(4x,x) + 37\zeta(3x,x) \tag{3.14}$$

for all $x \in \mathcal{G}$. Replacing (x, y) by $(2x, x)$ in (3.2), we obtain

$$\Big\| f(7x) - 9f(6x) + 36f(5x) - 84f(4x) + 126f(3x) - 126f(2x) \\ + 84f(x) - 36f(0) + 9f(-x) - f(-2x) - 9!f(x) \Big\| \leq \zeta(2x,x) \tag{3.15}$$

for all $x \in \mathcal{G}$. Multiplying by 93 on both sides of (3.15) and using oddness of f, one finds

$$\Big\| 93f(7x) - 837f(6x) + 3348f(5x) - 7812f(4x) \\ + 11718f(3x) - 11625f(2x) - 33740865f(x) \Big\| \leq 93\zeta(2x,x) \tag{3.16}$$

for all $x \in \mathcal{G}$. It follows from (3.14) and (3.16), we reach

$$\Big\| 162(6x) - 1248f(5x) + 4152(4x) - 7776f(3x) - 354006f(2x) + 50797440f(x) \Big\| \\ \leq \zeta(0,2x) + \zeta(5x,x) + 9\zeta(4x,x) + 37\zeta(3x,x) + 93\zeta(2x,x) \tag{3.17}$$

for all $x \in \mathcal{G}$. Dividing (3.17) by 2, we arrive at

$$\Big\| 81(6x) - 624f(5x) + 2076(4x) - 3888f(3x) - 177003f(2x) + 25398720f(x) \Big\| \\ \leq \frac{1}{2} \left[\zeta(0,2x) + \zeta(5x,x) + 9\zeta(4x,x) + 37\zeta(3x,x) + 93\zeta(2x,x) \right] \tag{3.18}$$

for all $x \in \mathcal{G}$. Replacing (x, y) by (x, x) in (3.2), we get

$$\Big\| f(6x) - 9f(5x) + 36f(4x) - 84f(3x) + 126f(2x) - 126f(x) + 84f(0) - 36f(-x) + 9f(-2x) - f(-3x) - 9!f(x)\Big\| \le \zeta(x, x) \tag{3.19}$$

for all $x \in \mathcal{G}$. Multiplying (3.19) by 81 and using oddness of f, one finds

$$\Big\| 81f(6x) - 729f(5x) + 2916f(4x) - 6723f(3x) + 9477f(2x) - 29400570f(x)\Big\| \le 81\zeta(x, x) \tag{3.20}$$

for all $x \in \mathcal{G}$. It follows from (3.18) and (3.20), we arrive at

$$\Big\| 105f(5x) - 840f(4x) + 2835f(3x) - 186480f(2x) + 54799290f(x)\Big\| \le \frac{1}{2}\left[\zeta(0, 2x) + \zeta(5x, x) + 9\zeta(4x, x) + 37\zeta(3x, x) + 93\zeta(2x, x)\right] + 81\zeta(x, x) \tag{3.21}$$

for all $x \in \mathcal{G}$. Replacing (x, y) by $(0, x)$ in (3.2), we obtain

$$\Big\| f(5x) - 9f(4x) + 36f(3x) - 84f(2x) + 126f(x) - 126f(0) + 84f(-x) - 36f(-2x) + 9f(-3x) - f(-4x) - 9!f(x)\Big\| \le \zeta(0, x) \tag{3.22}$$

for all $x \in \mathcal{G}$. Multiplying (3.22) by 105 and using oddness of f, one finds that

$$\Big\| 105f(5x) - 840f(4x) + 2835f(3x) - 5040f(2x) - 38097990f(x)\Big\| \le 105\zeta(0, x) \tag{3.23}$$

for all $x \in \mathcal{G}$. It follows from (3.21) and (3.23), we arrive at

$$\Big\| -181440f(2x) + 92897280f(x)\Big\| \le \frac{1}{2}\left[\zeta(0, 2x) + \zeta(5x, x) + 9\zeta(4x, x) + 37\zeta(3x, x) + 93\zeta(2x, x)\right] + 81\zeta(x, x) + 105\zeta(0, x) \tag{3.24}$$

for all $x \in \mathcal{G}$. It follows from (3.24), we reach

$$\Big\| f(2x) - 512f(x)\Big\| \le \frac{1}{181440}\Big\{\frac{1}{2}\left[\zeta(0, 2x) + \zeta(5x, x) + 9\zeta(4x, x) + 37\zeta(3x, x) + 93\zeta(2x, x)\right] + 81\zeta(x, x) + 105\zeta(0, x)\Big\} \tag{3.25}$$

for all $x \in \mathcal{G}$. Define

$$Z(x, x) = \frac{1}{181440}\Big\{\frac{1}{2}\left[\zeta(0, 2x) + \zeta(5x, x) + 9\zeta(4x, x) + 37\zeta(3x, x) + 93\zeta(2x, x)\right] + 81\zeta(x, x) + 105\zeta(0, x)\Big\} \tag{3.26}$$

for all $x \in \mathcal{G}$. From (3.26), we arrive

$$\left\| f(2x) - 2^9 f(x) \right\| \leq Z(x,x) \tag{3.27}$$

for all $x \in \mathcal{G}$. It follows from (3.27) that

$$\left\| \frac{f(2x)}{2^9} - f(x) \right\| \leq \frac{Z(x,x)}{2^9} \tag{3.28}$$

for all $x \in \mathcal{G}$. Now, replacing x by $2x$ and dividing by 2^9 in (3.28), we have

$$\left\| \frac{f(2^2x)}{2^{18}} - \frac{f(2x)}{2^9} \right\| \leq \frac{Z(2x,2x)}{2^{18}} \tag{3.29}$$

for all $x \in \mathcal{G}$. From (3.28) and (3.29), we obtain

$$\begin{aligned} \left\| \frac{f(2^2x)}{2^{18}} - f(x) \right\| &\leq \left\| \frac{f(2^2x)}{2^{18}} - \frac{f(2x)}{2^9} \right\| + \left\| \frac{f(2x)}{2^9} - f(x) \right\| \\ &\leq \frac{1}{2^9}\left[Z(x,x) + \frac{Z(2x,2x)}{2^9} \right] \end{aligned} \tag{3.30}$$

for all $x \in \mathcal{G}$. Generalizing, for a positive integer a, we reach

$$\left\| \frac{f(2^a x)}{2^{9a}} - f(x) \right\| \leq \frac{1}{2^9} \sum_{c=0}^{a-1} \frac{Z(2^c x, 2^c x)}{2^{9c}} \tag{3.31}$$

for all $x \in \mathcal{G}$. To prove the convergence of the sequence

$$\left\{ \frac{f(2^a x)}{2^{9a}} \right\},$$

replacing x by $2^d x$ and dividing by 2^{9d} in (3.31), for any $a, d > 0$, we get

$$\begin{aligned} \left\| \frac{f(2^{a+d}x)}{2^{9(a+d)}} - \frac{f(2^d x)}{2^{9d}} \right\| &= \frac{1}{2^{9d}} \left\| \frac{f(2^a \cdot 2^d x)}{2^{9a}} - f(2^d x) \right\| \\ &\leq \frac{1}{2^{9d}} \frac{1}{2^9} \sum_{c=0}^{a-1} \frac{Z(2^c \cdot 2^d x, 2^c \cdot 2^d x)}{2^{9c}} \\ &\leq \frac{1}{2^9} \sum_{c=0}^{\infty} \frac{Z(2^{c+d}x, 2^{c+d}x)}{2^{9(c+d)}} \\ &\to 0 \quad as \ \ d \to \infty \end{aligned}$$

for all $x \in \mathcal{G}$. Thus it follows that a sequence $\left\{ \frac{f(2^a x)}{2^{9a}} \right\}$ is a Cauchy in $\mathcal{H}$ and so it converges. Therefore, we see that a mapping $\mathcal{N}(x) : \mathcal{G} \to \mathcal{H}$ defined by

$$\mathcal{N}(x) = \lim_{a \to \infty} \frac{f(2^a x)}{2^{9a}}$$

is well defined for all $x \in \mathcal{G}$. In order to show that $\mathcal{N}$ satisfies (1.10), replacing (x, y) by $(2^a x, 2^a y)$ and dividing by 2^{9a} in (3.2), we have

$$\|\mathcal{N}(x, y)\| = \lim_{a\to\infty} \frac{1}{2^{9a}} \|Df_9(2^a x, 2^a y)\| \leq \lim_{a\to\infty} \frac{1}{2^{9a}} \zeta(2^a x, 2^a y)$$

for all $x, y \in \mathcal{G}$ and so the mapping $\mathcal{N}$ is Nonic. Taking the limit as a approaches to infinity in (3.31), we find that the mapping $\mathcal{N}$ is a Nonic mapping satisfying the inequality (3.3) near the approximate mapping $f : \mathcal{G} \to \mathcal{H}$ of equation (1.10). Hence, $\mathcal{N}$ satisfies (1.10), for all $x, y \in \mathcal{G}$.

To prove that $\mathcal{N}$ is unique, we assume now that there is $\mathcal{N}'$ as another Nonic mapping satisfying (1.10) and the inequality (3.3). Then it follows easily that

$$\mathcal{N}(2^a x) = 2^{9a}\mathcal{N}(x), \qquad \mathcal{N}'(2^a x) = 2^{9a}\mathcal{N}'(x)$$

for all $x \in \mathcal{G}$ and all $a \in \mathbb{N}$. Thus

$$\begin{aligned}
\|\mathcal{N}(x) - \mathcal{N}'(x)\| &= \frac{1}{2^{9a}} \|\mathcal{N}(2^a x) - \mathcal{N}'(2^a x)\| \\
&\leq \frac{1}{2^{9a}} \{\|\mathcal{N}(2^a x) - f(2^a x)\| + \|f(2^a x) - \mathcal{N}'(2^a x)\|\} \\
&\leq \frac{1}{2^8} \sum_{c=0}^{\infty} \frac{\zeta(2^{c+a}x, 2^{c+a}x)}{2^{9(c+a)}}
\end{aligned}$$

for all $x \in \mathcal{G}$. Therefore, as $a \to \infty$, in the above inequality, one establishes

$$\mathcal{N}(x) - \mathcal{N}'(x) = 0$$

for all $x \in \mathcal{G}$, completing the proof of the claimed uniqueness of $\mathcal{N}$. Hence the theorem holds for $b = 1$.

Case (ii): Assume $b = -1$.

Now replacing x by $\frac{x}{2}$ in (3.27), we get

$$\left\| f(x) - 2^9 f\left(\frac{x}{2}\right) \right\| \leq Z\left(\frac{x}{2}, \frac{x}{2}\right) \tag{3.32}$$

for all $x \in \mathcal{G}$. The rest of the proof is similar to that of case $b = 1$. Hence for $b = -1$ also the theorem holds. This completes the proof of the theorem. Q.E.D.

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.10).

Corollary 3.2. Let $Df_9 : \mathcal{G} \to \mathcal{H}$ be a mapping. If there exist real numbers ϑ and σ such that

$$\|Df_9(x, y)\| \leq \begin{cases} \vartheta, & \\ \vartheta \{||x||^\sigma + ||y||^\sigma\}, & \sigma \neq 9; \\ \vartheta ||x||^\sigma ||y||^\sigma, & 2\sigma \neq 9; \\ \vartheta \left\{||x||^\sigma ||y||^\sigma + \left\{||x||^{2\sigma} + ||y||^{2\sigma}\right\}\right\}, & 2\sigma \neq 9; \end{cases} \tag{3.33}$$

for all $x, y \in \mathcal{G}$, then there exists a unique Nonic function $\mathcal{N} : \mathcal{G} \to \mathcal{H}$ such that

$$\|f(x) - \mathcal{N}(x)\| \leq \begin{cases} \dfrac{\vartheta_C}{|2^9 - 1|}, \\ \dfrac{\vartheta_S ||x||^\sigma}{|2^9 - 2^\sigma|}, \\ \dfrac{\vartheta_P ||x||^{2\sigma}}{|2^9 - 2^{2\sigma}|}, \\ \dfrac{\vartheta_{SP} ||x||^{2\sigma}}{|2^9 - 2^{2\sigma}|} \end{cases} \tag{3.34}$$

where

$$\begin{aligned} \vartheta_C &= \frac{513\ \vartheta}{362880}, \\ \vartheta_S &= \frac{\vartheta\Big[5^\sigma + 9 \cdot 4^\sigma + 37 \cdot 3^\sigma + 94 \cdot 2^\sigma + 674\Big]}{362880}, \\ \vartheta_P &= \frac{\vartheta\Big[5^\sigma + 9 \cdot 4^\sigma + 37 \cdot 3^\sigma + 93 \cdot 2^\sigma + 162\Big]}{362880}, \\ \vartheta_{SP} &= \frac{\vartheta\Big[(5^{2\sigma} + 5^\sigma) + 9(4^{2\sigma} + 4^\sigma) + 37(3^{2\sigma} + 3^\sigma) + 93(2^{2\sigma} + 2^\sigma) + 2^{2\sigma} + 836\Big]}{362880} \end{aligned} \tag{3.35}$$

for all $x \in \mathcal{G}$.

3.2 Banach Space: Fixed Point Method

Throughout this section let $\mathcal{X}$ be a normed space and $\mathcal{Y}$ be a Banach space Define a mapping $Df_9 : \mathcal{X} \to \mathcal{Y}$ by

$$\begin{aligned} Df_9(x, y) =& f(x + 5y) - 9f(x + 4y) + 36f(x + 3y) - 84f(x + 2y) \\ &+ 126f(x + y) - 126f(x) + 84f(x - y) - 36f(x - 2y) \\ &+ 9f(x - 3y) - f(x - 4y) - 9!f(y), \end{aligned}$$

where $9! = 362880$ for all $x, y \in \mathcal{X}$. Using Theorem 1.1, we obtain the Hyers - Ulam stability of (1.10).

Theorem 3.3. Let $Df_9 : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\zeta : \mathcal{X}^2 \to [0, \infty)$ with the condition

$$\lim_{n \to \infty} \frac{1}{\hbar_i^{9n}} \zeta(\hbar_i^n x, \hbar_i^n y) = 0 \tag{3.36}$$

where

$$\hbar_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1 \end{cases} \tag{3.37}$$

such that the functional inequality

$$\|Df_9(x, y)\| \leq \zeta(x, y) \tag{3.38}$$

holds for all $x, y \in \mathcal{X}$. Assume that there exists $L = L(i)$ such that the function

$$x \to T(x,x) = Z\left(\frac{x}{2}, \frac{x}{2}\right)$$

where $Z(x,x)$ is defined in (3.26) with the property

$$\frac{1}{\hbar_i^9} T(\hbar_i x, \hbar_i x) = L\ T(x,x) \tag{3.39}$$

for all $x \in \mathcal{X}$. Then there exists a unique Nonic mapping $\mathcal{N} : \mathcal{X} \to \mathcal{Y}$ satisfying the functional equation (1.10) and

$$\| f(x) - \mathcal{N}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right) T(x,x) \tag{3.40}$$

for all $x \in \mathcal{X}$.

Proof. Consider the set

$$\Lambda = \{h/h : \mathcal{X} \to \mathcal{Y},\ h(0) = 0\}$$

and introduce the generalized metric on Λ,

$$\inf\{\rho \in (0,\infty) : \| h(x) - g(x) \| \leq \rho\ T(x,x), x \in \mathcal{X}\}. \tag{3.41}$$

It is easy to see that (3.41) is complete with respect to the defined metric. Define $J : \Lambda \to \Lambda$ by

$$Jh(x) = \frac{1}{\hbar_i^9} h(\hbar_i x),$$

for all $x \in \mathcal{X}$. Now, from (3.41) and $h, g \in \Lambda$, we arrive

$$\begin{aligned}
&\inf\{\rho \in (0,\infty) : \| h(x) - g(x) \| \leq \rho\ T(x,x), x \in \mathcal{X}\} \quad \textit{or} \\
&\inf\left\{\rho \in (0,\infty) : \left\| \frac{1}{\hbar_i^9} h(\hbar_i x) - \frac{1}{\hbar_i^9} g(\hbar_i x) \right\| \leq \frac{\rho}{\hbar_i^9} T(\hbar_i x, \hbar_i x), x \in \mathcal{X}\right\} \quad \textit{or} \\
&\inf\left\{L\rho \in (0,\infty) : \left\| \frac{1}{\hbar_i^9} h(\hbar_i x) - \frac{1}{\hbar_i^9} g(\hbar_i x) \right\| \leq L\rho T(x,x), x \in \mathcal{X}\right\} \quad \textit{or} \\
&\inf\{L\rho \in (0,\infty) : \| Jh(x) - Jg(x) \| \leq L\rho T(x,x), x \in \mathcal{X}\}.
\end{aligned}$$

This implies J is a strictly contractive mapping on Λ with Lipschitz constant L. It follows from (3.41),(3.27) and (3.39) for the case $i = 0$, we reach

$$\inf\left\{1 \in (0,\infty) : \left\| f(2x) - 2^9 f(x) \right\| \leq Z(x,x), x \in \mathcal{X}\right\} \quad \textit{or} \tag{3.42}$$

$$\begin{aligned}
&\inf\left\{1 \in (0,\infty) : \left\| \frac{f(2x)}{2^9} - f(x) \right\| \leq \frac{1}{2^9} Z(x,x), x \in \mathcal{X}\right\} \quad \textit{or} \\
&\inf\{L \in (0,\infty) : \|Jf(x) - f(x)\| \leq L\ T(x,x), x \in \mathcal{X}\} \quad \textit{or} \\
&\inf\left\{L^1 \in (0,\infty) : \|Jf(x) - f(x)\| \leq L\ T(x,x), x \in \mathcal{X}\right\} \quad \textit{or} \\
&\inf\left\{L^{1-0} \in (0,\infty) : \|Jf(x) - f(x)\| \leq L\ T(x,x), x \in \mathcal{X}\right\}.
\end{aligned} \tag{3.43}$$

Again replacing $x = \frac{x}{2}$ in (3.42) and (3.39) for the case $i = 1$, we get

$$\begin{aligned}
&\inf\left\{1 \in (0,\infty) : \left\| f(x) - 2^9 f\left(\frac{x}{2}\right)\right\| \leq Z\left(\frac{x}{2}, \frac{x}{2}\right), x \in \mathcal{X}\right\} \quad \textit{or} \\
&\inf\left\{1 \in (0,\infty) : \| f(x) - Jf(x)\| \leq T(x,x), x \in \mathcal{X}\right\} \quad \textit{or} \\
&\inf\left\{L^0 \in (0,\infty) : \| f(x) - Jf(x)\| \leq T(x,x), x \in \mathcal{X}\right\} \quad \textit{or} \\
&\inf\left\{L^{1-1} \in (0,\infty) : \| f(x) - Jf(x)\| \leq T(x,x), x \in \mathcal{X}\right\}. \qquad (3.44)
\end{aligned}$$

Thus, from (3.43) and (3.44), we arrive

$$\inf\left\{L^{1-i} \in (0,\infty) : \| f(x) - Jf(x)\| \leq L^{1-i} T(x,x), x \in \mathcal{X}\right\}. \qquad (3.45)$$

Hence property (FP1) holds. It follows from property (FP2) that there exists a fixed point $\mathcal{N}$ of J in Λ such that

$$\mathcal{N}(x) = \lim_{n\to\infty} \frac{1}{\hbar_i^{9n}} f(\hbar_i^n x) \qquad (3.46)$$

for all $x \in \mathcal{X}$. In order to show that $\mathcal{N}$ satisfies (1.10), replacing (x, y) by $(\hbar_i^n x, \hbar_i^n y)$ and dividing by $\hbar_i^{9n}$ in (3.38), we have

$$\|\mathcal{N}_9(x,y)\| = \lim_{n\to\infty} \frac{1}{\hbar_i^{9n}} \| Df_9(\hbar_i^n x, \hbar_i^n y)\| \leq \lim_{n\to\infty} \frac{1}{\hbar_i^{9n}} \zeta\left(\hbar_i^n x, \hbar_i^n y\right) = 0$$

for all $x, y \in \mathcal{X}$, and so the mapping $\mathcal{N}$ is Nonic. i.e., $\mathcal{N}$ satisfies the functional equation (1.10). By property (FP3), $\mathcal{N}$ is the unique fixed point of J in the set

$$\Delta = \{\mathcal{N} \in \Lambda : d(f, \mathcal{N}) < \infty\},$$

such that

$$\inf\left\{\rho \in (0,\infty) : \| f(x) - \mathcal{N}(x)\| \leq \rho T(x,x), x \in \mathcal{X}\right\}.$$

Finally by property (FP4), we obtain

$$\| f(x) - \mathcal{N}(x)\| \leq \| f(x) - Jf(x)\|,$$

implying

$$\| f(x) - \mathcal{N}(x)\| \leq \frac{L^{1-i}}{1-L},$$

which yields

$$\inf\left\{\frac{L^{1-i}}{1-L} \in (0,\infty) : \| f(x) - \mathcal{N}(x)\| \leq \left(\frac{L^{1-i}}{1-L}\right) T(x,x), x \in \mathcal{X}\right\}.$$

This completes the proof of the theorem. Q.E.D.

The following corollary is an immediate consequence of Theorem 3.3 concerning the stability of (1.10).

Corollary 3.4. Let $Df_9 : \mathcal{X} \to \mathcal{Y}$ be a mapping. If there exist real numbers ϑ and σ such that

$$\|Df_9(x,y)\| \leq \begin{cases} \vartheta, & \\ \vartheta\left\{||x||^{\sigma}+||y||^{\sigma}\right\} & \sigma \neq 9; \\ \vartheta||x||^{\sigma}||y||^{\sigma} & 2\sigma \neq 9; \\ \vartheta\left\{||x||^{\sigma}||y||^{\sigma}+\left\{||x||^{2\sigma}+||y||^{2\sigma}\right\}\right\} & 2\sigma \neq 9; \end{cases} \tag{3.47}$$

for all $x, y \in \mathcal{X}$, then there exists a unique Nonic function $\mathcal{N} : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - \mathcal{N}(x)\| \leq \begin{cases} \dfrac{\vartheta_1}{|2^9-1|}, \\ \dfrac{\vartheta_2}{|2^9-2^{\sigma}|}, \\ \dfrac{\vartheta_3}{|2^9-2^{2\sigma}|}, \\ \dfrac{\vartheta_4}{|2^9-2^{2\sigma}|} \end{cases} \tag{3.48}$$

where

$$\begin{aligned} \vartheta_1 &= \frac{513\vartheta}{362880}, \\ \vartheta_2 &= \frac{\vartheta||x||^{\sigma}}{362880\cdot 2^{\sigma}}\left[5^{\sigma}+9\cdot 4^{\sigma}+37\cdot 3^{\sigma}+93\cdot 2^{\sigma}+675\right] \\ \vartheta_3 &= \frac{\vartheta||x||^{2\sigma}}{362880\cdot 2^{2\sigma}}\left[5^{\sigma}+9\cdot 4^{\sigma}+37\cdot 3^{\sigma}+93\cdot 2^{\sigma}+162\right] \\ \vartheta_4 &= \frac{\vartheta||x||^{2\sigma}}{362880\cdot 2^{\sigma}}\left[5^{2\sigma}+5^{\sigma}+9(4^{2\sigma}+4^{\sigma})+37(3^{2\sigma}+3^{\sigma})+93(2^{2\sigma}+2^{\sigma})+837\right] \end{aligned} \tag{3.49}$$

for all $x \in \mathcal{X}$.

Proof. Let

$$\zeta(x,y) = \begin{cases} \vartheta, \\ \vartheta\left\{||x||^{\sigma}+||y||^{\sigma}\right\} \\ \vartheta||x||^{\sigma}||y||^{\sigma} \\ \vartheta\left\{||x||^{\sigma}||y||^{\sigma}+\left\{||x||^{2\sigma}+||y||^{2\sigma}\right\}\right\} \end{cases}$$

for all $x, y \in \mathcal{X}$. Now

$$\frac{1}{\hbar_i^{9n}}\zeta(\hbar_i^n x, \hbar_i^n y) = \begin{cases} \dfrac{\vartheta}{\hbar_i^{9n}}, \\ \dfrac{\vartheta}{\hbar_i^{9n}}\left\{||\hbar_i^n x||^{\sigma}+||\hbar_i^n y||^{\sigma}\right\}, \\ \dfrac{\vartheta}{\hbar_i^{9n}}||\hbar_i^n x||^{\sigma}\ ||\hbar_i^n y||^{\sigma} \\ \dfrac{\vartheta}{\hbar_i^{9n}}\left\{||\hbar_i^n x||^{\sigma}\ ||\hbar_i^n y||^{\sigma}+\left\{||\hbar_i^n x||^{2\sigma}+||\hbar_i^n y||^{2\sigma}\right\}\right\} \end{cases} \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (3.36) holds. But, we have

$$T(x,x) = Z\left(\frac{x}{2}, \frac{x}{2}\right)$$

has the property

$$\frac{1}{\hbar_i^9}T(\hbar_i x, \hbar_i x) = L\ T(x,x)$$

for all $x \in \mathcal{X}$. Hence,

$$\begin{aligned}
T(x,x) &= Z\left(\frac{x}{2},\frac{x}{2}\right)\\
&= \frac{1}{181440}\Big\{\frac{1}{2}\Big[\zeta(0,2\frac{x}{2})+\zeta(5\frac{x}{2},\frac{x}{2})+9\zeta(4\frac{x}{2},\frac{x}{2})+37\zeta(3\frac{x}{2},\frac{x}{2})+93\zeta(2\frac{x}{2},\frac{x}{2})\Big]\\
&\qquad\qquad +81\zeta(\frac{x}{2},\frac{x}{2})+105\zeta(0,\frac{x}{2})\Big\}\\
&= \begin{cases} \dfrac{513\vartheta}{362880}\\ \dfrac{\vartheta||x||^\sigma}{362880\cdot 2^\sigma}\Big[5^\sigma+9\cdot 4^\sigma+37\cdot 3^\sigma+93\cdot 2^\sigma+675\Big]\\ \dfrac{\vartheta||x||^{2\sigma}}{362880\cdot 2^{2\sigma}}\Big[5^\sigma+9\cdot 4^\sigma+37\cdot 3^\sigma+93\cdot 2^\sigma+162\Big]\\ \dfrac{\vartheta||x||^{2\sigma}}{362880\cdot 2^{\sigma}}\Big[5^{2\sigma}+5^\sigma+9(4^{2\sigma}+4^\sigma)+37(3^{2\sigma}+3^\sigma)+93(2^{2\sigma}+2^\sigma)+837\Big] \end{cases}
\end{aligned} \tag{3.50}$$

for all $x \in \mathcal{X}$. It follows from (3.50),

$$T(x,x) = Z\left(\frac{x}{2},\frac{x}{2}\right) = \begin{cases} \vartheta_1,\\ \vartheta_2,\\ \vartheta_3,\\ \vartheta_4, \end{cases} \tag{3.51}$$

where

$$\begin{aligned}
\vartheta_1 &= \frac{513\vartheta}{362880},\\
\vartheta_2 &= \frac{\vartheta||x||^\sigma}{362880\cdot 2^\sigma}\Big[5^\sigma+9\cdot 4^\sigma+37\cdot 3^\sigma+93\cdot 2^\sigma+675\Big],\\
\vartheta_3 &= \frac{\vartheta||x||^{2\sigma}}{362880\cdot 2^{2\sigma}}\Big[5^\sigma+9\cdot 4^\sigma+37\cdot 3^\sigma+93\cdot 2^\sigma+162\Big],\\
\vartheta_4 &= \frac{\vartheta||x||^{2\sigma}}{362880\cdot 2^{\sigma}}\Big[5^{2\sigma}+5^\sigma+9(4^{2\sigma}+4^\sigma)+37(3^{2\sigma}+3^\sigma)+93(2^{2\sigma}+2^\sigma)+837\Big].
\end{aligned} \tag{3.52}$$

for all $x \in \mathcal{X}$. Now, similarly by (3.50), we prove

$$\frac{1}{\hbar_i^9}T(\hbar_i x,\hbar_i x) = \begin{cases} \hbar_i^{-9}\ \vartheta_1,\\ \hbar_i^{\sigma-9}\vartheta_2,\\ \hbar_i^{2\sigma-9}\vartheta_3,\\ \hbar_i^{2\sigma-9}\vartheta_4. \end{cases}$$

Hence, the inequality (3.40) holds for

(i). $L = \hbar_i^{-9}$ if $i = 0$ and $L = \frac{1}{\hbar_i^{-9}}$ if $i = 1$, (ii). $L = \hbar_i^{\sigma-9}$ for $\sigma < 9$ if $i = 0$ and $L = \frac{1}{\hbar_i^{\sigma-9}}$ for $\sigma > 9$ if $i = 1$,

(iii). $L = \hbar_i^{2\sigma-9}$ for $2\sigma > 9$ if $i = 0$ and $L = \frac{1}{\hbar_i^{2\sigma-9}}$ for $2\sigma > 9$ if $i = 1$,

(iv). $L = \hbar_i^{2\sigma-9}$ for $2\sigma > 9$ if $i = 0$ and $L = \frac{1}{\hbar_i^{2\sigma-9}}$ for $2\sigma > 9$ if $i = 1$.

Now, from (3.40), we prove the following cases for condition (i).

$$\begin{array}{ll}
L = \hbar_i^{-9}, i = 0 & L = \frac{1}{\hbar_i^{-9}}, i = 1 \\
L = 2^{-9}, i = 0 & L = \frac{1}{2^{-9}}, i = 1 \\
L = 2^{-9}, i = 0 & L = 2^{9}, i = 1 \\
\parallel f(x) - \mathcal{N}(x) \parallel & \parallel f(x) - \mathcal{N}(x) \parallel \\
\quad \leq \left(\frac{L^{1-i}}{1-L}\right) T(x,x) & \quad \leq \left(\frac{L^{1-i}}{1-L}\right) T(x,x) \\
\quad = \left(\frac{(2^{-9})^{1-0}}{1-2^{-9}}\right) \vartheta_1 & \quad = \left(\frac{(2^{9})^{1-1}}{1-2^{9}}\right) \vartheta_1 \\
\quad = \left(\frac{2^{-9}}{1-2^{-9}}\right) \vartheta_1 & \quad = \left(\frac{1}{1-2^{9}}\right) \vartheta_1 \\
\quad = \left(\frac{\vartheta_1}{2^{9}-1}\right) & \quad = \left(\frac{\vartheta_1}{1-2^{9}}\right)
\end{array}$$

Also, from (3.40), we prove the following cases for condition (ii).

$$\begin{array}{ll}
L = \hbar_i^{\sigma-9}, \sigma < 9, i = 0 & L = \frac{1}{\hbar_i^{\sigma-9}}, \sigma > 9, i = 1 \\
L = 2^{\sigma-9}, \sigma < 9, i = 0 & L = \frac{1}{2^{\sigma-9}}, \sigma < 9, i = 1 \\
L = 2^{\sigma-9}, \sigma < 9, i = 0 & L = 2^{9-\sigma}, \sigma > 9, i = 1 \\
\parallel f(x) - \mathcal{N}(x) \parallel & \parallel f(x) - \mathcal{N}(x) \parallel \\
\quad \leq \left(\frac{L^{1-i}}{1-L}\right) T(x,x) & \quad \leq \left(\frac{L^{1-i}}{1-L}\right) T(x,x) \\
\quad = \left(\frac{(2^{\sigma-9})^{1-0}}{1-2^{\sigma-9}}\right) \vartheta_2 & \quad = \left(\frac{(2^{9-\sigma})^{1-1}}{1-2^{9-\sigma}}\right) \vartheta_2 \\
\quad = \left(\frac{2^{\sigma-9}}{1-2^{\sigma-9}}\right) \vartheta_2 & \quad = \left(\frac{1}{1-2^{9-\sigma}}\right) \vartheta_2 \\
\quad = \left(\frac{2^{\sigma}}{2^{9}-2^{\sigma}}\right) \vartheta_2 & \quad = \left(\frac{2^{\sigma}}{2^{\sigma}-2^{9}}\right) \vartheta_2
\end{array}$$

Again, from (3.40), we prove the following cases for condition (iii).

$$\begin{array}{ll}
L = \hbar_i^{2\sigma-9}, 2\sigma < 9, i = 0 & L = \frac{1}{\hbar_i^{2\sigma-9}}, 2\sigma > 9, i = 1 \\
L = 2^{2\sigma-9}, 2\sigma < 9, i = 0 & L = \frac{1}{2^{2\sigma-9}}, 2\sigma < 9, i = 1 \\
L = 2^{2\sigma-9}, 2\sigma < 9, i = 0 & L = 2^{9-2\sigma}, 2\sigma > 9, i = 1 \\
\parallel f(x) - \mathcal{N}(x) \parallel & \parallel f(x) - \mathcal{N}(x) \parallel \\
\quad \leq \left(\frac{L^{1-i}}{1-L}\right) T(x,x) & \quad \leq \left(\frac{L^{1-i}}{1-L}\right) T(x,x) \\
\quad = \left(\frac{(2^{2\sigma-9})^{1-0}}{1-2^{2\sigma-9}}\right) \vartheta_3 & \quad = \left(\frac{(2^{9-2\sigma})^{1-1}}{1-2^{9-2\sigma}}\right) \vartheta_3 \\
\quad = \left(\frac{2^{2\sigma-9}}{1-6^{2\sigma-9}}\right) \vartheta_3 & \quad = \left(\frac{1}{1-2^{9-2\sigma}}\right) \vartheta_3 \\
\quad = \left(\frac{2^{2\sigma}}{2^{9}-2^{2\sigma}}\right) \vartheta_3 & \quad = \left(\frac{2^{2\sigma}}{2^{2\sigma}-2^{9}}\right) \vartheta_3
\end{array}$$

Finally, to prove (3.40) for condition (iv), the proof is similar to that of condition (iii). Hence the proof is complete. Q.E.D.

4 Stability Results In Felbin's Type Spaces

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.10) in Felbin's type spaces using direct and fixed point methods.

Now, we recall the basic definitions and notations in Felbin's type spaces in Grantner [13], Lowen [25], Hoehle [15], Kaleva [21, 22, 23], Felbin [11] and Xiao and Zhu [50, 51]

Definition 4.1. [50] Let X be a real linear space, L and $\mathbb{R}$ (respectively, left norm and right norm) be symmetric and non-decreasing mappings from $[0,1] \times [0,1] \to [0,1]$ satisfying $L(0,0) = 0, \mathbb{R}(1,1) = 1$. Then $||\cdot||$ is called a fuzzy norm and $(X, ||\cdot||, L, \mathbb{R})$ is a Fuzzy Normed Linear Space (abbreviated to FNLS) if the mapping $||\cdot|| : X \to F^*(R)$ satisfies the following axioms, where $[||x||]_\alpha = [||x||_\alpha^-, ||x||_\alpha^+]$ for $x \in X$ and $\alpha \in (0,1]$:

$(A1)$ $||x|| = 0$ if and only if $x = 0$,

$(A2)$ $||rx|| = |r| \odot ||x||$ for all $x \in X$ and $r \in (-\infty, \infty)$,

$(A3)$ For all $x, y \in X$,

$(A3L)$ if $s \leq ||x||_1^-, t \leq ||y||_1^-$ and $s + t \leq ||x| + y||_1^-$,
then $||x + y||(s + t) \geq L(||x||(s), ||y||(t))$,

$(A3R)$ if $s \geq ||x||_1^-, t \geq ||y||_1^-$ and $s + t \geq ||x| + y||_1^-$,
then $||x + y||(s + t) \leq L(||x||(s), ||y||(t))$.

Theorem 4.2. [47] Let $(X, ||\cdot||, L, R)$ be an FNLS and $\lim\limits_{a \to 0^+} R(a, a) = 0$. Then $(X, ||\cdot||, L, R)$ is a Hausdorff topological vector space, whose neighborhood base of origin is $\{N(\varepsilon, \alpha) : \varepsilon > 0, \alpha \in (0,1]\}$, where $N(\varepsilon, \alpha) = \{x : ||x||_\alpha^+ \leq \varepsilon\}$.

Definition 4.3. Let $(X, ||\cdot||, L, R)$ be an FNLS. A sequence $\{x_n\}_{n=1}^\infty \subseteq X$ converges to $x \in X$, if $\lim\limits_{n\to\infty} \|x_n - x\|_\alpha^+$, for every $\alpha \in (0,1]$ denoted by $\lim\limits_{n\to\infty} x_n = x$.

Definition 4.4. Let $(X, ||\cdot||, L, R)$ be an FNLS. A sequence $\{x_n\}_{n=1}^\infty \subseteq X$ is called a Cauchy sequence if $\lim\limits_{m,n\to\infty} \|x_m - x_n\|_\alpha^+ = 0$ for every $\alpha \in (0,1]$.

Definition 4.5. Let $(X, ||\cdot||, L, R)$ be an FNLS. A subset $A \subseteq X$ is said to be complete if every Cauchy sequence in A, converges in A. The fuzzy normed space $(X, ||\cdot||, L, R)$ is said to be a fuzzy Banach space if it is complete.

Throughout this section let G be a linear space and $(H, |.|, L, R)$ be a fuzzy Banach space satisfying $(R-1)$. Define a mapping $Df_9 : G \to H$ by

$$\begin{aligned} Df_9(x,y) =& f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) \\ &+ 126f(x+y) - 126f(x) + 84f(x-y) - 36f(x-2y) \\ &+ 9f(x-3y) - f(x-4y) - 9!f(y), \end{aligned}$$

where $9! = 362880$ for all $x, y \in G$.

The proof of the following theorems and corollaries is similar to that of Theorems 3.1, 3.3 and Corollaries 3.2, 3.4. Hence the details of the proofs are omitted.

4.1 Felbin's Type Spaces: Direct Method

Theorem 4.6. Let $b = \pm 1$ and $\zeta, Z : G^2 \to F^*(R)$ be a function such that

$$\lim_{a\to\infty} \frac{\zeta\left(2^{ab}x, 2^{ab}y\right)_\alpha^+}{2^{9ab}} = 0 \tag{4.1}$$

for all $x, y \in G$. Let $Df_9 : G \to H$ be a function satisfying the inequality

$$\|Df_9(x,y)\|_\alpha^+ \ \preceq\ \zeta\,(x,y)_\alpha^+ \tag{4.2}$$

for all $x, y \in G$. Then there exists a unique Nonic function $N : G \to H$ which satisfies (1.10) and

$$\|f(x) - N(x)\|_\alpha^+ \preceq \frac{1}{2^9} \odot \sum_{c=\frac{1-b}{2}}^{\infty} \frac{Z\left(2^{cb}x, 2^{cb}x\right)_\alpha^+}{2^{9cb}} \tag{4.3}$$

where $Z\left(2^{cb}x, 2^{cb}x\right)_\alpha^+$ and $N(x)$ are defined by

$$\begin{aligned} Z\left(2^{cb}x, 2^{cb}x\right)_\alpha^+ = \frac{1}{181440}\Big\{\frac{1}{2}\Big[&\zeta(0, 2\cdot 2^{cb}x)_\alpha^+ + \zeta(5\cdot 2^{cb}x, 2^{cb}x)_\alpha^+ + 9\zeta(4\cdot 2^{cb}x, 2^{cb}x)_\alpha^+ \\ &+ 37\zeta(3\cdot 2^{cb}x, 2^{cb}x)_\alpha^+ + 93\zeta(2\cdot 2^{cb}x, 2^{cb}x)_\alpha^+\Big] \\ &+ 81\zeta(2^{cb}x, 2^{cb}x)_\alpha^+ + 105\zeta(0, 2^{cb}x)_\alpha^+\Big\} \end{aligned} \tag{4.4}$$

and

$$N(x) = \lim_{c\to\infty} \frac{f(2^{cb}x)}{2^{cb}} \tag{4.5}$$

for all $x \in G$, respectively.

Corollary 4.7. Let $Df_9 : G \to H$ be a mapping. If there exist real numbers ϑ and σ such that

$$\|Df_9(x,y)\|_\alpha^+ \ \preceq\ \begin{cases} \vartheta, & \\ \vartheta \otimes \{||x||^\sigma \oplus ||y||^\sigma\}, & \sigma \neq 9; \\ \vartheta \otimes ||x||^\sigma \otimes ||y||^\sigma, & \sigma \neq \frac{9}{2}; \\ \vartheta \otimes \left\{||x||^\sigma \otimes ||y||^\sigma + \left\{||x||^{2\sigma} \oplus ||y||^{2\sigma}\right\}\right\}, & \sigma \neq \frac{9}{2}; \end{cases} \tag{4.6}$$

for all $x, y \in G$, then there exists a unique Nonic function $N : G \to H$ such that

$$\|f(x) - N(x)\|_\alpha^+ \quad \preceq \quad \begin{cases} \dfrac{\vartheta_C}{|2^9 - 1|}, \\ \dfrac{\vartheta_S(||x||^\sigma)_\alpha^+}{|2^9 - 2^\sigma|}, \\ \dfrac{\vartheta_P(||x||^{2\sigma})_\alpha^+}{|2^9 - 2^{2\sigma}|}, \\ \dfrac{\vartheta_{SP}(||x||^{2\sigma})_\alpha^+}{|2^9 - 2^{2\sigma}|} \end{cases} \tag{4.7}$$

where

$$\begin{aligned}
\vartheta_C &= \frac{513 \otimes \vartheta_\alpha^+}{362880}, \\
\vartheta_S &= \frac{\vartheta_\alpha^+ \otimes \left[5^\sigma \oplus 9 \odot 4^\sigma \oplus 37 \odot 3^\sigma \oplus 94 \odot 2^\sigma \oplus 674\right]}{362880}, \\
\vartheta_P &= \frac{\vartheta_\alpha^+ \otimes \left[5^\sigma \oplus 9 \odot 4^\sigma \oplus 37 \odot 3^\sigma \oplus 93 \odot 2^\sigma \oplus 162\right]}{362880}, \\
\vartheta_{SP} &= \frac{\vartheta_\alpha^+ \otimes \left[(5^{2\sigma} \oplus 5^\sigma) \oplus 9(4^{2\sigma} \oplus 4^\sigma) \oplus 37(3^{2\sigma} \oplus 3^\sigma) \oplus 93(2^{2\sigma} \oplus 2^\sigma) \oplus 2^{2\sigma} \oplus 836\right]}{362880}
\end{aligned} \tag{4.8}$$

for all $x \in G$.

4.2 Felbin's Type Spaces: Fixed Point Method

Theorem 4.8. Let $Df_9 : G \to H$ be a mapping for which there exists a function $\zeta : G^2 \to F^*(R)$ with the condition

$$\lim_{n\to\infty} \frac{1}{\hbar_i^{9n}} \zeta(\hbar_i^n x, \hbar_i^n y)_\alpha^+ = 0 \tag{4.9}$$

where

$$\hbar_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1 \end{cases} \tag{4.10}$$

such that the functional inequality

$$\|Df_9(x,y)\|_\alpha^+ \ \preceq \ \zeta(x,y)_\alpha^+ \tag{4.11}$$

holds for all $x, y \in G$. Assume that there exists $L = L(i)$ such that the function

$$x \to T(x,x)_\alpha^+ = Z\left(\frac{x}{2}, \frac{x}{2}\right)_\alpha^+$$

where $Z(x,x)_\alpha^+$ is defined in (4.4) with the property

$$\frac{1}{\hbar_i^9} \odot T(\hbar_i x, \hbar_i x) = L \odot T(x,x)_\alpha^+ \tag{4.12}$$

for all $x \in G$. Then there exists a unique Nonic mapping $N : G \to H$ satisfying the functional equation (1.10) and

$$\| \, f(x) - N(x) \, \| \ \preceq \ \left(\frac{L^{1-i}}{1-L}\right) T(x,x)_\alpha^+ \tag{4.13}$$

for all $x \in G$.

Corollary 4.9. Let $Df_9 : G \to H$ be a mapping. If there exist real numbers ϑ and σ such that

$$\|Df_9(x,y)\|_\alpha^+ \ \preceq \ \begin{cases} \vartheta, & \\ \vartheta \otimes \{||x||^\sigma \oplus ||y||^\sigma\}, & \sigma \neq 9; \\ \vartheta \otimes ||x||^\sigma \otimes ||y||^\sigma, & \sigma \neq \frac{9}{2}; \\ \vartheta \otimes \{||x||^\sigma \otimes ||y||^\sigma + \{||x||^{2\sigma} \oplus ||y||^{2\sigma}\}\}, & \sigma \neq \frac{9}{2}; \end{cases} \tag{4.14}$$

for all $x, y \in G$, then there exists a unique Nonic function $N : G \to H$ such that

$$\|f(x) - N(x)\|_\alpha^+ \preceq \begin{cases} \dfrac{\vartheta_1}{|2^9 - 1|}, \\ \dfrac{\vartheta_2}{|2^9 - 2^\sigma|}, \\ \dfrac{\vartheta_3}{|2^9 - 2^{2\sigma}|}, \\ \dfrac{\vartheta_4}{|2^9 - 2^{2\sigma}|} \end{cases} \tag{4.15}$$

where

$$\begin{aligned} \vartheta_1 &= \frac{513 \otimes \vartheta_\alpha^+}{362880}, \\ \vartheta_2 &= \frac{\vartheta_\alpha^+(||x||^\sigma)_\alpha^+}{362880 \odot 2^\sigma} \otimes \left[5^\sigma \oplus 9 \odot 4^\sigma \oplus 37 \odot 3^\sigma \oplus 93 \odot 2^\sigma \oplus 675\right] \\ \vartheta_3 &= \frac{\vartheta_\alpha^+(||x||^{2\sigma})_\alpha^+}{362880 \odot 2^{2\sigma}} \otimes \left[5^\sigma \oplus 9 \odot 4^\sigma \oplus 37 \odot 3^\sigma \oplus 93 \odot 2^\sigma \oplus 162\right] \\ \vartheta_4 &= \frac{\vartheta_\alpha^+(||x||^{2\sigma})_\alpha^+}{362880 \odot 2^\sigma} \otimes \left[5^{2\sigma} \oplus 5^\sigma \oplus 9(4^{2\sigma} \oplus 4^\sigma) \oplus 37(3^{2\sigma} \oplus 3^\sigma) \oplus 93(2^{2\sigma} \oplus 2^\sigma) \oplus 837\right] \end{aligned} \tag{4.16}$$

for all $x \in G$.

5 Intuitionistic Fuzzy Normed Space: Stability Results

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.10) in IFNS: Intuitionistic Fuzzy Normed Space using direct and fixed point methods.

Now, we recall the basic definitions and notations in Intuitionistic Fuzzy Normed Space.

Definition 5.1. A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is said to be continuous $t-$ norm if $*$ satisfies the following conditions:

(1) $*$ is commutative and associative;

(2) $*$ is continuous;

(3) $a * 1 = a$ for all $a \in [0, 1]$;

(4) $a * b \le c * d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 5.2. A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is said to be continuous $t-$ conorm if $\diamond$ satisfies the following conditions:

(1') $\diamond$ is commutative and associative;

(2') $\diamond$ is continuous;

(3') $a \diamond 0 = a$ for all $a \in [0, 1]$;

(4') $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Using the notions of continuous $t-$ norm and $t-$ conorm, Saadati and Park [29] introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 5.3. The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an Intuitionistic Fuzzy Normed Spaces (for short, IFNS) if X is a vector space, $*1$ is a continuous $t-$ norm, $\diamond$ is a continuous $t-$ conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$

(IFN1) $\mu(x,t) + \nu(x,t) \leq 1$,

(IFN2) $\mu(x,t) > 0$,

(IFN3) $\mu(x,t) = 1$, if and only if $x = 0$.

(IFN4) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{\alpha}\right)$ for each $\alpha \neq 0$,

(IFN5) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,

(IFN6) $\mu(x, \cdot) : (0, \infty) \to [0,1]$ is continuous,

(IFN7) $\lim\limits_{t\to\infty} \mu(x,t) = 1$ and $\lim\limits_{t\to 0} \mu(x,t) = 0$,

(IFN8) $\nu(x,t) < 1$,

(IFN9) $\nu(x,t) = 0$, if and only if $x = 0$.

(IFN10) $\nu(\alpha x, t) = \nu\left(x, \frac{t}{\alpha}\right)$ for each $\alpha \neq 0$,

(IFN11) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$,

(IFN12) $\nu(x, \cdot) : (0, \infty) \to [0,1]$ is continuous,

(IFN13) $\lim\limits_{t\to\infty} \nu(x,t) = 0$ and $\lim\limits_{t\to 0} \nu(x,t) = 1$.

In this case, (μ, ν) is called an intuitionistic fuzzy norm.

Example 5.4. Let $(X, \|\cdot\|)$ be a normed space. Let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \leq 0; \end{cases} \qquad and \qquad \nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \leq 0. \end{cases}$$

Then $(X, \mu, \nu, *, \diamond)$ is an IFN-space.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [29].

Definition 5.5. Let $(X, \mu, \nu, *, \diamond)$ be an IFN-space. Then, a sequence The sequence $x = \{x_k\}$ is said to be *intuitionistic fuzzy convergent* to a point $L \in X$ if

$$\lim \quad \mu(x_k - L, t) = 1 \quad and \quad \lim \quad \nu(x_k - L, t) = 0$$

for all $t > 0$. In this case, we write

$$x_k \xrightarrow{IF} L \qquad as \qquad k \to \infty$$

Definition 5.6. Let $(X, \mu, \nu, *, \diamond)$ be an IFN-space. Then, $x = \{x_k\}$ is said to be *intuitionistic fuzzy Cauchy sequence* if

$$\mu(x_{k+p} - x_k, t) = 1 \qquad \textit{and} \qquad \nu(x_{k+p} - x_k, t) = 0$$

for all $t > 0$, and $p = 1, 2 \cdots$.

Definition 5.7. Let $(X, \mu, \nu, *, \diamond)$ be an IFN-space. Then $(X, \mu, \nu, *, \diamond)$ is said to be *complete* if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent $(X, \mu, \nu, *, \diamond)$.

Hereafter, throughout this section, assume that X be a linear space, (Z, μ', ν') be an intuitionistic fuzzy space and (Y, μ, ν) be a intuitionistic fuzzy Banach space. Now, we use the following notation for a given mapping $Df_9 : X \to Y$ such that

$$\begin{aligned} Df_9(x, y) =& f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) \\ &+ 126f(x+y) - 126f(x) + 84f(x-y) - 36f(x-2y) \\ &+ 9f(x-3y) - f(x-4y) - 9!f(y), \end{aligned}$$

where $9! = 362880$ for all $x, y \in X$.

5.1 IFNS: Direct Method

Theorem 5.8. Let $\tau \in \{1, -1\}$. Let $\zeta : X \times X \to Z$ be a function such that for some $0 < \left(\frac{p}{2}\right)^{\tau} < 1$,

$$\left.\begin{aligned} \mu'(\zeta(2^{n\tau}x, 2^{n\tau}y), r) \geq \mu'(p^{n\tau}\zeta(x, y), r) \\ \nu'(\zeta(2^{n\tau}x, 2^{n\tau}y), r) \leq \nu'(p^{n\tau}\zeta(x, y), r) \end{aligned}\right\} \tag{5.1}$$

for all $x \in X$ and all $r > 0$ and

$$\left.\begin{aligned} \lim_{n\to\infty} \mu'(\zeta(2^{\tau n}x, 2^{\tau n}y), 2^{\tau n}r) = 1 \\ \lim_{n\to\infty} \nu'(\zeta(2^{\tau n}x, 2^{\tau n}y), 2^{\tau n}r) = 0 \end{aligned}\right\} \tag{5.2}$$

for all $x, y \in X$ and all $r > 0$. Let $Df_9 : X \to Y$ be a function satisfies the inequality

$$\left.\begin{aligned} \mu(Df_9(x, y), r) \geq \mu'(\zeta(x, y), r) \\ \nu(Df_9(x, y), r) \leq \nu'(\zeta(x, y), r) \end{aligned}\right\} \tag{5.3}$$

for all $x, y \in X$ and all $r > 0$. Then there exists a unique Nonic mapping $N : X \to Y$ satisfying (1.10) and

$$\left.\begin{aligned} \mu(f(x) - N(x), r) \geq \mu'\left(Z_\mu(x, x), 2^9\wp|2^9 - p|r\right) \\ \nu(f(x) - N(x), r) \geq \nu'\left(Z_\nu(x, x), 2^9\wp|2^9 - p|r\right) \end{aligned}\right\} \tag{5.4}$$

where

$$Z_{\mu}(x,x) = \mu'\left(\zeta\left(0,2x\right),r\right) * \mu'\left(\zeta\left(5x,x\right),r\right) * \mu'\left(\zeta\left(\zeta(4x,x)\right),r\right) * \mu'\left(\zeta\left(3x,x\right),r\right) \\ * \mu'\left(\zeta\left(2x,x\right),r\right) * \mu'\left(\zeta\left(x,x\right),r\right) * \mu'\left(\zeta\left(0,x\right),r\right) \tag{5.5}$$

and

$$Z_{\nu}(x,x) = \nu'\left(\zeta\left(0,2x\right),r\right) \diamond \nu'\left(\zeta\left(5x,x\right),r\right) \diamond \nu'\left(\zeta\left(\zeta(4x,x)\right),r\right) \diamond \nu'\left(\zeta\left(3x,x\right),r\right) \\ \diamond \nu'\left(\zeta\left(2x,x\right),r\right) \diamond \nu'\left(\zeta\left(x,x\right),r\right) \diamond \nu'\left(\zeta\left(0,x\right),r\right) \tag{5.6}$$

for all $x \in X$ and all $r > 0$.

Proof. Let $\tau = 1$. Replacing (x,y) by $(0,2x)$ in (5.3), we get

$$\mu\left(f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - 362838f(2x), r\right) \geq \mu'\left(\zeta\left(0,2x\right),r\right) \tag{5.7}$$

for all $x \in X$ and all $r > 0$. Again setting (x,y) by $(5x,x)$ in (5.3), we obtain

$$\mu\Big(f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) - 126f(5x) \\ + 84f(4x) - 36f(3x) + 9f(2x) - 362881f(x), r\Big) \geq \mu'\left(\zeta\left(5x,x\right),r\right) \tag{5.8}$$

for all $x \in X$ and all $r > 0$. Combining (5.7), (5.8) and using (IFN5), we arrive

$$\begin{aligned} &\mu\Big(9f(9x) - 44f(8x) + 84f(7x) - 99f(6x) + 126f(5x) - 134f(4x) \\ &\qquad + 36f(3x) - 362847f(2x) + 362881f(x), 2r\Big) \\ &\geq \mu\left(f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - 362838f(2x), r\right) * \\ &\qquad \mu\Big(f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) - 126f(5x) \\ &\qquad + 84f(4x) - 36f(3x) + 9f(2x) - 362881f(x), r\Big) \\ &\geq \mu'\left(\zeta\left(0,2x\right),r\right) * \mu'\left(\zeta\left(5x,x\right),r\right) \end{aligned} \tag{5.9}$$

for all $x \in X$ and all $r > 0$. Replacing (x,y) by $(4x,x)$ in (5.3), we get

$$\mu\Big(f(9x) - 9f(8x) + 36f(7x) - 84f(6x) + 126f(5x) - 126f(4x) + 84f(3x) \\ - 36f(2x) + 9f(x) - f(0) - 9!f(x), r\Big) \geq \mu'\left(\zeta\left(\zeta(4x,x)\right),r\right) \tag{5.10}$$

for all $x \in X$ and all $r > 0$. It follows form (IFN4) and (5.10)

$$\mu\Big(9f(9x) - 81f(8x) + 324f(7x) - 756f(6x) + 1134f(5x) - 1134f(4x) \\ + 756f(3x) - 324f(2x) - 3265839f(x), 9r\Big) \geq \mu'\left(\zeta\left(\zeta(4x,x)\right),r\right) \tag{5.11}$$

for all $x \in X$ and all $r > 0$. It follows from (5.9), (5.11) and (IFN5), we arrive

$$\begin{aligned}\mu\Big(37f(8x) - 240f(7x) + 657f(6x) - 1008f(5x) + 1002f(4x)\\ - 720f(3x) - 362523f(2x) + 3628720f(x), 11r\Big)\\ \geq \mu'\left(\zeta\left(0, 2x\right), r\right) * \mu'\left(\zeta\left(5x, x\right), r\right) * \mu'\left(\zeta\left(\zeta(4x, x)\right), r\right)\end{aligned} \tag{5.12}$$

for all $x \in X$ and all $r > 0$. Letting (x, y) by $(3x, x)$ in (5.3), we have

$$\begin{aligned}\mu\Big(f(8x) - 9f(7x) + 36f(6x) - 84f(5x) + 126f(4x) - 126f(3x)\\ + 84f(2x) - 36f(x) + 9f(0) - f(-x) - 9!f(x), r\Big) \geq \mu'\left(\zeta\left(3x, x\right), r\right)\end{aligned} \tag{5.13}$$

for all $x \in X$ and all $r > 0$. Using oddness of f and (IFN4) in (5.13), we get

$$\begin{aligned}\mu\Big(37f(8x) - 333f(7x) + 1332f(6x) - 3108f(5x) + 46621f(4x)\\ - 46621f(3x) + 3108f(2x) - 13427855f(x), 37r\Big) \geq \mu'\left(\zeta\left(3x, x\right), r\right)\end{aligned} \tag{5.14}$$

for all $x \in X$ and all $r > 0$. It follows form (5.12), (5.14) and (IFN5), we arrive at

$$\begin{aligned}\mu\Big(93(7x) - 675f(6x) + 2100f(5x) - 3660(4x) + 3942f(3x)\\ - 365631f(2x) + 17056575f(x), 48r\Big)\\ \geq \mu'\left(\zeta\left(0, 2x\right), r\right) * \mu'\left(\zeta\left(5x, x\right), r\right) * \mu'\left(\zeta\left(\zeta(4x, x)\right), r\right) * \mu'\left(\zeta\left(3x, x\right), r\right)\end{aligned} \tag{5.15}$$

for all $x \in X$ and all $r > 0$. Replacing (x, y) by $(2x, x)$ in (5.3), we obtain

$$\begin{aligned}\mu\Big(f(7x) - 9f(6x) + 36f(5x) - 84f(4x) + 126f(3x) - 126f(2x) + 84f(x)\\ - 36f(0) + 9f(-x) - f(-2x) - 9!f(x), r\Big) \geq \mu'\left(\zeta\left(2x, x\right), r\right)\end{aligned} \tag{5.16}$$

for all $x \in X$ and all $r > 0$. It follows form (IFN4) and using oddness of f in (5.16), we get

$$\begin{aligned}\mu\Big(93f(7x) - 837f(6x) + 3348f(5x) - 7812f(4x) + 11718f(3x)\\ - 11625f(2x) - 33740865f(x), 93r\Big) \geq \mu'\left(\zeta\left(2x, x\right), r\right)\end{aligned} \tag{5.17}$$

for all $x \in X$ and all $r > 0$. It follows from (5.15), (5.17) and (IFN5), we reach

$$\begin{aligned}\mu\Big(162(6x) - 1248f(5x) + 4152(4x) - 7776f(3x) - 354006f(2x) + 50797440f(x), 141r\Big)\\ \geq \mu'\left(\zeta\left(0, 2x\right), r\right) * \mu'\left(\zeta\left(5x, x\right), r\right) * \mu'\left(\zeta\left(\zeta(4x, x)\right), r\right) * \mu'\left(\zeta\left(3x, x\right), r\right) * \mu'\left(\zeta\left(2x, x\right), r\right)\end{aligned} \tag{5.18}$$

for all $x \in X$ and all $r > 0$. It follows from (5.18) and (IFN4),

$$\mu\Big(81(6x) - 624f(5x) + 2076(4x) - 3888f(3x) - 177003f(2x) + 25398720f(x), \frac{141}{2}r\Big) \geq \mu'\left(\zeta\left(0,2x\right),r\right) * \mu'\left(\zeta\left(5x,x\right),r\right) * \mu'\left(\zeta\left(\zeta(4x,x)\right),r\right) * \mu'\left(\zeta\left(3x,x\right),r\right) * \mu'\left(\zeta\left(2x,x\right),r\right) \tag{5.19}$$

for all $x \in X$ and all $r > 0$. Replacing (x, y) by (x, x) in (5.3), we get

$$\mu\Big(f(6x) - 9f(5x) + 36f(4x) - 84f(3x) + 126f(2x) - 126f(x) + 84f(0) - 36f(-x) + 9f(-2x) - f(-3x) - 9!f(x), r\Big) \geq \mu'\left(\zeta\left(x,x\right),r\right) \tag{5.20}$$

for all $x \in X$ and all $r > 0$. Using oddness of f in (5.20) and (IFN4), we obtain

$$\mu\Big(81f(6x) - 729f(5x) + 2916f(4x) - 6723f(3x) + 9477f(2x) - 29400570f(x), 81r\Big) \geq \mu'\left(\zeta\left(x,x\right),r\right) \tag{5.21}$$

for all $x \in X$ and all $r > 0$. It follows from (5.19), (5.21) and (IFN5), we arrive at

$$\mu\Big(105f(5x) - 840f(4x) + 2835f(3x) - 186480f(2x) + 54799290f(x), \frac{303}{2}r\Big) \geq \mu'\left(\zeta\left(0,2x\right),r\right) * \mu'\left(\zeta\left(5x,x\right),r\right) * \mu'\left(\zeta\left(\zeta(4x,x)\right),r\right) * \mu'\left(\zeta\left(3x,x\right),r\right) * \mu'\left(\zeta\left(2x,x\right),r\right) * \mu'\left(\zeta\left(x,x\right),r\right) \tag{5.22}$$

for all $x \in X$ and all $r > 0$. Replacing (x, y) by $(0, x)$ in (5.3), we obtain

$$\mu\Big(f(5x) - 9f(4x) + 36f(3x) - 84f(2x) + 126f(x) - 126f(0) + 84f(-x) - 36f(-2x) + 9f(-3x) - f(-4x) - 9!f(x), r\Big) \geq \mu'\left(\zeta\left(0,x\right),r\right) \tag{5.23}$$

for all $x \in X$ and all $r > 0$. Using oddness of f in (5.23) and (IFN4), we get

$$\mu\Big(105f(5x) - 840f(4x) + 2835f(3x) - 5040f(2x) - 38097990f(x), 105r\Big) \geq \mu'\left(\zeta\left(0,x\right),r\right) \tag{5.24}$$

for all $x \in X$ and all $r > 0$. It follows from (5.22), (5.24) and (IFN5), we arrive at

$$\mu\Big(-181440f(2x) + 92897280f(x), \frac{513}{2}r\Big) \geq \mu'\left(\zeta\left(0,2x\right),r\right) * \mu'\left(\zeta\left(5x,x\right),r\right) * \mu'\left(\zeta\left(\zeta(4x,x)\right),r\right) * \mu'\left(\zeta\left(3x,x\right),r\right) * \mu'\left(\zeta\left(2x,x\right),r\right) * \mu'\left(\zeta\left(x,x\right),r\right) * \mu'\left(\zeta\left(0,x\right),r\right) \tag{5.25}$$

for all $x \in X$ and all $r > 0$. It follows from (5.25) and (IFN4), we reach

$$\begin{aligned}\mu\Big(f(2x) - 512f(x), \frac{513}{2\cdot 181440}r\Big) \\ \geq \mu'\left(\zeta\left(0, 2x\right), r\right) * \mu'\left(\zeta\left(5x, x\right), r\right) * \mu'\left(\zeta\left(\zeta(4x, x)\right), r\right) * \mu'\left(\zeta\left(3x, x\right), r\right) \\ * \mu'\left(\zeta\left(2x, x\right), r\right) * \mu'\left(\zeta\left(x, x\right), r\right) * \mu'\left(\zeta\left(0, x\right), r\right)\end{aligned} \tag{5.26}$$

for all $x \in X$ and all $r > 0$. Define

$$\begin{aligned}Z_\mu(x, x) = \mu'\left(\zeta\left(0, 2x\right), r\right) * \mu'\left(\zeta\left(5x, x\right), r\right) * \mu'\left(\zeta\left(\zeta(4x, x)\right), r\right) * \mu'\left(\zeta\left(3x, x\right), r\right) \\ * \mu'\left(\zeta\left(2x, x\right), r\right) * \mu'\left(\zeta\left(x, x\right), r\right) * \mu'\left(\zeta\left(0, x\right), r\right)\end{aligned} \tag{5.27}$$

and

$$\wp = \frac{513}{362880} \tag{5.28}$$

for all $x \in X$ and all $r > 0$. From (5.50), we arrive

$$\mu\Big(f(2x) - 2^9 f(x), \wp r\Big) \geq \mu'\left(Z_\mu(x, x), r\right) \tag{5.29}$$

for all $x \in X$ and all $r > 0$. Similarly, we prove

$$\nu\Big(f(2x) - 2^9 f(x), \wp r\Big) \leq \nu'\left(Z_\nu(x, x), r\right) \tag{5.30}$$

where

$$\begin{aligned}Z_\nu(x, x) = \nu'\left(\zeta\left(0, 2x\right), r\right) \diamond \nu'\left(\zeta\left(5x, x\right), r\right) \diamond \nu'\left(\zeta\left(\zeta(4x, x)\right), r\right) \diamond \nu'\left(\zeta\left(3x, x\right), r\right) \\ \diamond \nu'\left(\zeta\left(2x, x\right), r\right) \diamond \nu'\left(\zeta\left(x, x\right), r\right) \diamond \nu'\left(\zeta\left(0, x\right), r\right)\end{aligned} \tag{5.31}$$

for all $x \in X$ and all $r > 0$. It follows from (5.29) and (5.30),

$$\left.\begin{aligned}\mu\Big(f(2x) - 2^9 f(x), \wp r\Big) \geq \mu'\left(Z_\mu(x, x), r\right) \\ \nu\Big(f(2x) - 2^9 f(x), \wp r\Big) \leq \nu'\left(Z_\nu(x, x), r\right)\end{aligned}\right\} \tag{5.32}$$

for all $x \in X$ and all $r > 0$. Using (IFN4) in (5.32), we arrive

$$\left.\begin{aligned}\mu\Big(\tfrac{f(2x)}{2^9} - f(x), \tfrac{\wp r}{2^9}\Big) \geq \mu'\left(Z_\mu(x, x), r\right) \\ \nu\Big(\tfrac{f(2x)}{2^9} - f(x), \tfrac{\wp r}{2^9}\Big) \leq \nu'\left(Z_\nu(x, x), r\right)\end{aligned}\right\} \tag{5.33}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^n x$ in (5.33), we have

$$\left.\begin{aligned}\mu\Big(\tfrac{f(2^{n+1}x)}{2^9} - f(2^n x), \tfrac{\wp r}{2^9}\Big) \geq \mu'\left(Z_\mu(2^n x, 2^n x), r\right) \\ \nu\Big(\tfrac{f(2^{n+1}x)}{2^9} - f(2^n x), \tfrac{\wp r}{2^9}\Big) \leq \nu'\left(Z_\nu(2^n x, 2^n x), r\right)\end{aligned}\right\} \tag{5.34}$$

for all $x \in X$ and all $r > 0$. It is easy to verify from (5.34) and (IFN4) that

$$\left.\begin{aligned}\mu\left(\frac{f(2^{n+1}x)}{2^{9(n+1)}} - \frac{f(2^n x)}{2^{9n}}, \frac{\wp r}{2^9 . 2^{9n}}\right) \geq \mu'\left(Z_\mu(x,x), \frac{r}{p^n}\right)\\ \nu\left(\frac{f(2^{n+1}x)}{2^{9(n+1)}} - \frac{f(2^n x)}{2^{9n}}, \frac{\wp r}{2^9 . 2^{9n}}\right) \leq \nu'\left(Z_\nu(x,x), \frac{r}{p^n}\right)\end{aligned}\right\} \tag{5.35}$$

for all $x \in X$ and all $r > 0$. Replacing r by $p^n r$ in (5.35), we get

$$\left.\begin{aligned}\mu\left(\frac{f(2^{n+1}x)}{2^{9(n+1)}} - \frac{f(2^n x)}{2^{9n}}, \frac{p^n \wp r}{2^9 . 2^{9n}}\right) \geq \mu'\left(Z_\mu(x,x), r\right)\\ \nu\left(\frac{f(2^{n+1}x)}{2^{9(n+1)}} - \frac{f(2^n x)}{2^{9n}}, \frac{p^n \wp r}{2^9 . 2^{9n}}\right) \leq \nu'\left(Z_\nu(x,x), r\right)\end{aligned}\right\} \tag{5.36}$$

for all $x \in X$ and all $r > 0$. It is easy to see that

$$\frac{f(2^n x)}{2^{9n}} - f(x) = \sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{2^{9(i+1)}} - \frac{f(2^i x)}{2^{9i}} \tag{5.37}$$

for all $x \in X$. From equations (5.36) and (5.37), we have

$$\left.\begin{aligned}\mu\left(\frac{f(2^n x)}{2^{9n}} - f(x), \sum_{i=0}^{n-1} \frac{p^i \wp r}{2^9 . 2^{9i}}\right) = \mu\left(\sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{2^{9(i+1)}} - \frac{f(2^i x)}{2^{9i}}, \sum_{i=0}^{n-1} \frac{p^i \wp r}{2^9 . 2^{9i}}\right)\\ \nu\left(\frac{f(2^n x)}{2^{9n}} - f(x), \sum_{i=0}^{n-1} \frac{p^i \wp r}{2^9 . 2^{9i}}\right) = \nu\left(\sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{2^{9(i+1)}} - \frac{f(2^i x)}{2^{9i}}, \sum_{i=0}^{n-1} \frac{p^i \wp r}{2^9 . 2^{9i}}\right)\end{aligned}\right\} \tag{5.38}$$

for all $x \in X$ and all $r > 0$. From equations (5.36) and (5.39), we have

$$\left.\begin{aligned}\mu\left(\frac{f(2^n x)}{2^{9n}} - f(x), \sum_{i=0}^{n-1} \frac{p^i \wp r}{2^9 . 2^{9i}}\right) \geq \prod_{i=0}^{n-1} \mu\left(\frac{f(2^{i+1}x)}{2^{9(i+1)}} - \frac{f(2^i x)}{2^{9i}}, \frac{p^i \wp r}{2^9 . 2^{9i}}\right)\\ \nu\left(\frac{f(2^n x)}{2^{9n}} - f(x), \sum_{i=0}^{n-1} \frac{p^i \wp r}{2^9 . 2^{9i}}\right) \leq \coprod_{i=0}^{n-1} \nu\left(\frac{f(2^{i+1}x)}{2^{9(i+1)}} - \frac{f(2^i x)}{2^{9i}}, \frac{p^i \wp r}{2^9 . 2^{9i}}\right)\end{aligned}\right\} \tag{5.39}$$

where

$$\prod_{i=0}^{n-1} a_j = a_1 * a_2 * \cdots * a_n \qquad \textit{and} \qquad \coprod_{i=0}^{n-1} b_j = b_1 \diamond b_2 \diamond \cdots \diamond b_n$$

for all $x \in X$ and all $r > 0$. Hence

$$\left.\begin{aligned}\mu\left(\frac{f(2^n x)}{2^{9n}} - f(x), \sum_{i=0}^{n-1} \frac{p^i \wp r}{2^9 . 2^{9i}}\right) \geq \prod_{i=0}^{n-1} \mu'\left(Z_\mu(x,x), r\right) = \mu'\left(Z_\mu(x,x), r\right)\\ \nu\left(\frac{f(2^n x)}{2^{9n}} - f(x), \sum_{i=0}^{n-1} \frac{p^i \wp r}{2^9 . 2^{9i}}\right) \leq \coprod_{i=0}^{n-1} \nu'\left(Z_\nu(x,x), r\right) = \nu'\left(Z_\nu(x,x), r\right)\end{aligned}\right\} \tag{5.40}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^m x$ in (5.40) and using (5.2), $(IFN4)$, we obtain

$$\left.\begin{aligned}\mu\left(\frac{f(2^{n+m}x)}{2^{9(n+m)}} - \frac{f(2^m x)}{2^{9m}}, \sum_{i=0}^{n-1} \frac{p^i \wp r}{2^9 . 2^{9(i+m)}}\right) \geq \mu'\left(Z_\mu(2^m x, 2^m x), r\right) = \mu'\left(Z_\mu(x,x), \frac{r}{p^m}\right)\\ \nu\left(\frac{f(2^{n+m}x)}{2^{9(n+m)}} - \frac{f(2^m x)}{2^{9m}}, \sum_{i=0}^{n-1} \frac{p^i \wp r}{2^9 . 2^{9(i+m)}}\right) \leq \nu'\left(Z_\nu(2^m x, 2^m x), r\right) = \nu'\left(Z_\nu(x,x), \frac{r}{p^m}\right)\end{aligned}\right\} \tag{5.41}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Replacing r by $p^m r$ in (5.41), we get

$$\left.\begin{aligned} &\mu\Big(\tfrac{f(2^{n+m}x)}{2^{9(n+m)}} - \tfrac{f(2^m x)}{2^{9m}}, \textstyle\sum_{i=0}^{n-1} \tfrac{p^{i+m}\wp r}{2^9 . 2^{9(i+m)}}\Big) \geq \mu'\left(Z_\mu(x,x), r\right) \\ &\nu\Big(\tfrac{f(2^{n+m}x)}{2^{9(n+m)}} - \tfrac{f(2^m x)}{2^{9m}}, \textstyle\sum_{i=0}^{n-1} \tfrac{p^{i+m}\wp r}{2^9 . 2^{9(i+m)}}\Big) \leq \nu'\left(Z_\nu(x,x), r\right) \end{aligned}\right\} \tag{5.42}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. It follows from (5.43), that

$$\left.\begin{aligned} &\mu\Big(\tfrac{f(2^{n+m}x)}{2^{9(n+m)}} - \tfrac{f(2^m x)}{2^{9m}}, r\Big) \geq \mu'\left(Z_\mu(x,x), \tfrac{r}{\sum_{i=m}^{n-1} \frac{p^i \wp}{2^9 . 2^{9i}}}\right) \\ &\nu\Big(\tfrac{f(2^{n+m}x)}{2^{9(n+m)}} - \tfrac{f(2^m x)}{2^{9m}}, r\Big) \leq \nu'\left(Z_\nu(x,x), \tfrac{r}{\sum_{i=m}^{n-1} \frac{p^i \wp}{2^9 . 2^{9i}}}\right) \end{aligned}\right\} \tag{5.43}$$

holds for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Since $0 < p < 9$ and $\sum\limits_{i=0}^{n} \left(\frac{p}{9}\right)^i < \infty$. The Cauchy criterion for convergence in IFNS shows that $\left\{\dfrac{f(2^n x)}{2^{9n}}\right\}$ is a Cauchy sequence in (Y, μ, ν). Since (Y, μ, ν) is a complete IFN-space this sequence converges to some point $N(x) \in Y$. So, one can define the mapping $N : X \to Y$ by $\left\{\dfrac{f(2^n x)}{2^{9n}}\right\}$

$$\lim_{n\to\infty} \mu\left(\frac{f(2^n x)}{2^{9n}} - N(x), r\right) = 1, \lim_{n\to\infty} \nu\left(\frac{f(2^n x)}{2^{9n}} - N(x), r\right) = 0$$

for all $x \in X$ and all $r > 0$. Hence

$$\frac{f(2^n x)}{2^{9n}} \xrightarrow{IF} N(x), \quad as \quad n \to \infty.$$

Letting $m = 0$ in (5.43), we arrive

$$\left.\begin{aligned} &\mu\Big(\tfrac{f(2^n x)}{2^{9n}} - f(x), r\Big) \geq \mu'\left(Z_\mu(x,x), \tfrac{r}{\sum_{i=0}^{n-1} \frac{p^i \wp}{2^9 . 2^{9i}}}\right) \\ &\nu\Big(\tfrac{f(2^n x)}{2^{9n}} - f(x), r\Big) \leq \nu'\left(Z_\nu(x,x), \tfrac{r}{\sum_{i=0}^{n-1} \frac{p^i \wp}{2^9 . 2^{9i}}}\right) \end{aligned}\right\} \tag{5.44}$$

for all $x \in X$ and all $r > 0$. Letting n tend to infinity in (5.44), we have

$$\left.\begin{aligned} &\mu\Big(N(x) - f(x), r\Big) \geq \mu'\left(Z_\mu(x,x), 2^9 \wp r |2^9 - p|\right) \\ &\nu\Big(N(x) - f(x), r\Big) \leq \nu'\left(Z_\nu(x,x), 2^9 \wp r |2^9 - p|\right) \end{aligned}\right\} \tag{5.45}$$

for all $x \in X$ and all $r > 0$. To prove N satisfies (1.10), replacing (x, y) by $(2^n x, 2^n y)$ in (5.3) respectively, we obtain

$$\left.\begin{aligned} &\mu\left(\tfrac{1}{2^{9n}} Df_9(2^n x, 2^n y), r\right) \geq \mu'\left(\zeta(2^n x, 2^n y), 2^{9n} r\right) \\ &\nu\left(\tfrac{1}{2^{9n}} Df_9(2^n x, 2^n y), r\right) \leq \nu'\left(\zeta(2^n x, 2^n y), 2^{9n} r\right) \end{aligned}\right\} \tag{5.46}$$

for all $x \in X$ and all $r > 0$. Now,

$$\begin{aligned}
&\mu\Big(N(x+5y) - 9N(x+4y) + 36N(x+3y) - 84N(x+2y) + 126N(x+y) \\
&\quad - 126N(x) + 84N(x-y) - 36N(x-2y) + 9N(x-3y) - N(x-4y) - 9!N(y), r\Big) \\
&\geq \mu\Big(N(x+5y) - \frac{1}{2^{9n}} f(2^n(x+5y)), \frac{r}{12}\Big) * \mu\Big(-9\Big(N(x+4y) - \frac{1}{2^{9n}} f(2^n(x+4y))\Big), \frac{r}{12}\Big) * \\
&\quad \mu\Big(+36\Big(N(x+3y) - \frac{1}{2^{9n}} f(2^n(x+3y))\Big), \frac{r}{12}\Big) * \mu\Big(-84\Big(N(x+2y) - \frac{1}{2^{9n}} f(2^n(x+2y))\Big), \frac{r}{12}\Big) * \\
&\quad \mu\Big(+126\Big(N(x+y) - \frac{1}{2^{9n}} f(2^n(x+y))\Big), \frac{r}{12}\Big) * \mu\Big(-126\Big(N(x) - \frac{1}{2^{9n}} f(2^n(x))\Big), \frac{r}{12}\Big) * \\
&\quad \mu\Big(+84\Big(N(x-y) - \frac{1}{2^{9n}} f(2^n(x-y))\Big), \frac{r}{12}\Big) * \mu\Big(-36\Big(N(x-2y) - \frac{1}{2^{9n}} f(2^n(x-2y))\Big), \frac{r}{12}\Big) * \\
&\quad \mu\Big(+9\Big(N(x-3y) - \frac{1}{2^{9n}} f(2^n(x-3y))\Big), \frac{r}{12}\Big) * \mu\Big(-1\Big(N(x-4y) - \frac{1}{2^{9n}} f(2^n(x-4y))\Big), \frac{r}{12}\Big) * \\
&\quad \mu\Big(-9!\Big(N(y) - \frac{1}{2^{9n}} f(2^n(y))\Big), \frac{r}{12}\Big) * \mu\Big(\frac{1}{2^{9n}} Df_9(2^n x, 2^n y), \frac{r}{12}\Big) \\
&\nu\Big(N(x+5y) - 9N(x+4y) + 36N(x+3y) - 84N(x+2y) + 126N(x+y) \\
&\quad - 126N(x) + 84N(x-y) - 36N(x-2y) + 9N(x-3y) - N(x-4y) - 9!N(y), r\Big) \\
&\leq \nu\Big(N(x+5y) - \frac{1}{2^{9n}} f(2^n(x+5y)), \frac{r}{12}\Big) \diamond \nu\Big(-9\Big(N(x+4y) - \frac{1}{2^{9n}} f(2^n(x+4y))\Big), \frac{r}{12}\Big) \diamond \\
&\quad \nu\Big(+36\Big(N(x+3y) - \frac{1}{2^{9n}} f(2^n(x+3y))\Big), \frac{r}{12}\Big) \diamond \nu\Big(-84\Big(N(x+2y) - \frac{1}{2^{9n}} f(2^n(x+2y))\Big), \frac{r}{12}\Big) \diamond \\
&\quad \nu\Big(+126\Big(N(x+y) - \frac{1}{2^{9n}} f(2^n(x+y))\Big), \frac{r}{12}\Big) \diamond \nu\Big(-126\Big(N(x) - \frac{1}{2^{9n}} f(2^n(x))\Big), \frac{r}{12}\Big) \diamond \\
&\quad \nu\Big(+84\Big(N(x-y) - \frac{1}{2^{9n}} f(2^n(x-y))\Big), \frac{r}{12}\Big) \diamond \nu\Big(-36\Big(N(x-2y) - \frac{1}{2^{9n}} f(2^n(x-2y))\Big), \frac{r}{12}\Big) \diamond \\
&\quad \nu\Big(+9\Big(N(x-3y) - \frac{1}{2^{9n}} f(2^n(x-3y))\Big), \frac{r}{12}\Big) \diamond \nu\Big(-1\Big(N(x-4y) - \frac{1}{2^{9n}} f(2^n(x-4y))\Big), \frac{r}{12}\Big) \diamond \\
&\quad \nu\Big(-9!\Big(N(y) - \frac{1}{2^{9n}} f(2^n(y))\Big), \frac{r}{12}\Big) \diamond \nu\Big(\frac{1}{2^{9n}} Df_9(2^n x, 2^n y), \frac{r}{12}\Big) \qquad (5.47)
\end{aligned}$$

for all $x, y \in X$ and all $r > 0$. Since

$$\left.\begin{aligned}
\lim_{n\to\infty} \mu\Big(\tfrac{1}{2^{9n}} Df_9(2^n x, 2^n y), \tfrac{r}{12}\Big) &= 1 \\
\lim_{n\to\infty} \nu\Big(\tfrac{1}{2^{9n}} Df_9(2^n x, 2^n y), \tfrac{r}{12}\Big) &= 0
\end{aligned}\right\} \qquad (5.48)$$

for all $x \in X$ and all $r > 0$. Letting $n \to \infty$ in (5.47) and using (5.48), we observe that N fulfills (1.10). Therefore N is a Nonic mapping.

In order to prove $N(x)$ is unique, let $N'(x)$ be another Nonic functional equation satisfying

(1.10) and (5.4). Hence,

$$\begin{aligned}
\mu(N(x)-N'(x),r) &= \mu\left(\frac{N(2^n x)}{2^{9n}}-\frac{N'(2^n x)}{2^{9n}},r\right)\\
&\geq \mu\left(N(2^n x)-f(2^n x),\frac{r.2^{9n}}{2}\right)*\mu\left(f(2^n x)-N'(2^n x),\frac{r.2^{9n}}{2}\right)\\
&\geq \mu'\left(Z(2^n x,2^n x),\frac{2^9\wp r\ 2^{9n}|2^9-p|}{2}\right)\\
&\geq \mu'\left(Z(x,x),\frac{2^9\wp r\ 2^{9n}|2^9-p|}{2\cdot p^n}\right)\\
\nu(N(x)-N'(x),r) &= \nu\left(\frac{N(2^n x)}{2^{9n}}-\frac{N'(2^n x)}{2^{9n}},r\right)\\
&\leq \nu\left(N(2^n x)-f(2^n x),\frac{r.2^{9n}}{2}\right)\diamond\nu\left(f(2^n x)-N'(2^n x),\frac{r.2^{9n}}{2}\right)\\
&\leq \nu'\left(Z(2^n x,2^n x),\frac{2^9\wp\ 2^{9n}|2^9-p|}{2}\right)\\
&\leq \nu'\left(Z(x,x),\frac{2^9\wp\ 2^{9n}|2^9-p|}{2\cdot p^n}\right)
\end{aligned}$$

for all $x\in X$ and all $r>0$. Since

$$\lim_{n\to\infty}\frac{2^9\wp r\ 2^{9n}|2-a|}{2p^n}=\infty,$$

we obtain

$$\left.\begin{array}{l}
\lim\limits_{n\to\infty}\mu'\left(Z_\mu(x,x),\frac{2^9\wp r\ 2^{9n}|2^9-p|}{2\cdot p^n}\right)=1\\
\lim\limits_{n\to\infty}\nu'\left(Z(x,x),\frac{2^9\wp r\ 2^{9n}|2^9-p|}{2\cdot p^n}\right)=0
\end{array}\right\}$$

for all $x\in X$ and all $r>0$. Thus

$$\left.\begin{array}{l}
\mu(N(x)-N'(x),r)=1\\
\nu(N(x)-N'(x),r)=0
\end{array}\right\}$$

for all $x\in X$ and all $r>0$. Hence $N(x)=N'(x)$. Therefore $N(x)$ is unique.

For $\tau=-1$, we can prove the similar stability result. This completes the proof of the theorem.

Q.E.D.

The following corollary is an immediate consequence of Theorem 5.8, regarding the stability of (1.10)

Corollary 5.9. Suppose that a function $Df_9 : X \to Y$ satisfies the inequality

$$\left.\begin{array}{l} \mu\left(Df_9(x,y),r\right) \geq \left\{ \begin{array}{l} \mu\left(\lambda, r\right), \\ \mu\left(\lambda\left(||x||^s + ||y||^s\right), r\right), \\ \mu\left(\lambda\ ||x||^s||y||^s, r\right), \\ \mu\left(\lambda\left\{||x||^s\ ||y||^s + \left(||x||^{2s} + ||y||^{2s}\right)\right\}, r\right), \end{array}\right. \\ \nu\left(Df_9(x,y),r\right) \leq \left\{ \begin{array}{l} \nu\left(\lambda, r\right), \\ \nu\left(\lambda\left(||x||^s + ||y||^s\right), r\right), \\ \nu\left(\lambda\ ||x||^s||y||^s, r\right), \\ \nu\left(\lambda\left\{||x||^s\ ||y||^s + \left(||x||^{2s} + ||y||^{2s}\right)\right\}, r\right), \end{array}\right. \end{array}\right\} \tag{5.49}$$

for all $x, y \in X$ and all $r > 0$, where λ, s are constants with $\lambda > 0$. Then there exists a unique Nonic mapping $N : X \to Y$ such that

$$\left.\begin{array}{l} \mu\left(f(x) - N(x), r\right) \geq \left\{ \begin{array}{l} \mu\left(\lambda_1, 2^9\wp|2^9 - p|r\right), \\ \mu\left(\lambda_2||x||^s, 2^9\wp|2^9 - p|r\right), \\ \mu\left(\lambda_3\ ||x||^{2s}, 2^9\wp|2^9 - p|r\right), \\ \mu\left(\lambda_4||x||^{2s}, 2^9\wp|2^9 - p|r\right), \end{array}\right. \\ \nu\left(f(x) - N(x), r\right) \leq \left\{ \begin{array}{l} \nu\left(\lambda_1, 2^9\wp|2^9 - p|r\right), \\ \nu\left(\lambda_2\ ||x||^s, 2^9\wp|2^9 - p|r\right), \\ \nu\left(\lambda_3\ ||x||^{2s}, 2^9\wp|2^9 - p|r\right), \\ \nu\left(\lambda_4\ ||x||^{2s}, 2^9\wp|2^9 - p|r\right), \end{array}\right. \end{array}\right\} \tag{5.50}$$

where

$$\begin{aligned} \lambda_1 &= \frac{513\ \lambda}{362880}, \\ \lambda_2 &= \frac{\lambda\left[5^s + 9 \cdot 4^s + 37 \cdot 3^s + 94 \cdot 2^s + 674\right]}{362880}, \\ \lambda_3 &= \frac{\lambda\left[5^s + 9 \cdot 4^s + 37 \cdot 3^s + 93 \cdot 2^s + 162\right]}{362880}, \\ \lambda_4 &= \frac{\lambda\left[(5^{2s} + 5^s) + 9(4^{2s} + 4^s) + 37(3^{2s} + 3^s) + 93(2^{2s} + 2^s) + 2^{2s} + 836\right]}{362880} \end{aligned} \tag{5.51}$$

for all $x \in X$ and all $r > 0$.

5.2 IFNS: Fixed Point Method

Theorem 5.10. Let $Df_9 : X \to Y$ be a mapping for which there exist a function $\zeta : X \times X \to Z$ with the condition

$$\left.\begin{array}{l} \lim\limits_{n\to\infty} \mu'\left(\zeta\left(\chi_i^n x, \chi_i^n y\right), \chi^{9n} r\right) = 1 \\ \lim\limits_{n\to\infty} \nu'\left(\zeta\left(\chi_i^n x, \chi_i^n y\right), \chi^{9n} r\right) = 0 \end{array}\right\} \tag{5.52}$$

for all $x, y \in X$ and all $r > 0$ where

$$\chi_i = \left\{ \begin{array}{lll} 2 & if & i = 0 \\ \frac{1}{2} & if & i = 1 \end{array}\right. \tag{5.53}$$

and satisfying the functional inequality

$$\left.\begin{array}{l}\mu\left(Df_9(x,y),r\right) \geq \mu'\left(\zeta\left(x,y\right),r\right)\\[2ex] \nu\left(Df_9(x,y),r\right) \leq \nu'\left(\zeta\left(x,y\right),r\right)\end{array}\right\} \tag{5.54}$$

for all $x, y \in X$ and all $r > 0$. If there exists $L = L(i) > 0$ such that the function

$$x \to \rho(x) = Z\left(\frac{x}{2}, \frac{x}{2}\right),$$

has the property

$$\left.\begin{array}{l}\mu\left(L\frac{\rho(\chi_i x)}{\chi_i}, r\right) = \mu\left(\rho(x), r\right)\\ \nu\left(L\frac{\rho(\chi_i x)}{\chi_i}, r\right) = \nu\left(\rho(x), r\right)\end{array}\right\} \tag{5.55}$$

for all $x \in X$ and all $r > 0$, then there exists a unique Nonic function $N : X \to Y$ satisfying the functional equation (1.10) and

$$\left.\begin{array}{l}\mu\left(f(x) - N(x), r\right) \geq \mu'\left(\rho(x), \frac{L^{1-i}}{1-L}r\right)\\ \nu\left(f(x) - N(x), r\right) \leq \nu'\left(\rho(x), \frac{L^{1-i}}{1-L}r\right)\end{array}\right\} \tag{5.56}$$

for all $x \in X$ and all $r > 0$.

Proof. Consider the set

$$\Lambda = \{h/h : X \to Y,\ h(0) = 0\}$$

and introduce the generalized metric on Λ,

$$\left.\begin{array}{l}\inf\{K \in (0,\infty) : \mu(h(x) - g(x), r) \geq \mu'(\rho(x), Kr), x \in X\}\\ \inf\{K \in (0,\infty) : \nu(h(x) - g(x), r) \leq \nu'(\rho(x), Kr), x \in X\}\end{array}\right\} \tag{5.57}$$

It is easy to see that (5.57) is complete with respect to the defined metric. Define $J : \Lambda \to \Lambda$ by

$$Jh(x) = \frac{1}{\hbar_i^9} h(\hbar_i x),$$

for all $x \in \mathcal{X}$. The rest of the proof is similar to that of Theorem 3.3. Q.E.D.

The following corollary is an immediate consequence of Theorem 5.10, regarding the stability of (1.10).

Corollary 5.11. Suppose that a function $Df_9 : X \to Y$ satisfies the inequality

$$\left.\begin{array}{l}\mu\left(Df_9(x,y),r\right) \geq \left\{\begin{array}{l}\mu\left(\lambda, r\right),\\ \mu\left(\lambda\left(||x||^s + ||y||^s\right), r\right),\\ \mu\left(\lambda\ ||x||^s||y||^s, r\right),\\ \mu\left(\lambda\left\{||x||^s\ ||y||^s + \left(||x||^{2s} + ||y||^{2s}\right)\right\}, r\right),\end{array}\right.\\ \nu\left(Df_9(x,y),r\right) \leq \left\{\begin{array}{l}\nu\left(\lambda, r\right),\\ \nu\left(\lambda\left(||x||^s + ||y||^s\right), r\right),\\ \nu\left(\lambda\ ||x||^s||y||^s, r\right),\\ \nu\left(\lambda\left\{||x||^s\ ||y||^s + \left(||x||^{2s} + ||y||^{2s}\right)\right\}, r\right),\end{array}\right.\end{array}\right\} \tag{5.58}$$

for all $x, y \in X$ and all $r > 0$, where λ, s are constants with $\lambda > 0$. Then there exists a unique Nonic mapping $N : X \to Y$ such that

$$\left. \begin{array}{l} \mu\left(f(x) - N(x), r\right) \geq \left\{ \begin{array}{l} \mu\left(\lambda_1, 2^9 \wp |2^9 - p| r\right), \\ \mu\left(\lambda_2 ||x||^s, 2^9 \wp |2^9 - p| r\right), \\ \mu\left(\lambda_3 \ ||x||^{2s}, 2^9 \wp |2^9 - p| r\right), \\ \mu\left(\lambda_4 ||x||^{2s}, 2^9 \wp |2^9 - p| r\right), \end{array} \right. \\ \nu\left(f(x) - N(x), r\right) \leq \left\{ \begin{array}{l} \nu\left(\lambda_1, 2^9 \wp |2^9 - p| r\right), \\ \nu\left(\lambda_2 \ ||x||^s, 2^9 \wp |2^9 - p| r\right), \\ \nu\left(\lambda_3 \ ||x||^{2s}, 2^9 \wp |2^9 - p| r\right), \\ \nu\left(\lambda_4 \ ||x||^{2s}, 2^9 \wp |2^9 - p| r\right), \end{array} \right. \end{array} \right\} \tag{5.59}$$

where

$$\begin{aligned} \lambda_1 &= \frac{513\lambda}{362880}, \\ \lambda_2 &= \frac{\lambda}{362880 \cdot 2^s}\left[5^s + 9 \cdot 4^s + 37 \cdot 3^s + 93 \cdot 2^s + 675\right] \\ \lambda_3 &= \frac{\lambda}{362880 \cdot 2^{2s}}\left[5^s + 9 \cdot 4^s + 37 \cdot 3^s + 93 \cdot 2^s + 162\right] \\ \lambda_4 &= \frac{\lambda}{362880 \cdot 2^s}\left[5^{2s} + 5^s + 9(4^{2s} + 4^s) + 37(3^{2s} + 3^s) + 93(2^{2s} + 2^s) + 837\right] \end{aligned} \tag{5.60}$$

for all $x \in X$ and all $r > 0$.

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