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Spatial Asymptotic Profiles of Solutions to the Navier-Stokes System in a Rotating Frame with Fast Decaying Data

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Abstract. The nonstationary Navier-Stokes system for a viscous, incompressible fluid influenced by a Coriolis force in the whole space $\mathbb{R}^3$ is considered at large distances. The solvability of the corresponding integral equations of these equations in weighted L^∞-spaces is established. Furthermore, the leading terms of the asymptotic profile of the solution at fixed time $t > 0$ for $|x| > t$ and far from the axis of rotation are investigated.

Key words: Rotating Navier-Stokes equations, Coriolis operator, mild solutions, weighted L^∞-spaces, rate of spatial decay.

1. Introduction

In this paper we study the 3-dimensional "rotating" Navier-Stokes equations

$$
\text{(NSC)}\begin{cases} u_t - \Delta u + u \cdot \nabla u + \Omega e_3 \times u + \nabla p = f & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ u(0) = u_0 & \text{in } \mathbb{R}^3, \end{cases}
$$

with a given constant Coriolis parameter $\Omega \neq 0$, initial data u_0 and external force f. The unknowns $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $p = p(x,t)$ denote the velocity vector field and the pressure of the fluid at the point $(x,t) \in \mathbb{R}^3 \times [0,T)$, respectively. Here e_3 denotes the unit vector $(0,0,1)$, the term $\Omega e_3 \times u$ describes the Coriolis force, and u_0 denotes a solenoidal initial velocity field. These equations above are also referred to as the *Navier-Stokes-Coriolis equations*.

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One of the most important features that distinguishes flows in fluid dynamics from those in ocean and atmospherical dynamics is the influence of the rotation of the earth. The equations (NSC) describe the motion of rotating fluids influenced by the Coriolis force. Almost all of the models of oceanography and meteorology dealing with large-scale phenomena include a Coriolis force. Many important features of oceanic flows, e.g. the intensification of the Gulf stream near the Gulf of Mexico, can be explained by the rotation of the earth. Although precise explanations require models including temperature effects, boundaries describing the sea ground or coasts. A first step is to neglect these additional influences and to investigate the simplified equations (NSC).

The investigation of the spatial behaviour of the velocity field at large distances and of the leading asymptotic term is an important research topic, e.g. in the error analysis of numerical approximations. Bae, Brandolese and Vigneron found out the leading terms in the non-rotating case, see [2], [3]. For the spatial behaviour of the Boussinesq system including heat convection the reader can find in [8].

To investigate the spatial behaviour of the solutions of the rotating Navier-Stokes system it will be helpful to consider the solvability of these equations in weighted L^∞-spaces. In the case of slow decay of u_0 the solution decreases almost in the same way as the initial velocity. However, Brandolese, Vigneron and Bae have already proved in the case of the non-rotating Navier-Stokes equations that in general we can not expect a faster decay behavior than $|x|^{-4}$ or even $|x|^{-3}$ if the flow is influenced by an external force.

The present Navier-Stokes equations in a rotational frame have been investigated by several authors, see e.g. [1], [4], [9], [10], [11], [12], [15]. In particular, the technique for proving the global regularity has recently been developed in Sobolev space setting in [14]. However, up to now the spatial asymptotics is almost disregarded.

Recently there is also an intensive research on Navier-Stokes flow around a rotating obstacle which leads to an additional linear term *not* subordinate to the Laplacian. In particular Farwig, Galdi, Hishida and Kyed considered the asymptotic structure of stationary solutions, see e.g. [5], [6], [7], [13], [16].

Let us introduce some elementary concepts. Using the Riesz transforms

$$\mathcal{R}_j = \partial_j(-\Delta)^{-1/2} \quad \text{with symbol } \hat{\mathcal{R}}_j(\xi) = i\frac{\xi_j}{|\xi|},\ 1 \le j \le 3, \tag{1.1}$$

and the Riesz vector $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ the Helmholtz projection is given by

$$\mathbb{P} = \mathbf{I} + \mathcal{R} \otimes \mathcal{R} = \big(\delta_{j,h} + \mathcal{R}_j\mathcal{R}_h\big)_{j,h=1}^{3}.$$

Furthermore, with the matrix

$$J := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

characterising the linear map $J : \mathbb{R}^3 \to \mathbb{R}^3, Jv = e_3 \times v$ and the Coriolis operator $\mathcal{C} = \mathbb{P}J\mathbb{P}$ we transform the first equation of (NSC) into the abstract equation

$$u_t + A_\Omega u + \mathbb{P}(u \cdot \nabla u) = \mathbb{P}f \qquad \text{in } \mathbb{R}^3 \times (0,T); \tag{1.2}$$

here $A_\Omega := -\mathbb{P}\Delta + \Omega\mathcal{C}$ is the so-called *Stokes-Coriolis operator* combining the Stokes operator $A = -\mathbb{P}\Delta$ and the Coriolis operator. We also define the Riesz symbol

$$\mathbf{R}(\xi) = \big(\mathbf{R}_{i,j}(\xi)\big)_{i,j=1}^{3} = \frac{1}{|\xi|}\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}, \tag{1.3}$$

i.e., $\mathbf{R}$ represents the linear map $a \mapsto \mathbf{R}a = (\xi \times a)/|\xi|$ for $a \in \mathbb{R}^3$ and $\mathbf{R}_{i,j}(\xi) = (1-\delta_{i,j})(-1)^{1/2+|1/2+i-j|}(\xi_{6-i-j})/|\xi|$. The symbol $\mathcal{C}(\xi)$ of $\mathcal{C}$ is nothing but $(\xi_3/|\xi|)\mathbf{R}(\xi)$ and thus

$$\mathcal{C} = \mathcal{R}_3\begin{pmatrix} 0 & \mathcal{R}_3 & -\mathcal{R}_2 \\ -\mathcal{R}_3 & 0 & \mathcal{R}_1 \\ \mathcal{R}_2 & -\mathcal{R}_1 & 0 \end{pmatrix}.$$

Note that we used the Fourier transform, e.g. of a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^3)$, in the form

$$\mathcal{F}(\phi)(\xi) := \int_{\mathbb{R}^3} \phi(x) e^{-2\pi i x\cdot\xi}\,\mathrm{d}x.$$

Using this notation, (1.2) leads to the integral equation

$$u(t) = e^{-tA_\Omega}u_0 - \int_0^t e^{-(t-\tau)A_\Omega}\mathbb{P}(u\cdot\nabla u - f)(\tau)\,\mathrm{d}\tau, \tag{1.4}$$

where the semigroup e^{-tA_Ω} associated with the linearised problem of (NSC) is given explicitly by the symbol

$$e^{-4\pi^2 t|\xi|^2}\left(\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{I} - \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{R}(\xi)\right), \tag{1.5}$$

see [10]. A solution u of (1.4) is called a *mild solution.*

To describe the spatial asymptotic structure in terms of u_0 and of the external force $f = (f_1, f_2, f_3)$ we have to take into account the Helmholtz projection involved in (1.4), i.e., it is necessary to control not only e^{-tA_Ω}, but also $\mathbb{P}e^{-tA_\Omega} = e^{-tA_\Omega}\mathbb{P}$ the symbol of which is given by

$$\begin{aligned}
&\big(\mathbf{I} + \hat{\mathcal{R}}\otimes\hat{\mathcal{R}}\big)e^{-4\pi^2 t|\xi|^2}\left(\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{I} - \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{R}(\xi)\right)\\
&\quad = e^{-4\pi^2 t|\xi|^2}\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)\big(\mathbf{I} + \hat{\mathcal{R}}\otimes\hat{\mathcal{R}}\big)\\
&\qquad - e^{-4\pi^2 t|\xi|^2}\sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{R}(\xi);
\end{aligned} \tag{1.6}$$

here we used that $\hat{\mathcal{R}}\mathbf{R} = 0$ with the vector of symbols $\hat{\mathcal{R}} = (\hat{\mathcal{R}}_1, \hat{\mathcal{R}}_2, \hat{\mathcal{R}}_3)$, see (1.1).

This paper is organized as follows. In Section 2 we present the main results, Theorem 2.1 on existence and uniqueness of mild solutions to (1.4), and on their asymptotic spatial decay for fixed $t > 0$, see Theorem 2.2. In contrast to previous works e.g. [12], [14], Theorem 2.1 concerns with the solvability of (1.4) in weighted L^∞-spaces, which are useful for the investigation of the spatial asymptotic profile. Next, in Section 3 we present several auxiliary lemmata to be proved in Section 4. Whereas the proof of Theorem 2.1 is already sketched in Section 3, the proof of Theorem 2.2 will be postponed to Section 5.

2. Main Results

In this paper we assume that the initial data u_0 and the external force f belong to weighted L^∞-spaces. The Banach space $L^\infty_\mu(\mathbb{R}^3)$, $\mu > 0$, is defined as the set of all measurable functions h on $\mathbb{R}^3$ such that

$$\|h\|_{L^\infty_\mu} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} (1+|x|)^\mu |h(x)| < \infty.$$

Its solenoidal subspace is denoted by

$$L^\infty_{\mu,\sigma}(\mathbb{R}^3) = \{u \in L^\infty_\mu(\mathbb{R}^3)^3 : \operatorname{div} u = 0\}.$$

Furthermore, for any $\kappa > 0$, we introduce the space

$$\begin{aligned} L^\infty_\kappa([0,T]; L^\infty_\mu(\mathbb{R}^3)) := \{u : \mathbb{R}^3 \times (0,T] \to \mathbb{R} \mid u \text{ is measurable},\\ \operatorname{ess\,sup}_{0<t<T} t^\kappa \|u(t)\|_{L^\infty_\mu} < \infty\}.\end{aligned}$$

Using Banach's fixed point theorem we get the following existence theorem of mild solutions in spaces of weakly*-continuous functions in time with values in weighted L^∞-spaces.

For simplicity we assume the external force f to be independent of time.

Theorem 2.1 (Existence and Uniqueness of Mild Solutions) *Let $\varepsilon \in (0,1/3)$ and $\mu \in (0,3]$. For every initial velocity $u_0 \in L^\infty_{\mu+\varepsilon,\sigma}(\mathbb{R}^3)$ and external force $f \in L^\infty_{\mu+\varepsilon}(\mathbb{R}^3)^3$ there exists a constant $T_0 > 0$ and a unique solution*

$$u \in L^\infty_\kappa\big([0,T_0]; L^\infty_\mu(\mathbb{R}^3)^3\big) \cap C_\omega\big((0,T_0]; L^\infty_\mu(\mathbb{R}^3)^3\big)$$

to the integral equation (1.4) *for all $\kappa \in (0, 3\varepsilon^2/2(1+\varepsilon^2))$. In particular, with the bound $C_0 = C_0(\Omega)$ for the operator norms in Lemma* 3.2 *below, T_0 is estimated from below as*

$$10C_0\left(\frac{10}{9}C_0\|u_0\|_{L^\infty_{\mu+\varepsilon}} + \big(\|u_0\|_{L^\infty_{\mu+\varepsilon}} + \|f(t)\|_{L^\infty_{\mu+\varepsilon}}\big)^{1/2}\right)\big(T_0^{1/2-\kappa} + T_0^{6-\kappa}\big) < 1.$$

The space $C_\omega((0,T]; L^\infty_\mu)$ denotes all L^∞_μ-valued weakly*-continuous functions $v(t)$ defined in $(0,T]$, i.e., $v(t')$ converges to $v(t)$ in the weak* sense on L^∞_μ as $t' \to t$ for all $t \in (0,T]$. The necessity for working in the space C_ω lies in the fact that already for the heat semigroup $e^{t\Delta}$ the term $e^{t\Delta}h$, with $h \in L^\infty_\mu$, does not converge to h in L^∞_μ as $t \searrow 0$, but only weakly*. But at least, since the Stokes operator $A = \mathbb{P}\Delta$ generates a bounded analytic semigroup e^{-tA} on $L^p_\sigma(\mathbb{R}^3)$ for all $p \in (1,\infty)$, by perturbation theory, see [18, Chapter 3], we conclude that the operator A_Ω generates also an analytic semigroup e^{-tA_Ω} on $L^p_\sigma(\mathbb{R}^3)$, since the Coriolis operator $\mathcal{C} = \mathbb{P}J\mathbb{P}$ is bounded on $L^p_\sigma(\mathbb{R}^3)$.

In view of the result $u(t) \in L^\infty_\mu(\mathbb{R}^3)^3$ with $\mu \in (0,3]$ for mild solutions in Theorem 2.1 it is an interesting question whether the upper bound 3 for μ is optimal in some sense. Actually, the decay $|x|^{-4}$ is optimal for solutions to the non-rotating Navier-Stokes equations without external forces; see Brandolese and Vigneron [3] who proved that the result of Theorem 2.1 can not be true for $\mu > 4$. Analogously, we will see that one can not get the same results for $\mu > 3$ in Theorem 2.2 below.

For our purpose it is useful to introduce Bessel functions J_ν for $\nu \in \mathbb{N}_0$, which can be represented as the series

$$J_\nu(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+\nu)!}\left(\frac{z}{2}\right)^{2n+\nu}, \quad z \in \mathbb{R}. \tag{2.1}$$

They are analytic functions, and behave asymptotically as

$$J_\nu(z) \sim (\operatorname{sgn} z)^\nu \sqrt{\frac{2}{\pi|z|}} \cos\left(|z| - \frac{\pi}{4}(2\nu+1)\right) \quad |z| \to \infty, \tag{2.2}$$

$$J_0(z) \sim 1 - \left(\frac{z}{2}\right)^2, \; J_1(z) \sim \frac{z}{2} - \frac{1}{2}\left(\frac{z}{2}\right)^3,$$

$$J_2(z) \sim \frac{1}{2}\left(\frac{z}{2}\right)^2 - \frac{1}{6}\left(\frac{z}{2}\right)^4 \quad |z| \to 0,$$

see e.g. [19, Chapter 3, Chapter 7]. In our main result these three functions play a crucial role by describing the asymptotic profile. Hence we define the vector $V := (V_0, V_1, V_2, V_3, V_4, V_5)$ by

$$V(x,t) = \left(J_0, \frac{J_1}{\eta}, \frac{J_2}{\eta^2}\right)(\eta)$$

$$\times \begin{pmatrix} \zeta - 1 & -1 & \Omega t - 3\dfrac{\zeta}{\Omega t} & 3\Omega t & 3 & 0 \\ -(\eta^2 + \zeta) & -4\zeta & \dfrac{\zeta\eta^2}{\Omega t} & \Omega t\zeta & 6\zeta & -3(\Omega t)^2 \\ 0 & -\zeta^2 & 0 & 0 & \zeta^2 & -(\Omega t)^2\zeta\eta \end{pmatrix} \tag{2.3}$$

where $\eta(x,t) := |\Omega|t(|x'|/|x|)$, $\zeta(x,t) := (\Omega t(x_3/|x|))^2$ and $x' = (x_1, x_2, 0)$. Comparing z, h introduced in the proof of Lemma 3.5 below there holds $\eta = \sqrt{z+h}$ and $\zeta = -h$. Moreover, we define the vector $A := (A_0, A_1, A_2, A_3, A_4, A_5)$ as the integral over time of V, i.e., $A_i(x,t) = \int_0^t V_i(x,\tau)\,\mathrm{d}\tau$, $i = 0, 1, \dots, 5$. We remark that the formal singularities of V for $|\eta| \to 0$ as $|\Omega t| \to 0$ or $|x'| \to 0$ actually do not appear, hence J_ν/η^ν is continuously well-defined, see (2.2).

Theorem 2.2 (Spatial Asymptotic Profile) *For an initial velocity $u_0 \in L^\infty_{\mu,\sigma}(\mathbb{R}^3)$, $\mu > 4$, and an external force $f \in L^\infty_\mu(\mathbb{R}^3)^3$, let u be the mild solution of Theorem 2.1. Then the following profile holds for almost all $|x| \gg \sqrt{t}$:*

$$\begin{aligned} u(x,t) = \frac{1}{4\pi|x|^3} \Bigg[&\begin{pmatrix} A_0 & A_2 & 0 \\ -A_2 & A_0 & 0 \\ 0 & 0 & A_1 \end{pmatrix}(x,t) \int_{\mathbb{R}^3} f(y)\,\mathrm{d}y \\ &+ A_3(x,t)\frac{x_3}{|x|^2} x' \times \int_{\mathbb{R}^3} f(y)\,\mathrm{d}y + A_4(x,t)\frac{x \otimes x}{|x|^2} \int_{\mathbb{R}^3} f(y)\,\mathrm{d}y \\ &+ A_5(x,t)\frac{x_3}{|x|^2} \begin{pmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ x_1 & x_2 & 0 \end{pmatrix} \int_{\mathbb{R}^3} f(y)\,\mathrm{d}y \Bigg] + \mathcal{O}_t(|x|^{-4}). \end{aligned}$$

As long as the external force f belongs to $L^\infty_\mu(\mathbb{R}^3)^3$ for $\mu > 4$, having non-zero mean, this theorem shows that in general we only expect an $|x|^{-3}$-decay of the velocity. It is remarkable that if the external force f has vanishing mean value, the solution even decays as $|x|^{-4}$, although the convolution kernel, see (3.1) below, on which the study of spatial asymptotics is based has $|x|^{-3}$-decay. In contrast, the Coriolis force does not affect the rate of decay, essentially the structure of the leading terms, cf. [2]. In particular, no matter how small the external force f is, it has a significant effect at large scale analysis. Although the general formula given in Theorem 2.2 does not look manageable, for some special cases it simplifies to better understandable representations in what follows.

For $\Omega = 0$ the equations (NSC) turn to the usual Navier-Stokes equations. In this case the vector V, see (2.3), equals to $V(x,t)|_{\Omega=0} = (-1,-1,0,0,3,0)$. However, Theorem 2.2 yields the asymptotic representation

$$u(x,t) = -\frac{t}{4\pi}\nabla\frac{x}{|x|^3}\cdot\int_{\mathbb{R}^3} f(y)\,\mathrm{d}y + \mathcal{O}_t(|x|^{-4}) \qquad \text{for } |x| \gg \sqrt{t},$$

see [3] or [8, Theorem 2.3], i.e., we obtain the spatial asympotic profile of the Navier-Stokes equations with external forces.

For the asymptotic profile along the x_3-axis, i.e., $|x_3| \gg \sqrt{t}$ and $|x'| = 0$ the components of vector A can be simplified to polynomials with respect to Ω and t:

$$A_0 = -t + \frac{1}{6}\Omega^2 t^3, \qquad A_3 = \frac{3}{2}\Omega t^2 + \frac{1}{8}\Omega^3 t^4,$$

$$A_1 = -t - \frac{2}{3}\Omega^2 t^3 - \frac{1}{10}\Omega^4 t^5, \qquad A_4 = 3t + \Omega^2 t^3 + \frac{1}{10}\Omega^4 t^5,$$

$$A_2 = \frac{1}{2}(\Omega - 3|\Omega|)t^2, \qquad A_5 = -\frac{1}{2}\Omega^2 t^3.$$

Finally, let us consider the asymptotic profile that arises in some sense far from the rotating axis, i.e., for $|x| \gg \sqrt{t}$ there exists $\varepsilon \in (0,1)$ such that $|x_3| \le |x|^\varepsilon$. This case leads to $\zeta(x,t) = \mathcal{O}_t(|x|^{2\varepsilon-2})$ and the leading terms V' in V, see (2.3), are given by

$$V'(x,t) = \left(J_0, \frac{J_1}{\eta}\right)(\eta) \times \begin{pmatrix} -1 & -1 & \Omega t & 3\Omega t & 3 & 0 \\ -\eta^2 & 0 & 0 & 0 & 0 & -3(\Omega t)^2 \end{pmatrix}.$$

Due to the function ζ, the remaining terms of the asymptotic profile of u with higher decay only behave like $|x|^{-\min\{3+2(1-\varepsilon),4\}}$ instead of $|x|^{-4}$.

3. Preliminaries

In this section we analyse the spatial asymptotics of the matrix-valued convolution kernel $\mathbf{K} = (K_{i,j})_{i,j=1}^3$ defined by

$$\mathbf{K}(x,t) = \int_{\mathbb{R}^3} e^{-4\pi^2 t|\xi|^2 + 2\pi i x\cdot\xi} \times \left[\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)\left(\mathbf{I} + \hat{\mathcal{R}} \otimes \hat{\mathcal{R}}\right) - \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{R}(\xi)\right]\mathrm{d}\xi \tag{3.1}$$

which corresponds to the operator $\mathbb{P}e^{-tA_\Omega}$, see (1.6); here $\mathbf{I}$ denotes the 3×3-unit matrix and $\mathcal{R}$, $\mathbf{R}(\xi)$ have been defined in (1.1), (1.3), respectively. In [17, Proposition 11.1] the operator $\mathbb{P}e^{t\Delta}$ which deals with the non-rotating Navier-Stokes equations was treated as a pseudo-differential operator and the derivatives of the corresponding convolution kernel satisfy certain decay properties. Unfortunately, in the case of a rotating frame the convolution kernel of the semigroup e^{-tA_Ω} does not belong to $L^1(\mathbb{R}^3)$ for $t > 0$, since the symbol (1.5) is not continuous at $\xi = 0$. Giga et al. [10] proved that this convolution kernel decays like $|x|^{-3}$ and thus lies in $L^p(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$, $p \in (1, \infty]$, since the symbol (1.5) is integrable. However, as in [17] we need a more precise investigation about the derivatives of this kernel given by the next Lemma.

Lemma 3.1 *Given the Riesz transforms $\mathcal{R}_j$ let $\mathcal{K}_t = (\mathcal{K}_{h,k,t})_{h,k=1}^3$ denote the matrix-valued kernel related to the pseudo-differential operators $(\mathcal{R}_h\mathcal{R}_k e^{-tA_\Omega})_{h,k=1}^3$, i.e., $\mathcal{R}_h\mathcal{R}_k e^{-tA_\Omega} f = \mathcal{K}_{h,k,t} * f$ for $t > 0$. Then $\mathcal{K}_t(x) = t^{-3/2}\mathcal{K}(x/\sqrt{t}, \Omega t)$ with a smooth function $\mathcal{K}(\cdot,\cdot)$ on $\mathbb{R}^3 \times \mathbb{R}_+$ satisfying*

$$\left[(x,t) \mapsto (1 + (|\Omega|t)^{4+|\alpha|})^{-1}(1+|x|)^{3+|\alpha|}\partial_x^\alpha \mathcal{K}(x,\Omega t)\right] \in L^\infty(\mathbb{R}^3 \times \mathbb{R}_+)$$

for all multi-indices $\alpha \in \mathbb{N}_0^3$.

Since $e^{-tA_\Omega} = -\sum_{j=1}^3 \mathcal{R}_j^2 e^{-tA_\Omega}$, the above Lemma 3.1 implies that the derivatives of the corresponding kernel $\tilde{\mathbf{K}}(x,t) := -\sum_{j=1}^3 \mathcal{K}_{j,j,t}(x)$ decay similarly to those of $\mathcal{K}$. This Lemma also leads to the estimate

$$\left|\partial_x^\alpha \mathcal{K}_t(x)\right| \le t^{-3/2}\left|\partial_x^\alpha \mathcal{K}\left(\frac{x}{\sqrt{t}}, \Omega t\right)\right| \lesssim (1 + (|\Omega|t)^{4+|\alpha|})\left(\sqrt{t} + |x|\right)^{-3-|\alpha|}$$

for all $\alpha \in \mathbb{N}_0^3$.

To construct a unique mild solution of (1.4) for given initial data $u_0 \in L^\infty_{\mu+\varepsilon,\sigma}(\mathbb{R}^3)^3$ and given external force $f \in L^\infty_{\mu+\varepsilon}(\mathbb{R}^3)^3$, it is useful to study the integral operators

$$\mathcal{B}(u_1, u_2)(t) := -\int_0^t e^{-(t-s)A_\Omega}\mathbb{P}\nabla \cdot (u_1 \otimes u_2)(s)\,\mathrm{d}s \tag{3.2}$$

$$\mathcal{D}(f)(t) := \int_0^t e^{-(t-s)A_\Omega}\mathbb{P}f\,\mathrm{d}s. \tag{3.3}$$

Lemma 3.2 *Let $T > 0$, $\varepsilon > 0$, $\mu \in (0, 4]$. Then the operators*

$$\begin{aligned}\mathcal{B}(\cdot,\cdot) &: L^\infty\big([0,T]; L^\infty_\mu\big) \times L^\infty([0,T]; L^\infty) \to C_\omega\big([0,T]; L^\infty_\mu\big),\\ \mathcal{D}(\cdot) &: L^\infty_{\mu+\varepsilon} \to C_\omega\big([0,T]; L^\infty_\mu\big),\end{aligned}$$

defined by (3.2) *and* (3.3), *are continuous with operator norm* $\mathcal{O}(\sqrt{T} + T^6)$.

Thanks to the previous result, the estimates for $\mathcal{B}$ and $\mathcal{D}$ can be proved in a similar way as in [17, Proposition 25.1].

Sketch of the proof of Theorem 2.1. The existence and uniqueness of mild solutions to (1.4) base on the abstract formulation of a solution u as a fixed point of the coupled system cf. (1.4)

$$u(t) = e^{-tA_\Omega}u_0 + \mathcal{B}(u, u)(t) + \mathcal{D}(f)(t)$$

in the Banach space $L^\infty_\kappa([0, T_0]; L^\infty_\mu(\mathbb{R}^3)^3) \cap C_\omega((0, T_0]; L^\infty_\mu(\mathbb{R}^3)^3)$. With the help of Lemmata 3.1 and 3.2 the result is easily proved by Banach's fixed point theorem. □

Similarly to [3], [8] we proceed to get an asymptotic profile of the solutions of the rotating Navier-Stokes equations and have to handle mainly the terms of the integral equation (1.4). Due to the symbol (1.5) of e^{-tA_Ω} the present issue exacerbates this method by dealing with an infinite sum of Riesz operators applied to the heat kernel

$$\mathcal{G}_t(x) := \frac{1}{(4\pi t)^{3/2}} e^{-|x|^2/4t}.$$

The next statements are useful to manage this difficulty. At first we easily obtain the following Lemma by induction.

Lemma 3.3 *Let $\mathcal{R} := (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ be the vector of Riesz transforms, and $\alpha \in \mathbb{N}_0^3$ be any multi-index of even order, i.e. $|\alpha| = 2n$ with $n \in \mathbb{N}$. Then for all $x \in \mathbb{R}^3$ and $t > 0$ there holds*

$$\mathcal{R}^\alpha \mathcal{G}_t(x) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \partial_x^\alpha \mathcal{G}_s(x)\,\mathrm{d}s.$$

The purpose of Lemmata 3.4, 3.5 below is to ascertain the leading terms of $\mathbb{P}e^{-tA_\Omega}$. To handle high order derivatives of the heat kernel it is reasonable to introduce *Hermite polynomials*

$$H_n(y) := e^{y^2} \frac{d^n}{\mathrm{d}y^n} e^{-y^2} = n! \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{l+n}}{l!(n-2l)!} (2y)^{n-2l}, \qquad y \in \mathbb{R},\ n \in \mathbb{N}_0, \tag{3.4}$$

using the floor function $\lfloor . \rfloor$. Consider any mixed derivative $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ of order $|\alpha| = 2n$, and define the multivariate Hermite polynomial

$$H^\alpha(y) = H_{\alpha_1}(y_1) H_{\alpha_1}(y_2) H_{\alpha_3}(y_3), \quad y = (y_1, y_2, y_3).$$

Substituting $y_j = x_j/\sqrt{4t}$ in the identity $(\mathrm{d}/\,\mathrm{d}y_j)^{\alpha_j} e^{-y_j^2} = H_{\alpha_j}(y_j) e^{-y_j^2}$, we get that

$$\partial_x^\alpha \mathcal{G}_s(x) = \frac{1}{(4s)^n} H^\alpha\left(\frac{x}{\sqrt{4s}}\right) \mathcal{G}_s(x). \tag{3.5}$$

Since the series

$$e^{-4\pi^2 t|\xi|^2} \left(\sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} \left(\frac{\xi_3}{|\xi|}\Omega t\right)^{2n} \left(\mathbf{I} + \frac{i\xi \otimes i\xi}{|\xi|^2}\right) - \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} \left(\frac{\xi_3}{|\xi|}\Omega t\right)^{2n+1} \mathbf{R} \right)$$

converges for fixed $t > 0$ to the symbol of (1.6) in L^1, the corresponding inverse Fourier transform of this series converges uniformly to the kernel $\mathbf{K}(x,t)$, see (3.1). Thanks to Lemma 3.3 and (3.5) we get for $x \in \mathbb{R}^3$ three converging infinite series consisting of terms

$$\frac{(\Omega t)^{2n}}{(2n)!}\,\mathcal{R}^\alpha\mathcal{G}_t(x) \qquad \text{for} \quad \begin{cases} |\alpha| = 2n \text{ or} \\ |\alpha| = 2n+2,\ 2n \le \alpha_3 \le 2n+2, \end{cases} \tag{3.6}$$

$$\frac{(\Omega t)^{2n+1}}{(2n+1)!}\,\mathcal{R}^\alpha\mathcal{G}_t(x) \qquad \text{for} \quad |\alpha| = 2n+2,\ 2n+1 \le \alpha_3 \le 2n+2. \tag{3.7}$$

For a typical term $\mathcal{R}^\alpha\mathcal{G}_t(x)$ with $|\alpha| = 2n$ we will use the change of variables $\lambda = |x|/\sqrt{4s}$ to get that

$$\begin{aligned}
\mathcal{R}^\alpha\mathcal{G}_t(x) &= \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\partial^\alpha\mathcal{G}_s(x)\,\mathrm{d}s \\
&= \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\,\frac{1}{(4s)^n}H^\alpha\Big(\frac{x}{\sqrt{4s}}\Big)\mathcal{G}_s(x)\,\mathrm{d}s \\
&= \int_0^{|x|/\sqrt{4t}} \frac{1}{(n-1)!}\Big(\frac{|x|^2}{4\lambda^2}-t\Big)^{n-1}\Big(\frac{\lambda}{|x|}\Big)^{2n}H^\alpha\Big(\frac{\lambda x}{|x|}\Big)\frac{e^{-\lambda^2}}{2\pi^{3/2}|x|}\,\mathrm{d}\lambda.
\end{aligned}$$

To find the leading term we simplify the above identity, split the term $(|x|^2/4\lambda^2)^{n-1}$ from $(|x|^2/4\lambda^2 - t)^{n-1}$ and decompose the last integral as follows:

$$\mathcal{R}^\alpha\mathcal{G}_t(x) = \frac{4^{-n}}{\pi^{3/2}|x|^3}\int_0^\infty \lambda^2 H^\alpha\Big(\frac{\lambda x}{|x|}\Big)e^{-\lambda^2}\,\mathrm{d}\lambda + \cdots,$$

see [3, Lemma 2.1]. This equation is crucial to obtain the leading terms of the integrals in (3.6) and (3.7) and of the asymptotic profile.

To this aim, we introduce the functions

$$L_n^{(0)}(x) := \frac{(2n)!}{\pi^{3/2}}\sum_{l=0}^{n}\frac{(-1)^l\Gamma(n-l+3/2)}{4^l l!(2n-2l)!}\Big(\frac{x_3}{|x|}\Big)^{2(n-l)}, \tag{3.8}$$

$$L_{i,n}^{(1)}(x) := \frac{(2n-1)!}{\pi^{3/2}}\,\frac{x_i}{|x|}\sum_{l=0}^{n-1}\frac{(-1)^l\Gamma(n-l+3/2)}{4^l l!(2n-1-2l)!}\Big(\frac{x_3}{|x|}\Big)^{2n-1-2l}, \tag{3.9}$$

$$L_{i,j,n}^{(2)}(x) := \frac{(2n-2)!}{2\pi^{3/2}}\sum_{l=0}^{n-1}\frac{(-1)^l\Gamma(n-l+1/2)}{4^l l!(2n-2-2l)!}\Big(\frac{x_3}{|x|}\Big)^{2n-2-2l}$$

$$\times \left[2\frac{x_i x_j}{|x|^2}\left(n - l + \frac{1}{2}\right) - \delta_{i,j}\right], \tag{3.10}$$

for all $i, j \leq 2$ and establish the next result as a first step.

Lemma 3.4 *Let $n \in \mathbb{N}$, $i, j = 1, 2, 3$, $i \leq j$ and $|x|^2 \gg t$. Then there holds*

$$\int_t^\infty (s-t)^{n-1}\partial_i\partial_j\partial_3^{2n-2}\mathcal{G}_s(x)\,\mathrm{d}s = |x|^{-3}L_{i,j,n}(x) + |x|^{-3}\Psi_{i,j,n}\left(\frac{x}{\sqrt{t}}\right)$$

with leading term

$$L_{i,j,n} = \begin{cases} L_n^{(0)}, & \text{if } i = j = 3, \\ L_{i,n}^{(1)}, & \text{if } i < j = 3, \\ L_{i,j,n}^{(2)}, & \text{if } i \leq j < 3 \end{cases}$$

defined in (3.8), (3.9) *or* (3.10), *respectively, and a remainder term* $\Psi_{i,j,n}(y) = \mathcal{O}(|y|^{-2})$ *as* $|y| \to \infty$.

Due to the crucial Lemma 3.4 and (2.3) we obtain the spatial asymptotics of the considered kernel $\mathbf{K}$ in Lemma 3.5 and 3.6 below. For this let

$$\tilde{V}_0 := \zeta J_0(\eta) - \eta J_1(\eta) \tag{3.11}$$

with $\eta(x,t) := |\Omega|t(|x'|/|x|)$, $\zeta(x,t) := (\Omega t(x_3/|x|))^2$. The kernel $\tilde{\mathbf{K}} = (\tilde{K}_{i,j})_{i,j=1}^3$ corresponding to the semigroup e^{-tA_Ω} is

$$\tilde{\mathbf{K}}(x,t) = \int_{\mathbb{R}^3} e^{-4\pi^2 t|\xi|^2 + 2\pi i x\cdot\xi}\left[\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)I - \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{R}(\xi)\right]\mathrm{d}\xi,$$

and hence only differs from the kernel $\mathbf{K}$ of $\mathbb{P}e^{-tA_\Omega}$ in the additional term

$$\mathbf{K}^+(x,t) := \int_{\mathbb{R}^3} e^{-4\pi^2 t|\xi|^2 + 2\pi i x\cdot\xi}\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)\hat{\mathcal{R}}\otimes\hat{\mathcal{R}}\mathrm{d}\xi,$$

i.e., $\mathbf{K} = \tilde{\mathbf{K}} + \mathbf{K}^+$. Therefore, it is reasonable to investigate the spatial asymptotic behaviour of $\tilde{\mathbf{K}}$ and of $\mathbf{K}^+$ apart:

Lemma 3.5 *For $|x|^2 \gg t$ the convolution kernel $\tilde{\mathbf{K}}$ has the decomposition $\tilde{\mathbf{K}}(x,t) = \tilde{\mathbf{K}}^*(x,t) + |x|^{-3}\tilde{\mathbf{\Psi}}(x,t)$ with a remainder $\tilde{\mathbf{\Psi}}(x,t) = \mathcal{O}_t(|x|^{-2})$. Moreover, the leading term $\tilde{\mathbf{K}}^*(x,t)$ has the form*

$$\tilde{K}^*_{i,j}(x,t) = \frac{1}{4\pi|x|^3}\left[\delta_{i,j}\tilde{V}_0(x,t) + s_{i,j}\frac{x_{6-i-j}x_3}{|x|^2}V_3(x,t)\right]$$

when $i+j \neq 3$; here $s_{i,j} = (1-\delta_{i,j})(-1)^{1/2+|1/2+i-j|}$. Finally,

$$\tilde{K}^*_{i,j}(x,t) = (-1)^{1/2+|1/2+j-i|}\frac{1}{4\pi|x|^3}V_2(x,t)$$

when $i+j=3$.

The proof of the next lemma follows the same line as of the previous Lemma 3.5. Since the remaining term $\mathbf{K}^+$ corresponds to (3.6) where $|\alpha| = 2n+2$ and $2n \leq \alpha_3 \leq 2n+2$, three distinct cases have to be considered.

Lemma 3.6 *For $|x|^2 \gg t$ the convolution kernel $\mathbf{K}^+$ has the decomposition $\mathbf{K}^+(x,t) = \mathbf{K}^{+,*}(x,t) + |x|^{-3}\mathbf{\Psi}^+(x,t)$ with a remainder $\mathbf{\Psi}^+(x,t) = \mathcal{O}_t(|x|^{-2})$. Moreover, the leading term $\mathbf{K}^{+,*}(x,t)$ has the form*

$$K^{+,*}_{i,j}(x,t) = \frac{1}{4\pi|x|^3}\Big[\delta_{i,j}\big((1-\delta_{i,3})(V_0-\tilde{V}_0)(x,t) + \delta_{i,3}(V_1-\tilde{V}_0)(x,t)\big) + \frac{x_ix_j}{|x|^2}\big(V_4(x,t) + (\delta_{i,3}+\delta_{j,3}-2\delta_{i,3}\delta_{j,3})V_5(x,t)\big)\Big]$$

when $i,j \leq 3$.

Combining Lemma 3.5 and 3.6 we obtain the decomposition $\mathbf{K}(x,t) = \mathbf{K}^*(x,t) + |x|^{-3}\mathbf{\Psi}(x,t)$ with $\mathbf{K}^* := \tilde{\mathbf{K}}^* + \mathbf{K}^{+,*}$ and $\mathbf{\Psi} := \tilde{\mathbf{\Psi}} + \mathbf{\Psi}^+$.

4. Proof of Lemmata 3.1, 3.4 and 3.5

First of all let us briefly introduce some notation for the Littlewood-Paley decomposition needed in the next proof. Let $\phi \in \mathcal{S}(\mathbb{R}^3)$ denote a non-negative Schwartz function supported in the annulus $\{\xi \in \mathbb{R}^3 : 1/2 \leq |\xi| \leq 2\}$ such that $\sum_{k=-\infty}^{\infty}\phi(2^{-k}\xi) = 1$ for all $\xi \neq 0$. We define for all $j \in \mathbb{Z}$ a function φ_j as follows

$$\hat{\varphi}_j(\xi) := \phi(2^{-j}\xi) \quad \text{for all } \xi \neq 0.$$

Further we define

$$\psi_0(\xi) := \mathcal{F}^{-1}\left(1 - \sum_{j=0}^{\infty} \hat{\varphi}_j(\xi)\right) \quad \text{for all } \xi \in \mathbb{R}^3.$$

Proof of Lemma 3.1. We define $\mathcal{K}$ by

$$\mathcal{K}(\cdot, \Omega t) := -\mathcal{F}^{-1}\left(\frac{\xi \otimes \xi}{|\xi|^2}\mathbf{g}(\xi, \Omega t)e^{-4\pi^2|\xi|^2}\right)$$

with the bounded matrix-valued function

$$\mathbf{g}(\xi, \Omega t) := \cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{I} - \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{R}(\xi).$$

By this definition and (1.5) there holds

$$\mathcal{K}_t(x) = t^{-3/2}\mathcal{K}\left(\frac{x}{\sqrt{t}}, \Omega t\right).$$

In what follows, we write shortly $\mathcal{K}$ instead of $\mathcal{K}(\cdot, \Omega t)$ for fixed Ωt. Thanks to its rapid decay, the symbol

$$\frac{\xi \otimes \xi}{|\xi|^2}\mathbf{g}(\xi, \Omega t)\, e^{-4\pi^2|\xi|^2}$$

is integrable when multiplied by any polynomial. Thus $\partial^\alpha \mathcal{K} \in C_0(\mathbb{R}^3)$ for all $\alpha \in \mathbb{N}^3$ and fixed Ωt. For $|x| \geq 1$ we use the Littlewood-Paley decomposition and write

$$\mathcal{K} = (\mathcal{K} - \psi_0 * \mathcal{K}) + \sum_{l<0} \varphi_l * \mathcal{K}$$

with $\mathcal{K} - \psi_0 * \mathcal{K} \in \mathcal{S}(\mathbb{R}^3)$. By substituting $\xi = 2^l\tilde{\xi}$ we have for all $l < 0$

$$\begin{aligned}\varphi_l * \mathcal{K}(\cdot) &= \mathcal{F}^{-1}\big(\phi(2^{-l}\xi)\hat{\mathcal{K}}(\xi)\big)(\cdot)\\ &= 2^{3l}\mathcal{F}^{-1}\big(\phi(\tilde{\xi})\hat{\mathcal{K}}(2^l\tilde{\xi})\big)(2^l\cdot) \in \mathcal{S}(\mathbb{R}^3).\end{aligned}$$

Since the set of functions $\{\mathcal{F}^{-1}\big(\phi(\tilde{\xi})\hat{\mathcal{K}}(2^l\tilde{\xi})\big) : l < N\}$, $N \in \mathbb{N}$, is bounded in $\mathcal{S}(\mathbb{R}^3)$, we get for each $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^3$ a constant $C_{N,\alpha} > 0$ such that

$$\big(1 + 2^l|x|\big)^N 2^{-l(3+|\alpha|)}\ |\partial^\alpha(\varphi_l * \mathcal{K})(x)| \le C_N(1 + |\Omega|t)^N.$$

This gives for $N := 4 + |\alpha|$ and $|x| > 1$

$$\begin{aligned} &|\partial^\alpha(\psi_0 * \mathcal{K})(x)| \\ &\quad\le C(1 + |\Omega|t)^N \Bigg(\sum_{l:2^l|x|\le 1} 2^{l(3+|\alpha|)} + \sum_{l:2^l|x|>1} 2^{l(3+|\alpha|-N)}|x|^{-N}\Bigg) \\ &\quad\le C(1 + |\Omega|t)^N|x|^{-3-|\alpha|}. \end{aligned}$$

Since $\mathcal{K} - \psi_0 * \mathcal{K} \in \mathcal{S}(\mathbb{R}^3)$ Lemma 3.1 is proved. □

Proof of Lemma 3.4. First let $i = j = 3$. Then by (3.5) we have

$$\partial_3^{2n}\mathcal{G}_s(x) = \frac{1}{(4s)^n}H_{2n}\bigg(\frac{x_3}{\sqrt{4s}}\bigg) \cdot \mathcal{G}_s(x)$$

and thus by (3) and the substitution $\lambda = |x|/\sqrt{4s}$ we get

$$\begin{aligned} &\int_t^\infty (s-t)^{n-1}\partial_3^{2n}\mathcal{G}_s(x)\,\mathrm{d}s = \int_t^\infty (s-t)^{n-1}\frac{1}{(4s)^n}H_{2n}\bigg(\frac{x_3}{\sqrt{4s}}\bigg) \cdot \mathcal{G}_s(x)\,\mathrm{d}s \\ &\quad= (2n)!\sum_{l=0}^n \frac{(-1)^l}{l!(2n-2l)!}\int_t^\infty \frac{(s-t)^{n-1}}{(4s)^n}\bigg(\frac{2x_3}{\sqrt{4s}}\bigg)^{2(n-l)} \cdot \mathcal{G}_s(x)\,\mathrm{d}s \\ &\quad= \frac{(2n)!}{2\pi^{3/2}}\sum_{l=0}^n \frac{(-1)^l}{l!(2n-2l)!} \\ &\qquad\times \int_0^{|x|/\sqrt{4t}} \bigg(\frac{|x|^2}{4\lambda^2} - t\bigg)^{n-1}\bigg(\frac{\lambda}{|x|}\bigg)^{2n}\bigg(2\frac{x_3}{|x|}\lambda\bigg)^{2(n-l)}|x|^{-1}e^{-\lambda^2}\,\mathrm{d}\lambda. \end{aligned}$$

Therefore, we obtain

$$\int_t^\infty (s-t)^{n-1}\partial_3^{2n}\mathcal{G}_s(x)\,\mathrm{d}s$$

$$= \frac{(2n)!}{\pi^{3/2}|x|^3} \sum_{l=0}^{n} \frac{(-1)^l}{l!(2n-2l)!} \sum_{k=0}^{n-1} \binom{n-1}{k} 2^{1+2k-2l} \frac{(-t)^k}{|x|^{2k}} \left(\frac{x_3}{|x|}\right)^{2(n-l)}$$
$$\times \int_0^{|x|/\sqrt{4t}} \lambda^{2+2n+2k-2l} e^{-\lambda^2}\,\mathrm{d}\lambda. \tag{4.1}$$

By the definition of the gamma function there holds

$$\int_0^{|x|/\sqrt{4t}} \lambda^{2+2n+2k-2l} e^{-\lambda^2} d\lambda$$
$$= \frac{1}{2}\Gamma\left(\frac{3+2n+2k-2l}{2}\right) - \int_{|x|/\sqrt{4t}}^{\infty} \lambda^{2+2n+2k-2l} e^{-\lambda^2}\,\mathrm{d}\lambda$$

for all $k, l = 0, \ldots, n$. Now in (4.1) we fix $k = 0$ and get for each $l = 0 \ldots, n$

$$\frac{(-1)^l (2n)! 2^{1-2l}}{\pi^{3/2} l!(2n-2l)!|x|^3} \left(\frac{x_3}{|x|}\right)^{2(n-l)} \int_0^{|x|/\sqrt{4t}} \lambda^{2+2n-2l} e^{-\lambda^2}\,\mathrm{d}\lambda$$
$$= \frac{(-1)^l (2n)!}{\pi^{3/2} l!(2n-2l)!|x|^3} \left[4^{-l}\Gamma\left(n-l+\frac{3}{2}\right)\left(\frac{x_3}{|x|}\right)^{2(n-l)} + \Psi^{(1)}_{3,n,l}\left(\frac{x}{\sqrt{t}}\right)\right]$$

with the exponentially decaying remainder function

$$\Psi^{(1)}_{3,3,n,l}(y) := -2^{1-2l}\left(\frac{y_3}{|y|}\right)^{2(n-l)} \int_{|y|/2}^{\infty} \lambda^{2+2n-2l} e^{-\lambda^2} d\lambda,$$

which satisfies the estimate

$$\left|\Psi^{(1)}_{3,3,n,l}(y)\right| \le 2^{1-2l} e^{-|y|^2/8} \int_{|y|/2}^{\infty} \lambda^{2+2n-2l} e^{-\lambda^2/2} d\lambda$$
$$\le 2^{3/2+n-3l}(n-l+1)! e^{-|y|^2/8}.$$

To establish the above estimate we applied the equality

$$\int_0^{\infty} \tau^d e^{-\tau^2/2}\,\mathrm{d}\tau = 2^{(d-1)/2}\Gamma\left(\frac{d+1}{2}\right), \quad d > -1. \tag{4.2}$$

This yields for

$$\Psi^{(1)}_{3,3,n}(y) := \frac{(2n)!}{\pi^{3/2}} \sum_{l=0}^{n} \frac{(-1)^l}{l!(2n-2l)!} \Psi^{(1)}_{3,3,n,l}(y)$$

the exponential decay

$$\left|\Psi^{(1)}_{3,3,n}(y)\right| \leq \frac{\sqrt{8}e^{1/4}}{\pi^{3/2}}(n+1)(2n)!e^{-|y|^2/8}$$

since $(n-l+1)! \leq (n+1)2^{l-n}(2n-2l)!$ and $\sum_{l=0}^{n}(1/l!)2^{-2l} \leq e^{1/4}$.

In the same way we see that for all $k \neq 0$ and $l = 0, \dots, n$ the term

$$\begin{aligned}\left(\frac{x_3}{|x|}\right)^{2(n-l)} \int_0^{|x|/\sqrt{4t}} \lambda^{2+2n+2k-2l} e^{-\lambda^2}\, d\lambda &\leq \frac{1}{2}\Gamma\left(\frac{3+2n+2k-2l}{2}\right) \\ &\leq \frac{(2n-l+1)!}{2n}\end{aligned}$$

uniformly in $x \in \mathbb{R}^3$. Thus we put the left terms into the remainder $\Psi^{(2)}_{3,3,n}$ defined as

$$\begin{aligned}\Psi^{(2)}_{3,3,n}\left(\frac{x}{\sqrt{t}}\right) := \frac{(2n)!}{\pi^{3/2}} \sum_{l=0}^{n} \frac{(-1)^l}{l!(2n-2l)!} \sum_{k=1}^{n-1} \binom{n-1}{k} 2^{1+2k-2l} \frac{(-t)^k}{|x|^{2k}} \left(\frac{x_3}{|x|}\right)^{2(n-l)} \\ \times \int_0^{|x|/\sqrt{4t}} \lambda^{2+2n+2k-2l} e^{-\lambda^2}\, d\lambda\end{aligned}$$

and get for $y = x/\sqrt{t}$ with $|y| \geq 1$ the estimate

$$\left|\Psi^{(2)}_{3,3,n}(y)\right| \leq \frac{(2n)!}{n\pi^{3/2}|y|^2} \sum_{l=0}^{n} 2^{-2l} \frac{(2n-l+1)!}{l!(2n-2l)!} \sum_{k=1}^{n-1} \binom{n-1}{k} 2^{2k}.$$

Since the inner sum over k equals 5^{n-1} it suffices to estimate the term

$$\sum_{l=0}^{n} 2^{-2l} \frac{(2n-l+1)!}{l!(2n-2l)!} = \sum_{l=0}^{n} \frac{2^{-2l}}{B(l+1, 2n-2l+1)} \tag{4.3}$$

where B denotes the beta function. Since

$$B(l+1, 2n-2l+1) = \int_0^1 \tau^l (1-\tau)^{2n-2l}\, \mathrm{d}\tau$$
$$\geq \int_0^1 \tau^{2l} (1-\tau)^{2n-2l}\, \mathrm{d}\tau = \frac{(2l)!(2n-2l)!}{(2n+1)!},$$

we see that the term in (4.3) can be estimated by

$$\leq (2n+1) \sum_{l=0}^{n} \binom{2n}{2l} 2^{-2l} \leq (2n+1) \sum_{j=0}^{2n} \binom{2n}{j} 2^{-j} = (2n+1) \left(\frac{3}{2}\right)^{2n}.$$

Finally, with $\Psi_{3,3,n} := \Psi^{(1)}_{3,3,n} + \Psi^{(2)}_{3,3,n}$, (4.1) and a summary of the previous estimates yield

$$\int_t^\infty (s-t)^{n-1} \partial_3^{2n} \mathcal{G}_s(x)\, \mathrm{d}s = |x|^{-3} L_n^{(0)}(x) + |x|^{-3} \Psi_{3,3,n}\left(\frac{x}{\sqrt{t}}\right),$$

where $L_n^{(0)}$ has been defined in (3.8) and, using that $6^n (3/2)^{2n} \leq 14^n$,

$$|\Psi_{3,3,n}(y)| \lesssim 14^n (2n)! |y|^{-2} \tag{4.4}$$

for all $n \in \mathbb{N}$ and $|x| \gg \sqrt{t}$.

Now, let $i < j = 3$. In this case there holds

$$\partial_i \partial_3^{2n-1} \mathcal{G}_s(x) = -\frac{x_i}{2s} (4s)^{-(2n-1)/2} H_{2n-1}\left(\frac{x_3}{\sqrt{4s}}\right) \cdot \mathcal{G}_s(x).$$

Repeating the previous procedure we get with $L^{(1)}_{i,n}$ as in (3.9)

$$\int_t^\infty (s-t)^{n-1} \partial_i \partial_3^{2n-1} \mathcal{G}_s(x)\, \mathrm{d}s = |x|^{-3} L^{(1)}_{i,n}(x) + |x|^{-3} \Psi_{i,3,n}\left(\frac{x}{\sqrt{t}}\right),$$

with the remainder

$$\Psi_{i,3,n}(y) := 2\pi^{-3/2} (2n-1)! \frac{y_i}{|y|} \sum_{l=0}^{n-1} \frac{(-1)^l 4^{-l}}{l!(2n-1-2l)!}$$
$$\times \Bigg[\sum_{k=1}^{n-1} \binom{n-1}{k} (-1)^k 4^k |y|^{-2k} \left(\frac{y_3}{|y|}\right)^{2(n-l)-1}$$

$$\times \int_0^{|y|/2} \lambda^{2+2n+2k-2l} e^{-\lambda^2}\, d\lambda$$
$$- \left(\frac{y_3}{|y|}\right)^{2(n-l)-1} \int_{|y|/2}^{\infty} \lambda^{2+2n-2l} e^{-\lambda^2}\, d\lambda\Bigg].$$

The term corresponding to (4.3) now reads

$$\sum_{l=0}^{n-1} 2^{-2l} \frac{(2n-l)!}{l!(2n-2l-1)!} = \sum_{l=0}^{n-1} \frac{2^{-2l}}{B(l+1,2n-2l)},$$

which comes from the inequality $B(l+1,2n-2l)^{-1} \geq B(2l+1,2n-2l)^{-1} = 2n\binom{2n-1}{2l}$ bounded by $2n(3/2)^{2n-1}$. Hence $\Psi_{i,3,n}$ satisfies the estimate $|\Psi_{i,3,n}(y)| \lesssim 14^n(2n)!|y|^{-2}$, see (4.4).

Finally, let $i \leq j < 3$. Since there holds

$$\partial_i\partial_j\partial_3^{2n-2}\mathcal{G}_s(x) = \left[\frac{x_ix_j}{4s^2} - \frac{\delta_{i,j}}{2s}\right](4s)^{-(n-1)} H_{2n-2}\left(\frac{x_3}{\sqrt{4s}}\right) \cdot \mathcal{G}_s(x)$$

we obtain in the same line as above

$$\int_t^\infty (s-n)^{n-1}\partial_i\partial_j\partial_3^{2n-2}\mathcal{G}_s(x)\,\mathrm{d}s = |x|^{-3}L^{(2)}_{i,j,n}(x) + |x|^{-3}\Psi_{i,j,n}\left(\frac{x}{\sqrt{t}}\right),$$

with the remainder

$$\Psi_{i,j,n}(y) := \pi^{-3/2}(2n-2)! \sum_{l=0}^{n-1} \frac{(-1)^l 4^{-l}}{l!(2n-2-2l)!}\Bigg[\sum_{k=1}^{n-1}\binom{n-1}{k}(-1)^k 4^k |y|^{-2k}$$
$$\times \left(\frac{y_3}{|y|}\right)^{2(n-l)-2} \int_0^{|y|/2} \left(2\frac{y_iy_j}{|y|^2}\lambda^2 - \delta_{i,j}\right)\lambda^{2n+2k-2l} e^{-\lambda^2}\, d\lambda$$
$$- \left(\frac{y_3}{|y|}\right)^{2(n-l)-2} \int_{|y|/2}^\infty \left(2\frac{y_iy_j}{|y|^2}\lambda^2 - \delta_{i,j}\right)\lambda^{2n-2l} e^{-\lambda^2}\, d\lambda\Bigg].$$

Again the remainder $\Psi_{i,j,n}$ satisfies $|\Psi_{i,j,n}(y)| \lesssim 14^n(2n)!|y|^{-2}$, see (4.4). $\square$

Proof of Lemma 3.5. Applying Lemma 3.4 to (3.6), we get for $|\alpha| = 2n$

$$\begin{aligned}
&\mathcal{F}^{-1}\biggl(\cos\biggl(\frac{\xi_3}{|\xi|}\Omega t\biggr)e^{-4\pi^2 t|\xi|^2}\biggr)\\
&\quad= \mathcal{G}_t(x) + \frac{1}{|x|^3}\sum_{n=1}^{\infty}\frac{(\Omega t)^{2n}}{(2n)!(n-1)!}\biggl[L_n^{(0)}(x) + \Psi_{3,3,n}\biggl(\frac{x}{\sqrt{t}}\biggr)\biggr]\\
&\quad= \frac{1}{|x|^3}\sum_{n=1}^{\infty}\frac{(\Omega t)^{2n}}{(2n)!(n-1)!}L_n^{(0)}(x) + \frac{1}{|x|^3}\Psi^{(0)}(x,t),
\end{aligned}$$

where $\Psi^{(0)}(x,t) := |x|^3\mathcal{G}_t(x) + \sum_{n=1}^{\infty}((\Omega t)^{2n}/(2n)!(n-1)!)\Psi_{3,3,n}(x/\sqrt{t})$. Since the heat kernel $\mathcal{G}_t(x)$ decays for fixed $t > 0$ exponentially fast as $|x| \to \infty$, we now pay attention to the absolutely convergent series

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(\Omega t)^{2n}}{(2n)!(n-1)!}L_n^{(0)}(x)\\
&\quad= \frac{1}{\pi^{3/2}}\sum_{n=1}^{\infty}\sum_{l=0}^{n}\frac{(-1)^{n+l}\Gamma(l+3/2)}{4^{n-l}(n-1)!(n-l)!(2l)!}(\Omega t)^{2n}\biggl(\frac{x_3}{|x|}\biggr)^{2l}\\
&\quad= \frac{1}{\pi}\sum_{l=1}^{\infty}\frac{l+1/2}{l!}\biggl(\frac{x_3}{|x|}\biggr)^{2l}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+l-1)!n!}\biggl(\frac{\Omega t}{2}\biggr)^{2n+2l}\\
&\qquad+ \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n n!(n-1)!}(\Omega t)^{2n}\\
&\quad= \frac{1}{\pi}\sum_{l=1}^{\infty}\frac{l+1/2}{l!}\biggl(\frac{x_3}{|x|}\biggr)^{2l}\biggl(\frac{\Omega t}{2}\biggr)^{l+1}J_{l-1}(\Omega t) - \frac{1}{4\pi}\Omega t J_1(\Omega t).
\end{aligned}$$

By Lommel's expansion [19, Section 5.22] of $(z+h)^{-\nu/2}J_\nu(\sqrt{z+h})$ with $h := -(\Omega t)^2(x_3/|x|)^2$, $z := (\Omega t)^2$ and $\nu \in \{0,-1\}$ we obtain the equation

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(\Omega t)^{2n}}{(2n)!(n-1)!}L_n^{(0)}(x)\\
&\quad= \frac{1}{\pi}\sum_{l=0}^{\infty}\frac{1}{l!}\biggl(\frac{x_3}{|x|}\biggr)^{2l+2}\biggl(\frac{\Omega t}{2}\biggr)^{l+2}(\operatorname{sgn}\Omega)^l J_l(|\Omega|t)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2\pi}\sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{x_3}{|x|}\right)^{2l}\left(\frac{\Omega t}{2}\right)^{l+1}(\operatorname{sgn}\Omega)^{l-1}J_{l-1}(|\Omega|t)\\
&-\frac{1}{2\pi}\frac{\Omega t}{2}J_{-1}(\Omega t)-\frac{\sqrt{\pi}}{4}\Omega tJ_1(\Omega t)\\
=&\ \frac{1}{4\pi}\left(\Omega t\frac{x_3}{|x|}\right)^2 J_0\big(\sqrt{z+h}\big)+\frac{1}{4\pi}\sqrt{z+h}J_{-1}\big(\sqrt{z+h}\big)\\
&-\frac{1}{4\pi}\Omega tJ_{-1}(\Omega t)-\frac{1}{4\pi}\Omega tJ_1(\Omega t).
\end{aligned}
$$

Actually, since $J_{-1}=-J_1$ we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(\Omega t)^{2n}}{(2n)!(n-1)!}L_n^{(0)}(x)\\
&=\frac{1}{4\pi}\left[\left(\Omega t\frac{x_3}{|x|}\right)^2 J_0\big(\sqrt{z+h}\big)-\sqrt{z+h}J_1\big(\sqrt{z+h}\big)\right]
\end{aligned}
$$

and thus by (3.11)

$$
\mathcal{F}^{-1}\left(\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)e^{-4\pi^2 t|\xi|^2}\right)=\frac{1}{4\pi|x|^3}\tilde{V}_0(x,t)+|x|^{-3}\Psi^{(0)}(x,t).
$$

Let us now apply Lemma 3.4 to (3.7)

$$
\begin{aligned}
&\mathcal{F}^{-1}\left(-\frac{\xi_i}{|\xi|}\sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)e^{-4\pi^2 t|\xi|^2}\right)\\
&=|x|^{-3}\sum_{n=0}^{\infty}\frac{(\Omega t)^{2n+1}}{(2n+1)!n!}\left[L_{i,3,n+1}(x)+\Psi_{i,3,n+1}\left(\frac{x}{\sqrt{t}}\right)\right].
\end{aligned}
$$

We have to differ the cases if either $i=3$ or $i\neq 3$ to analyse the second term further. Let $i=3$, then we obtain again by Lommel's expansions

$$
\begin{aligned}
&\mathcal{F}^{-1}\left(-\frac{\xi_i}{|\xi|}\sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)e^{-4\pi^2 t|\xi|^2}\right)\\
&=\pi^{-3/2}|x|^{-3}\sum_{n=0}^{\infty}\sum_{l=0}^{n+1}\frac{(-1)^l\Gamma(n+1-l+3/2)(2n+2)}{4^l n!l!(2n+2-2l)!}
\end{aligned}
$$

$$
\begin{aligned}
&\times (\Omega t)^{2n+1}\left(\frac{x_3}{|x|}\right)^{2(n+1-l)} \\
&+ |x|^{-3}\sum_{n=0}^{\infty}\frac{(\Omega t)^{2n+1}}{(2n+1)!n!}\Psi_{i,3,n+1}\left(\frac{x}{\sqrt{t}}\right) \\
&= -\frac{1}{4\pi|x|^3}V_2(x,t) + |x|^{-3}\Psi_i^{(1)}(x,t),
\end{aligned}
$$

where we define $\Psi_i^{(1)}(x,t) := \sum_{n=0}^{\infty}((\Omega t)^{2n+1}/(2n+1)!n!)\Psi_{i,3,n+1}(x/\sqrt{t})$. Now let $i \neq 3$. This case implies

$$
\begin{aligned}
&\mathcal{F}^{-1}\left(-\frac{\xi_i}{|\xi|}\sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)e^{-4\pi^2 t|\xi|^2}\right) \\
&= \pi^{-3/2}\frac{x_i x_3}{|x|^5}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-1)^l\Gamma(n-l+5/2)}{4^l n!l!(2n+1-2l)!}(\Omega t)^{2n+1}\left(\frac{x_3}{|x|}\right)^{2(n-l)} \\
&+ |x|^{-3}\sum_{n=0}^{\infty}\frac{(\Omega t)^{2n+1}}{(2n+1)!n!}\Psi_{i,3,n+1}\left(\frac{x}{\sqrt{t}}\right) \\
&= \frac{x_i x_3}{4\pi|x|^5}V_3(x,t) + |x|^{-3}\Psi_i^{(1)}(x,t).
\end{aligned}
$$

Now, by (3.1), for $i + j \neq 3$

$$
\begin{aligned}
\tilde{K}_{i,j}(x,t) &= (-1)^{1/2+|1/2+i-j|}(1-\delta_{i,j})\frac{x_{6-i-j}x_3}{4\pi|x|^5}V_3(x,t) \\
&\quad + \frac{1}{4\pi|x|^3}\delta_{i,j}\tilde{V}_0(x,t) + |x|^{-3}\tilde{\Psi}_{i,j}(x,t),
\end{aligned}
$$

and for $i + j = 3$

$$
\tilde{K}_{i,j}(x,t) = (-1)^{1/2+|1/2+j-i|}\frac{1}{4\pi|x|^3}V_2(x,t) + |x|^{-3}\tilde{\Psi}_{i,j}(x,t).
$$

Here the remainder term $\tilde{\Psi}_{i,j}$ is defined by

$$
\tilde{\Psi}_{i,j}(x,t) := \delta_{i,j}\Psi^{(0)}(x,t) + (-1)^{1/2+|1/2+i-j|}(1-\delta_{i,j})\Psi^{(1)}_{6-i-j}(x,t).
$$

It remains to verify the decay of this function. Therefore, due to (4.4) we obtain

$$\begin{aligned}\left|\tilde{\Psi}_{i,j}(x,t)\right| &\le \left|\Psi^{(0)}(x,t)\right| + \left|\Psi^{(1)}_{6-i-j}(x,t)\right| \\ &\lesssim |x|^3\mathcal{G}_t(x) + |x|^{-2}t(1+|\Omega|t)\left[\sum_{n=1}^{\infty}\frac{16^n}{(n-1)!}(\Omega t)^{2n}+1\right] \\ &\lesssim |x|^3\mathcal{G}_t(x) + |x|^{-2}t(1+|\Omega|t)^3\left[1+e^{16(\Omega t)^2}\right] \qquad (4.5)\end{aligned}$$

and the proof is complete. □

Note that due to (4.5) the remainder term $\tilde{\Psi}_{i,j}$ may glow up at most exponentially in time at $t\to\infty$. In fact, also $\Psi^{+}_{i,j}$, see Lemma 3.6, can be estimated by (4.5).

5. Proof of Theorem 2.2

In this section we strongly refer to [3]. Since a mild solution solves (1.4) we can rewrite each component of the velocity as

$$\begin{aligned}u_i(t) = \sum_{j=1}^{3}\tilde{K}_{i,j}(t)*u_{0,j} - \sum_{j,h=1}^{3}\int_0^t \partial_h K_{i,j}(t-s)*(u_ju_h)(s)\,\mathrm{d}s \\ +\sum_{j=1}^{3}\int_0^t K_{i,j}(t-s)*f_j\,\mathrm{d}s,\end{aligned}$$

where $\tilde{K}_{i,j}$ and $K_{i,j}=(\delta_{i,j}+\mathcal{R}_i\mathcal{R}_j)\tilde{K}_{i,j}$ denotes the corresponding components of the convolution operator e^{-tA_Ω} and $\mathbb{P}e^{-tA_\Omega}$, see (3.1), respectively. Assuming $\mu>4$ we obtain with Lemma 3.1

$$\int_0^t \big(\partial_h(\delta_{i,j}+\mathcal{R}_i\mathcal{R}_j)K_{i,j}(t-s)*(u_ju_h)(s)\big)(x)\,\mathrm{d}s \lesssim \big(\sqrt{t}+t^6\big)(1+|x|)^{-4}$$

for all $j,h=1,2,3$, i.e.

$$\int_0^t e^{-(t-\tau)A_\Omega}\mathbb{P}(u\cdot\nabla u)(\tau)\,\mathrm{d}\tau = \mathcal{O}_t(|x|^{-4}). \qquad (5.1)$$

Thus it suffices to analyse only the terms dealing either with the initial data or with the external force more precisely. Note that the profile of $(K_{i,j})_{i,j=1}^3$ has already been investigated in Section 3.

Let us define the auxiliary functions v_j and w_j, $j = 1, 2, 3$, as follows:

$$u_{0,j} = \mathcal{G}_1(x) \int_{\mathbb{R}^3} u_{0,j}(y)\ \mathrm{d}y + v_j(x), \tag{5.2}$$

$$f_j = \mathcal{G}_1(x) \int_{\mathbb{R}^3} f_j(y)\ \mathrm{d}y + w_j(x). \tag{5.3}$$

This leads us to

$$\begin{aligned}(K_{i,j}(t) * f_j)(x) &= (K_{i,j}(t) * \mathcal{G}_1)(x) \int_{\mathbb{R}^3} f_j(y)\,\mathrm{d}y + (K_{i,j}(t) * w_j)(x) \\ &= K^*_{i,j}(x,t) \int_{\mathbb{R}^3} f_j(y)\,\mathrm{d}y + |x|^{-3} (\Psi^{\mathcal{G}_1}_{i,j})(x,t) \int_{\mathbb{R}^3} f_j(y)\,\mathrm{d}y \\ &\quad + (K_{i,j}(t) * w_j)(x)\end{aligned}$$

and the Fourier transform yields

$$\begin{aligned}&(K_{i,j}(t) * \mathcal{G}_1)(x) \\ &= \mathcal{F}^{-1}\left(e^{-4\pi^2(t+1)|\xi|^2}\left[\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)(\delta_{i,j} + \mathcal{R}_i\mathcal{R}_j) - \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{R}_{i,j}(\xi)\right]\right).\end{aligned}$$

Due to Lemma 3.4 we get the same leading term $L_{i,j,n}$ independent on time for the shifted integral

$$\int_{t+1}^{\infty} (s-t+1)^{n-1} \partial_i \partial_j \partial_3^{2n-2} \mathcal{G}_s(x)\,\mathrm{d}s = |x|^{-3}\left[L_{i,j,n}(x) + \Psi_{i,j,n}\left(\frac{x}{\sqrt{t+1}}\right)\right].$$

Dealing with $K_{i,j}(t) * \mathcal{G}_1$ instead of $K_{i,j}(t)$ thus only requires a slight modification on the remainder terms $\Psi_{i,j}$. Since, e.g.

$$\begin{aligned}\tilde{\Psi}^{\mathcal{G}_1}_{i,j}(x,t) &:= \delta_{i,j}|x|^3\mathcal{G}_{t+1}(x) + \delta_{i,j}\sum_{n=1}^{\infty}\frac{(\Omega t)^{2n}}{(2n)!(n-1)!}\Psi_{3,3,n}\left(\frac{x}{\sqrt{t+1}}\right) \\ &\quad + (-1)^{1/2+|1/2+i-j|}(1-\delta_{i,j})\sum_{n=0}^{\infty}\frac{(\Omega t)^{2n+1}}{(2n+1)!n!}\end{aligned}$$

$$\times\, \Psi_{6-i-j,3,n+1}\left(\frac{x}{\sqrt{t+1}}\right)$$

decays like $|x|^{-2}$, comparing (4.5), it remains to investigate the decay of the convolution $K_{i,j}(t) * w_j$. Since $\int_{\mathbb{R}^3} w_j(y)\,\mathrm{d}y = 0$ the Taylor formula yields

$$\begin{aligned}\big|(K_{i,j}(t) * w_j)(x)\big| \le \int_{|y|\le|x|/2} &|y||w_j(y)|\,\mathrm{d}y \sup_{|z|\le|x|/2} |\nabla K_{i,j}(x+z,t)| \\ &+ \int_{|y|>|x|/2} |w_j(y)|\,\mathrm{d}y |K_{i,j}(x,t)| \\ &+ \int_{|y|>|x|/2} |w_j(y)||K_{i,j}(x-y,t)|\,\mathrm{d}y.\end{aligned}$$

Let $0 < \varepsilon < (\mu-4)/3$. The definition of w_j, see (5.3), implies

$$|w_j(y)| \le |f_j(y)| + \mathcal{G}_1(y)\|f_j\|_1 \lesssim (1+|y|)^{-\mu}.$$

Applying Lemma 3.1 for any $\kappa > 0$ as well as the Hölder inequality yield

$$\begin{aligned}\big|(K_{i,j}(t) * w_j)(x)\big| &\lesssim \big(1+(|\Omega|t)^5\big)\big(\sqrt{t}+|x|\big)^{-4} \int_{|y|\le|x|/2} |y||w_j(y)|\,\mathrm{d}y \\ &\quad + (1+(|\Omega|t)^4)\big(\sqrt{t}+|x|\big)^{-3} \int_{|y|>|x|/2} |w_j(y)|\,\mathrm{d}y \\ &\quad + \|K_{i,j}(t)\|_{1+\varepsilon}\bigg(\int_{|y|>|x|/2} |w_j(y)|^{(1+\varepsilon)/\varepsilon}\,\mathrm{d}y\bigg)^{\varepsilon/(1+\varepsilon)} \\ &\lesssim \big(1+(|\Omega|t)^5 + \|K_{i,j}(t)\|_{1+\varepsilon}\big)\big(\min\{\sqrt{t},1\}+|x|\big)^{-4}.\end{aligned}$$

Note that due to the choice of ε there holds $(\mu-4)((1+\varepsilon)/\varepsilon) > 3$ which ensures the convergence of the integral $\int_{\mathbb{R}^3}(1+|y|)^{(\mu-4)((1+\varepsilon)/\varepsilon)}\,\mathrm{d}y$.

Due to the assumptions on the solenoidal initial data $u_0 \in L^\infty_4(\mathbb{R}^3)^3$ we obtain that the Fourier transform $\mathcal{F}(u_0)$ is continuous and $\xi\cdot\mathcal{F}(u_0)(\xi) = 0$ for all $\xi \in \mathbb{R}^3$. By continuity this implies $\mathcal{F}(u_0)(\xi) = 0$ for all $\xi \in \mathbb{R}^3$ and thus in particular

$$\int_{\mathbb{R}^3} u_0(y)\, \mathrm{d}y = 0.$$

Therefore, repeating the above procedure with $\tilde{K}_{i,j} * v_j$, see (5.2), we finally get for $i = 1, 2$ the asymptotic profile

$$\begin{aligned} u_i(x,t) = \frac{1}{4\pi|x|^3} \Bigg[& A_0(x,t) \int_{\mathbb{R}^3} f_i(y)\,\mathrm{d}y + (-1)^{i+1} A_2(x,t) \int_{\mathbb{R}^3} f_{3-i}(y)\,\mathrm{d}y \\ & + \left((-1)^{i+1} \frac{x_{3-i}x_3}{|x|^2} A_3(x,t) + \frac{x_i x_3}{|x|^2} A_5(x,t) \right) \int_{\mathbb{R}^3} f_3(y)\,\mathrm{d}y \\ & + \sum_{k=1}^{3} \frac{x_i x_k}{|x|^2} A_4(x,t) \int_{\mathbb{R}^3} f_k(y)\,\mathrm{d}y \Bigg] + \mathcal{O}_t(|x|^{-4}), \\ u_3(x,t) = \frac{1}{4\pi|x|^3} \Bigg[& A_1(x,t) \int_{\mathbb{R}^3} f_3(y)\,\mathrm{d}y + \left(\frac{x_1 x_3}{|x|^2} A_5(x,t) - \frac{x_2 x_3}{|x|^2} A_3(x,t) \right) \\ & \times \int_{\mathbb{R}^3} f_1(y)\,\mathrm{d}y + \left(\frac{x_1 x_3}{|x|^2} A_3(x,t) + \frac{x_2 x_3}{|x|^2} A_5(x,t) \right) \int_{\mathbb{R}^3} f_2(y)\,\mathrm{d}y \\ & + \sum_{k=1}^{3} \frac{x_3 x_k}{|x|^2} A_4(x,t) \int_{\mathbb{R}^3} f_k(y)\,\mathrm{d}y \Bigg] + \mathcal{O}_t(|x|^{-4}), \end{aligned}$$

which is the component-wise presentation of the assertion. This completes the proof of Theorem 2.2. □

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