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Moving frames and conservation laws of a Lagrangian invariant under the Hyperbolic Rotation-Translation group

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Abstract. Noether's First Theorem guarantees conservation laws provided that the Lagrangian is invariant under a Lie group action. In this paper, via the concept of Killing vector fields and the Minkowski metric, we first construct an important Lie group, known as Hyperbolic Rotation-Translation group. Then, according to Gonçalves and Mansfield's method, we obtain the invariantized Euler-Lagrange equations and the space of conservation laws in terms of vectors of invariants and the adjoint representation of a moving frame, for Lagrangians, which are invariant under Hyperbolic Rotation-Translation (or HRT) group action, in the case where the independent variables are not invariant.

Key words: Conservation laws; Moving frames; Differential invariants; Normalized differential invariants; Syzygies; Killing vector fields.

1. Introduction

The vast significance of the concept of conservation laws in a large number of applications in physics and mechanics is beyond any doubt. In 1918 Emmy Noether in seminal paper [8], proved that for every system arising from a variational principle, conservation laws of the system come from Lie group actions that leave the Lagrangian invariant. (See Theorem 4.29. in page 272 of [9].)

Recently in [4], [6], [7], Gonçalves and Mansfield considered diverse Lagrangians, which are invariant under a Lie group action, where independent variables are invariant. They presented the mathematical structure behind both the Euler-Lagrange equations and the set of conservation laws, and they showed Noether's conservation laws can be displayed as the product of adjoint representation of a right moving frame, which is equivariant, and a matrix where the columns are vectors of invariants. These results were presented in [6] for all three inequivalent $SL(2)$ actions in the complex plane, and in [4] for the standard $SE(2)$ and $SE(3)$ actions.

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In recent works [5], Gonçalves and Mansfield considered invariant Lagrangians under a Lie group action, where independent variables are no longer invariant. They proved in this case, Noether's conservation laws have an analogous form as ones mentioned in [6], but with an additional term – the matrix representing the group action on the space of $(p-1)$-forms, where p is the number of independent variables.

In this paper, first we take a relevant moving frame, for Hyperbolic Rotation-Translation group action, and obtain differential relations or syzygies between normalized differential invariants, then according to [5], we calculate conservation laws associated to special case of the Monge–Ampère equation, which Lagrangian is invariant under Hyperbolic Rotation-Translation group action, and the two independent variables are not invariant.

In Section 2, we briefly recall concepts of the Minkowski metric, Lorentzian manifold and killing vector fields, that will motivate us to create a Lie group, known as Hyperbolic Rotation-Translation group. The main goal of this section is to find a group action, via the Lorentzian metric and Killing vector fields on a pseudo-Riemannian manifold known as Lorentzian manifold.

In Section 3, we will briefly give some background on moving frames, differential invariants of a group action, invariant differentiation operators and invariant forms. Throughout Section 3 we will use the group action of Hyperbolic Rotation-Translation on the space $(x, y, u(x, y))$.

In Section 4, we concentrate on invariant calculus of variations and find the adjoint representation associated to Hyperbolic Rotation-Translation group action. Then we end this section with the calculation of Noether's conservation laws associated to a case of the Monge–Ampère equation, in terms of vectors of invariants, the adjoint representation of the moving frame and a matrix which represents the group action on the 1-forms.

2. The structure of Lie group

In this section, we recall some motivational concepts, to create the Hyperbolic Rotation-Translation group action. Here we present some vital concepts, which leads to structure of a group action, the Lie group that we will use in this paper.

Recall that, if V be an n-dimensional real vector space, we define a

Lorentzian scalar product on V as a non-degenerate symmetric bilinear form $\langle . , . \rangle$ of index 1, i.e. one can find a basis $e_1, \ldots, e_n$ of V such that

$$\langle e_i, e_j \rangle = \begin{cases} -1 & i = j = 1 \\ 1 & i = j = 2, \ldots, n \\ 0 & \text{otherwise} \end{cases} .$$

Definition 2.1 In cartesian coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}^n$, the *Minkowski metric* is defined by $g = -(dx_1)^2 + (dx_2)^2 + \cdots + (dx_n)^2$.

Definition 2.2 A *pseudo-Riemannian* manifold (or *semi-Riemannian* manifold) (M, g) with $\dim(M) \geqslant 2$, is a differentiable manifold M equipped with a non-degenerate, smooth, symmetric metric tensor g, so that, the signature of this metric is (p, q), where both p and q are non-negative.

Definition 2.3 A *Lorentzian manifold* is a pair (M, g) where M is an n-dimensional smooth manifold and g associates with each point $p \in M$ a Lorentzian scalar product g_p on the tangent space T_pM, it means that, g is a Lorentzian metric. In other words, a Lorentzian manifold is a pseudo-Riemannian manifold in which the signature of the metric is $(1, n-1)$ (or, equivalently, $(n-1, 1)$).

We now state the major definition of this section, known as *Killing vector field*, that via it we obtain an important Lie group, known as the Hyperbolic Rotation-Translation group, and group action associated to it. For more details of the Killing vector fields, see [1]. We know the Lie derivative, describes the action of a vector field on tensors such as one forms, the metric or another vector field. We now investigate the vector fields which are symmetries of the metric.

Definition 2.4 Let σ_t be the integral curve of the vector field $\mathbf{v}$ on a Riemannian or Lorentzian manifold (M, g). Then $\mathbf{v}$ is called a *Killing vector field* if σ_t is an isometry, i.e. it leaves the metric invariant ${\sigma_t}^*(g) = g$. This means that, the Lie derivative of the metric g along $\mathbf{v}$ vanishes, $\mathcal{L}_{\mathbf{v}} g = 0$.

Equivalently, if ∇_μ is the covariant derivative defined by the Christoffel connection of the metric g, and $\mathbf{v}_\mu = g_{\mu\nu}\mathbf{v}^\nu$ is the dual vector corresponding to the vector field $\mathbf{v}^\mu$, then $\mathbf{v}^\mu$ is a *Killing vector field* if and only if it solves the Killing equations:

$$\nabla_\mu \mathbf{v}_\nu + \nabla_\nu \mathbf{v}_\mu = \mathcal{L}_{\mathbf{v}} g_{\mu\nu} = 0.$$

Remark 2.5 The set of Killing vectors of metric g, form a Lie algebra using the commutator of vector fields as the Lie bracket.

We know that, n-dimensional Minkowski spaces, are equipped with $n(n+1)/2$ Killing vector fields. Now if we set $n = 2$, the Minkowski metric (or Lorentzian metric) is given by

$$(g)_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

then the line element is $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -(dx^1)^2 + (dx^2)^2$, or $ds^2 = -(dx)^2 + (dy)^2$. Therefore, we obtain the Killing vector fields as follows

$$v_1 = -\partial_x, \qquad v_2 = \partial_y, \qquad v_3 = y\partial_x + x\partial_y. \tag{1}$$

Remark 2.6 These three linearly independent Killing vector fields on the hyperbolic plane are the generators of translations and rotation.

Definition 2.7 A group action of G on M is a map

$$G \times M \longrightarrow M, \qquad (g, z) \longmapsto \tilde{z} = g.z,$$

which satisfies either $g.(h.z) = (gh).z$, called a *left action*, or $g.(h.z) = (hg).z$, called a *right action*.

Remark 2.8 Any linear combination of two Killing vector fields is a Killing vector field. Also, the set of vector fields $\{\mathbf{v}_1 = v_1 + v_2, \mathbf{v}_2 = -v_1 + v_2, \mathbf{v}_3 = v_3\}$ forms a basis for the same space as the one generated by $\{v_1, v_2, v_3\}$.

The action of the Lie group associated to Killing vector fields (1) on a 2-dimensional manifold M with coordinates (x, y), is given as follows

$$\tilde{x} = -a + b + x\cosh\theta + y\sinh\theta, \qquad \tilde{y} = a + b + x\sinh\theta + y\cosh\theta, \tag{2}$$

where a, b and θ are constants that parametrize the group action. Thus, we consider the Killing vector fields

$$\mathbf{v}_1 = -\partial_x + \partial_y, \qquad \mathbf{v}_2 = \partial_x + \partial_y, \qquad \mathbf{v}_3 = y\partial_x + x\partial_y. \tag{3}$$

Therefore, in next sections we consider the Lie group associated to vector fields (3), and we call this group the *Hyperbolic Rotation-Translation.*

3. Moving frames, differential invariants of a group action and invariant forms

In this section, we will introduce some concepts regarding moving frames, differential invariants of a group action, invariant differential operators and invariant forms as formulated by Fels and Olver [2], [3], Gonçalves and Mansfield [4], [5], [6] and Mansfield [7]. We will use the Hyperbolic Rotation-Translation action on the space as our applied problem.

Suppose $M = J^n(X \times U)$ is the n-th jet bundle, with coordinates

$$z = (x_1, \ldots, x_p, u^1, \ldots, u^q, u^1_1, \ldots),$$

where X and U are the space of independent variables with coordinates $\mathbf{x} = (x_1, \ldots, x_p)$ and the space of dependent variables with coordinates $\mathbf{u} = (u^1, \ldots, u^q)$ respectively. On this space, the *total differentiation operator* is defined by

$$D_i = \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \sum_{K} u^{\alpha}_{Ki} \frac{\partial}{\partial u^{\alpha}_{K}}$$

where

$$u^{\alpha}_{K} = \frac{\partial^{|K|} u^{\alpha}}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_p^{k_p}}$$

is the derivatives of u^{α} with a multi-index notation, and the p-tuple $K = (k_1, \ldots k_p)$, is a multi-index of differentiation of order $|K| = k_1 + \cdots + k_p$.

Definition 3.1 We say two smooth surfaces $\mathcal{K}$ and $\mathcal{O}$ contained in $\mathbb{R}^n$, such that, $\dim(\mathcal{K}) = \alpha$, $\dim(\mathcal{O}) = \beta$, $0 \leqslant \alpha, \beta \leqslant n$, $\alpha + \beta \geqslant n$, intersect *transversally* if for every $x \in \mathcal{K} \cap \mathcal{O}$, the tangent spaces $T_x\mathcal{K}$ and $T_x\mathcal{O}$, as subspaces of $T_x\mathbb{R}^n$, satisfy

$$T_x\mathcal{K} + T_x\mathcal{O} = T_x\mathbb{R}^n.$$

Suppose G is a Lie group which acts freely and regularly on some domain Ω in smooth manifold M, then as given in page 115 of [7], for every $x \in \Omega$, there is a neighbourhood $\mathcal{U}$ of x such that the following hold.

- The group orbits all have the same dimension of the group and foliate $\mathcal{U}$.
- There is a surface $\mathcal{K} \subset \mathcal{U}$ that crosses these orbits transversally at a single point. This surface is called the *cross section*.
- If $\mathcal{O}(z)$ represents the orbit through z, then the element $g \in G$ taking $z \in \mathcal{U}$ to $\{k\} = \mathcal{O}(z) \cap \mathcal{K}$ is unique.

By above conditions, a *right moving frame* is defined as the map $\rho : \mathcal{U} \to G$ that sends an element $z \in \mathcal{U}$ to the unique group element $g = \rho(z)$ such that

$$\rho(z).z = k, \qquad \{k\} = \mathcal{O}(z) \cap \mathcal{K}.$$

For obtaining the right moving frame, according to [7] in page 117, first we define the cross section $\mathcal{K}$ as the locus of the set of equations $\psi_j(z) = 0$, for $j = 1, \ldots, r = \dim(G)$. Then, to obtain the group element that takes z to k, we solve the so called *normalization equations*

$$\psi_j(\tilde{z}) = \psi_j(g.z) = 0, \qquad j = 1, \ldots, r,$$

for the r group parameters that describe the Lie group near its identity element, which yields the frame ρ in parametric form.

We now consider the action of obtained group in previous section on the space $(x, y, u(x, y))$, associated to transformation (2), that u is invariant.

Example 3.2 Consider the *Hyperbolic Rotation-Translation* group action on the space $(x, y, u(x, y))$ as follows

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b - a \\ b + a \end{pmatrix}, \qquad \tilde{u} = u, \tag{4}$$

where a, b and θ are constants that parametrize the group action. The prolonged action on u_x and u_y is given explicitly by

$$g.u_x = \tilde{u}_x = \tilde{D}_x\tilde{u}, \qquad g.u_y = \tilde{u}_y = \tilde{D}_y\tilde{u}.$$

The transformed total differentiation operators $\tilde{D}_i$ are defined by

$$\tilde{D}_i = \frac{d}{d\tilde{x}_i} = \sum_{k=1}^{p} \left(\frac{d\tilde{x}}{dx}\right)^{-T}_{ik} D_k,$$

where $(d\tilde{x}/dx)$ is the Jacobian matrix. Therefore,

$$\tilde{u}_x = u_x \cosh\theta - u_y \sinh\theta, \qquad \tilde{u}_y = -u_x \sinh\theta + u_y \cosh\theta.$$

If we take M to be the space with coordinates $(x, y, u, u_x, u_y, u_{x^2}, u_{xy}, u_{y^2}, \dots)$, then the action is locally free near the identity of group Hyperbolic Rotation-Translation and regular. Therefore, if we take the normalization equations to be $\tilde{x} = 0$, $\tilde{y} = 0$ and $\tilde{u}_x = 0$, we obtain

$$a = \frac{1}{2}\frac{(-y+x)(u_x - u_y)}{\sqrt{-{u_x}^2 + {u_y}^2}}, \quad b = -\frac{1}{2}\frac{\sqrt{-{u_x}^2 + {u_y}^2}(x+y)}{u_x - u_y},$$
$$\theta = \frac{1}{2}\ln\left(\frac{u_x + u_y}{-u_x + u_y}\right), \tag{5}$$

as the frame in parametric form.

Theorem 3.3 *Let $\rho(z)$ be a right moving frame. Then the quantity $I(z) = \rho(z).z$ is an invariant of the group action.* [2]

According to above Theorem 3.3, as given in page 128 of [7], if $\mathbf{z} = (z_1, \dots, z_n) \in M$, and the normalization equations are $\tilde{z}_i = c_i$ for $i = 1, \dots, r = \dim(G)$, then the components of

$$\rho(\mathbf{z}).\,\mathbf{z} = (c_1, \dots, c_r, I(z_{r+1}), \dots, I(z_n)),$$

where

$$I(z_k) = g.\,z_k|_{g=\rho(z)}, \qquad k = r+1, \dots, n,$$

are all invariants.

Definition 3.4 For any prolonged action in the jet space $M = J^n(X \times U)$, the invariantized jet coordinates known as the *normalized differential invariants* are denoted as

$$J^i = I(x_i) = \tilde{x}_i\big|_{g=\rho(z)}, \qquad I_k^\alpha = I(u_k^\alpha) = \tilde{u}_k^\alpha\big|_{g=\rho(z)},$$

which is the original M. Fels and P.J. Olver notation [3]. According to Replacement Theorem (Theorem 10.3 in page 38 of [3]), any invariant is a function of the $I(z_k)$. Particularly, the set $\{J^i, I_k^\alpha\}$ is a complete set of differential invariants for a prolonged action.

Now we turn our attention to considering the invariants for the Example 3.2:

Example 3.2. (cont.) The normalized differential invariants up to order two are as follows

$$\begin{aligned} g.z &= (\tilde{x}, \tilde{y}, \tilde{u}, \tilde{u}_x, \tilde{u}_y, \tilde{u}_{xx}, \tilde{u}_{xy}, \tilde{u}_{yy})\big|_{g=\rho(z)} \\ &= (J^x, J^y, I^u, I_1^u, I_2^u, I_{11}^u, I_{12}^u, I_{22}^u) \\ &= \Bigg(0, 0, u, 0, -\sqrt{-{u_x}^2 + {u_y}^2}, -\frac{u_{xx}{u_y}^2 - 2u_{xy}u_xu_y + u_{yy}{u_x}^2}{{u_x}^2 - {u_y}^2}, \\ &\quad -\frac{-u_{xx}u_xu_y + u_{xy}{u_x}^2 + u_{xy}{u_y}^2 - u_{yy}u_xu_y}{{u_x}^2 - {u_y}^2}, \\ &\quad -\frac{u_{xx}{u_x}^2 - 2u_{xy}u_xu_y + u_{yy}{u_y}^2}{{u_x}^2 - {u_y}^2}\Bigg). \end{aligned}$$

The first, second and fourth components correspond to the normalization equations and are known as the *phantom invariants*.

Definition 3.5 The *invariant differential operators* denoted as $\mathcal{D}_i = \tilde{D}_i\big|_{g=\rho(z)}$, where

$$\tilde{D}_i = \frac{d}{d\tilde{x}_i} = \sum_{k=1}^{p} \left(\frac{d\tilde{x}}{d\tilde{x}}\right)_{ik}^{-T} D_k.$$

According to Example 4.5.1 in [7], we know in general $\mathcal{D}_i I_k^\alpha \neq I_{ki}^\alpha$. This fact motivates the definition of the *invariant differentiation* and *syzygy* that will be required in the next section.

Definition 3.6 As defined in [7], we define the *invariant differentiation* of the jet coordinates, J^i and I_k^α, by

$$\mathcal{D}_j J^i = \delta_{ij} + N_{ij}, \qquad \mathcal{D}_j I^\alpha_K = I^\alpha_{Kj} + M^\alpha_{Kj},$$

where N_{ij} and M^α_{Kj} are the *correction terms*, and δ_{ij} is the Kronecker delta. For more information on correction terms see Section 4.5 of [7].

Now let I^α_J and I^α_L be two generating differential invariants, and let $JK = LM$ such that $I^\alpha_{JK} = I^\alpha_{LM}$. Thus, as given in [7], we will have the so called *syzygies* or *differential identities*

$$\mathcal{D}_K I^\alpha_J - \mathcal{D}_M I^\alpha_L = M^\alpha_{JK} - M^\alpha_{LM}.$$

To obtain the correction terms, we define the *infinitesimals of the prolonged group action* with respect to the group parameters a_j, evaluated at the identity element e, as

$$\xi^i_j = \left.\frac{\partial \tilde{x}_i}{\partial a_j}\right|_{g=e}, \qquad \varphi^\alpha_{K,j} = \left.\frac{\partial \tilde{u}^\alpha_K}{\partial a_j}\right|_{g=e}.$$

Now let the normalization equations be $\{\psi_\lambda(z) = 0,\ \lambda = 1, \ldots, r = \dim(G)\}$ and suppose the n variables actually occurring in the $\psi_\lambda(z)$ are $\zeta_1, \ldots, \zeta_n$ such that m of these are independent variables and $n-m$ of them are dependent variables and their derivatives. Let $\mathbf{T}$ denote the invariant $p \times n$ total derivative matrix

$$\mathbf{T}_{ij} := I\left(\frac{D}{Dx_i}\zeta_j\right).$$

Also, define φ to be the $r \times n$ matrix as follows,

$$\varphi_{ij} := \left(\left.\frac{\partial(g.\zeta_j)}{\partial g_i}\right|_{g=e}\right)(I).$$

Moreover, define $\mathbf{J}$ to be the $n \times r$ matrix

$$\mathbf{J}_{ij} := \frac{\partial \psi_j(I)}{\partial I(\zeta_i)},$$

that is, transpose of the Jacobian matrix of the normalization equations $\psi_1, \ldots, \psi_r$, with invariantised arguments.

Using the above defined matrices, the correction terms can be obtained as follows, that has been proved in [7].

Theorem 3.7 *The formulae for the correction terms are*

$$N_{ij} = \sum_{l=1}^{r} \mathbf{K}_{jl} \xi_l^i(I), \qquad M_{Kj}^{\alpha} = \sum_{l=1}^{r} \mathbf{K}_{jl} \varphi_{K,l}^{\alpha}(I),$$

where l is the index for the group parameters, $r = \dim(G)$, and the $p \times r$ correction matrix $\mathbf{K}$, is given by $\mathbf{K} = -\mathbf{TJ}(\varphi \mathbf{J})^{-1}$.

Now we calculate the invariant differentiation of the jet coordinates and the syzygies of the transformation (4) in Example 3.2.

Example 3.2. (cont.) If we set $u = u(x, y, t)$ and $\tilde{t} = t$ and take the normalization equations as before, we obtain

$$\tilde{u}_t\big|_{g=\rho(z)} = I_3^u = u_t,$$

$$\tilde{u}_y\big|_{g=\rho(z)} = I_2^u = -\sqrt{-{u_x}^2 + {u_y}^2},$$

$$\tilde{u}_{xx}\big|_{g=\rho(z)} = I_{11}^u = -\frac{u_{xx}{u_y}^2 - 2u_{xy}u_xu_y + u_{yy}{u_x}^2}{{u_x}^2 - {u_y}^2},$$

$$\tilde{u}_{xy}\big|_{g=\rho(z)} = I_{12}^u = -\frac{-u_{xx}u_xu_y + u_{xy}{u_x}^2 + u_{xy}{u_y}^2 - u_{yy}u_xu_y}{{u_x}^2 - {u_y}^2},$$

$$\tilde{u}_{yy}\big|_{g=\rho(z)} = I_{22}^u = -\frac{u_{xx}{u_x}^2 - 2u_{xy}u_xu_y + u_{yy}{u_y}^2}{{u_x}^2 - {u_y}^2},$$

$$\tilde{u}_{xxy}\big|_{g=\rho(z)} = I_{112}^u = \frac{\begin{array}{c} -u_{xxy}{u_y}^3 + u_{xyy}{u_x}^3 + 2u_{xyy}u_x{u_y}^2 \\ -u_{yyy}{u_x}^2u_y + u_{xxx}u_x{u_y}^2 - 2u_{xxy}{u_x}^2u_y \end{array}}{(-{u_x}^2 + {u_y}^2)^{3/2}},$$

$$\tilde{u}_{xyy}\big|_{g=\rho(z)} = I_{122}^u = \frac{\begin{array}{c} u_{xxy}{u_x}^3 - u_{xyy}{u_y}^3 - 2u_{xyy}{u_x}^2u_y \\ +u_{yyy}u_x{u_y}^2 - u_{xxx}{u_x}^2u_y + 2u_{xxy}u_x{u_y}^2 \end{array}}{(-{u_x}^2 + {u_y}^2)^{3/2}}.$$

According to Theorem 3.7 we obtain the invariant differentiation of the jet coordinates as follows

$$\mathcal{D}_x I_2^u = I_{12}^u, \qquad \mathcal{D}_y I_2^u = I_{22}^u, \qquad \mathcal{D}_t I_2^u = I_{23}^u,$$

$$\mathcal{D}_x I_{11}^u = I_{111}^u - \frac{2I_{11}^u I_{12}^u}{I_2^u}, \qquad \mathcal{D}_x I_{22}^u = I_{122}^u - \frac{2I_{11}^u I_{12}^u}{I_2^u},$$

$$\mathcal{D}_y I_{11}^u = I_{112}^u - \frac{2(I_{12}^u)^2}{I_2^u}, \qquad \mathcal{D}_y I_{22}^u = I_{222}^u - \frac{2(I_{12}^u)^2}{I_2^u},$$

$$\mathcal{D}_t I_{11}^u = I_{113}^u - \frac{2I_{12}^u I_{13}^u}{I_2^u}, \qquad \mathcal{D}_t I_{22}^u = I_{223}^u - \frac{2I_{12}^u I_{13}^u}{I_2^u},$$

$$\mathcal{D}_x I_{12}^u = I_{112}^u - \frac{I_{11}^u}{I_2^u}(I_{11}^u + I_{22}^u), \qquad \mathcal{D}_y I_{12}^u = I_{122}^u - \frac{I_{12}^u}{I_2^u}(I_{11}^u + I_{22}^u),$$

$$\mathcal{D}_t I_{12}^u = I_{123}^u - \frac{I_{13}^u}{I_2^u}(I_{11}^u + I_{22}^u).$$

We know that there are two ways to reach I_{112}^u and since both ways must be equal, we get the following syzygy between I^u and I_{11}^u:

$$\mathcal{D}_2 I^u \big((\mathcal{D}_1)^2 \mathcal{D}_2 I^u - \mathcal{D}_2 I_{11}^u\big) + (I_{11}^u)^2 + I_{11}^u (\mathcal{D}_2)^2 I^u - 2(\mathcal{D}_1 \mathcal{D}_2 I^u)^2 = 0. \quad (6)$$

Similarly, there are two possibilities to obtain I_{113}^u. Thus, we get a syzygy between I_3^u and I_{11}^u and the syzygy is:

$$\mathcal{D}_3 I_{11}^u = \left((\mathcal{D}_1)^2 - \frac{2I_{12}^u \mathcal{D}_1}{I_2^u} + \frac{I_{11}^u \mathcal{D}_2}{I_2^u}\right) I_3^u, \quad (7)$$

and likewise, the syzygy between I_3^u and I_{22}^u is:

$$\mathcal{D}_3 I_{22}^u = \left((\mathcal{D}_2)^2 - \frac{I_{12}^u \mathcal{D}_1}{I_2^u}\right) I_3^u. \quad (8)$$

Finally, there are two syzygies between I_3^u and I_{12}^u, which are as follows:

$$\mathcal{D}_3 I_{12}^u = \left(\mathcal{D}_1 \mathcal{D}_2 - \frac{I_{22}^u \mathcal{D}_1}{I_2^u}\right) I_3^u, \quad (9)$$

$$\mathcal{D}_3 I_{12}^u = \left(\mathcal{D}_2 \mathcal{D}_1 + \frac{I_{12}^u \mathcal{D}_2}{I_2^u} - \frac{I_{11}^u \mathcal{D}_1}{I_2^u} - \frac{I_{22}^u \mathcal{D}_1}{I_2^u}\right) I_3^u. \quad (10)$$

From Equations (9) and (10) we can verify that the invariant operators $\mathcal{D}_x$ and $\mathcal{D}_y$ do not commute. In general, the invariant total differentiation operators do not commute. In fact, we have the following Theorem [3]:

Theorem 3.8 *Denote the invariantized derivatives of the infinitesimals* ξ_l^k, *for* $k, i = 1, \ldots, p$ *and* $l = 1, \ldots, r$, *by* $\Xi_{li}^k = \tilde{D}_i \xi_l^k(\tilde{z})\big|_{g=\rho(z)}$, *then the commutators are given by*

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{p} A_{ij}^k \mathcal{D}_k, \quad A_{ij}^k = \sum_{l=1}^{r} K_{jl} \Xi_{li}^k - K_{il} \Xi_{lj}^k.$$

We now define invariant one-forms that will be required in the next section.

Definition 3.9 The *invariant one-forms* are denoted as

$$I(dx_i) = d\tilde{x}_i\big|_{g=\rho(z)} = \left(\sum_{j=1}^{p} D_j(\tilde{x}_i) dx_j \right)_{g=\rho(z)}.$$

By Theorem 3.10 below, we see that an invariant total differentiation operator $\mathcal{D}_i$ sends invariant differential forms to invariant differential forms. In fact, if $\mathcal{D}_i$ is the invariant differentiation operator and ω is a form, then $\mathcal{D}_i(\omega)$ denote as a *Lie derivative.* For more details see [5].

Theorem 3.10 *Consider the set of invariant total differentiation operators,* $\{\mathcal{D}_i\}$, *and the set of invariant one-forms,* $\{I(dx_j)\}$. *Therefore, if*

$$\mathcal{D}_i(I(dx_j)) = \sum_{k=1}^{p} B_{ij}^k I(dx_k),$$

then $B_{ki}^j = A_{jk}^i$.

Finally, in the end of this section, from the above Theorem 3.10 we obtain the Lie derivatives of $I(dx_j)$ with respect to $\mathcal{D}_i$ for the Hyperbolic Rotation-Translation group action on (x, y, t), that has been given in Example 3.2.

Example 3.2. (cont.) Recall that $g \in G$ (the it Hyperbolic Rotation-Translation group) act on (x, y, t), where t is an invariant dummy indepen-

Table 1. Lie derivatives of the $I(dx_j)$ with respect to the $\mathcal{D}_i$.

Lie derivative	$I(dx)$	$I(dy)$	$I(dt)$
$\mathcal{D}_x$	$\frac{I^u_{11}}{I^u_2}I(dy)$	$-\frac{I^u_{12}}{I^u_2}I(dy)-\frac{I^u_{13}}{I^u_2}I(dt)$	0
$\mathcal{D}_y$	$-\frac{I^u_{11}}{I^u_2}I(dx)-\frac{I^u_{13}}{I^u_2}I(dt)$	$\frac{I^u_{12}}{I^u_2}I(dx)$	0
$\mathcal{D}_t$	$\frac{I^u_{13}}{I^u_2}I(dy)$	$\frac{I^u_{13}}{I^u_2}I(dx)$	0

dent variable introduced to effect variation. Therefore, the Lie derivatives of $I(dx_j)$ with respect to $\mathcal{D}_i$ are as shown in the Table 1.

4. Invariant calculus of variations and structure of Noether's conservation laws

In this section, we will use the concept of invariant calculus of variations as formulated by Gonçalves and Mansfield [4], [5], [6] and Mansfield [7]. Suppose the Lagrangian $\bar{L}[\mathbf{u}]$ of the variational problem $\bar{\varphi}[\mathbf{u}] = \int \bar{L}[\mathbf{u}]\, d\mathbf{x}$ is a smooth function of $\mathbf{x} = (x_1, \ldots, x_p)$, $\mathbf{u} = (u^1, \ldots, u^q)$ and finite number of derivatives of u^α, where $\bar{\varphi}[\mathbf{u}]$ is invariant under some group action with finite set of generators $\{\kappa_1, \ldots, \kappa_N\}$. Thus, as given in [5], we can rewrite $\bar{\varphi}[\mathbf{u}]$ as $\varphi[\kappa] = \int L[\kappa]\, I(d\mathbf{x})$, in which $I(d\mathbf{x}) = I(dx_1) \ldots I(dx_p)$ denotes the invariant volume form and $d\mathbf{x} = dx_1 \ldots dx_p$ is the standard volume form. Now we suppose the functional $\bar{\varphi}[\mathbf{u}]$ be extremized by $\mathbf{x} \to (\mathbf{x}, \mathbf{u}(\mathbf{x}))$, then for a small perturbation of $\mathbf{u}$

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \bar{\varphi}[\mathbf{u} + \varepsilon \mathbf{v}] \\ &= \int \sum_{\alpha=1}^{q} \left[E^\alpha(\bar{L}) v^\alpha + \sum_{i=1}^{p} \frac{d}{dx_i} \left(\frac{\partial \bar{L}}{\partial u_i^\alpha} v^\alpha + \cdots \right) \right] d\mathbf{x}, \end{aligned}$$

where $$E^\alpha = \sum_K (-1)^K \frac{D^{|K|}}{Dx_1^{k_1} Dx_2^{k_2} \cdots Dx_p^{k_p}} \frac{\partial}{\partial u_K^\alpha},$$

is the Euler operator with respect to the dependent variable u^α, and symbolically,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\bar{\varphi}[\mathbf{u}+\varepsilon\mathbf{v}] = \frac{d}{dt}\bigg|_{u_t=v}\bar{\varphi}[\mathbf{u}].$$

According to [5], we have

$$\begin{aligned}0 &= \mathcal{D}_t\int L[\kappa]\, I(d\mathbf{x})\\ &= \mathcal{D}_{p+1}\int L[\kappa]\, I(d\mathbf{x})\\ &= \int\Bigg(\sum_\alpha E^\alpha(L) I^\alpha_\tau I(dx)\\ &\qquad + \sum_{i=1}^{p}\mathcal{D}_i\Bigg(\sum_{j=1}^{p+1}F_{ij}I(dx_1)\ldots\widehat{I(dx_j)}\ldots I(dx_{p+1})\Bigg)\Bigg),\end{aligned}$$

where $E^\alpha(L)$ are the invariantized Euler-Lagrange equations, F_{ij} depend on $I^\alpha_{K,p+1}$ and I^α_J with K and J multi-indices of differentiation with respect to x_i, for $i = 1, \ldots, p$, and

$$I(dx_1)\ldots\widehat{I(dx_j)}\ldots I(dx_{p+1}) = I(dx_1)\ldots I(dx_{j-1})I(dx_{j+1})\ldots I(dx_{p+1}).$$

Theorem 4.1 *The process of calculating the invariantized Euler-Lagrange equations produces boundary terms*

$$\int\sum_{i=1}^{p}\mathcal{D}_i\Bigg(\sum_{j=1}^{p+1}F_{ij}I(dx_1)\ldots\widehat{I(dx_j)}\ldots I(dx_{p+1})\Bigg),$$

that can be written as

$$\int\sum_{i=1}^{p} d\Bigg((-1)^{i-1}\Bigg\{\sum_{K,\alpha}I^\alpha_{K,\tau}C^\alpha_{K,i}\Bigg\}I(dx_1)\ldots\widehat{I(dx_j)}\ldots I(dx_{p+1})\Bigg),$$

where K is a multi-index of differentiation with respect to x_i, for $i = 1, \ldots, p$, and $C^\alpha_{K,i}$ are functions of I^α_J, with J a multi-index of differentiation with respect to x_i. [5]

Now in this section we consider a variational problem associated to an spacial case of an important equation known as the Monge–Ampère equation.

Example 4.2 Consider the variational problem

$$\iint \left[u(u_{x^2}u_{y^2} - u_{xy}^2) + u_{x^2} - u_{y^2}\right] dxdy, \tag{11}$$

associated to a type of the Monge–Ampère equation, which is invariant under the action presented in Example 3.2. To find the invariantized Euler-Lagrange equation, introduce a dummy invariant independent variable t to effect the variation, and set $u = u(x, y, t)$, therefore $\tilde{u}_t\big|_{g=\rho(z)} = I_3^u = u_t$. Rewriting the above variational problem in terms of the invariants of the group action yields

$$\iint \left[I^u(I_{11}^u I_{22}^u - (I_{12}^u)^2) + I_{11}^u - I_{22}^u\right] I(dx)\, I(dy). \tag{12}$$

To obtain the invariantized Euler-Lagrange equation and boundary terms, after differentiating (12) under the integral sign we obtain

$$\begin{aligned}
\mathcal{D}_t \iint & \left[I^u(I_{11}^u I_{22}^u - (I_{12}^u)^2) + I_{11}^u - I_{22}^u\right] I(dx)I(dy) \\
= \iint & \Big[\big(\mathcal{D}_t(I^u)(I_{11}^u I_{22}^u - (I_{12}^u)^2) + I^u I_{22}^u \mathcal{D}_t I_{11}^u + I^u I_{11}^u \mathcal{D}_t I_{22}^u \\
& \quad - 2I^u I_{12}^u \mathcal{D}_t I_{12}^u + \mathcal{D}_t I_{11}^u - \mathcal{D}_t I_{22}^u\big) I(dx)I(dy) \\
& \quad + \big(I^u(I_{11}^u I_{22}^u - (I_{12}^u)^2) + I_{11}^u - I_{22}^u\big)\mathcal{D}_t(I(dx)I(dy))\Big].
\end{aligned}$$

Using Table 1 we see that $\mathcal{D}_t(I(dx)I(dy)) = 0$. Then substituting $\mathcal{D}_t I_{11}^u$, $\mathcal{D}_t I_{22}^u$, and $\mathcal{D}_t I_{12}^u$ by (7), (8), and (9), respectively, and performing integration by parts yields

$$\begin{aligned}
&\iint 3\big(I_{11}^u I_{22}^u - (I_{12}^u)^2\big) I_3^u\, I(dx)\, I(dy) \\
&\quad + \iint \Bigg[\mathcal{D}_x\Bigg(\Bigg\{\Bigg(\frac{I^u I_{11}^u I_{12}^u}{I_2^u} + \frac{I^u I_{22}^u I_{12}^u}{I_2^u} - I^u I_{122}^u\Bigg) I_3^u + I^u I_{22}^u I_{13}^u
\end{aligned}$$

$$
\begin{aligned}
&\quad + I^u_{13} - 2I^u I^u_{12} I^u_{23}\Big\} I(dx)\, I(dy) \\
&\quad + \left\{ \frac{I^u I^u_{22} I^u_{13} I^u_3}{I^u_2} + \frac{I^u_{13} I^u_3}{I^u_2} \right\} I(dx)\, I(dt) \Bigg) \\
&+ \mathcal{D}_y \Bigg(\Bigg\{ \left(I^u I^u_{112} - I^u_2 I^u_{11} - \frac{I^u (I^u_{11})^2}{I^u_2} - \frac{I^u I^u_{11} I^u_{22}}{I^u_2} \right) I^u_3 \\
&\quad + I^u I^u_{11} I^u_{23} - I^u_{23} \Bigg\} I(dx)\, I(dy) \\
&- \frac{2I^u I^u_{12} I^u_{13} I^u_3}{I^u_2} I(dx)\, I(dt) + \left\{ \frac{I^u_{13} I^u_3}{I^u_2} - \frac{I^u I^u_{11} I^u_{13} I^u_3}{I^u_2} \right\} I(dy)\, I(dt) \Bigg) \Bigg],
\end{aligned}
$$

where all forms involving $I(dt)$, because there is no integration along t, have been discarded. Thus, we obtain the invariantized Euler-Lagrange equation

$$E^u(L) = 3(I^u_{11} I^u_{22} - (I^u_{12})^2) = 3(u_{x^2} u_{y^2} - u^2_{xy}).$$

Therefore, according to Theorem 4.1 the boundary terms can be written as

$$
\begin{aligned}
\iint d\Bigg(&\Bigg\{ \left(\frac{I^u I^u_{12}(I^u_{11} + I^u_{22})}{I^u_2} - I^u I^u_{122} \right) I^u_3 + (I^u I^u_{22} + 1) I^u_{13} - 2I^u I^u_{12} I^u_{23} \Bigg\} I(dy) \\
&- \Bigg\{ \left(I^u I^u_{112} - I^u_2 I^u_{11} - \frac{I^u I^u_{11}(I^u_{11} + I^u_{22})}{I^u_2} \right) I^u_3 \\
&\quad + (I^u I^u_{11} - 1) I^u_{23} \Bigg\} I(dx) \Bigg). \qquad (13)
\end{aligned}
$$

Theorem 4.3 ([5]) *Let $\int L(k_1, k_2, \dots)\, I(d\mathbf{x})$ be invariant under $G \times M \to M$, where $M = J^n(X, U)$, with generating invariants κ_j, for $j = 1, \dots, N$. Introduce a dummy invariant variable t to effect the variation and then integration by parts yields*

$$
\begin{aligned}
&\mathcal{D}_t \int L(k_1, k_2, \dots)\, I(d\mathbf{x}) \\
&\quad = \int \Bigg[\sum_\alpha E^\alpha(L) I^\alpha_t I(d\mathbf{x}) \\
&\qquad + \sum_{k=1}^{p} d\Bigg((-1)^{k-1} \Bigg(\sum_{J,\alpha} I^\alpha_{Jt} C^\alpha_{J,k} \Bigg) I(dx_1) \dots \widehat{I(dx_k)} \dots I(dx_{p+1}) \Bigg) \Bigg],
\end{aligned}
$$

where this defines the vectors $\mathbf{C}_k^\alpha = (C_{J,k}^\alpha)$.

Recall that $E^\alpha(L)$ are the invariantized Euler- Lagrange equations and $I_{Jt}^\alpha = I(u_{Jt}^\alpha)$, where J is a multi-index of differentiation with respect to the variables x_i, for $i = 1, \ldots, p$. Let $(a_1, \ldots, a_r)$ be the coordinates of G near the identity e, and $\mathbf{v}_i$, for $i = 1, \ldots, r$, the associated infinitesimal vector fields. Furthermore, let $\mathrm{Ad}(g)$ be the Adjoint representation of G with respect to these vector fields. For each dependent variable, define the matrices of characteristics to be

$$\mathcal{Q}^\alpha(\tilde{z}) = \left(\widetilde{D_K(Q_i^\alpha)}\right), \qquad \alpha = 1, \ldots, q,$$

where K is a multi-index of differentiation with respect to the x_k, and

$$Q_i^\alpha = \varphi_i^\alpha - \sum_{k=1}^p \xi_i^k u_k^\alpha = \left.\frac{\partial \widetilde{u^\alpha}}{\partial a_i}\right|_{g=e} - \sum_{k=1}^p \left.\frac{\partial \widetilde{x_k}}{\partial a_i}\right|_{g=e} u_k^\alpha$$

are the components of the q-tuple $\mathbf{Q}_i$ known as the characteristic of the vector field $\mathbf{v}_i$. Let $\mathcal{Q}^\alpha(J, I)$, for $\alpha = 1, \ldots, q$, be the invariantization of the above matrices. Then, the r conservation laws obtained via Noether's Theorem can be written in the form

$$d(\mathrm{Ad}(\rho)^{-1}(\upsilon_1, \ldots, \upsilon_p) M_{\mathcal{J}} d^{p-1}\hat{\mathbf{x}}) = 0,$$

where

$$\upsilon_k = \sum_\alpha (-1)^{k-1} (\mathcal{Q}^\alpha(J, I)\mathbf{C}_k^\alpha + L(\Xi(J, I))_k),$$

are the vectors of invariants, with $(\Xi(J, I))_k$ the k^{th} column of $\Xi(J, I)$, $M_{\mathcal{J}}$ is the matrix of first minors of the Jacobian matrix evaluated at the frame, $\mathcal{J} = (d\tilde{x}/dx)|_{g=\rho(z)}$, and

$$d^{p-1}\hat{x} = \begin{pmatrix} \widehat{dx_1} dx_2 \ldots dx_p \\ dx_1 \widehat{dx_2} dx_3 \ldots dx_p \\ \vdots \\ dx_1 \ldots dx_{p-1} \widehat{dx_p} \end{pmatrix} = \begin{pmatrix} dx_2 dx_3 \ldots dx_p \\ dx_1 dx_3 \ldots dx_p \\ \vdots \\ dx_1 dx_2 \ldots dx_{p-1} \end{pmatrix}.$$

Lemma 4.4 *The inverse of the Adjoint representation of the Hyperbolic Rotation-Translation group with respect to its generating vector fields evaluated at the frame* (5) *is*

$$Ad(\rho(z))^{-1} = \begin{pmatrix} \frac{u_x - u_y}{\sqrt{{u_y}^2 - {u_x}^2}} & 0 & 0 \\ 0 & -\frac{u_x + u_y}{\sqrt{{u_y}^2 - {u_x}^2}} & 0 \\ \frac{1}{2}\frac{(x-y)(u_x - u_y)}{\sqrt{{u_y}^2 - {u_x}^2}} & -\frac{1}{2}\frac{(x+y)(u_x + u_y)}{\sqrt{{u_y}^2 - {u_x}^2}} & 1 \end{pmatrix}. \tag{14}$$

Proof. Consider the action (4) and let it act on the infinitesimal vector fields generating the Hyperbolic Rotation-Translation group,

$$\mathbf{v}_1 = -\partial_x + \partial_y, \qquad \mathbf{v}_2 = \partial_x + \partial_y, \qquad \mathbf{v}_3 = y\partial_x + x\partial_y,$$

as follows

$$\begin{aligned}
&g.(\alpha(-\partial_x + \partial_y) + \beta(\partial_x + \partial_y) + \gamma(y\partial_x + x\partial_y)) \\
&= \alpha(-\partial_{\tilde{x}} + \partial_{\tilde{y}}) + \beta(\partial_{\tilde{x}} + \partial_{\tilde{y}}) + \gamma(\tilde{y}\partial_{\tilde{x}} + \tilde{x}\partial_{\tilde{y}}) \\
&= \alpha(-\cosh(\theta)\partial_x + \sinh(\theta)\partial_y - \sinh(\theta)\partial_x + \cosh(\theta)\partial_y) \\
&\quad + \beta(\cosh(\theta)\partial_x - \sinh(\theta)\partial_y - \sinh(\theta)\partial_x + \cosh(\theta)\partial_y) \\
&\quad + \gamma((a + b + x\sinh(\theta) + y\cosh(\theta))(\cosh(\theta)\partial_x - \sinh(\theta)\partial_y) \\
&\quad + (-a + b + x\cosh(\theta) + y\sinh(\theta))(-\sinh(\theta)\partial_x + \cosh(\theta)\partial_y)) \\
&= \alpha(\cosh(\theta)(-\partial_x + \partial_y) + \sinh(\theta)(-\partial_x + \partial_y)) \\
&\quad + \beta(\cosh(\theta)(\partial_x + \partial_y) - \sinh(\theta)(\partial_x + \partial_y)) \\
&\quad + \gamma([(a + b)\cosh(\theta) + (a - b)\sinh(\theta)]\partial_x \\
&\quad + [(-a + b)\cosh(\theta) - (a + b)\sinh(\theta)]\partial_y + (y\partial_x + x\partial_y)) \\
&= \alpha(\cosh(\theta)(-\partial_x + \partial_y) + \sinh(\theta)(-\partial_x + \partial_y)) \\
&\quad + \beta(\cosh(\theta)(\partial_x + \partial_y) - \sinh(\theta)(\partial_x + \partial_y)) \\
&\quad + \gamma([-a\cosh(\theta) - a\sinh(\theta)](-\partial_x + \partial_y) \\
&\quad + [b\cosh(\theta) - b\sinh(\theta)](\partial_x + \partial_y) + (y\partial_x + x\partial_y))
\end{aligned}$$

$$= (\alpha\ \beta\ \gamma) \begin{pmatrix} \cosh(\theta)+\sinh(\theta) & 0 & 0 \\ 0 & \cosh(\theta)-\sinh(\theta) & 0 \\ -a(\cosh(\theta)+\sinh(\theta)) & b(\cosh(\theta)-\sinh(\theta)) & 1 \end{pmatrix} \begin{pmatrix} -\partial_x+\partial_y \\ \partial_x+\partial_y \\ y\partial_x+x\partial_y \end{pmatrix},$$

where the above 3×3 matrix, $\mathrm{Ad}(g)$, is the Adjoint representation of G with respect to its generating infinitesimal vector fields. Thus, $\mathrm{Ad}(g)^{-1}$ is

$$\mathrm{Ad}(g)^{-1} = \begin{pmatrix} \cosh\theta-\sinh\theta & 0 & 0 \\ 0 & \cosh\theta+\sinh\theta & 0 \\ a & -b & 1 \end{pmatrix}.$$

Now evaluating $\mathrm{Ad}(g)$ and $\mathrm{Ad}(g)^{-1}$ at the frame (3.2), leads

$$\begin{pmatrix} -\dfrac{u_x+u_y}{\sqrt{{u_y}^2-{u_x}^2}} & 0 & 0 \\ 0 & \dfrac{u_x-u_y}{\sqrt{{u_y}^2-{u_x}^2}} & 0 \\ -\dfrac{1}{2}x+\dfrac{1}{2}y & -\dfrac{1}{2}x-\dfrac{1}{2}y & 1 \end{pmatrix},$$

$$\begin{pmatrix} \dfrac{u_x-u_y}{\sqrt{{u_y}^2-{u_x}^2}} & 0 & 0 \\ 0 & -\dfrac{u_x+u_y}{\sqrt{{u_y}^2-{u_x}^2}} & 0 \\ \dfrac{1}{2}\dfrac{(x-y)(u_x-u_y)}{\sqrt{{u_y}^2-{u_x}^2}} & -\dfrac{1}{2}\dfrac{(x+y)(u_x+u_y)}{\sqrt{{u_y}^2-{u_x}^2}} & 1 \end{pmatrix},$$

respectively. □

We now calculate the Noether's conservation laws of Euler-Lagrange equations for the variational problem (11), associated to the Monge–Ampère equation.

Theorem 4.5 *The three Noether's conservation laws of Euler-Lagrange equations for the variational problem*

$$\iint \left[u(u_{x^2}u_{y^2}-u_{xy}^2)+u_{x^2}-u_{y^2}\right] dxdy$$

are

$$
d\left(\begin{pmatrix} \dfrac{u_x-u_y}{\sqrt{{u_y}^2-{u_x}^2}} & 0 & 0 \\ 0 & -\dfrac{u_x+u_y}{\sqrt{{u_y}^2-{u_x}^2}} & 0 \\ \dfrac{1}{2}\dfrac{(x-y)(u_x-u_y)}{\sqrt{{u_y}^2-{u_x}^2}} & -\dfrac{1}{2}\dfrac{(x+y)(u_x+u_y)}{\sqrt{{u_y}^2-{u_x}^2}} & 1 \end{pmatrix}\right.
$$

$$
\times \begin{pmatrix} I^u(-I^u_{11}I^u_{12}+I^u_2 I^u_{122}-(I^u_{12})^2)-I^u_{12}+I^u_{22} & I^u I^u_{11}(-I^u_{11}-I^u_{12}-I^u_{22})+I^u(I^u_2 I^u_{112}+(I^u_{12})^2)-I^u_{11}(I^u_2)^2+I^u_{12}-I^u_{11} \\ I^u(-I^u_{11}I^u_{12}+I^u_2 I^u_{122}+(I^u_{12})^2)-I^u_{12}-I^u_{22} & I^u I^u_{11}(-I^u_{11}+I^u_{12}-I^u_{22})+I^u(I^u_2 I^u_{112}+(I^u_{12})^2)-I^u_{11}(I^u_2)^2-I^u_{12}-I^u_{11} \\ -I^u I^u_2 I^u_{22}-I^u_2 & 0 \end{pmatrix}
$$

$$
\left.\times \begin{pmatrix} \dfrac{-u_y}{\sqrt{{u_y}^2-{u_x}^2}} & \dfrac{-u_x}{\sqrt{{u_y}^2-{u_x}^2}} \\ \dfrac{-u_x}{\sqrt{{u_y}^2-{u_x}^2}} & \dfrac{-u_y}{\sqrt{{u_y}^2-{u_x}^2}} \end{pmatrix}\begin{pmatrix} dy \\ dx \end{pmatrix}\right) = 0.
$$

Proof. According to Theorem 4.3 the elements of $\mathbf{C}^u_1, \mathbf{C}^u_2$ correspond to the coefficients of the I^α_{Jt} in (13), respectively, as follows

$$
\begin{pmatrix} I^u I^u_{12}(I^u_{11}+I^u_{22})/I^u_2 - I^u_1 I^u_{22} - I^u I^u_{122} \\ I^u I^u_{22}+1 \\ -2I^u I^u_{12} \end{pmatrix},
$$

$$
\begin{pmatrix} I^u I^u_{112} - I^u_2 I^u_{11} - I^u I^u_{11}(I^u_{11}+I^u_{22})/I^u_2 \\ 0 \\ I^u I^u_{11}-1 \end{pmatrix},
$$

and the $(\Xi(J,I))_i$, for $i=1,2$, are

$$
(\Xi(J,I))_1 = \begin{array}{c c} & \xi^x \\ \begin{array}{c} a \\ b \\ \theta \end{array} & \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \end{array}, \qquad (\Xi(J,I))_2 = \begin{array}{c c} & \xi^y \\ \begin{array}{c} a \\ b \\ \theta \end{array} & \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \end{array}.
$$

Since $I_1^u = 0$, the invariantized matrix of characteristics is

$$
\mathcal{Q}^u(J,I) = \begin{array}{c c} & \begin{array}{ccc} Q^u & D_x(Q^u) & D_y(Q^u) \end{array} \\ \begin{array}{c} a \\ b \\ \theta \end{array} & \begin{pmatrix} -I_2^u & I_{11}^u - I_{12}^u & I_{12}^u - I_{22}^u \\ -I_2^u & -I_{11}^u - I_{12}^u & -I_{12}^u - I_{22}^u \\ 0 & -I_2^u & 0 \end{pmatrix} \end{array},
$$

thus, the vectors of invariants are

$$
\upsilon_1 = \begin{pmatrix} I^u(-I_{11}^u I_{12}^u + I_2^u I_{122}^u - (I_{12}^u)^2) - I_{12}^u + I_{22}^u \\ I^u(-I_{11}^u I_{12}^u + I_2^u I_{122}^u + (I_{12}^u)^2) - I_{12}^u - I_{22}^u \\ -I^u I_2^u I_{22}^u - I_2^u \end{pmatrix},
$$

$$
\upsilon_2 = \begin{pmatrix} I^u I_{11}^u(-I_{11}^u - I_{12}^u - I_{22}^u) + I^u(I_2^u I_{112}^u + (I_{12}^u)^2) - I_{11}^u (I_2^u)^2 + I_{12}^u - I_{11}^u \\ I^u I_{11}^u(-I_{11}^u + I_{12}^u - I_{22}^u) + I^u(I_2^u I_{112}^u + (I_{12}^u)^2) - I_{11}^u (I_2^u)^2 - I_{12}^u - I_{11}^u \\ 0 \end{pmatrix},
$$

and according to Lemma 4.4, the inverse of the Adjoint representation $\mathrm{Ad}(\rho)^{-1}$ is as (14). Finally, the Jacobian matrix $\mathcal{J}$ is

$$
\begin{aligned}
\mathcal{J} &= \begin{pmatrix} \left.\dfrac{\partial \tilde{x}}{\partial x}\right|_{g=\rho(z)} & \left.\dfrac{\partial \tilde{x}}{\partial y}\right|_{g=\rho(z)} \\ \left.\dfrac{\partial \tilde{y}}{\partial x}\right|_{g=\rho(z)} & \left.\dfrac{\partial \tilde{y}}{\partial y}\right|_{g=\rho(z)} \end{pmatrix} = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{-u_y}{\sqrt{{u_y}^2 - {u_x}^2}} & \dfrac{-u_x}{\sqrt{{u_y}^2 - {u_x}^2}} \\ \dfrac{-u_x}{\sqrt{{u_y}^2 - {u_x}^2}} & \dfrac{-u_y}{\sqrt{{u_y}^2 - {u_x}^2}} \end{pmatrix},
\end{aligned}
$$

and its matrix of first minors, $M_{\mathcal{J}}$, is

$$M_{\mathcal{J}} = \begin{pmatrix} \dfrac{-u_y}{\sqrt{{u_y}^2 - {u_x}^2}} & \dfrac{-u_x}{\sqrt{{u_y}^2 - {u_x}^2}} \\ \dfrac{-u_x}{\sqrt{{u_y}^2 - {u_x}^2}} & \dfrac{-u_y}{\sqrt{{u_y}^2 - {u_x}^2}} \end{pmatrix}.$$

Thus, the conservation laws are

$$d\big(Ad(\rho)^{-1}.(\upsilon_1\ \upsilon_2).M_{\mathcal{J}}.d^1\hat{x}\big) = 0, \qquad \text{where} \qquad d^1\hat{x} = \begin{pmatrix} dy \\ dx \end{pmatrix}. \qquad \square$$

5. Concluding remarks

We see that the three Noether's conservation laws are in terms of vectors of invariants, the adjoint representation of the moving frame and a matrix which represents the group action on the 1-forms. Also, we notice that for calculation of (13) in Example 4.2 if we substitute $\mathcal{D}_t I^u_{12}$ by Equation (10) instead of Equation (9), or we use a combination of the two; in any case the conservation laws are equivalent.

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References

[1] Berestovskii V. N. and Nikonorov Yu. G., *Killing vector fields of constant length on Riemannian manifolds.* Siberian Mathematical Journal **49** (2008), 395–407.

[2] Fels M. and Olver P. J., *Moving coframes: I. A practical algorithm.* Acta Applicandae Mathematicae **51** (1998), 161–213.

[3] Fels M. and Olver P. J., *Moving coframes: II. Regularization and theoretical foundations.* Acta Applicandae Mathematicae **55** (1999), 127–208.

[4] Gonçalves T. M. N. and Mansfield E. L., *Moving frames and conservation laws for Euclidean invariant Lagrangians.* Studies in Applied Mathematics **130** (2013), 134–166.

[5] Gonçalves T. M. N. and Mansfield E. L., *Moving frames and Noether's conservation laws - the general case.* Forum of Mathematics, Sigma., Cambridge Journals, 2015.

[6] Gonçalves T. M. N. and Mansfield E. L., *On moving frames and Noether's*

conservation laws. Studies in Applied Mathematics **128** (2012), 1–29.

[7] Mansfield E. L., *A practical guide to the invariant calculus*. Vol. 26. Cambridge University Press, 2010.

[8] Noether E., *Invariante variationsprobleme*. Nachrichten von der Gesellschaft der Wissenschaften zu Gottingen, mathematisch-physikalische Klasse **1918** (1918), 235–257.

[9] Olver P. J., *Applications of Lie groups to differential equations*. Vol. 107, Springer, 2000.

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