On three dimensional quasi-Sasakian manifolds

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Abstract

Let M be a 3-dimensional quasi-Sasakian manifold. Olszak [6] proved that M is conformally flat with constant scalar curvature and hence its structure function β is constant. We have shown that in such M, a second order symmetric parallel tensor is a constant multiple of the associated metric tensor. A necessary and sufficient condition for such a manifold to be minimal has been obtained. Finally if such M satisfies R(X,Y).S =0, then, S has two different non-zero eigen values.

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1 Introduction

In 1926, Levi [4] proved that a second order symmetric parallel non singular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma [7] generalised Levi's result. In this paper, we have considered a 3-dimensional quasi-Sasakain manifold. Olszak [6] proved that such a space is conformally flat with constant scalar curvature and hence the structure function β is constant. In this paper we have shown that in a 3-dimensional quasi- Sasakian manifold with constant scalar curvature, a second order symmetric parallel tensor is a constant multiple of the associated metric tensor. In the next section, we consider 3-dimensional quasi-Sasakian manifold with constant scalar curvature which are hypersurfaces of a Riemannian manifold of constant curvature 1. A necessary and sufficent condition for such a manifold to be minimal has been obtained. Lastly,if a three dimensional quasi-Sasakian manifold with constant scalar curvature satisfies R(X,Y).S=0, then it is proved that the symmetric endomorphism Q of the tangent space corresponding to S has two different non-zero eigen values. Let M be an almost contact metric manifold of dimension (2n+1) with an almost contact metric structure (φ,ξ,η,g)[2] where φ,ξ,η are tensor fields of type (1,1),(1,0),(0,1) respectively and g is a Riemannian metric on M such that

$$\begin{aligned}&\varphi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \varphi\xi = 0,\ \eta o\varphi = 0,\ \eta(X) = g(X, \xi),\\ &g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\end{aligned} \tag{1.1}$$

$\forall$ X,Y $\in$ TM.

M is said to be quasi-Sasakian , if it is normal and the fundamenatal 2-form Φ is closed ($d\Phi = 0$, $\Phi(X, Y) = g(X, \varphi Y)$ [1]). It has been proved [5] that an almost contact metric manifold M of dimension 3 is quasi-Sasakian if and only if

$$\nabla_X \xi = -\beta\varphi X \tag{1.2}$$

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for a certain function β on M such that $\xi\beta$=0. Hence

$$(\nabla_X\varphi)Y = \beta(g(X,Y)\xi - \eta(Y)X). \tag{1.3}$$

Now by Theorem 3.6 of [6], such a space is conformally flat with constant scalar curvature. Consequently, if R,S denote the curvature tensor and the Ricci tensor of M, then

$$R(X,Y)\xi = \beta^2(\eta(Y)X - \eta(X)Y) \tag{1.4}$$

$$S(X,Y) = (\frac{r}{2} - \beta^2)g(X,Y) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Y) \tag{1.5}$$

S(X,Y)=g(QX, Y) where Q is the symmetric endomorphism of the tangent space of M.

$$S(\varphi X, \varphi Y) = S(X,Y) - 2\beta^2\eta(X)\eta(Y) \tag{1.6}$$

$$S(X,\xi) = 2\beta^2\eta(X) \tag{1.7}$$

$$R(\xi,X)\xi = \beta^2(\eta(X)\xi - X). \tag{1.8}$$

The above results will be used in the next sections.

2 3-dimensional quasi-Sasakian manifolds with second order symmetric parallel tensor

Let T denote a (0,2) tensor field on a 3-dimensional quasi-Sasakian manifold such that ∇T=0. Then

$$T(R(W,X)Y,Z) + T(Y,R(W,X)Z) = 0 \tag{1.9}$$

for arbitrary vector fields X,Y,Z,W on M.
Taking Y=Z=W=ξ in 1.9)we get

$$T(R(\xi,X)\xi,\xi) + T(\xi,R(\xi,X)\xi) = 0. \tag{1.10}$$

Using 1.8) in 1.10) we have

$$g(X,\xi)T(\xi,\xi) - T(X,\xi) = 0 \tag{1.11}$$

as T is symmetric. Differentiating 1.11) along Y, we get

$$\begin{aligned}\{g(\nabla_Y X,\xi) + g(X,\nabla_Y\xi)\}T(\xi,\xi) + 2g(X,\xi)T(\nabla_Y\xi,\xi)\\ -T(\nabla X,\xi) - T(\nabla_Y\xi,X) = 0.\end{aligned} \tag{1.12}$$

as T is symmetric.
Putting $X = \nabla_Y X$ in 1.11) we find

$$g(\nabla_Y X,\xi)T(\xi,\xi) - T(\nabla_Y X,\xi) = 0. \tag{1.13}$$

Using 1.12) in 1.13) we find, on using 1.2)

$$g(X,\varphi Y)T(\xi,\xi) + 2g(X,\xi)T(\varphi Y,\xi) - T(\varphi Y,X) = 0. \tag{1.14}$$

Replacing X by φY in 1.11) we find, on using 1.1)

$$T(\varphi Y, \xi) = 0. \tag{1.15}$$

From 1.14) and 1.15) we obtain

$$g(X, \varphi Y)T(\xi, \xi) - T(\varphi Y, X) = 0. \tag{1.16}$$

Replacing Y by φY and using 1.1) and 1.11) we obtain

$$T(X, Y) = T(\xi, \xi)g(X, Y). \tag{1.17}$$

The fact that T(ξ,ξ) is a constant can be checked by differentiating it along any vector field on M. Thus we state

Theorem 2.1. On a 3-dimensional quasi-Sasakian manifold with constant scalar curvature, a second order symmetric parallel tensor is a constant multiple of the associated metric tensor.

3 3-dimensional quasi-Sasakian manifolds which are hypersurfaces of a Riemannian manifold of constant curvature

Let M be a 3-dimensional quasi-Sasakain manifolds which is isometrically immersed in a Riemannian manifold of dimension 4 of constant curvature 1. Then we have the Gauss and Coddazi equations [3]

$$R(X, Y) = X \wedge Y + AX \wedge AY \tag{1.18}$$

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(AY, Z)AX - g(AX, Z)AY \tag{1.19}$$

$$(\nabla_X A)Y = (\nabla_Y A)X \tag{1.20}$$

where A is a (1,1) tensor field associated with the second fundamental form B by B(X,Y)=g(X,AY). Here A is symmetric with respect to g and when the trace of A vanishes, the immersion is called minimal. The rank of A is called the type number of immersion. Since the Ricci curvature tensor S is given by

$$S(X, Y) \to trace[W \to R(X, W)Y].$$

By 1.19) we have

$$S(X, Y) = (3-1)g(X, Y) + (traceA)g(AX, Y) - g(AAX, Y). \tag{1.21}$$

Replacing X and Y by φ X and φ Y in 1.21),we find

$$S(\varphi X, \varphi Y) = 2g(\varphi X, \varphi Y) + \theta g(A\varphi X, \varphi Y) - g(AA\varphi X, \varphi Y) \tag{1.22}$$

where θ is the trace of A. Again,

$$g(A\varphi X, \varphi Y) = -g(\varphi A\varphi X, Y),$$

$$g(AA\varphi X, \varphi Y) = -g(\varphi AA\varphi X, Y).$$

Using 1.1), and 1.6), it follows from 1.22)

$$S(X,Y) = 2g(X,Y) + 2(\beta^2-1)\eta(X)\eta(Y) - \theta g(\varphi A\varphi X,Y) + g(\varphi AA\varphi X,Y). \tag{1.23}$$

Now 1.21) and 1.23) imply

$$\theta AX - AAX + 2(1-\beta^2)\eta(X)\xi + \theta\varphi A\varphi X - \varphi AA\varphi X = 0. \tag{1.24}$$

Now if trace of A vanishes , then from 1.24) we get

$$2(1-\beta^2)\eta(X)\xi = AAX + \varphi AA\varphi X. \tag{1.25}$$

If 1.25) holds, then from 1.24),θ=0. Hence we can state the following theorem

Theorem 3.1. A necessary and sufficient condition for a 3-dimensional quasi-Sasakian manifold with constant scalar curvature, to be minimal is that 1.25) holds.

4 3-dimensional quasi-Sasakian manifolds with R(X,Y).S=0

It is known that for a conformally flat Riemannian manifold

$$R(X,Y)Z = \{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \tag{1.26}$$

Now let

$$R(X,Y).S = 0 \tag{1.27}$$

where R(X,Y) is considered as derivation of the tensor algebra at each point of the manifold for tangent vectors X,Y and S is the non-zero Ricci tensor such that

$$g(QX,Y) = S(X,Y) \tag{1.28}$$

where Q is the symmetric endomorphosm of the tangent space of M. From 1.27) we have

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = 0. \tag{1.29}$$

Using 1.26) in 1.29) and taking Y=Z we have

$$\begin{aligned} &g(Z,Z)S(QX,W) - g(X,Z)S(QZ,W) + g(Z,W)S(Z,QX) \\ &- g(X,W)S(Z,QZ) - \frac{r}{2}\{g(Z,Z)S(X,W) - g(X,Z)S(Z,W) \\ &+ g(Z,W)S(Z,X) - g(X,W)S(Z,Z)\} = 0. \end{aligned} \tag{1.30}$$

Let us put Z=e_i where { e_i :i= 1,2,3}is the set of orthonormal basis of tangent space at each point of M and summing for i=1,2,3 we get,

$$3S(QX,W) - g(QX,QW) - \frac{r}{2}\{4g(QX,W) - g(QW,X) - rg(X,W)\} = 0. \tag{1.31}$$

Let λ be any eigen value of the endomorphism Q corresponding to the eigen vector X. Then

$$QX = \lambda X. \tag{1.32}$$

Using 1.32) in 1.31) and applying 1.28) we get

$$2\lambda^2 g(X,W) - \frac{r}{2}(3\lambda - r)g(X,W) = 0. \tag{1.33}$$

That is,

$$\lambda^2 - \frac{3r}{4}\lambda + \frac{r^2}{2} = 0. \tag{1.34}$$

We denote two roots of 1.33) byλ_1 and λ_2. We can write

$$r = m\lambda_1 + (3-m)\lambda_2 \tag{1.35}$$

as r is the trace of Q and m is a positive integer which is the multiplicity of λ_1 and hence the multiplicity of λ_2 must be $(3-m)$. From 1.33) we write

$$\lambda_1 + \lambda_2 = \frac{3r}{4}, \qquad \lambda_1.\lambda_2 = \frac{r^2}{2}. \tag{1.36}$$

Solving 1.34) and 1.35)

$$\lambda_1 = \frac{3m-5}{(2m-3).4} r$$

and

$$\lambda_2 = \frac{3m-4}{(2m-3).4} r.$$

Thus we state

Theorem 4.1. If a three dimensional quasi-Sasakian manifold with constant scalar curvature satifies R(X,Y).S=0, then, the symmetric endomorphism Q of the tangent space corresponding to the Ricci tensor S has two different non zero eigen values.

References

[1] D. E. Blair, *The theory of quasi-Sasakian structures*, J. Differential Geometry **1**(1967), 331–345.

[2] D. E. Blair,*Contact manifolds in Riemannian Geometry* Lecture notes in Math. vol. **509**, Springer-Verlag 1976.

[3] B. Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, New York, 1973.

[4] H. Levi, *Symmetric tensors of the second order whose covariant derivative vanishes*, Ann. of Math. **27**(1926), 91–98.

[5] Z. Olszak,*Normal almost contact metric manifolds of dimension three*, Ann. Pol. Math. **47**(1986), 41–50.

[6] Z. Olszak, *On three-dimensional conformally flat quasi-Sasakian manifolds*, Periodica Mathematica Hungarica, **33(2)**, 1996, 105–113.

[7] R. Sharma, *Second order parallel tensor in real and complex space forms*, Inter. J. Math. Sci. **12**(1989), 787–790.