Generalized hybrid $B-(b,\rho,\theta,\tilde{p},\tilde{r})-$invexities and efficiency conditions for multiobjective fractional programming

Ram U. Verma

Department of Mathematics, Texas State University, San Marcos, Texas 78666, USA
E-mail: verma99@msn.com

Abstract

This communication deals with first introducing the exponential type hybrid $B-(b,\rho,\theta,\tilde{p},\tilde{r})$-invexities and then establishing a class of the $\varepsilon-$ efficiency conditions applying to multiobjective fractional programming problems. The exponential type hybrid $B-(b,\rho,\theta,\tilde{p},\tilde{r})$-invexities encompass most of the existing generalized higher order invexities as well as the exponential type generalized invexities, including the Antczak type first order $B-(\tilde{p},\tilde{r})-$invexities. To the best of our knowledge, the obtained results seem to be most advanced on generalized invexities available in the literature, while offer more suitable applications to other fields and beyond.

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1 Introduction

Recently, Zalmai [38] has generalized the exponential type invexity to the case of the Hanson-Antczak type $(\alpha,\beta,\gamma,\xi,\eta,\rho,\theta)-$V-invexity, and then established a series of results on global parametric sufficient efficiency conditions for a semiinfinite multiobjective fractional programming problem. Verma [28] introduced a higher order exponential type generalization - $B-(\rho,\eta,\theta,\tilde{p},\tilde{r})$-invexities - to exponential type first order $B-(\tilde{p},\tilde{r})-$invexities by Antczak [1], and applied to explore parametric sufficient efficiency conditions to semiinfinite minimax fractional programming problems, while Verma [27] introduced and investigated second order $(\Phi,\Psi,\rho,\eta,\theta)-$invexities to the context of parametric sufficient optimality conditions in semiinfinite discrete minimax fractional programming problems. The contribution of Antczak [1-3] on first order $B-(p,r)-$invexities is enormous to the context of nonlinear mathematical programming problems, which have been applied to a class of global parametric sufficient optimality conditions based on first order $B-(p,r)-$invexities for semiinfinite discrete minimax fractional programming problems. This was followed by Zalmai [37, 38] who generalized $B-(p,r)-$invexities introduced by Antczak [1-3], and applied to a class of global parametric sufficient optimality criteria using various assumptions for semiinfinite discrete minimax fractional programming problems. Verma [25] also developed a general framework for a class of $(\rho,\eta,\theta)-$invex functions to examine some parametric sufficient efficiency conditions for multiobjective fractional programming problems for weakly $\varepsilon-$efficient solutions, while Kim et al. [8] have established some $\varepsilon-$optimality conditions for multiobjective fractional optimization problems. Motivated by the recent advanced research contributions, we introduce a higher order exponential type generalization - $B-(b,\rho,\theta,\tilde{p},\tilde{r})$-invexities - a major generalization to the exponential type first order $B-(\tilde{p},\tilde{r})-$invexities - well-explored in the literature, and establish some advanced results on the $\varepsilon-$efficiency conditions based on the higher order exponential type generalization - $B-(b,\rho,\theta,\tilde{p},\tilde{r})$-invexities. The obtained results in this communication encompass most

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of the results in the literature primarily because of the enormous generality power of the higher order exponential type hybrid $B-(b,\rho,\theta,\tilde{p},\tilde{r})$-invexities, which may not be limited to applications to just $\varepsilon-$efficiency conditions and further applications.

We consider under the generalized framework of the second order $B-(b,\rho,\theta,\tilde{p},\tilde{r})-$invexities of functions, the following multiobjective fractional programming problem:
(P)

$$Minimize\Big(\frac{f_1(x)}{g_1(x)},\frac{f_2(x)}{g_2(x)},\cdots,\frac{f_p(x)}{g_p(x)}\Big)$$

$$\text{subject to } x\in Q=\{x\in X: H_j(x)\leq 0, j\in\{1,2,\cdots,m\}\},$$

where X is a nonempty subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), f_i and g_i for $i\in\{1,\cdots,p\}$ and H_j for $j\in\{1,\cdots,m\}$ are real-valued functions defined on X such that $f_i(x)\geq 0$, $g_i(x)>0$ for $i\in\{1,\cdots,p\}$ and for all $x\in Q$. Here Q denotes the feasible set of (P).

Next, we observe that problem (P) is equivalent to the nonfractional programming problem:
(Pλ)

$$Minimize\Big(f_1(x)-\lambda_1 g_1(x),\cdots,f_p(x)-\lambda_p g_p(x)\Big)$$

$$\text{subject to } x\in Q \text{ with}$$

$$\lambda=\Big(\lambda_1,\lambda_2,\cdots,\lambda_p\Big)=\Big(\frac{f_1(x^*)}{g_1(x^*)},\frac{f_2(x^*)}{g_2(x^*)},\cdots,\frac{f_p(x^*)}{g_p(x^*)}\Big),$$

where x^* is an efficient solution to (P).

The general theory of nonlinear programming serves a great purpose, not just in terms of the theory, but also in terms of applications to various fields, including decision and management sciences, game theory, statistical analysis, engineering design (including design of control systems, design of earthquakes-resistant structures, digital filters, and electronic circuits), random graphs, boundary value problems, wavelet analysis, environmental protection planning, optimal control problems, continuum mechanics, robotics, and data envelopment analysis. For more details, we refer the reader [1- 41].

2 Preliminaries

Recently, Verma [28] generalized the notion of the first order Antczak type $B-(\tilde{p},\tilde{r})$-invexiies [1] to the case of the second order B-(ρ, η, θ, $\tilde{p}$, $\tilde{r}$)- invexities. These notions of the second order invexity encompass most of the existing notions in the literature. Let f be a twice continuously differentiable real-valued function defined on X. Furthermore, let $\rho: X\times X\to\mathbb{R}$ and $\theta: X\times X\to\mathbb{R}^n$ be functions on $X\times X$.

Definition 2.1. The function f is said to be exponential type hybrid B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$) - invex at $x^* \in X$ if there exist a function $b : X \times X \to [0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} & b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) \geq \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*)+\frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}z}-1\big\rangle\Big) \\ + \quad & \rho(x,x^*)\|\theta(x,x^*)\|^2 \ \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0, \end{aligned}$$

$$\begin{aligned} & b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) \geq \Big(\big\langle \nabla f(x^*)+\frac{1}{2}\nabla^2 f(x^*)z, z\big\rangle\Big) \\ + \quad & \rho(x,x^*)\|\theta(x,x^*)\|^2 \ \text{ for } \tilde{p} = 0 \text{ and } \tilde{r} \neq 0, \end{aligned}$$

$$\begin{aligned} & b(x,x^*)\Big(f(x)-f(x^*)\Big) \geq \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*)+\frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}z}-1\big\rangle\Big) \\ + \quad & \rho(x,x^*)\|\theta(x,x^*)\|^2 \ \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} = 0, \end{aligned}$$

$$\begin{aligned} & b(x,x^*)\Big(f(x)-f(x^*)\Big) \geq \Big(\big\langle \nabla f(x^*)+\frac{1}{2}\nabla^2 f(x^*)z, z\big\rangle\Big) \\ + \quad & \rho(x,x^*)\|\theta(x,x^*)\|^2 \ \text{ for } \tilde{p} = 0 \text{ and } \tilde{r} = 0. \end{aligned}$$

Definition 2.2. The function f is said to be exponential type hybrid B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex at $x^* \in X$ if there exist a function $b : X \times X \to [0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} & \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*)+\frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}z}-1\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0 \\ \Rightarrow \quad & b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) \geq 0 \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0, \end{aligned}$$

$$\begin{aligned} & \Big(\big\langle \nabla f(x^*)+\frac{1}{2}\nabla^2 f(x^*)z, z\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0 \\ \Rightarrow \quad & b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) \geq 0 \text{ for } \tilde{p} = 0 \text{ and } \tilde{r} \neq 0, \end{aligned}$$

$$\begin{aligned} & \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*)+\frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}z}-1\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0 \\ \Rightarrow \quad & b(x,x^*)\Big(f(x)-f(x^*)\Big) \geq 0 \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} = 0, \end{aligned}$$

$$\begin{aligned} & \Big(\big\langle \nabla f(x^*)+\frac{1}{2}\nabla^2 f(x^*)z, z\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0 \\ \Rightarrow \quad & b(x,x^*)\Big(f(x)-f(x^*)\Big) \geq 0 \text{ for } \tilde{p} = 0 \text{ and } \tilde{r} = 0. \end{aligned}$$

We remark that based on the above definitions, the following definitions also hold for the special cases when $\tilde{p} = 0$ or $\tilde{r} = 0$ or both, so we include only the case for $\tilde{p} \neq 0$ and $\tilde{r} \neq 0$.

Definition 2.3. The function f is said to be strictly exponential type hybrid B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex at $x^* \in X$ if there exist a function $b : X \times X \to [0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} & \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}z} - 1\big\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0 \\ \Rightarrow \quad & b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) > 0 \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0, \end{aligned}$$

equivalently,

$$\begin{aligned} & b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \leq 0 \\ \Rightarrow \quad & \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}z} - 1\big\rangle\Big) \\ + \quad & \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0 \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0. \end{aligned}$$

Definition 2.4. The function f is said to be prestrictly exponential type hybrid B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex at $x^* \in X$ if there exist a function $b : X \times X \to [0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} & \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}z} - 1\big\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 > 0 \\ \Rightarrow \quad & b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \geq 0 \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0. \end{aligned}$$

Definition 2.5. The function f is said to be exponential type hybrid B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$)-quasiinvex at $x^* \in X$ if there exist a function $b : X \times X \to [0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} & b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \leq 0 \\ \Rightarrow \quad & \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}z} - 1\big\rangle\Big) \\ + \quad & \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0 \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0. \end{aligned}$$

Definition 2.6. The function f is said to be prestrictly exponential type hybrid B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$)-quasiinvex at $x^* \in X$ if there exist a function $b : X \times X \to [0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,

$$
\begin{aligned}
& b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big)<0 \\
\Rightarrow\quad & \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*)+\frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}z}-1\big\rangle\Big) \\
+\quad & \rho(x,x^*)\|\theta(x,x^*)\|^2 \le 0 \text{ for } \tilde{p}\neq 0 \text{ and } \tilde{r}\neq 0,
\end{aligned}
$$

equivalently,

$$
\begin{aligned}
& \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*)+\frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}z}-1\big\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2>0 \\
\Rightarrow\quad & b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big)\ge 0 \text{ for } \tilde{p}\neq 0 \text{ and } \tilde{r}\neq 0.
\end{aligned}
$$

We remark that sometimes the equivalent forms for the definitions turn out to be more suitable during the proofs of the theorems.

Now we consider the $\varepsilon-$efficiency solvability conditions for (P) and (Pλ) problems motivated by the publications (see Verma [28]) and (Kim et al. [8]), where they have investigated the $\varepsilon-$efficiency as well as the weak $\varepsilon-$efficiency conditions for multiobjective fractional programming problems under constraint qualifications. Based on these developments in the literature, we plan to establish some parametric sufficient efficiency conditions for multiobjective fractional programming problem (P) under this framework of the exponential type hybrid B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$)-invexities. We need to recall some auxiliary results crucial to the problem on hand.

Definition 2.7. A point $x^*\in Q$ is an $\varepsilon-$efficient solution to (P) if there exists no $x\in Q$ such that

$$
\frac{f_i(x)}{g_i(x)}\le\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\,\forall\, i=1,\cdots,p,
$$

$$
\frac{f_j(x)}{g_j(x)}<\frac{f_j(x^*)}{g_j(x^*)}-\varepsilon_j \text{ for some } j\in\{1,\cdots,p\},
$$

where $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_p)$ with $\varepsilon_i\ge 0$ for $i=1,\cdots,p$.

Next to this context, we have the following auxiliary problem:

(P$\bar{\lambda}$)

$$
minimize_{x\in Q}\,(f_1(x)-\bar{\lambda}_1 g_1(x),\cdots,f_p(x)-\bar{\lambda}_p g_p(x)),
$$

$$
\text{subject to } x\in Q,
$$

where $\bar{\lambda}_i$ for $i\in\{1,\cdots,p\}$ are parameters, and $\bar{\lambda}_i=\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i$.

Next, we introduce the $\varepsilon-$efficient solvability conditions for (P$\bar{\lambda}$) problem.

Definition 2.8. A point $x^* \in Q$ is an $\bar{\varepsilon}-$efficient solution to (P$\bar{\lambda}$) if there exists no $x \in Q$ such that

$$f_i(x) - \bar{\lambda} g_i(x) \leq f_i(x^*) - \bar{\lambda} g_i(x^*) - \bar{\varepsilon}_i \, \forall \, i = 1, \cdots, p,$$

$$f_j(x) - \bar{\lambda} g_j(x) < f_j(x^*) - \bar{\lambda} g_j(x^*) - \bar{\varepsilon}_j \text{ for some } j \in \{1, \cdots, p\},$$

where $\bar{\lambda_i} = \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i$, $\bar{\varepsilon_i} = \varepsilon_i g(x^*)$ with $\varepsilon_i \geq 0$ for $i = 1, \cdots, p$, and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_p)$ with $\varepsilon_i \geq 0$ for $i = 1, \cdots, p$.

Lemma 2.9. Let $x^* \in Q$, and $f_i(x^*) \geq \varepsilon_i g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:

(i) x^* is an $\varepsilon-$efficient solution to (P).

(ii) x^* is an ε^*-efficient solution to (P$\bar{\lambda}$), where

$$\bar{\lambda} = (\frac{f_1(x^*)}{g_1(x^*)} - \varepsilon_1, \cdots, \frac{f_p(x^*)}{g_p(x^*)} - \varepsilon_p),$$

where $\varepsilon^* = (\varepsilon_1 g_1(x^*), \cdots, \varepsilon_p g_p(x^*))$.

Lemma 2.10. Let $x^* \in Q$, and $f_i(x^*) \geq \varepsilon_i g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:

(i) x^* is an $\varepsilon-$efficient solution to (P).

(ii) There exists $c = (c_1, \cdots, c_p) \in \Re^p_+ \setminus \{0\}$ such that

$$\begin{aligned} & \Sigma_{i=1}^p c_i [f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) g_i(x)] \geq 0 \\ = \; & \Sigma_{i=1}^p c_i [f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) g_i(x^*)] - \Sigma_{i=1}^p c_i \varepsilon_i g_i(x^*) \end{aligned}$$

for any $x \in Q$.

Now, we need recall the following result (Verma [28]) that is crucial to developing the results for the next section based on second order hybrid B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$)-invexities.

Theorem 2.11. [28] Let $x^* \in \mathbb{F}$, $\lambda^* = \varphi(x^*)$ for each $i \in \underline{p}$, f_i and g_i be continuously differentiable at x^* for each $j \in \underline{q}$, the function $\zeta \to G_j(\zeta, t)$ be continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $\zeta \to H_k(\zeta, s)$ be continuously differentiable at x^* for all $s \in S_k$.

If x^* is an efficient solution of (P), the generalized Guignard constraint qualification holds at x^*, and if for any critical direction y, the set cone

$$\begin{aligned} & \{\Big(\nabla G_j(x^*,t), \langle y, \nabla^2 G_j(x^*,t)y\rangle\Big) : t \in \hat{T}_j(x^*), j \in \underline{q}\} \\ + \quad & span\{\Big(\nabla H_k(x^*,s), \langle y, \nabla^2 H_k(x^*,s)y\rangle\Big) : s \in S_k, k \in \underline{r}\}, \\ & where\ \hat{T}_j(x^*) \equiv \{t \in T_j \ : \ G_j(x^*,t) = 0\} \end{aligned}$$

is closed, then there exist $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0,\ \sum_{i=1}^{p} u_i = 1\}$ and integers ν_0^* and ν^*, with $0 \leq \nu_0^* \leq \nu^* \leq n+1$ such that there exist ν_0^* indices j_m with $1 \leq j_m \leq q$ together with ν_0^* points $t^m \in \hat{T}_{j_m}(x^*),\ m \in \underline{\nu_0^*},\ \nu^* - \nu_0^*$ indices k_m, with $1 \leq k_m \leq r$ together with $\nu^* - \nu_0^*$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu^*}\backslash\underline{\nu_0^*}$, and $\overline{\nu^*}$ real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0^*}$, with the property that

$$\begin{aligned} & \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^*[\nabla G_{j_m}(x^*,t^m) \\ + \quad & \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_k(x^*,s^m) = 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} & \langle y, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^* \nabla^2 G_{j_m}(x^*,t^m) \\ + \quad & \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla^2 H_k(x^*,s^m)\Big]y\rangle \geq 0, \end{aligned} \tag{2.2}$$

where $\hat{T}_{j_m}(x^*) = \{t \in T_{j_m} : G_{j_m}(x^*,t) = 0\}$, $U = \{u \in \mathbb{R}^p : u \geq 0, \sum_{i=1}^{p} u_i = 1\}$, and $\underline{\nu^*}\backslash\underline{\nu_0^*}$ is the complement of the set $\underline{\nu_0^*}$ relative to the set $\underline{\nu^*}$.

3 Second order sufficient efficiency conditions

This section deals with some parametric sufficient efficiency conditions for problem (P) under the generalized frameworks of second order hybrid B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$)- invexities for generalized invex functions. We start with real-valued functions $E_i(., x^*, u^*)$ and $B_j(., v)$ defined by

$$E_i(x, x^*, u^*) = u_i[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)g_i(x)],\ i \in \{1, \cdots, p\}$$

and

$$B_j(., v) = v_j H_j(x),\ j = 1, \cdots, m.$$

Theorem 3.1. Let $x^* \in Q$, f_i, g_i for $i \in \{1,\cdots,p\}$ with $\varphi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1,\cdots,m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \mathbb{R}^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in \mathbb{R}_+^m$ such that

$$\Sigma_{i=1}^p u_i^* [\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) \bigtriangledown g_i(x^*)] + \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*) = 0, \tag{3.1}$$

$$\Big\langle z, \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0, \tag{3.2}$$

where $z \in \mathbb{R}^n$, and

$$v_j^* H_j(x^*) = 0,\ j \in \{1,\cdots,m\}. \tag{3.3}$$

Suppose, in addition, that any one of the following assumptions holds:

(a) (i) $E_i(.\,; x^*, u^*)\ \forall i \in \{1,\cdots,p\}$ are exponential type hybrid B-($\bar{b}$, $\bar{\rho}$, θ, $\tilde{p}$, $\tilde{r}$)- pseudoinvex at $x^* \in X$ if there exist a function $\bar{b} : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$, $\bar{b}(x, x^*) > 0$.

(ii) $B_j(.\,, v^*)\ \forall j \in \{1,\cdots,m\}$ are second order hybrid $B-(b, \rho, \theta, \tilde{p}, \tilde{r})$-quasiinvex at $x^* \in X$ if there exist a function $b : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ for all $x \in X$.

(iii) $\rho(x, x^*) + \bar{\rho}(x, x^*) \geq 0\ \forall x \in Q$.

(b) (i) $E_i(.\,; x^*, u^*)\ \forall i \in \{1,\cdots,p\}$ are exponential type hybrid B-($\bar{b}$, ρ_1, θ, $\tilde{p}$, $\tilde{r}$)- pseudoinvex at x^* if there exist a function $\bar{b} : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$, $\bar{b}(x, x^*) > 0$.

(ii) $B_j(.\,, v^*)\ \forall j \in \{1,\cdots,m\}$ are exponential type hybrid B-(b, ρ_2, θ, $\tilde{p}$, $\tilde{r}$)-quasiinvex at $x^* \in X$ if there exist a function $b : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ for all $x \in X$.

(iii) $\rho_1(x, x^*) + \rho_2(x, x^*) \geq 0$.

(c) (i) $E_i(.\,; x^*, u^*)\ \forall i \in \{1,\cdots,p\}$ are prestrictly exponential type hybrid B-($\bar{b}$, $\bar{\rho}$, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex at $x^* \in X$ if there exist a function $\bar{b} : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$, $\bar{b}(x, x^*) > 0$.

(ii) $B_j(.\,, v^*)\ \forall j \in \{1,\cdots,m\}$ are strictly exponential hybrid B-(b,ρ, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex at $x^* \in X$ if there exist a function $b : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$.

(iii) $\rho(x, x^*) + \bar{\rho}(x, x^*) \geq 0$.

(d) (i) $E_i(.\,; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ are prestrictly exponential type hybrid B-($\bar{b}$, $\bar{\rho}$, θ, $\tilde{p}$, $\tilde{r}$)-quasiinvex at $x^* \in X$ if there exist a function $\bar{b} : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$, $\bar{b}(x, x^*) > 0$.

(ii) $B_j(.\,, v^*)$ $\forall j \in \{1, \cdots, m\}$ are strictly exponential type hybrid B-($\bar{b}$, ρ, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex at $x^* \in X$ if there exist a function $b : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$.

(iii) $\rho(x, x^*) + \bar{\rho}(x, x^*) \geq 0$.

(e) (i) For each $i \in \{1, \cdots, p\}$, f_i is exponential type hybrid B-($\bar{b}$, ρ_1, θ, $\tilde{p}$, $\tilde{r}$)-invex and $-g_i$ is second order hybrid B-($\bar{b}$, ρ_2, θ, $\tilde{p}$, $\tilde{r}$)-invex at x^* with $\bar{b}(x, x^*) > 0$.

(ii) $H_j(.\,, v^*)$ $\forall j \in \{1, \cdots, m\}$ is exponential type hybrid B-(b, ρ_3, θ, $\tilde{p}$, $\tilde{r}$)-quasi-invex at x^*.

(iii) $\Sigma_{j=1}^m v_j^* \rho_3 + \rho^* \geq 0$ for $\rho^* = \Sigma_{i=1}^p u_i^*(\rho_1 + \varphi(x^*)\rho_2)$ and for $\varphi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i$.

Then x^* is an $\varepsilon-$efficient solution to (P).

Proof. If (a) holds, and $x^* \in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned}
& \frac{1}{\tilde{p}}\langle \Sigma_{i=1}^p u_i^*[\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla g_i(x^*)] + \Sigma_{j=1}^m v_j^* \nabla H_j(x^*) \\
+ \quad & \frac{1}{2}\Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*)] \\
+ \quad & \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big] z, e^{\tilde{p}z} - 1\rangle \geq 0. \qquad (3.4)
\end{aligned}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^m v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^m v_j^* H_j(x^*),$$

and so

$$b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x) - H_j(x^*)]} - 1\big)\Big) \leq 0,$$

which follows from $\tilde{r} \neq 0$ and $b(x, x^*) \geq 0$ for all $x \in Q$. In light of the hybrid $B - (b, \rho, \theta, \tilde{p}, \tilde{r})$-quasiinvexity of $B_j(., v^*)$ at x^*, we have

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla H_j(x^*) + \frac{1}{2}\nabla^2 H_j(x^*) z, e^{\tilde{p}z} - 1\big\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0,$$

and hence,

$$\frac{1}{\tilde{p}}\Big(\Sigma_{j=1}^{m}\big\langle\nabla H_j(x^*)+\frac{1}{2}\nabla^2 H_j(x^*)z, e^{\tilde{p}z}-1\big\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2\leq 0. \tag{3.5}$$

It follows from (3.4), (3.5) and (iii) that

$$\begin{aligned} & \frac{1}{\tilde{p}}\Big(\langle\Sigma_{i=1}^{p}u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\bigtriangledown g_i(x^*)] \\ + \quad & \frac{1}{2}\sum_{i=1}^{p}u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)z], e^{\tilde{p}z}-1\Big\rangle\Big) \\ \geq \quad & -\bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2. \end{aligned} \tag{3.6}$$

Applying B-($\bar{b}$, $\bar{\rho}$, η, θ, $\tilde{p}$, $\tilde{r}$)- pseudo-invexity at x^* to (3.6), we find

$$\frac{1}{\tilde{r}}\bar{b}(x,x^*)\big(e^{\tilde{r}[E_i(x,x^*,u^*)-E_i(x^*,x^*,u^*)]}-1\big)\geq 0. \tag{3.7}$$

Next, based on $\bar{b}(x,x^*)>0$, (3.7) implies

$$\begin{aligned} & \Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x)] \\ \geq \quad & \Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x^*)] \\ \geq \quad & \Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x^*)]-\Sigma_{i=1}^{p}u_i^*\varepsilon_i g_i(x^*)=0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x)]\geq 0. \tag{3.8}$$

Under the assumption $u_i^*>0$ for each $i\in\{1,\cdots,p\}$, we conclude that there does not exist an $x\in Q$ such that

$$\frac{f_i(x)}{g_i(x)}-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\leq 0\ \forall\, i=1,\cdots,p,$$

$$\frac{f_j(x)}{g_j(x)}-(\frac{f_j(x^*)}{g_j(x^*)}-\varepsilon_j)<0 \text{ for some } j\in\{1,\cdots,p\}.$$

Hence, x^* is an $\varepsilon-$efficient solution to (P).

The proof for (b) is similar to that of (a), but we include for the sake of the completeness. If (b) holds, and $x^*\in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned} & \frac{1}{\tilde{p}}\langle\Sigma_{i=1}^{p}u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\bigtriangledown g_i(x^*)], e^{\tilde{p}z}-1\rangle \\ + \quad & \frac{1}{\tilde{p}}\langle\Sigma_{j=1}^{m}v_j^*\bigtriangledown H_j(x^*), e^{\tilde{p}z}-1\rangle=0\,\forall\, x\in Q, \end{aligned} \tag{3.9}$$

$$\frac{1}{2\tilde{p}}\Big\langle e^{\tilde{p}z}-1,\Big[\sum_{i=1}^{p}u_i^*[\nabla^2 f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)]$$
$$+\quad \sum_{j=1}^{m}v_j^*\nabla^2 H_j(x^*)\Big]z\Big\rangle\geq 0. \tag{3.10}$$

Applying $v^*\geq 0$, $x\in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m}v_j^*H_j(x)\leq 0=\Sigma_{j=1}^{m}v_j^*H_j(x^*),$$

and so

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x)-H_j(x^*)]}-1\big)\Big)\leq 0$$

using $\tilde{r}\neq 0$ and $b(x,x^*)\geq 0$ for all $x\in Q$. Now the hybrid B-(b, ρ_2, θ, $\tilde{p}$, $\tilde{r}$)-quasiinvexity of $B_j(.,v^*)$ at x^* implies

$$\frac{1}{\tilde{p}}\Big(\big\langle\nabla H_j(x^*)+\frac{1}{2}\nabla^2 H_j(x^*)z,e^{\tilde{p}z}-1\big\rangle\Big)$$
$$+\rho_2(x,x^*)\|\theta(x,x^*)\|^2\leq 0.$$

This results in

$$\frac{1}{\tilde{p}}\Big(\Sigma_{j=1}^{m}\big\langle\nabla H_j(x^*)+\frac{1}{2}\Sigma_{j=1}^{M}\nabla^2 H_j(x^*)z,e^{\tilde{p}z}-1\rangle\Big)$$
$$+\quad \rho_2(x,x^*)\|\theta(x,x^*)\|^2\leq 0. \tag{3.11}$$

It follows from (3.9), (3.10), (3.11) and (iii) that

$$\frac{1}{\tilde{p}}\Big(\langle\Sigma_{i=1}^{p}u_i^*[\triangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\triangledown g_i(x^*)]$$
$$+\quad \frac{1}{2}\sum_{i=1}^{p}u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)z],e^{\tilde{p}z}-1\Big\rangle\Big)$$
$$\geq\quad -\rho_1(x,x^*)\|\theta(x,x^*)\|^2. \tag{3.12}$$

Next, applying the hybrid B-($\bar{b}$, ρ_1, θ, $\tilde{p}$, $\tilde{r}$)-pseudo-invexity at x^* to (3.12), we have

$$\frac{1}{\tilde{r}}\bar{b}(x,x^*)\big(e^{\tilde{r}[E_i(x,x^*,u^*)-E_i(x^*,x^*,u^*)]}-1\big)\geq 0, \tag{3.13}$$

which further, using $\bar{b}(x,x^*)>0$, implies

$$\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x)]$$
$$\geq\quad \Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x^*)])\geq 0.$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)g_i(x)] \geq 0. \tag{3.14}$$

Using $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) \leq 0 \;\forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \varepsilon_j) < 0 \text{ for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\varepsilon-$efficient solution to (P).

Next, we prove (c) as follows: if (c) holds, and $x^* \in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned} & \frac{1}{\tilde{p}}\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) \bigtriangledown g_i(x^*)], e^{\tilde{p}z} - 1\rangle \\ + \quad & \frac{1}{\tilde{p}}\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), e^{\tilde{p}z} - 1\rangle = 0, \end{aligned} \tag{3.15}$$

$$\begin{aligned} & \frac{1}{2\tilde{p}}\Big\langle e^{\tilde{p}z} - 1, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*)] \\ + \quad & \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big]z\Big\rangle \geq 0. \end{aligned} \tag{3.16}$$

As $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

which implies

$$b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x) - H_j(x^*)]} - 1\big)\Big) \leq 0.$$

Then, in light of the strict hybrid B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvexity of $B_j(., v^*)$ at x^*, we have

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla H_j(x^*) + \frac{1}{2}\nabla^2 H_j(x^*)z, e^{\tilde{p}z} - 1\big\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0. \tag{3.17}$$

It follows from (3.15), (3.16), (3.17) and (iii) that

$$\begin{aligned} & \frac{1}{\tilde{p}}\Big(\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) \bigtriangledown g_i(x^*)], e^{\tilde{p}\eta(x,x^*)} - 1\rangle \\ + \quad & \frac{1}{2}\Big\langle e^{\tilde{p}z} - 1, \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*)z]\Big\rangle\Big) \\ > \quad & -\bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2. \end{aligned} \tag{3.18}$$

As a result, applying the prestrict hybrid B-($\bar{b}$, $\bar{\rho}$, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvexity at x^* to (3.18), we have

$$\Big(\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)} - \varepsilon_i)g_i(x)] - \Sigma_{i=1}^p u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)g_i(x^*)]\Big) \geq 0,$$

which implies

$$\begin{aligned} & \Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)g_i(x)] \\ \geq \ & \Sigma_{i=1}^p u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)g_i(x^*)]) \geq 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)g_i(x)] \geq 0. \tag{3.19}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) \leq 0 \ \forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \varepsilon_j) < 0 \text{ for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\varepsilon-$efficient solution to (P).

The proof applying (d) is similar to that of (c), but still we include it as follows: if $x^* \in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned} & \frac{1}{\tilde{p}}\langle \Sigma_{i=1}^p u_i^*[\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) \nabla g_i(x^*)], e^{\tilde{p}z} - 1\rangle \\ + \ & \frac{1}{\tilde{p}}\langle \Sigma_{j=1}^m v_j^* \nabla H_j(x^*), e^{\tilde{p}z} - 1\rangle = 0, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} & \frac{1}{2\tilde{p}}\Big\langle e^{\tilde{p}z} - 1, \Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*)] \\ + \ & \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0. \end{aligned} \tag{3.21}$$

Again, since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^m v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^m v_j^* H_j(x^*),$$

which implies

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x)-H_j(x^*)]}-1\big)\Big)\le 0.$$

Then, in light of the equivalent form for the strict hybrid B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvexity of $B_j(.,v^*)$ at x^*, we have

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla H_j(x^*)+\frac{1}{2}\nabla^2 H_j(x^*)z, e^{\tilde{p}z}-1\big\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2<0.$$

It follows from the above inequality, (3.20), (3.21) and (iii) that

$$\begin{aligned}
& \frac{1}{\tilde{p}}\Big(\langle \Sigma_{i=1}^p u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\bigtriangledown g_i(x^*)], e^{\tilde{p}z}-1\rangle \\
+ \quad & \frac{1}{2}\Big\langle e^{\tilde{p}z}-1, \sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)z]\Big\rangle\Big) \\
> \quad & -\bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2. \qquad (3.22)
\end{aligned}$$

As a result, applying the equivalent form for the prestrict hybrid B-($\bar{b}$, $\bar{\rho}$, θ, $\tilde{p}$, $\tilde{r}$)-quasiinvexity of $E_i(.;x^*,u^*)$ at x^* to (3.22), we have

$$\Big(\Sigma_{i=1}^p u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(g^*)}-\varepsilon_i)g_i(x)]-\Sigma_{i=1}^p u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x^*)]\Big)\ge 0,$$

which implies

$$\begin{aligned}
& \Sigma_{i=1}^p u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x)] \\
\ge \quad & \Sigma_{i=1}^p u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x^*)]) \ge 0.
\end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^p u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x)]\ge 0. \qquad (3.23)$$

Since $u_i^*>0$ for each $i\in\{1,\cdots,p\}$, we conclude that there does not exist an $x\in Q$ such that

$$\frac{f_i(x)}{g_i(x)}-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\le 0\ \forall\, i=1,\cdots,p,$$

$$\frac{f_j(x)}{g_j(x)}-(\frac{f_j(x^*)}{g_j(x^*)}-\varepsilon_j)<0 \text{ for some } j\in\{1,\cdots,p\}.$$

Hence, x^* is an $\varepsilon-$efficient solution to (P).

Finally, we prove (e) as follows: since $x^*\in Q$, it follows that
$H_j(x)\le H_j(x^*)$, which (in conjunction with $b(x,x^*)\ge 0$) implies

$$b(x,x^*)\Big(H_j(x)-H_j(x^*)\Big)\le 0.$$

Then applying the hybrid B-(b, ρ_3, θ, $\tilde{p}$, $\tilde{r}$)-quasi-invexity of H_j at x^* and $v^* \in R_+^m$, we have

$$\begin{aligned} & \frac{1}{\tilde{p}}\Big(\langle \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*) + \frac{1}{2}\Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*)z, e^{\tilde{p}z}-1\Big\rangle\Big) \\ \leq\ & -\Sigma_{j=1}^m v_j^* \rho_3 \|\theta(x,x^*)\|^2. \end{aligned}$$

Based on the assumptions, $u^* \geq 0$ and $\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \geq 0$, it follows from hybrid B-($\bar{b}$, ρ_1, θ, $\tilde{p}$, $\tilde{r}$)-invexity of f_i and hybrid B-($\bar{b}$, ρ_2, θ, $\tilde{p}$, $\tilde{r}$)-invexity of $-g_i$ that

$$\begin{aligned} & \bar{b}(x,x^*)\frac{1}{\tilde{r}}\Big(e^{\tilde{r}\Sigma_{i=1}^p u_i^*\big([f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x)]-[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x^*)]\big)}-1\Big) \\ =\ & \bar{b}(x,x^*)\frac{1}{\tilde{r}}\Big(e^{\tilde{r}\Sigma_{i=1}^p u_i^*\{[f_i(x)-f_i(x^*)]-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)[g_i(x)-g_i(x^*)]\}}-1\Big) \\ \geq\ & \frac{1}{\tilde{p}}\Big(\Sigma_{i=1}^p u_i^*\{\langle \bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\bigtriangledown g_i(x^*), e^{\tilde{p}z}-1\rangle\} \\ +\ & \frac{1}{2}\langle e^{\tilde{p}z}-1, \Sigma_{i=1}^p u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)z\rangle]\Big) \\ +\ & \Sigma_{i=1}^p u_i^*[\rho_1+\varphi(x^*)\rho_2]\|\theta(x,x^*)\|^2 \\ \geq\ & -\frac{1}{\tilde{p}}\big[\langle \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*), e^{\tilde{p}z}-1\rangle \\ +\ & \frac{1}{2}\Big\langle e^{\tilde{p}z}-1, \Sigma_{j=1}^m v_j^*\nabla^2 H_j(x^*)z\Big\rangle\big]\Big) + \Sigma_{i=1}^p u_i^*[\rho_1+\varphi(x^*)\rho_2]\|\theta(x,x^*)\|^2 \\ \geq\ & (\Sigma_{j=1}^m v_j^*\rho_3 + \Sigma_{i=1}^p u_i^*[\rho_1+\varphi(x^*)\rho_2])\|\theta(x,x^*)\|^2 \\ =\ & (\Sigma_{j=1}^m v_j^*\rho_3 + \rho^*)\|\theta(x,x^*)\|^2 \geq 0, \end{aligned}$$

where $\varphi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i$ and $\rho^* = \Sigma_{i=1}^p u_i^*(\rho_1 + \varphi(x^*)\rho_2)$. Q.E.D.

Next, we present a certain specialization to Theorem 3.1 relating to the second order hybrid $B-(b,\rho,\theta,\tilde{p},\tilde{r})$-invexities when $\tilde{p}=0$ and $\tilde{r}=0$.

Theorem 3.2. Let $x^* \in Q$, f_i, g_i for $i \in \{1,\cdots,p\}$ with $\varphi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1,\cdots,m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \mathbb{R}^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$, $z \in \mathbb{R}^n$ and $v^* \in \mathbb{R}_+^m$ such that

$$\Sigma_{i=1}^p u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\bigtriangledown g_i(x^*)] + \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*) = 0 \tag{3.24}$$

$$\Big\langle z, \Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^*\nabla^2 H_j(x^*)\Big]z\Big\rangle \geq 0, \tag{3.25}$$

$$v_j^* H_j(x^*) = 0,\ j \in \{1,\cdots,m\}. \tag{3.26}$$

Suppose, in addition, that any one of the following assumptions holds:

(a) (i) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are second order hybrid B-($\bar{b}$, $\bar{\rho}$, θ)- pseudoinvex at $x^* \in X$ if there exists a function $\bar{b} : X \times X \to [0, \infty)$ such that for all $x \in X$, $\bar{b}(x, x^*) > 0$.

(ii) $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are second order hybrid B-(b, ρ, θ)- quasiinvex at $x^* \in X$ if there exists a function $b : X \times X \to [0, \infty)$.

(iii) $\rho(x, x^*) + \bar{\rho}(x, x^*) \geq 0$ for all $x \in X$.

(b) (i) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are second order hybrid B-($\bar{b}$, ρ_1, θ)- pseudoinvex at $x^* \in X$ if there exists a function $\bar{b} : X \times X \to [0, \infty)$ such that $\bar{b}(x, x^*) > 0$.

(ii) $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are second order hybrid B-(b, ρ_2, θ)-quasiinvex at $x^* \in X$ if there exists a function $b : X \times X \to [0, \infty)$.

(iii) $\rho_1(x, x^*) + \rho_2(x, x^*) \geq 0$.

(c) (i) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are second order prestrictly hybrid B-($\bar{b}$, $\bar{\rho}$, θ)- pseudoinvex at $x^* \in X$ if there exists a function $\bar{b} : X \times X \to [0, \infty)$ such that $\bar{b}(x, x^*) > 0$.

(ii) $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are second order strictly hybrid B-(b, ρ, θ)-pseudoinvex at $x^* \in X$ if there exists a function $b : X \times X \to [0, \infty)$.

(iii) $\rho(x, x^*) + \bar{\rho}(x, x^*) \geq 0$.

(d) (i) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are second order prestrictly hybrid B-($\bar{b}$, $\bar{\rho}$, θ)- quasi-invex at $x^* \in X$ if there exists a function $\bar{b} : X \times X \to [0, \infty)$ such that for all $x \in X$, $\bar{b}(x, x^*) > 0$.

(ii) $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are second order strictly hybrid B-(b, ρ, θ)- pseudoinvex at $x^* \in X$ if there exists a function $b : X \times X \to [0, \infty)$.

(iii) $\rho(x, x^*) + \bar{\rho}(x, x^*) \geq 0$.

(e) (i) For each $i \in \{1, \cdots, p\}$, f_i is second order hybrid B-($\bar{b}$, ρ_1, θ)- invex and $-g_i$ is second order hybrid B-($\bar{b}$, ρ_2, θ)- invex at x^* such that $\bar{b}(x, x^*) > 0$.

(ii) $H_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ is hybrid B-(b, ρ_3, θ)- quasi-invex at x^*.

(iii) $\Sigma_{j=1}^{m} v_j^* \rho_3 + \rho^* \geq 0$ for $\rho^* = \Sigma_{i=1}^{p} u_i^* (\rho_1 + \varphi(x^*)\rho_2)$ and for $\varphi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i$.

Then x^* is an $\varepsilon-$efficient solution to (P).

Proof. Consider the proof for (a) as follows: if (a) holds, and $x^* \in Q$, then based on (3.24) and (3.25), we have

$$\begin{aligned} & \langle \Sigma_{i=1}^{p} u_i^* [\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) \bigtriangledown g_i(x^*)], z\rangle \\ + \quad & \langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), z\rangle = 0, \end{aligned} \tag{3.27}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0. \tag{3.28}$$

As $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

and so

$$b(x, x^*)\big([H_j(x) - H_j(x^*)]\big) \leq 0$$

for $b(x, x^*) \geq 0$ for all $x \in Q$. In light of the hybrid B-(b, ρ, θ)-quasiinvexity of $B_j(., v^*)$ at x^*, it follows that

$$\big\langle \nabla H_j(x^*) + \frac{1}{2}\nabla^2 H_j(x^*) z, z\big\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0,$$

and hence,

$$\big\langle \Sigma_{j=1}^{m} \nabla H_j(x^*) + \frac{1}{2}\Sigma_{j=1}^{m} \nabla^2 H_j(x^*) z, z\big\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0. \tag{3.29}$$

It follows from (3.27), (3.28), (3.29) and (iii) that

$$\begin{aligned} & \langle \Sigma_{i=1}^{p} u_i^* [\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) \bigtriangledown g_i(x^*)] \\ + \quad & \frac{1}{2}\sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*) z], z\rangle \geq -\bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2. \end{aligned} \tag{3.30}$$

Applying hybrid B-($\bar{b}$, $\bar{\rho}$, θ)-pseudoinvexity at x^* to (3.30), we have

$$\bar{b}(x, x^*)\big([E_i(x, x^*, u^*) - E_i(x^*, x^*, u^*)]\big) \geq 0. \tag{3.31}$$

Next, using $\bar{b}(x, x^*) > 0$, (3.31) implies

$$\begin{aligned} & \Sigma_{i=1}^{p} u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) g_i(x)] \\ \geq \quad & \Sigma_{i=1}^{p} u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) g_i(x^*)]) \geq 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)g_i(x)] \geq 0. \tag{3.32}$$

Now, since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) \leq 0 \; \forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \varepsilon_j) < 0 \text{ for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\varepsilon-$efficient solution to (P).

(b) - (e): The proofs are similar to that of (a). Q.E.D.

4 Concluding remarks

As we investigated the multiobjective fractional programming problem (P) based on the generalized framework of the second order B-(b, ρ, θ, $\tilde{p}$, $\tilde{r}$)- invexities of functions, the *semiinfinite* aspects of the problem (P) using the generalized second order invexities are relatively less explored to the context of mathematical programming in the literature. Note that multiobjective programming problems like (P) but with a finite number of constraints, that is, when the functions G_j are independent of t, and the functions H_k are independent of s, have been the subject of numerous investigations during the past decades, including several classes of static and dynamic optimization problems with multiple fractional objective functions have been studied and, as a result, a number of sufficient efficiency and duality results are currently available for these problems in the related literature. A close observation of research publications on multiobjective mathematical programming and other related sources reveals that despite a phenomenal proliferation of publications in several areas of multiobjective programming, so far *semiinfinite nonlinear multiobjective fractional programming problems* have not received much attention in the area of mathematical programming. As a matter of fact, until very recently there were almost no publications dealing with any kind of semiinfinite multiobjective programming problems that made substantial application of any class of generalized convex functions in establishing sufficient efficiency conditions or duality results.

Example 4.1. We consider a significant semiinfinite multiobjective fractional programming problem:

(P^*) Minimize $\varphi(x) = \big(\varphi_1(x), \ldots, \varphi_p(x)\big) = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right)$

subject to

$$G_j(x,t) \leqq 0 \;\text{ for all } t \in T_j, \;\; j \in \underline{q},$$
$$H_k(x,s) = 0 \;\text{ for all } s \in S_k, \;\; k \in \underline{r},$$
$$x \in X,$$

where p, q, and r are positive integers, X is a nonempty subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), for each $j \in \underline{q} \equiv \{1, 2, \ldots, q\}$ and $k \in \underline{r}$, T_j and S_k are compact subsets of complete metric spaces, for each $i \in \underline{p}$, f_i and g_i are real-valued functions defined on X, for each $j \in \underline{q}$, $G_j(\cdot, t)$ is a real-valued function defined on X for all $t \in T_j$, for each $k \in \underline{r}$, $H_k(\cdot, s)$ is a real-valued function defined on X for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $G_j(x, \cdot)$ and $H_k(x, \cdot)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all x satisfying the constraints of (P^*).

The results established applying the hybrid $B-(b, \rho, \theta, \tilde{p}, \tilde{r})$-invexities for the multiobjective fractional programming problem (P) can be generalized to the case of the semiinfinite multiobjective fractional programming problem (P^*) as well.

Next, we remark that the notion of the hybrid invexity seems to be application-oriented in the sense of managing calculations, while generalizes and unifies most of the generalized invexity concepts in the literature. We may also agree that the hybrid $B-(b, \rho, \theta, \tilde{p}, \tilde{r})$-invexities can be upgraded to the case of the hybrid $B-(b, \rho, \eta, \omega, \theta, \tilde{p}, \tilde{r})$-invexities by introducing functions $\eta, \omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, which unifies most of the generalized invexities as follows:

Definition 4.2. The function f is said to be exponential type hybrid B-(b, ρ, η, ω, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex at $x^* \in X$ if there exist a function $b : X \times X \to [0, \infty)$, real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ $(x \neq x^*)$, and $z \in \mathbb{R}^n$,

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\big\rangle + \frac{1}{2}\big\langle \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - 1\big\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0$$

$$\Rightarrow \quad b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \geq 0 \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0.$$

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