CONTRIBUTIONS TO THE THEORY OF MEROMORPHIC FUNCTIONS IN THE UNIT-CIRCLE

By

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This paper is divided into two parts. In part 1, a class of meromorphic functions in the unit-circle is considered and some results analogous to K. YOSIDA's[1] are obtained. Next, in part 2, a definition is given to a locally univalent function, regular in the unit-circle, which is distinct from that of P. MONTEL[2] and some theorems concerning such a function are enunciated.

PART 1. ON A CLASS OF MEROMORPHIC FUNCTIONS IN THE UNIT-CIRCLE

Suppose that $y=f(x)$ is meromorphic in the unit-circle and consider the family $\{f_a(z)\}$ formed by all the functions

$$(1) \qquad f_a(z) \equiv f\left(\frac{z-a}{\bar{a}z-1}\right),$$

where a varies throughout inside the unit-circle $|z|<1$. Then, let us say, after K. YOSIDA[3], that *the function* $y=f(x)$ *belongs to class* (A), if $\{f_a(z)\}$ is a normal family for $|z|<1$ in MONTEL's sense. The object of this part is to study some properties of class (A). For this purpose, use is made of the pseudo-distance

$$(2) \qquad D(a_1, a_2) = \left|\frac{a_1-a_2}{\bar{a}_1 a_2 - 1}\right|$$

(1) K. YOSIDA: On a class of meromorphic functions, Proc. Phys.-Math. Soc. Japan, 3. ser., 16 (1934), pp. 227-235.

(2) P. MONTEL: Sur les fonctions localement univalentes ou multivalentes, Ann. Éc. Norm., (3), 54 (1937).

(3) Let $y=\varphi(x)$ be a meromorphic function in $|x|<\infty$ such that the family $\varphi_a(z) \equiv \varphi(z+a)$, $|a|<\infty$, is normal for $|z|<\infty$. Then, $y=\varphi(x)$ is called a function of class (A) (of the parabolic case). K. YOSIDA has obtained some interesting theorems on this class. Cf. K. YOSIDA: loc. cit.

of two points of a_1 and a_2 lying inside the circle $|z|<1$ and the locus

$$D(z, a) = \rho \qquad (0<\rho<1, \quad |a|<1)$$

which shall be called a *pseudo-circle* $C_\rho(a)$ with *pseudo-centre* a of *pseudo-radius* ρ, the interior of $C_\rho(a)$ being denoted by $I[C_\rho(a)]$. Of course, the pseudo-distance and the pseudo-circle are invariant under any transformation of the form

$$x = e^{i\theta}\frac{z-a}{\bar{a}z-1}. \tag{3}$$

First the following will be proved.

Theorem 1. *In order that $y=f(x)$ may belong to class (A), it is necessary and sufficient that there should exist a positive number K such that*

$$\frac{(1-|x|^2)|f'(x)|}{1+|f(x)|^2}<K \qquad \text{for} \qquad |x|<1. \tag{4}$$

Suppose that the inequality (4) holds. It is easily shown that, nside the unit-circle,

$$\frac{(1-|z|^2)|f'_a(z)|}{1+|f_a(z)|^2} = \frac{(1-|x|^2)|f'(x)|}{1+|f(x)|^2} \qquad (|a|<1), \tag{5}$$

as a consequence of the invariant form

$$\frac{|dx|}{1-|x|^2} = \frac{|dz|}{1-|z|^2}$$

for any transformation (3). By MARTY's criterion[1] the family $\{f_a(z)\}$ is normal in $|z|<1$ and $y=f(x)$ belongs to class (A). If, on the contrary,

$$\operatorname*{upper\ limit}_{|x|<1} \frac{(1-|x|^2)|f'(x)|}{1+|f(x)|^2} = +\infty,$$

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(1) MARTY has proved the following theorem: in order that a family $\{f(z)\}$, formed of meromorphic functions in a domain D, should be normal in D, it is necessary and sufficient that there should exist a positive number $K(D_1)$ such that, inside D_1, $\frac{|f'(z)|}{1+|f(z)|^2}<K(D_1)$ for any function $f(z)$ of the family, where D_1 denotes any closed domain, lying entirely within the domain D. See F. MARTY: Recherches sur la répartition des valeurs d'une fonction méromorphe, Ann. Fac. Univ. Toulouse, (3), 23 (1931).

then, there exists a sequence of points a_n inside $|x|<1$ such that

$$\lim_{n\to\infty}\frac{(1-|a_n|^2)\,|f'(a_n)|}{1+|f(a_n)|^2}=+\infty\ .$$

Associate a function

$$f_{a_n}(z)=f\left(\frac{z-a_n}{\bar{a}_n z-1}\right)$$

with each a_n, then we have

$$\lim_{n\to\infty}\frac{|f'_{a_n}(0)|}{1+|f_{a_n}(0)|^2}=\lim_{n\to\infty}\frac{(1-|a_n|^2)\,|f'(a_n)|}{1+|f(a_n)|^2}=+\infty\ .$$

Consequently the sequence $\{f_{a_n}(z)\}$ is not normal for $|z|<\rho$, ρ being any positive number less than unity.

Remark. The above theorem corresponds to theorem 1(1) in YOSIDA's paper.

It is well known that the characteristic function, $T(r)$, of $y=f(x)$ is given by the formula

$$T(r)=S(r)+O(1)\ ,$$

where

$$S(r)=\int_0^r\frac{A(t)}{t}dt \quad\text{and}\quad A(r)=\int_0^r\int_0^{2\pi}\frac{|f'(x)|^2}{(1+|f(x)|^2)^2}r\,dr\,d\theta,\ (x=re^{i\theta}).$$

Thus, from the above theorem, we obtain

Theorem 2. *Suppose that $y=f(x)$ is a meromorphic function of class (A). Then*

$$T(r)=O\left(\log\frac{1}{1-r}\right).$$

In particular if $y=f(x)$ is regular in $|x|<1$, then

$$\log M(r)=O\left(\frac{1}{1-r}\log\frac{1}{1-r}\right),$$

$M(r)$ denoting the quantity $\max\limits_{|x|=r}|f(x)|$.

Remark. This theorem corresponds to corollary 2(2) in YOSIDA's paper.

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(1) K. YOSIDA: loc. cit.
(2) K. YOSIDA: loc. cit.

Now let us consider the asymptotic values of a function $y=f(x)$ of class (A). Let L: $\zeta=\zeta(t)$ $(0\leqq t<1)$ be a continuous curve lying inside the unit-circle such that $\zeta(0)=0$ and $\lim_{t\to 1}|\zeta(t)|=1$. We call α an asymptotic value of $y=f(x)$ along L, provided that $\lim_{t\to 1} f[\zeta(t)]=\alpha$. Suppose that $y=f(x)$, a function of class (A), has an asymptotic value α along the curve L. We consider any increasing sequence t_n such that $0<t_n<1$ and $t_n\to 1$ and put $\zeta_n=\zeta(t_n)$ $(n=1, 2, 3, \ldots)$. Since $y=f(x)$ belongs to class (A), we may select from the sequence $\{\psi_n(z)\}$ of functions

$$\psi_n(z)\equiv f_{\zeta_n}(z)\equiv f\left(\frac{z-\zeta_n}{\bar{\zeta}_n z-1}\right)$$

a subsequence $\{\psi_{n_\nu}(z)\}$ which converges uniformly to the limiting function $f_0(z)$ in the interior of $|z|<1$[1]. Clearly the image, in the x-plane, of the circle $|z|=\rho$ (ρ fixed, $0<\rho<1$) formed by the transformation

$$z=\frac{x-\zeta_{n_\nu}}{\bar{\zeta}_{n_\nu}x-1}\,, \qquad \left(\zeta_{n_\nu}=\zeta(t_{n_\nu})\right)$$

is a pseudo-circle $C_\rho(\zeta_{n_\nu})$ with pseudo-centre $\zeta(t_{n_\nu})$ of pseudo-radius ρ. Denoting by $\zeta(t'_{n_\nu})$ the furthest point from the origin of the points of intersection of L with $C_\rho(\zeta_{n_\nu})$ and putting

$$z'_{n_\nu}=\frac{\zeta(t'_{n_\nu})-\zeta_{n_\nu}}{\bar{\zeta}_{n_\nu}\cdot\zeta(t'_{n_\nu})-1}\,,$$

we have

$$\lim_{\nu\to\infty}\psi_{n_\nu}(z'_{n_\nu})=\lim_{\nu\to\infty}f\left[\zeta(t'_{n_\nu})\right]=\alpha\,.$$

Consequently the limiting function $f_0(z)$ has at least one α-point on $|z|=\rho$, since all z'_{n_ν} lie on $|z|=\rho$. Remembering that ρ can be taken arbitrarily, it can be asserted that $f_0(z)$ is identical with a constant α. Hence, it follows without difficulty that the original sequence $\{f_{\zeta_n}(z)\}$ converges to the constant limit α. Thus, it is concluded that for any positive number ε, we may find a positive number $\delta=\delta(\varepsilon)$ such that $|f_{\zeta(t)}(z)-\alpha|<\varepsilon$ in $|z|\leqq\rho<1$, ρ being

(1) "To converge uniformly in the interior of D" means "to converge uniformly in any closed domain inside D".

arbitrarily fixed, provided that $t \geqq 1-\delta$. In other words, we can enunciate

Theorem 3. *Suppose that $y=f(x)$ is a meromorphic function of class (A). Let $L: \zeta=\zeta(t)$, $0 \leqq t < 1$ be a continuous curve inside the unit-circle such that $\zeta(0)=0$ and $\lim_{t \to 1} |\zeta(t)|=1$ and denote by Δ the domain consisting of all points interior to any pseudo-circle $C_\rho(a)$ (ρ being fixed, $0<\rho<1$) where the pseudo-centre a describes the curve L. Then, if $y=f(x)$ has an asymptotic value α along L, it may be asserted that $y=f(x)$ converges uniformly to α inside the domain Δ, as the modulus of the variable point x tends to unity, and moreover that the normal family $\{f_a(z)\}$ generated by $y=f(x)$ admits at least one constant limit.*

Remark. Compare this theorem with those of E. ULLRICH and others.[1]

The following theorem was enunciated by F. IVERSEN[2]: *any meromorphic function $f(x)$ with no asymptotic value takes every value infinitely often.* However this theorem also holds good when $f(x)$ is meromorphic in the unit-circle. An elementary proof for it will be given here. Suppose that $y=f(x)$ is a function, meromorphic in the unit-circle, with no asymptotic value. First, it must be shown that for any point ω ($|\omega|=1$), the cluster set S_ω of $y=f(x)$ contains all complex numbers. By definition, S_ω is the set of all limiting values β such that $f(x_n) \to \beta$ as $x_n \to \omega$. If α does not belong to[3] S_ω, then we may find two positive numbers ρ and ε such that $|f(x)-\alpha| \geqq \varepsilon$ for $|x-\omega|<\rho$ inside the unit-circle. Let us denote by Δ' the domain bounded by two circles $|x-\omega|=\rho$ and $|x|=1$. Mapping the domain Δ', by $x=\varphi(z)$, conformally on $|z|<1$ and considering the function $\psi(z)=\varepsilon \cdot [f(\varphi(z))-\alpha]^{-1}$ bounded in $|z|<1$, we can apply a well-known theorem due to FATOU. Thus it is seen without difficulty, that there must exist a continuous curve L, lying inside Δ' except the end-point on $|x|=1$, along which $y=f(x)$ has an asymptotic value. It contradicts the above assumption. Next,

(1) Analogous results to theorem 3 have been obtained by some authors. See E. ULLRICH: Über eine Anwendung des Verzerrungssatzes auf meromorphe Funktionen, Journ. f. Math., 166 (1932), Satz 8, p. 232 and also T. SHIMIZU: On the paths of determination and indetermination of integral functions, Proc. Phys.-Math. Soc. of Japan, (3), 12 (1930), p. 127.

(2) Cf. R. NEVANLINNA: Eindeutige analytische Funktionen, Berlin, 1936, p. 274.

(3) For brevity, let us suppose α to be finite. In the other case, we have only slightly to modify our argument.

suppose α to be an exceptional value[1] in PICARD's sense and let all the α-points be denoted by x_i ($i = 1, 2, \ldots, m$). If a sufficiently small circle: $|y-\alpha| = \delta$ be described on the y-plane, there are m JORDAN closed domains Δ_i, incluaed entirely inside $|x| < 1$ and each containing a point x_i, such that $y = f(x)$ takes in each Δ_i every value of $|y-\alpha| \leqq \delta$ exactly the same number of times. Clearly it may be assumed that any two of the domains Δ_i have no point in common. On the other hand, outside all Δ_i, we can find such a point x_0 that $y_0 = f(x_0)$ lies within $|y-\alpha| < \delta$, by the just proved fact. Denoting by e_{x_0} the inverse element obtained from the expansion of $y = f(x)$ at $x = x_0$ and continuing e_{x_0} along the segment $\overline{y_0\alpha}$, we see that there exists a path L, starting from $x = x_0$ and approaching indefinitely the circumference $|x| = 1$, along which $y = f(x)$ has certain asymptotic value. This also contradicts the assumption.

Thus is proved the following

Theorem 4. *Let $y = f(x)$ be a meromorphic function in the unit-circle. If $y = f(x)$ has no asymptotic value, $f(x)$ takes every value infinitely often.*

After K. YOSIDA, all the function of class (A) can be divided into two categories: if the normal family $\{f_a(z)\}$ generated by $y = f(x)$ admits no constant limit, $y = f(x)$ is a member of *the first category* and all the functions, not belonging to the first category, form *the second*. By combining theorem 4 with theorem 3, we obtain at once

Theorem 5. *Let $y = f(x)$ be a function of class (A). If $y = f(x)$ belongs to the first category, $f(x)$ has no asymptotic value and so takes every value infinitely often.*

To answer under what condition it can be asserted that $y = f(x)$, a function of class (A), belongs to the first category, we denote by $A[C_\rho(a)]$ the area of the RIEMANNian image of the interior $I[C_\rho(a)]$ of $C_\rho(a)$, mapped by $y = f(x)$ on the RIEMANN's sphere of radius $\frac{1}{2}$, touching the y-plane: in other words, by $A[C_\rho(a)]$ is denoted the quantity

$$\iint_{I[C_\rho(a)]} \frac{|f'(x)|^2}{(1+|f(x)|^2)^2} d\omega$$

($d\omega$ = the area element on the x-plane).

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(1) It is the same as the foot-note (3) in page 153.

It is known that if a sequence $\{f_{a_n}(z)\}$ converges uniformly to the limiting function $f_0(z)$ in the interior of $|z|<1$, then $\dfrac{|f'_{a_n}(z)|}{1+|f_{a_n}(z)|^2}$ converges uniformly to 0 or $\dfrac{|f'_0(z)|}{1+|f_0(z)|^2}$ for $|z|<\rho<1$, ρ being fixed, according as $f_0(z)$ is constant or not. Consequently

$$\lim_{n\to\infty} A[C_\rho(a_n)] = \lim_{n\to\infty}\iint_{|z|<\rho}\frac{|f'_{a_n}(z)|^2}{(1+|f_{a_n}(z)|^2)^2}r\,dr\,d\theta \qquad (z=re^{i\theta})$$

$$=\begin{cases} 0 & \text{, if } f_0(z) \text{ is constant,} \\ \displaystyle\iint_{|z|<\rho}\frac{|f'_0(z)|^2}{(1+|f_0(z)|^2)^2}r\,dr\,d\theta, & \text{if } f_0(z) \text{ is not constant.}\end{cases}$$

Hence the following

Theorem 6. *In order that $y=f(x)$ of class (A) should belong to the first category, it is necessary and sufficient that*

$$\underset{\text{for all } a,\ |a|<1}{\textit{lower limit}}\ A[C_\rho(a)]>0\ ,$$

for any fixed positive number ρ less than 1.

From the equality (5), it is easily seen that if $y=f(x)$ belongs to class (A), for any number a ($|a|<1$) the function $f_a(z)$ has the same property. From the fact that the pseudo-distance of two points is invariant under any transformation (3) and by applying the just proved theorem, it is shown that $y=f(x)$ and $f_a(z)$ must belong to the same category of class (A) and moreover any limiting function $f_0(z)$ of a sequence $\{f_{a_n}(z)\}$, each member being a function of the first category, belongs also to the same category of class (A). From these remarks, we get

Theorem 7. *Let $y=f(x)$ be a function of the first category of class (A). Then such a positive number ρ (<1) can be found that $y=f(x)$ takes every value at least once in the interior of the pseudo-circle $C_\rho(a)$, a denoting any point in the unit-circle.*

Suppose that the assertion is false. Then we may find a sequence of points $\{a_n\}$ ($|a_n|<1$) and that of positive numbers $\{\rho_n\}$ ($\rho_n<1$, $\rho_n\to 1$) such that for any natural number n, $y=f(x)$ admits an exceptional value, say α_n, in the interior of the pseudo-circle $C_{\rho_n}(a_n)$. Hence the function $f_{a_n}(z)$, obtained from $y=f(x)$ by formula (1), does not take α_n for $|z|<\rho_n$. Without loss of generality, it may be supposed that $\{f_{a_n}(z)\}$ converges uniformly to the limiting function

$f_0(z)$ in the interior of $|z|<1$ and a_n tends to α[1]. By the above remark, $f_0(z)$ belongs to the first category of class (A). Let z_0 be an α-point inside $|z|<1$. Then, from HURWITZ's theorem, it follows that for sufficiently large n, $f_{a_n}(z)$ has an α_n-point in any neighbourhood of z_0. Hence, there is a contradiction, for α_n is exceptional for $f_{a_n}(z)$ inside $|z|<\rho_n$ and ρ_n tends to 1.

Remark 1. This theorem corresponds to theorem 6 in YOSIDA's paper.

Remark 2. It can be easily shown that SCHWARZ's triangle functions belong to the first category of class (A), provided that the closure of one of its fundamental domains lies completely within the unit-circle. This fact corresponds to the fact that the doubly periodic functions belong to the first category of class (A) considered by K. YOSIDA[2].

PART 2. ON LOCALLY UNIVALENT FUNCTIONS.

P. MONTEL[3] called $f(x)$ a locally univalent function of modulus ρ, if there is a positive number ρ such that $f(x)$, regular in a finite domain D, is univalent (*schlicht*) in any circular domain of the form: $|x-a|<\rho$ included within D. Locally univalent functions are not necessarily univalent in the ordinary sense, for example e^x is a locally univalent function of modulus π. P. MONTEL investigated locally univalent (and further multivalent) functions and obtained some interesting results.

In this part, let $y=f(x)$ be supposed regular in the unit-circle and $y=f(x)$ be called a locally univalent function *of pseudo-modulus* ρ, provided that $y=f(x)$ is univalent in the interior of any pseudo-circle $C_\rho(a)$, a varying throughout in $|x|<1$ and ρ being a fixed positive number <1. By an elementary calculation, it is seen that the EUCLIDean radius of $C_\rho(a)$ is given by $R=\dfrac{\rho(1-|a|^2)}{1-|a|^2\rho^2}$, R attaining its maximum value ρ at $a=0$. Hence, a locally univalent function in $|x|<1$ of modulus ρ is necessarily a locally univalent function of pseudo-modulus ρ, while its converse is not necessarily true. Our definition gives directly

(1) For brevity, we consider only the case in which α is finite.
(2) K. YOSIDA: loc. cit.
(3) P. MONTEL: loc. cit.

Theorem 8. *Suppose that $y=f(x)$ is regular in the unit-circle. Then, in order that $y=f(x)$ should be a locally univalent function of pseudo-modulus ρ, it is necessary and sufficient that all the functions $f_a(z)$ of form (1), that is: $f_a(z)\equiv f\left(\frac{z-a}{\bar{a}z-1}\right)$ should be univalent in $|z|<\rho$.*

Consequently we can apply a classical method[1] due to R. NEVANLINNA on the univalent functions in the unit-circle. As $f_a(z)$ is expanded into TAYLOR's series for $|z|<1$:

$$f_a(z)=f(a)-(1-|a|^2)\,f'(a)z+\frac{1}{2}\Big[f''(a)\;(1-|a|^2)+ +2\bar{a}(1-|a|^2)\,f'(a)\Big]z^2+\cdots,$$

by putting $z=\rho\zeta$, the function

$$-\frac{f_a(\rho\zeta)-f(a)}{\rho(1-|a|^2)f'(a)}=\zeta-\frac{\rho}{2}\left[\frac{f''(a)}{f'(a)}(1-|a|^2)+2\bar{a}\right]\zeta^2+\cdots$$

is univalent in the unit-circle. Consequently, BIEBERBACH's inequality gives

$$\left|\frac{f''(a)}{f'(a)}+\frac{2\bar{a}}{1-|a|^2}\right|\leqq\frac{4}{\rho(1-|a|^2)},$$

whence follows as usual

Theorem 9. *If $y=f(x)$ is a locally univalent function of pseudo-modulus ρ in the unit-circle, then it follows, putting $|x|=r$,*

$$\frac{(1-r)^{2\rho^{-1}-1}}{(1+r)^{2\rho^{-1}+1}}\leqq\left|\frac{f'(x)}{f'(0)}\right|\leqq\frac{(1+r)^{2\rho^{-1}-1}}{(1-r)^{2\rho^{-1}+1}}\qquad(\textit{Verzerrungssatz}),$$

$$\left|\arg\frac{f'(x)}{f'(0)}\right|\leqq\frac{2}{\rho}\log\frac{1+r}{1+r}\qquad(\textit{Drehungssatz}).$$

From this follows immediately

Theorem 10. *If $\{f_n(x)\}$ is a sequence of locally univalent regular functions of modulus ρ inside $|x|<1$ and if $\{f_n(x)\}$ converges uniformly to a non-constant limit $f_0(x)$ in the interior of $|x|<1$, then $f_0(x)$ is also locally univalent, its modulus being ρ. Moreover, the family $\{f(x)\}$, each $f(x)$ being a locally univalent regular function of*

(1) For example, see P. MONTEL: Leçons sur les fonctions univalentes ou multivalentes, Paris, 1933, p. 51.

modulus ρ inside $|x|<1$, such that $f(0)=0$ and $f'(0)=1$, is normal in $|x|<1$ and any limiting function belongs to the family.

Suggested by E. LANDAU's proof for A. BLOCH's theorem, the following will be added here.

Theorem 11. *Suppose that $y=f(x)$ is regular and non-constant for $|x|\leqq 1$. Then, there is a numerical constant R $(0<R<1)$ such that $f(x)$ is univalent in the interior of a pseudo-circle $C_R(a)$, provided that its pseudo-centre is suitably chosen.*

After E. LANDAU[1], we consider a function

$$\Phi(x)=(1-|x|^2)\,|f'(x)|\,,$$

such that $\Phi(x)$ is continuous, non-negative in $|x|\leqq 1$ and vanishes on $|x|=1$. Denote by N the maximum value of $\Phi(x)$ for $|x|\leqq 1$ and let the maximum be attained at a. It is clear that

$$N=\Phi(a)=(1-|a|^2)\,|f'(a)|>0 \quad \text{and} \quad 0\leqq|a|<1\,.$$

Let $f_a(z)$ be the function defined by formula (1), then we have

$$(1-|z|^2)\,|f_a'(z)|=(1-|x|^2)\,|f'(x)|=\Phi(x)\,,$$

where

$$x=\frac{z-a}{\bar{a}z-1}\,.$$

Hence it holds that

$$|f_a'(z)|\leqq\frac{N}{1-|z|^2} \qquad \text{for} \qquad |z|<1 \qquad \text{and}$$

$$|f_a'(0)|=|f'(a)|\,(1-|a|^2)=N\,.$$

If we put $\varphi_a(z)\equiv\dfrac{f_a(z)}{N}$, then we have

$$|\varphi_a'(0)|=1 \qquad \text{and} \qquad |\varphi_a'(z)|\leqq\frac{4}{3} \qquad \text{for} \qquad |z|<\frac{1}{2}\,.$$

Here we can apply a known fact[2]: Let $F(z)=a_0+a_1z+\cdots$, $|a_1|=1$, be regular for $|z|<R$ and $|F'(z)|\leqq M$ for $|z|<R$. Then $F(z)$

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(1) E. LANDAU: Über die BLOCHsche Konstante und zwei verwandte Weltkonstanten, Math. Zeit., vol. 30 (1929), p. 618.

(2) K. NOSHIRO: Proc. Imp. Acad. vol. 8 (1932), p. 275.

is univalent for $|z| < \frac{R}{M}$. By putting $R = \frac{1}{2}$, $M = \frac{4}{3}$, we see that $\varphi_a(z)$ is univalent for $|z| < \frac{3}{8}$. Thus, for the present theorem it is sufficient to make R equal to $\frac{3}{8}$.

Remark. The theorem just obtained above may also be stated as follows: Suppose that $y = f(x)$ is regular in the closed circle $|x| \leqq 1$ and non-constant. Then *there is a numerical constant R^* such that if the circle $|x| < 1$ is regarded as a non-*EUCLID*ean plane, there exists a non-*EUCLID*ean circle, with non-*EUCLID*ean radius R^*, in the interior of which $y = f(x)$ is univalent.* This is deduced from the fact that between the non-EUCLIDean distance $D^*(a_1, a_2)$ and the pseudo-distance $D(a_1, a_2)$ of two points a_1, a_2 in $|x| < 1$, there holds a well-known equality

$$D^*(a_1, a_2) = \frac{1}{2} \log \frac{1 + D(a_1, a_2)}{1 - D(a_1, a_2)}.$$

It seems very difficult to find the greatest value R and, in consequence, R^*.

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