ON INTRINSIC THEORIES IN THE MANIFOLD OF SURFACE-ELEMENTS OF HIGHER ORDER

By

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Introduction. It is well known that the space in which a measure of a hypersurface: $x^i = x^i(u^1, u^2, \cdots, u^{n-1})$, $i=1, 2, \cdots, n$ is given by the $(n-1)$-ple integral: $\int_{(n-1)} F(x^i, \partial x^i/\partial u^\alpha)\, du^1 \cdots du^{n-1}$ is called a CARTAN space. As it is shown by CARTAN, this space is to be regarded as a manifold of hyperplane-elements $(x^i, \partial x^i/\partial u^\alpha)$. The geometry of CARTAN space were discussed by E. CARTAN [1][(1)] and L. BERWALD [6][7] at large. Thereafter, T. OHKUBO [9] and the present author [10][11] extended this theory to the $(n-1)$-ple integral of higher order of special forms. Recently, the present author [12] have established a geometry of an $(n-1)$-ple integral of the second order in general form, but the space in which the theories are discussed was regarded as a manifold of hypersurface-elements of the third order. On the other hand the theory of K-spreads in an n-dimensional manifold which are concerned with a system of partial differential equations of the second order was studied at first by J. DOUGLAS, and the theory was treated in the manifold of all K-dimensional surface-elements of order 1. Thereafter A. KAWAGUCHI and H. HOMBU [5] studied the theory of K-spreads of the m-th order $(m \geqq 2)$, and the manifold of all K-dimensional surface-elements of the $(m-1)$-th order was based in this case. In this paper we aim to establish the foundation of differential geometries in the manifold of K-dimensional surface-elements of higher order under the transformation group of the surface-elements which is deduced from the groups of arbitrary transformations of coordinates and parameters, and treat of the geometry of multiple integral of higher order in detail.

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§ 1. The manifold $F_n^{(m)}$ and notations. In an n-dimensional space X_n with point coordinates $x^1, x^2, \cdots, x^n$ a K-dimensional surface is defined analytically by the parametric equations

(1) Numbers in brackets refer to the references at the end of the paper.

$$x^i = x^i(u^\alpha), \quad \alpha = \dot{1}, \dot{2}, \cdots, \dot{K},$$

where u^α are K essential parameters for the K-dimensional surface.

At every point on this K-dimensional surface a K-dimensional surface element of the m-th order can be determined by

$$x^i = x^i(u^\alpha), p^i_{\alpha(1)} \equiv p^i_{\alpha_1} = \frac{\partial x^i}{\partial u^{\alpha_1}}, \quad p^i_{\alpha(2)} \equiv p^i_{\alpha_1\alpha_2} = \frac{\partial^2 x^i}{\partial u^{\alpha_1}\partial u^{\alpha_2}},$$

$$\cdots\cdots\cdots\cdots, p^i_{\alpha(m)} \equiv p^i_{\alpha_1\alpha_2\cdots\alpha_m} = \frac{\partial^m x^i}{\partial u^{\alpha_1}\partial u^{\alpha_2}\cdots\partial u^{\alpha_m}}.$$

Now, adjoining arbitrary system of values x^i, $p^i_{\alpha(1)}, \cdots, p^i_{\alpha(m)}$ to every point in X_n, we have the $n\binom{K+m}{K}$-dimensional manifold $F^{(m)}_n$. We shall name the quantity which is transformed according to the tensor law under the transformation groups of coordinates and parameters:

$$(1.1) \qquad x^{i'} = x^{i'}(x^1, x^2, \cdots\cdots, x^n),$$

$$(1.2) \qquad u^{\alpha'} = u^{\alpha'}(u^{\dot{1}}, u^{\dot{2}}, \cdots\cdots u^{\dot{K}})$$

the intrinsic quantity according to E. Bortolotti.

We can speak of x-transformations or u-transformations alone, and of x-tensors or u-tensors accordingly. Tensor will mean, unless otherwise mentioned, a geometrical object which has the proper law of transformation for both sorts of indices.

Throughout this paper we shall use the notations

$$X^i_{i'} = \frac{\partial x^i}{\partial x^{i'}}, \quad X^{i'}_i = \frac{\partial x^{i'}}{\partial x^i}, \quad X^i_{i'(2)} = \frac{\partial^2 x^i}{\partial x^{i'_1}\partial x^{i'_2}}, \cdots\cdots\cdots,$$

$$U^\alpha_{\alpha'} = \frac{\partial u^\alpha}{\partial u^{\alpha'}}, \quad U^{\alpha'}_\alpha = \frac{\partial u^{\alpha'}}{\partial u^\alpha}, \quad U^\alpha_{\alpha'(2)} = \frac{\partial^2 u^\alpha}{\partial u^{\alpha'_1}\partial u^{\alpha'_2}}, \cdots\cdots\cdots.$$

and

$$F_{;\alpha} = \frac{\partial F}{\partial u^\alpha}, \quad F_{;i} = \frac{\partial F}{\partial x^i}, \quad F^{\alpha(s)}_{;\ i} = \frac{\bar{\partial}}{\bar{\partial} p^i_{\alpha(s)}} F = \frac{l_1!\, l_2! \cdots l_t!}{s!} \frac{\partial}{\partial p^i_{\alpha(s)}} F,$$

when the indices $\alpha_1, \alpha_2, \cdots\cdots, \alpha_s$ consist of $l_1, l_2, \cdots\cdots, l_t$ blocks of the same indices.

Moreover we shall often use the notations for indices in the form

$$I_{(\alpha_1\alpha_2\cdots\cdots\alpha_r} J_{\alpha_{r+1}\cdots\cdots\alpha_t)} \equiv I_{(\alpha(r)} J_{\alpha(t-r))}.$$

§ 2. **Transformation laws of the various quantities.** Under the transformation (1.1) the surface-elements in $F^{(m)}_n$ has the laws of transformation as the forms:

$$x^i = x^i(x^{i'}), \qquad p^i_\alpha = X^i_{i'} p^{i'}_\alpha ,$$
$$p^i_{\alpha(2)} = X^i_{i'} p^{i'}_{\alpha(2)} + X^i_{j'k'} p^{j'}_{\alpha_1} p^{k'}_{\alpha_2} ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$
$$p^i_{\alpha(m)} = X^i_{i'} p^{i'}_{\alpha(m)} + R^i_{\alpha(m)}(x', p'_{(1)}, p'_{(2)}, \cdots\cdots, p'_{(m-1)}),$$

so that $dp^i_{\alpha(t)} = \sum_{s=0}^{t} \frac{\partial p^i_{\alpha(t)}}{\partial p^{i'}_{\beta(s)}} dp^{i'}_{\beta(s)}$, putting $dp^i_{\alpha(0)} = dx^i$.

It is well known that

$$\text{(2.1)} \qquad \frac{\partial p^i_{\alpha(t)}}{\partial p^{i'}_{\beta(s)}} = \frac{t!\,(s-r)!}{s!\,(t-r)!}\,\delta^{\beta(r)}_{(\alpha(r)}\frac{\partial p^i_{\alpha(t-r))}}{\partial p^{i'}_{\beta(s-r)}}$$
$$= \binom{t}{s}\delta^{\beta_1}_{(\alpha_1}\cdots\delta^{\beta_s}_{\alpha_s}\frac{\partial p^i_{\alpha_{s+1}\cdots\alpha_t)}}{\partial x^{i'}} = \binom{t}{s}\delta^{\beta(s)}_{(\alpha(s)}(X^i_{i'})_{;\alpha(t-s))} \text{ [5]}$$
$$(t \geqq s \geqq r),$$

putting $\delta^{\beta(r)}_{\alpha(r)} = \delta^{\beta_1}_{(\alpha_1}\delta^{\beta_2}_{\alpha_2}\cdots\cdots\delta^{\beta_r}_{\alpha_r)}$.

On the other hand, by the transformations (1.2) the partial derivatives $f_{;\alpha(r)} = \frac{\partial^r f}{\partial u^{\alpha_1}\partial u^{\alpha_2}\cdots\partial u^{\alpha_r}}$ $(r = 1, 2, \cdots\cdots, m)$ are transformed in the manners

$$f_{;\alpha'} = U^\alpha_{\alpha'} f_{;\alpha}, \quad f_{;\alpha'(2)} = U^{\alpha_1}_{\alpha'_1} U^{\alpha_2}_{\alpha'_2} f_{;\alpha(2)} + U^\beta_{\alpha'_1\alpha'_2} f_{;\beta}, \cdots\cdots\cdots$$

and in general

$$\text{(2.2)} \qquad f_{;\alpha'(s)} = \sum_{t=0}^{s} A^{\alpha(t)}_{\alpha'(s)} f_{;\alpha(t)},$$

so that, we have

$$\text{(2.3)} \qquad p^i_{\alpha'(s)} = \sum_{t=0}^{s} A^{\alpha(t)}_{\alpha'(s)} p^i_{\alpha(t)} \qquad (s = 0, 1, \cdots\cdots\cdots, m)$$

and consequently

$$dp^i_{\alpha'(s)} = \sum_{t=0}^{s} A^{\alpha(t)}_{\alpha'(s)} dp^i_{\alpha(t)} \qquad (s = 0, 1, \cdots\cdots\cdots, m),$$

putting $A^{\alpha(0)}_{\alpha'(0)} = 1$, $A^{\alpha(0)}_{\alpha'(1)} = A^{\alpha(0)}_{\alpha'(2)} = \cdots\cdots = 0$.

It is easily seen that the quantities $A^{\alpha(t)}_{\alpha'(s)}$ are polynomials of the derivatives $U^\beta_{\alpha'_1}, U^\beta_{\alpha'_1\alpha'_2}, \cdots\cdots, U^\beta_{\alpha_1'\alpha_2'\cdots\cdots\alpha'_{s-t+1}}$, and determined from the recurring formulae

$$\text{(2.4)} \qquad A^{\alpha(t)}_{\alpha'(s)} = \begin{cases} U^{\alpha(s)}_{\alpha'(s)} = U^{(\alpha_1}_{\alpha_1'} U^{\alpha_2}_{\alpha'_2}\cdots\cdots U^{\alpha_s)}_{\alpha'_s} & t = s, \\ A^{(\alpha\,(t-1)}_{\alpha'(s-1)} U^{\alpha_t)}_{\alpha'_s} + A^{\alpha(t)}_{\alpha'(s-1);\alpha'_s} & s > t > 1, \\ U^\alpha_{\alpha'_1\alpha'_2\cdots\cdots\alpha'_s} & t = 1. \end{cases}$$

Now, we shall prove the formula

$$(2.5)\qquad \binom{r}{t} A^{\alpha(r)}_{\alpha'(s)} = \sum_{u=t}^{s-r+t} \binom{s}{u} A^{\alpha(t)}_{(\alpha'(u)} A^{\alpha(r-t)}_{\alpha'(s-u))}\,.$$

Let ϕ and ψ be any two functions of the parameters u^{α}'s, then it follows that

$$(\phi\cdot\psi)_{;\alpha'(s)} = \sum_{u=0}^{s} \binom{s}{u} (\phi_{;(\alpha'(u)}\psi_{;\alpha'(s-u))}) \text{ [5]}$$

which is known as the generalized Leibniz formula. Therefore, by the transformation (1.2) we have

$$\sum_{r=0}^{s} (\phi\cdot\psi)_{;\alpha(r)} A^{\alpha(r)}_{\alpha'(s)} = \sum_{u=0}^{s} \binom{s}{u} \Big(\sum_{v=0}^{u} \sum_{l=0}^{s-u} \phi_{;\alpha(v)}\psi_{;\beta(l)} A^{\alpha(v)}_{(\alpha'(u)} A^{\beta(l)}_{\alpha'(s-u))} \Big)$$

or

$$\sum_{r=0}^{s} \sum_{t=0}^{r} \binom{r}{t} \phi_{;(\alpha(t)}\psi_{;\alpha(r-t))} A^{\alpha(r)}_{\alpha'(s)}$$

$$= \sum_{u=v}^{s} \sum_{v=0}^{s} \sum_{l=0}^{s-u} \binom{s}{u} \phi_{;\alpha(v)}\psi_{;\beta(l)} A^{\alpha(v)}_{(\alpha'(u)} A^{\beta(l)}_{\alpha'(s-u))}\,.$$

Comparing the coefficients of $\phi_{;\alpha(t)}\psi_{;\alpha(r-t)}$ on both sides, we have the formula (2.5).

Let us consider an operator $P^{\alpha(l)}$ applied to any quantity of the manifold $F^{(m)}_{n}$, that is

$$P^{\alpha(l)}(L) = \sum_{t=l}^{m} \binom{t}{l} L_{;}{}^{\alpha(l)\beta(t-l)}_{i}\, dp^{i}_{\beta(t-l)} \text{ [5]},$$

then we have

Theorem 1. *Under the transformation group* (1.1) *and* (1.2) *the operator* $P^{\alpha(l)}$ *has the law of transformation*:

$$(2.6)\qquad P^{\alpha(l)}(L) = \sum_{u=l}^{m} A^{\alpha(l)}_{\alpha'(u)} P^{\alpha'(u)}(L)\,.$$

Proof. If we effect the x-transformations alone, it follows that

$$P^{\alpha(l)}(L) = \sum_{t=l}^{m} \binom{t}{l} \sum_{s=t}^{m} L_{;}{}^{\gamma(s)}_{i'} \frac{\partial p^{i'}_{\gamma(s)}}{\partial p^{i}_{\alpha(l)\beta(t-l)}} dp^{i}_{\beta(t-l)}$$

$$= \sum_{t=l}^{m} \binom{t}{l} \sum_{s=t}^{m} L_{;}{}^{\gamma(s)}_{i'} \frac{s!\,(t-l)!}{t!\,(s-l)!} \delta^{\alpha(l)}_{(\gamma(l)} \frac{\partial p^{i'}_{\gamma(s-l))}}{\partial p^{i}_{\beta(t-l)}} dp^{i}_{\beta(t-l)}$$

$$= \sum_{s=l}^{m} \binom{s}{l} L_{;}{}^{\gamma(s)}_{i'} \sum_{t=l}^{s} \delta^{\alpha(l)}_{(\gamma(l)} \frac{\partial p^{i'}_{\gamma(s-l))}}{\partial p^{i}_{\beta(t-l)}} dp^{i}_{\beta(t-l)}$$

$$= \sum_{s=l}^{m} \binom{s}{l} L_{;}{}^{\alpha(l)\gamma(s-l)}_{i'} dp^{i'}_{\gamma(s-l)} = P^{\alpha(l)}(L)\,.$$

By effecting the u-transformations we have

$$\begin{aligned}
P^{\alpha(l)}(L) &= \sum_{t=l}^{m}\binom{t}{l}\sum_{s=t}^{m} L_{;}{}^{\gamma'(s)}_{i'} \frac{\partial p^{i'}_{\gamma'(s)}}{\partial p^{j'}_{\alpha(l)\beta(t-l)}} dp^{j'}_{\beta(t-l)} \\
&= \sum_{s=t}^{m}\sum_{t=l}^{m}\binom{t}{l} L_{;}{}^{\gamma'(s)}_{i'} A^{\alpha(l)\beta(t-l)}_{\gamma'(s)} dp^{i'}_{\beta(t-l)} \\
&= \sum_{s=t}^{m}\sum_{t=l}^{m} L_{;}{}^{\gamma'(s)}_{i'} \sum_{u=l}^{s-t+l}\binom{s}{u} A^{\alpha(l)}_{(\gamma'(u)}A^{\beta(t-l)}_{\gamma'(s-u))} dp^{i'}_{\beta(t-l)} \\
&= \sum_{s=l}^{m}\sum_{r=0}^{s-l}\sum_{u=l}^{s-r}\binom{s}{u} L_{;}{}^{\gamma'(s)}_{i'} A^{\beta(r)}_{\gamma'(s-u)} dp^{i'}_{\beta(r)} A^{\alpha(l)}_{\gamma'(u)} \\
&= \sum_{s=l}^{m}\sum_{u=l}^{s}\binom{s}{u} L_{;}{}^{\gamma'(s)}_{i'} \sum_{r=0}^{s-u} A^{\beta(r)}_{\gamma'(s-u)} dp^{i'}_{\beta(r)} A^{\alpha(l)}_{\gamma'(u)} \\
&= \sum_{u=l}^{m}\left(\sum_{s=u}^{m}\binom{s}{u} L_{;}{}^{\gamma'(s)}_{i'} dp^{i'}_{\gamma'(s-u)}\right) A^{\alpha(l)}_{\gamma'(u)} \\
&= \sum_{u=l}^{m} P^{\gamma'(u)}(L)\, A^{\alpha(l)}_{\gamma'(u)}\,.
\end{aligned}$$

Theorem 2. *If T^A be an intrinsic quantity of $F^{(m)}_n$ whose transformation law under the transformations* (1.1) *and* (1.2) *is $T^A = \mathfrak{A}^A_{A'}T^{A'}$, then $P^{\alpha(l)}(T^A)$ obeys the transformation law*

$$P^{\alpha(l)}(T^A) = \sum_{u=l}^{m} A^{\alpha(l)}_{\alpha'(u)}\mathfrak{A}^A_{A'}P^{\alpha'(u)}(T^{A'})\,. \tag{2.7}$$

Theorem 3. *If w^i be any vector of $F^{(m)}_n$ and L be a scalar of $F^{(m)}_n$, the quantity*

$$D_j(L)w^j = \sum_{r=0}^{m} L_{;}{}^{\beta(r)}_{j} w^j{}_{/\beta(r)} \ [5] \tag{2.8}$$

is invariant under the transformations (1.1) *and* (1.2), *where $/\beta$ denotes the total differentiation with respect to u^β, that is*

$$w^j{}_{/\beta} = w^j{}_{;\beta} + \sum_{s=0}^{m} w^j{}_{;}{}^{\alpha(s)}_{i} p^i_{\alpha(s)\beta}\,.$$

Proof. If we effect the x-transformations alone, it follows that

$$\begin{aligned}
\sum_{r=0}^{m} L_{;}{}^{\beta(r)}_{j} w^j{}_{/\beta(r)} &= \sum_{r=0}^{m}\sum_{s=r}^{m} L_{;}{}^{\alpha(s)}_{i'} \frac{\partial p^{i'}_{\alpha(s)}}{\partial p^j_{\beta(r)}} w^j{}_{/\beta(r)} \\
&= \sum_{r=0}^{m}\sum_{s=r}^{m} L_{;}{}^{\alpha(s)}_{i'}\binom{s}{r}\delta^{\beta(r)}_{(\alpha(r)}\left(\frac{\partial x^{i'}}{\partial x^j}\right)_{/\alpha(s-r))} w^j{}_{/\beta(r)} \\
&= \sum_{r=0}^{s}\sum_{s=0}^{m} L_{;}{}^{\alpha(s)}_{i'}\binom{s}{r}\left(\frac{\partial x^{i'}}{\partial x^j}\right)_{/\alpha(s-r)} w^j{}_{/\alpha(r)} \\
&= \sum_{s=0}^{m} L_{;}{}^{\alpha(s)}_{i'}\left(\frac{\partial x^{i'}}{\partial x^j} w^j\right)_{/\alpha(s)} = \sum_{s=0}^{m} L_{;}{}^{\alpha(s)}_{i'} w^{i'}{}_{/\alpha(s)}\,.
\end{aligned}$$

By effecting the u-transformations we have

$$\sum_{s=0}^{m} L_{;\ i'}^{\alpha(s)} w^{i'}{}_{/\alpha(s)} = \sum_{s=0}^{m}\sum_{r=s}^{m} L_{;\ i'}^{\beta(s)} \frac{\partial p^{i'}_{\beta'(r)}}{\partial p^{j'}_{\alpha(s)}} w^{j'}{}_{/\alpha(s)}$$

$$= \sum_{s=0}^{m}\sum_{r=s}^{m} L_{;\ i'}^{\beta'(r)} A^{\alpha(s)}_{\beta'(r)} w^{i'}{}_{/\alpha(s)}$$

$$= \sum_{r=0}^{m}\sum_{s=0}^{r} L_{;\ i'}^{\beta'(r)} A^{\alpha(s)}_{\beta'(r)} w^{i'}{}_{/\alpha(s)} = \sum_{r=0}^{m} L_{;\ i'}^{\beta'(r)} w^{i'}_{\beta'(r)} .$$

Theorem 4. *Let T^A be an intrinsic quantity of $F^{(m)}_n$ whose transformations law under the transformations* (1.1) *and* (1.2) *is $T^A = \mathfrak{A}^A_{A'} T^{A'}$, and v^i be any vector of $F^{(m)}_n$, then the quantities*

$$D^{\alpha(u)}_{j}(T^A)\, v^j = \sum_{r=u}^{m} \binom{r}{u} T^{A\ \alpha(u)\beta(r-u)}_{;\ \ j} v^j{}_{/\beta(r-u)}$$

$$(u = 1, 2, \cdots\cdots, m)$$

are transformed by the transformations (1.1) *and* (1.2) *in the manners*

$$D^{\alpha(u)}_{j}(T^A) v^j = \mathfrak{A}^A_{A'} \sum_{t=u}^{m} A^{\alpha(u)}_{\lambda'(t)} D^{\lambda'(t)}_{j'}(T^{A'})\, v^{j'} . \tag{2.9}$$

Proof. When we put $L = T^A$ and $w^j = \phi v^j$ into (2.8), ϕ being any scalar of $F^{(m)}_n$, one obtains the intrinsic quantity

$$D_j(T^A)\,\phi v^j = \sum_{r=0}^{m} T^{A\ \beta(r)}_{;\ \ j} (\phi v^j)_{/\beta(r)}$$

$$= \sum_{r=0}^{m} T^{A\ \beta(r)}_{;\ \ j} \sum_{u=0}^{r} \binom{r}{u} \phi_{/\beta(u)} v^j{}_{/\beta(r-u)}$$

$$= \sum_{u=0}^{m} \Big(\sum_{r=u}^{m} \binom{r}{u} T^{A\ \beta(r)}_{;\ \ j} v^j_{/\beta(r-u)} \Big) \phi_{/\beta(u)} ,$$

so that we can conclude (2.9) by virtue of (2.2).

§ 3. **Intrinsic operators and intrinsic PFAFFian form.** Let $f(x^i, p^i_{\alpha(1)}, \cdots\cdots, p^i_{\alpha(m)})$ be any scalar of $F^{(m)}_n$, then it is seen from theorem 2 and the above mensioned theorem that the quantities

$$\sum_{l=1}^{m} f_{/\alpha(l)} P^{\alpha(l)}_{j}(T^A) , \tag{3.1a}$$

$$\sum_{l=1}^{m} f_{/\alpha(l)} D^{\alpha(l)}_{j}(T^A)\, v^j \tag{3.1b}$$

are tensors of the same kind with T^A.

Suppose now that we have the quantity $G^\alpha_{\beta\gamma}$ whose transformation law under the transformations (1.1) and (1.2) is the same as that of coefficient of the affine connection of u-tensor, that is

$$U^{\alpha'}_{\beta\gamma} = G^{\alpha}_{\beta\gamma} U^{\alpha'}_{\alpha} - G^{\alpha'}_{\beta'\gamma'} U^{\beta'}_{\beta} U^{\gamma'}_{\gamma}\,, \tag{3.2}$$

then we can derive from $f_{/\alpha(1)}$, $f_{/\alpha(2)}, \cdots\cdots, f_{/\alpha(t)}$ the intrinsic quantities $f_{\beta(s)}$ $(s=1, 2, \cdots\cdots, t)$ in the following way.

First of all we see that $f_{\beta(1)} = f_{/\alpha_1}\delta^{\alpha_1}_{\beta_1}$ is an intrinsic quantity. Assume that there are the quantities $'K^{\alpha(l)}_{\beta(s-1)}$ $(l=1, 2, \cdots\cdots, s-1)$ such that $f_{\beta(s-1)} = \sum_{l=1}^{s-1} f_{/\alpha(l)} {'K}^{\alpha(l)}_{\beta(s-1)}$ is intrinsic, namely

$$\sum_{l=1}^{s-1} f_{/\alpha(l)} {'K}^{\alpha(l)}_{\beta s-1)} = \Big(\sum_{l=1}^{s-1} f_{/\alpha'(l)} {'K}^{\alpha'(l)}_{\beta'(s-1)}\Big) U^{\beta'(s-1)}_{\beta(s-1)}\,. \tag{3.3}$$

Differentiating the above equation with respect to u^{β_s} and symmetrizing with respect to the indices $\beta_1, \beta_2, \cdots\cdots, \beta_s$ one gets

$$\begin{aligned}
&\sum_{l=1}^{s-1} f_{/\alpha(l)(\beta_s} {'K}^{\alpha(l)}_{\beta(s-1))} + \sum_{l=1}^{s-1} f_{/\alpha(l)} {'K}^{\alpha(l)}_{(\beta(s-1)/\beta_s)} \\
&\quad = \Big(\sum_{l=1}^{s-1} f_{/\alpha'(l)(\beta'_s} {'K}^{\alpha'(l)}_{\beta'(s-1))} + \sum_{l=1}^{s-1} f_{/\alpha'(l)} {'K}^{\alpha(l)}_{(\beta'(s-1)/\beta'_s)}\Big) U^{\beta'(s)}_{\beta(s)} \\
&\quad + (s-1)\Big(\sum_{l=1}^{s-1} f_{/\alpha'(l)} {'K}^{\alpha'(l)}_{\beta'(s-1)}\Big) U^{\beta'_1}_{(\beta_1}\cdots\cdots U^{\beta'_{s-2}}_{\beta_{s-2}} U^{\beta'_{s-1}}_{\beta_{s-1}\beta_s)}\,.
\end{aligned} \tag{3.4}$$

Eliminating $U^{\beta'_{s-1}}_{\beta_{s-1}\beta_s}$ from the above equation and (3.2) putting $\alpha'=\beta'_{s-1}$, $\beta = \beta_{s-1}$, $\gamma = \beta_s$, we have under the consideration of (3.3)

$$\begin{aligned}
&\sum_{l=1}^{s-1} f_{/\alpha(l)(\beta_s} {'K}^{\alpha(l)}_{\beta(s-1))} + \sum_{l=1}^{s-1} f_{/\alpha(l)} {'K}^{\alpha(l)}_{(\beta(s-1)/\beta_s))} \\
&\qquad - (s-1)\sum_{l=1}^{s-1} f_{/\alpha(l)} {'K}^{\alpha(l)}_{(\beta(s-2)|\gamma|} G^{\gamma}_{\beta_{s-1}\beta_s)} \\
&\quad = \Big(\sum_{l=1}^{s-1} f_{/\alpha'(l)(\beta'_s} {'K}^{\alpha'(l)}_{\beta'(s-1))} + \sum_{l=1}^{s-1} f_{/\alpha'(l)} {'K}^{\alpha'(l)}_{(\beta'(s-1)/\beta'_s)} \\
&\qquad - (s-1)\sum_{l=1}^{s-1} f_{/\alpha'(l)} {'K}^{\alpha'(l)}_{(\beta'(s-2)|\gamma'|} G^{\gamma'}_{\beta'_{s-1}\beta'_s)}\Big) U^{\beta'(s)}_{\beta(s)}\,.
\end{aligned}$$

Therefore, if we put

$$\delta^{\alpha_l}_{(\beta_s} K^{\alpha(l-1)}_{\beta(s-1))} + {'K}^{\alpha(l)}_{(\beta(s-1)/\beta_s)} - (s-1)\,{'K}^{\alpha(l)}_{(\beta((s-2)|\gamma|} G^{\gamma}_{\beta_{s-1}\beta_s)} = {'K}^{\alpha(l)}_{\beta(s)}\,, \tag{3.5}$$

the quantity $f_{\beta(s)} = \sum_{l=1}^{s} f_{/\alpha(l)} {'K}^{\alpha(l)}_{\beta(s)}$ is also an intrinsic quantity. Thus we can see that there exist the intrinsic quantities $f_{\beta(s)} = \sum_{l=1}^{s} f_{/\alpha(l)} {'K}^{\alpha(l)}_{\beta(s)}$ $(s = 1, 2, \cdots\cdots, m)$ whose coefficients are determined from the recurring formula (3.5) putting ${'K}^{\alpha(1)}_{\beta(1)} = \delta^{\alpha}_{\beta_1}$.

It is easily seen from (3.5) that

$${'K}^{\alpha(s)}_{\beta(s)} = \delta^{\alpha(s)}_{\beta(s)}\,. \tag{3.6}$$

Moreover we can find the quantities $K^{\beta(s)}_{\gamma(t)}$ $(1 \leqq s \leqq t \leqq m)$ such that the relations

$$(3.7) \qquad \sum_{s=l}^{t} {}'K^{\alpha(l)}_{\beta(s)} K^{\beta(s)}_{\gamma(t)} = \delta^{\alpha(l)}_{\gamma(t)} \qquad (1 \leqq l \leqq t \leqq m)$$

hold. Specially, we have from (3.5), (3.6) and the above relations

$$(3.8) \qquad K^{\gamma(l)}_{\alpha(l)} = \delta^{\gamma(l)}_{\alpha(l)},\ K^{\gamma(l-1)}_{\alpha(l)} = -{}'K^{\gamma(l-1)}_{\alpha(l)} = (\text{linear form of } G^{\gamma}_{\alpha\beta}).$$

Therefore, from (3.1a) and (3.1b) we have the following theorems:

Theorem 5. *The operators* $\mathfrak{P}^{\beta(s)}$ *defined by*

$$\mathfrak{P}^{\beta(s)} = \sum_{t=s}^{m} K^{\beta(s)}_{\gamma(t)} P^{\gamma(t)} \qquad (s = 1, 2, \cdots\cdots, m)$$

are intrinsic operators, that is

$$(3.9) \qquad \mathfrak{P}^{\beta(s)}(L) = U^{\beta(s)}_{\beta'(s)} \mathfrak{P}^{\beta'(s)}(L),$$

and if T^A *be an intrinsic quantity of* $F^{(m)}_n$ *whose transformation law is* $T^A = \mathfrak{A}^A_{A'} T^{A'}$, *then* $\mathfrak{P}^{\beta(s)}(T^A)$ $(s = 1, 2, \cdots\cdots, m)$ *are also intrinsic quantitis whose transformation laws are*

$$(3.10) \qquad \mathfrak{P}^{\beta(s)}(T^A) = \mathfrak{A}^A_{A'} U^{\beta(s)}_{\beta'(s)} \mathfrak{P}^{\beta'(s)}(T^{A'}).$$

Theorem 6. *Let* T^A *be an intrinsic quantity of* $F^{(m)}_n$ *and* v^i *be any vector, then*

$$\mathfrak{D}^{\beta(l)}(T^A) v^i = \sum_{r=l}^{m} K^{\beta(l)}_{\alpha(r)} D^{\alpha(r)}_{i}(T^A) v^i \qquad (l = 1, 2, \cdots\cdots, m)$$

are also intrinsic quantities.

Moreover we have

Theorem 7. *If the* PFAFF*ian form* $\sum_{r=0}^{m} P^{J\alpha(r)}_{i} dp^i_{\alpha(r)}$ *defined in the manifold* $F^{(m)}_n$ *has the tensor character with respect to the index J, and* v^i *be any vector of* $F^{(m)}_n$,

$$P^{J\alpha(m)}_{i} dv^i + \sum_{r=0}^{m} P^{J\beta(r)}_{i\ ;\ j}{}^{\alpha(m)} v^j dp^i_{\beta(r)}$$

has also the tensor character with respect to the index J and $\alpha_{(m)}$. [5].

Suppose now that we have the quantity $\Gamma^i_{j\beta}$ whose transformation law under the transformations (1.1) and (1.2) is

$$\Gamma^i_{j\beta} = \Gamma^{i'}_{j'\beta'} X^i_{i'} X^{j'}_{j} U^{\beta'}_{\beta} - X^i_{j'k'} X^{j'}_{j} p^{k'}_{\beta'} U^{\beta'}_{\beta},$$

then we have

Theorem 8. *If* $\omega^i_{\beta(s)}(d) = \sum_{r=0}^{s} P^i_{\beta(s)}{}^{\alpha(r)}_{k} dp^k_{\alpha(r)}$ *be an intrinsic* PFAFF*ian* *form, then*

$$\omega^i_{\beta(s+1)}(d) = \sum_{r=0}^{s} P^i_{(\beta(s)}{}^{\alpha(r)}_{|k|}dp^k_{|\alpha(r)|\beta_{s+1})} + \sum_{r=0}^{s}(P^i_{(\beta(s)}{}^{\alpha(r)}_{|k|/\beta_{s+1})}$$
$$+ \Gamma^i_{j(\beta_{s+1}}P^j_{\beta(s))}{}^{\alpha(r)}_{k} - sG^{\gamma}_{(\beta_1\beta_2}P^i_{\beta_3\cdots\cdots\beta_{s+1})\gamma}{}^{\alpha(r)}_{k})\,dp^k_{\alpha(r)}$$

is also an intrinsic PFAFF*ian form.*

Proof. It has been proved in the work [5] of A. KAWAGUCHI and H. HOMBU that this theorem is true under the x-transformations alone. We shall now prove it under the u-transformations (x fixed). Since

$$P^i_{\beta(s)}{}^{\alpha(r)}_{k} = U^{\beta'(s)}_{\beta(s)}\sum_{t=r}^{s} P^i_{\beta'(s)}{}^{\alpha'(t)}_{j}\frac{\partial p^j_{\alpha'(t)}}{\partial p^k_{\alpha(r)}},$$

we have

$$\sum_{r=0}^{s} P^i_{\beta(s)}{}^{\alpha(r)}_{k}dp^k_{\alpha(r)\beta_{s+1}} = U^{\beta'(s)}_{\beta(s)}\sum_{r=0}^{s}\sum_{t=r}^{s} P^i_{\beta'(s)}{}^{\alpha'(t)}_{j}\frac{\partial p^j_{\alpha'(t)}}{\partial p^k_{\alpha(r)}}dp^k_{\alpha(r)\beta_{s+1}},$$

$$\sum_{r=0}^{s} P^i_{\beta(s)}{}^{\alpha(r)}_{k/\beta_{s+1}}dp^k_{\alpha(r)} = \sum_{r=0}^{s}\sum_{t=r}^{r}\left(U^{\beta'(s)}_{\beta(s)}P^i_{\beta'(s)}{}^{\alpha'(t)}_{j}\frac{\bar\partial p^j_{\alpha'(t)}}{\bar\partial p^k_{\alpha(r)}}\right)_{/\beta_{s+1}}dp^k_{\alpha(r)}$$
$$= U^{\beta'(s)}_{\beta(s)/\beta_{s+1}}\sum_{r=0}^{s}\sum_{t=r}^{s} P^i_{\beta'(s)}{}^{\alpha'(t)}_{j}\frac{\bar\partial p^j_{\alpha'(t)}}{\bar\partial p^k_{\alpha(r)}}dp^k_{\alpha(r)}$$
$$+ U^{\beta'(s)}_{\beta(s)}\sum_{r=0}^{s}\sum_{t=r}^{s} P^i_{\beta'(s)}{}^{\alpha'(t)}_{j/\beta'_{s+1}}U^{\beta'_{s+1}}_{\beta_{s+1}}\frac{\bar\partial p^j_{\alpha'(t)}}{\bar\partial p^k_{\alpha(r)}}dp^k_{\alpha(r)}$$
$$+ U^{\beta'(s)}_{\beta(s)}\sum_{r=0}^{s}\sum_{t=r}^{s} P^i_{\beta'(s)}{}^{\alpha'(t)}_{j}\left(\frac{\bar\partial p^j_{\alpha'(t)}}{\bar\partial p^k_{\beta(r)}}\right)_{/\beta_{s+1}}dp^k_{\alpha(r)}$$
$$= U^{\beta'(s)}_{\beta(s)/\beta_{s+1}}\sum_{t=0}^{s} P^i_{\beta'(s)}{}^{\alpha'(t)}_{j}dp^j_{\alpha'(t)}$$
$$+ U^{\beta'(s)}_{\beta(s)}U^{\beta'_{s+1}}_{\beta_{s+1}}\sum_{t=0}^{s} P^i_{\beta'(s)}{}^{\alpha'(t)}_{j/\beta'_{s+1}}dp^j_{\alpha'(t)}$$
$$+ U^{\beta'(s)}_{\beta(s)}\sum_{r=0}^{s}\sum_{t=r}^{s} P^i_{\beta'(s)}{}^{\alpha'(t)}_{j}\left(\frac{\bar\partial p^j_{\alpha'(t)}}{\bar\partial p^k_{\alpha(r)}}\right)_{/\beta_{s+1}}dp^k_{\alpha(r)},$$

$$\Gamma^i_{j\beta_{s+1}}P^j_{\beta(s)}{}^{\alpha(r)}_{k}dp^k_{\alpha(r)} = U^{\beta'(s)}_{\beta(s)}U^{\beta'_{s+1}}_{\beta_{s+1}}\Gamma^i_{j\beta'_{s+1}}P^j_{\beta'(s)}{}^{\alpha'(r)}_{k}dp^k_{\alpha'(r)}$$

and

$$\sum_{r=0}^{s} sG^{\gamma}_{(\beta_1\beta_2}P^i_{\beta_3\cdots\cdots\beta_{s+1})\gamma}{}^{\alpha(r)}_{k}dp^k_{\alpha(r)}$$
$$= s\,(G^{\gamma'}_{\beta'_1\beta'_2}U^{\gamma}_{\gamma'}U^{\beta'_1}_{(\beta_1}U^{\beta'_2}_{\beta_2} + U^{\gamma'}_{(\beta_1\beta_2}U^{\gamma}_{|\gamma'|})U^{\beta'_3}_{\beta_3}\cdots U^{\beta'_{s+1}}_{\beta_{s+1})}U^{\gamma'}_{\gamma}$$

$$\times \sum_{r=0}^{s}\sum_{t=r}^{s} P^{i}_{\beta'_3\cdots\cdots\beta'_{s+1}\nu'}{}^{\alpha'(t)}_{j} \frac{\partial p^{j}_{\alpha'(t)}}{\partial p^{k}_{\alpha(r)}} dp^{k}_{\alpha(r)}$$

$$= sU^{\beta'(s+1)}_{\beta(s+1)} \sum_{t=0}^{s} G^{\gamma'}_{\beta'_1\beta'_2} P^{i}_{\beta'_3\cdots\cdots\beta'_{s+1}\gamma'}{}^{\alpha'(t)}_{j} dp^{j}_{\alpha'(t)}$$

$$+ U^{\beta'(s)}_{(\beta(s)/\beta_{s+1})} \sum_{t=0}^{s} P^{i}_{\beta'(s)}{}^{\alpha'(t)}_{j} dp^{j}_{\alpha'(t)} \,.$$

On the other hand we have

$$U^{\beta'(s)}_{\beta(s)} \sum_{r=0}^{s}\sum_{t=r}^{s} P^{i}_{\beta'(s)}{}^{\alpha'(t)}_{j} \left(\frac{\partial p^{j}_{\alpha'(t)}}{\partial p^{k}_{\alpha(r)}} dp^{k}_{\alpha(r)\beta_{s+1}} + \left(\frac{\partial p^{j}_{\alpha'(t)}}{\partial p^{k}_{\alpha(r)}}\right)_{/\beta_{s+1}} dp^{k}_{\alpha(r)} \right)$$

$$= U^{\beta'(s)}_{\beta(s)} \sum_{r=0}^{s}\sum_{t=r}^{s} P^{i}_{\beta'(s)}{}^{\alpha'(t)}_{j} \left(A^{\alpha(r)}_{\alpha'(t)} \sum_{l=1}^{r+1} A^{\gamma'(l)}_{\alpha(r)\beta_{s+1}} dp^{j}_{\gamma'(l)} \right.$$

$$\left. + A^{\alpha(r)}_{\alpha'(t)/\beta_{s+1}} \sum_{l=1}^{r} A^{\gamma'(l)}_{\alpha(r)} dp^{j}_{\gamma'(l)} \right)$$

$$= U^{\beta'(s)}_{\beta(s)} \sum_{r=0}^{s}\sum_{t=r}^{s} P^{i}_{\beta'(s)}{}^{\alpha'(t)}_{j} \left(A^{\alpha(r)}_{\alpha'(t)} \sum_{l=1}^{r+2} A^{\gamma'(l-1)}_{\alpha(r)} U^{\gamma'_l}_{\beta_{s+1}} dp^{j}_{\gamma'(l)} \right.$$

$$\left. + A^{\alpha(r)}_{\alpha'(t)} \sum_{l=1}^{r} A^{\gamma'(l)}_{\alpha(r)/\beta_{s+1}} dp^{j}_{\gamma'(l)} + A^{\alpha(r)}_{\alpha'(t)/\beta_{s+1}} \sum_{l=1}^{r} A^{\gamma'(l)}_{\alpha(r)} dp^{j}_{\gamma'(l)} \right)$$

$$= U^{\beta'(s)}_{\beta(s)} \sum_{l=1}^{s+1}\sum_{t=r}^{s}\sum_{r=l-1}^{s} P^{i}_{\beta'(s)}{}^{\alpha'(t)}_{j} A^{\alpha(r)}_{\alpha'(t)} A^{\gamma'(l-1)}_{\alpha(r)} U^{\gamma'_l}_{\beta_{s+1}} dp^{j}_{\gamma'(l)}$$

$$+ U^{\beta'(s)}_{\beta(s)} \sum_{l=1}^{s}\sum_{t=r}^{s}\sum_{r=l}^{s} P^{i}_{\beta'(s)}{}^{\alpha'(t)}_{j} \left(A^{\alpha(r)}_{\alpha'(t)} A^{\gamma'(l)}_{\alpha(r)/\beta_{s+1}} + A^{\alpha(r)}_{\alpha'(t)/\beta_{s+1}} A^{\gamma'(l)}_{\alpha(r)}\right) dp^{j}_{\gamma'(l)}$$

$$= U^{\beta'(s)}_{\beta(s)} \sum_{l=1}^{s+1}\sum_{t=l-1}^{s}\sum_{r=l-1}^{t} P^{i}_{\beta'(s)}{}^{\alpha'(t)}_{j} A^{\alpha(r)}_{\alpha'(t)} A^{\gamma'(l-1)}_{\alpha(r)} U^{\gamma'_l}_{\beta_{s+1}} dp^{j}_{\gamma'(l)}$$

$$= U^{\beta'(s)}_{\beta(s)} \sum_{l=1}^{s+1}\sum_{t=l-1}^{s} P^{i}_{\beta'(s)}{}^{\alpha'(t)}_{j} \delta^{\gamma'(l-1)}_{\alpha'(t)} U^{\gamma'_l}_{\beta_{s+1}} dp^{j}_{\gamma'(l)}$$

$$= U^{\beta'(s)}_{\beta(s)} U^{\beta_{s+1}}_{\beta_{s+1}} \sum_{l=0}^{s} P^{i}_{\beta'(s)}{}^{\gamma'(l)}_{j} dp^{j}_{\gamma'(l)\beta'_{s+1}} \,,$$

consequently, we have the intrinsic PFAFFian form

$$\omega^{i}_{\beta(s+1)}(d) = \sum_{r=0}^{s+1} P^{i}_{\beta(s+1)}{}^{\alpha(r)}_{k} dp^{k}_{\alpha(r)} \,,$$

putting

$$\begin{aligned} & P^{i}_{\beta(s+1)}{}^{\alpha(s+1)}_{k} = P^{i}_{(\beta(s)}{}^{\alpha(s)}_{k} \delta^{\alpha_{s+1}}_{\beta_{s+1})} \,, \\ (3.11) \qquad & P^{i}_{\beta(s+1)}{}^{\alpha(r)}_{k} = P^{i}_{(\beta(s)}{}^{\alpha(r-1)}_{|k|} \delta^{\alpha_r}_{\beta_{s+1})} + P_{i(\beta(s)}{}^{\alpha(r)}_{|k|/\beta_{s+1})} \\ & \qquad + \Gamma^{i}_{j(\beta_{s+1}} P^{j}_{\beta(s))}{}^{\alpha(r)}_{k} - sG^{\gamma}_{(\beta_{s+1}\beta_s} P^{i}_{\beta_{s-1}\cdots\cdots\beta_1)\gamma}{}^{\alpha(r)}_{k} \,. \end{aligned}$$

§ 4. ***K*–ple integral of the *m*–th order.** We shall now proceed to discuss the geometry of the *K*–ple integral of the *m*–th order:

$$(4.1) \qquad \int_{(K)} F(x^i, p^i_{a(1)}, \cdots\cdots, p^i_{a(m)})\, du^1 \cdots\cdots du^K \qquad (m > 1)$$

by using of the results obtained in the preceding paragraphs, where the function $F(x^i, p^i_{a(1)}, \cdots\cdots, p^i_{a(m)})$ is differentiable to sufficient order with respect to its arguments.

If the integral (4.1) be regarded as defining a measure of K-dimensional surface in an n-dimensional manifold, it is adequate to suppose that the integral (4.1) is invariant under any parameter transformations. In order this it is necessary and sufficient that the function F is transformed under the parameter transformations (1.2) in the manner

$$(4.2) \qquad F(x^i, p^i_{a'(1)}, \cdots\cdots, p^i_{a'(m)}) = \Delta F(x^i, p^i_{a(1)}, \cdots\cdots, p^i_{a(m)}), \ \Delta = |U^a_{a'}|.$$

From this one has the well known relations [5]

$$(4.3a) \qquad \sum_{s=1}^{m} s p^i_{\gamma\beta(s-1)} F;^{a\beta(s-1)}_{i} = \delta^a_\gamma F,$$

$$(4.3b) \qquad \sum_{s=t}^{m} \binom{s}{t} p^i_{\gamma\beta(s-t)} F;^{a(t)\beta(s-t)}_{i} = 0 \qquad (m \geq t > 1).$$

When $t = m$, (4.3b) becomes

$$F;^{a(m)}_{i} p^i_\gamma = 0 \qquad (\gamma = \dot{1}, \dot{2}, \cdots\cdots, \dot{K}).$$

Differentiating with respect to $p^j_{\lambda(m)}$ we have

$$F;^{a(m)}_{i};^{\lambda(m)}_{j} p^i_\gamma = 0 \qquad (\gamma = \dot{1}, \dot{2}, \cdots\cdots, \dot{K}),$$

so that

$$(4.4) \qquad F;^{(a(m)}_{[i_1};^{(\lambda(m)}_{[j_1} F;^{\beta(m)}_{i_2};^{\mu(m)}_{j_2} \cdots\cdots F;^{\gamma(m))}_{i_{n-K}]};^{\nu(m))}_{j_{n-K}]} p^{i_1}_\gamma = 0$$
$$(\gamma = \dot{1}, \dot{2}, \cdots\cdots, \dot{K}),$$

$$(4.5) \qquad F;^{(a(m)}_{[i_1};^{(\lambda(m)}_{[j_1} F;^{\beta(m)}_{i_2};^{\mu(m)}_{j_2} \cdots\cdots F;^{\gamma(m))}_{i_{n-K}]};^{\nu(m))}_{j_{n-K}]} p^{j_1}_\gamma = 0$$
$$(\gamma = \dot{1}, \dot{2}, \cdots\cdots, \dot{K}).$$

On the other hand it is evident that

$$(4.6) \qquad \varepsilon_{i_1 i_2 \cdots\cdots i_{n-K} i_{n-K+1} \cdots\cdots i_n} p^{i_{n-K+1}}_{\dot{1}} \cdots\cdots p^{i_n}_{\dot{K}} p^{i_1}_\gamma = 0$$
$$(\gamma = \dot{1}, \dot{2}, \cdots\cdots, \dot{K}),$$

where $\varepsilon_{i_1 i_2 \cdots\cdots i_n} = n!\,\delta^1_{[i_1} \cdots \delta^n_{i_n]}$.

We can see from (4.4), (4.5) and (4.6) that there is one system of the quantities $\rho^{(a_1 a_2 \cdots\cdots a_N)(\lambda_1 \lambda_2 \cdots\cdots \lambda_N)}$ such that

$$F;^{(a(m)}_{[i_1};^{(\lambda(m)}_{[j_1} \cdots\cdots F;^{a(m))}_{i_{n-K}]};^{\nu(m))}_{j_{n-K}]} = \varepsilon_{i_1 \cdots\cdots i_n} \varepsilon_{j_1 \cdots\cdots j_n}$$
$$\times\, p^{i_{n-K+1}}_{\dot{1}} \cdots\cdots p^{i_n}_{\dot{K}} p^{j_{n-K+1}}_{\dot{1}} \cdots\cdots p^{j_n}_{\dot{K}} \rho^{(a_1 \cdots\cdots a_N)(\lambda_1 \cdots\cdots \lambda_N)},$$

where $(\alpha_1\, \alpha_2 \cdots\cdots \alpha_N)$ and $(\lambda_1\, \lambda_2 \cdots\cdots \lambda_N)$ represent $(\alpha_1 \cdots \alpha_m \beta_1 \cdots \beta_m \cdots \gamma_1 \cdots \gamma_m)$ and $(\lambda_1 \cdots \lambda_m \mu_1 \cdots \mu_m \cdots \nu_1 \cdots \nu_m)$ respectively, and consequently $N = m(n-K)$.

It is easily seen from the definition that under the transformations (1.1) and (1.2) the quantity $\rho^{\alpha(N)\lambda(N)} = \rho^{(\alpha_1 \cdots\cdots \alpha_N)(\lambda_1 \cdots\cdots \lambda_N)}$ obeys the transformation law

$$\rho^{\alpha'(N)\lambda'(N)} = \Delta^{(m-K)-2} D^2 \rho^{\alpha(N)\lambda(N)} U^{\alpha'(N)}_{\alpha(N)} U^{\lambda'(N)}_{\lambda(N)}$$

or

$$\rho^{\alpha'(N)\lambda'(N)} = \Delta^{(n-K)-2} D^2 \sum_{\alpha(N),\lambda(N)} \rho^{\alpha(N),\lambda(N)}\, {}'U^{\alpha'(N)}_{\alpha(N)}\, {}'U^{\lambda'(N)}_{\lambda(N)}$$

where $\sum\limits_{\alpha(N),\lambda(N)}$ denotes the summation for the all different combinations of N indices $\alpha_1, \alpha_2, \cdots\cdots, \alpha_N$ and the all different combinations of N indices $\lambda_1, \lambda_2, \cdots\cdots, \lambda_N$, and ${}'U^{\alpha'(N)}_{\alpha(N)}$ denotes $\dfrac{N!}{l_1!\, l_2! \cdots l_t!}\, U^{\alpha'(N)}_{\alpha(N)}$, when $\alpha_1, \alpha_2, \cdots \cdots, \alpha_N$ consist of $l_1, l_2, \cdots\cdots, l_t$ blocks of the same indices.

Consequently, the $\binom{K+N-1}{N}$-rowed determinant $\rho = |\rho^{\alpha(N)\lambda(N)}|$ is transformed by the transformations (1.1) and (1.2) as follows:

$$\rho' = \Delta^x D^y |{}'U^{\alpha'(N)}_{\alpha(N)}|^2 \rho\,,$$

where $x = \binom{K+N-1}{N}(n-K-2)$, $y = 2\binom{K+N-1}{N}$ and $|{}'U^{\alpha'(N)}_{\alpha(N)}|$ represents the $\binom{K+N-1}{N}$-rowed determinant.

In the same manner it shoud be obtained that

$$\rho = \Delta^{-x} D^{-y}\, |{}'U^{\alpha(N)}_{\alpha'(N)}|\;\; \rho'\,,$$

so that $|{}'U^{\alpha'(N)}_{\alpha(N)}|^2 |{}'U^{\alpha(N)}_{\alpha'(N)}|^2 = 1$. Hence we can conclude that $|{}'U^{\alpha'(N)}_{\alpha(N)}|$ is a power of Δ multiplying a suitable constant, since $|{}'U^{\alpha'(N)}_{\alpha(N)}|$ is a homogeneous function of $U^{\alpha'}_{\alpha}$'s. Under some consideration we see that

$$|{}'U^{\alpha(N)}_{\alpha'(N)}| = \Delta^{\binom{K+N-1}{N-1}}$$

and consequently we have

$$\rho' = \Delta^r D^y \rho\,,$$

putting $\quad r = \binom{K+N-1}{N}(n-K-2) - 2\binom{K+N-1}{N-1}.$

When we put

$$P^{\alpha(N)\lambda(N)} = F^{-2\frac{N}{K}} \rho^{\frac{-2}{y}} \rho^{\alpha(N)\lambda(N)}$$

and denote by $P_{\alpha(N)\lambda(N)}$ the inverse of $P^{\alpha(N)\lambda(N)}$, that is $P^{\alpha(N)\lambda(N)}P_{\alpha(N)\mu(N)} = \delta^{\lambda(N)}_{\mu(N)}$, assuming that the $\binom{K+N-1}{N}$-rowed determinant $P = |P^{\alpha(N)\lambda(N)}|$ does not vanish, these are transformed by the transformations (I.1) and (1.2) in the manners

$$P^{\alpha'(N)\lambda'(N)} = P^{\alpha(N)\lambda(N)} U^{\alpha'(N)}_{\alpha(N)} U^{\lambda'(N)}_{\lambda(N)},$$
$$P_{\alpha'(N)\lambda'(N)} = P_{\alpha(N)\lambda(N)} U^{\alpha(N)}_{\alpha'(N)} U^{\lambda(N)}_{\lambda'(N)}.$$

§ 5. **Determination of the quantities $G^{\alpha}_{\beta\gamma}$ and $\Gamma^{i}_{j\gamma}$.** We shall now attempt to determine the quantity $G^{\alpha}_{\beta\gamma}$ as mentioned in § 3 in use of $P^{\alpha(N)\lambda(N)}$ and $P_{\alpha(N)\mu(N)}$. If we put

$$Q^{\lambda(N)}_{\mu(N)\nu} = P^{\lambda(N)\alpha(N)} (P_{\mu(N)\alpha(N)/\nu}),$$

under the transformation (1.1) and (1.2) it follows that

$$\begin{aligned} Q^{\lambda'(N)}_{\mu'(N)\nu'} &= P^{\lambda'(N)\alpha'(N)} (P_{\mu(N)\alpha(N)} U^{\mu(N)}_{\mu'(N)} U^{\alpha(N)}_{\alpha'(N)})_{/\nu'} \\ &= Q^{\lambda(N)}_{\mu(N)\nu} U^{\lambda'(N)}_{\lambda(N)} U^{\mu(N)}_{\mu'(N)} U^{\nu}_{\nu'} + P^{\lambda'(N)\alpha'(N)} (U^{\mu(N)}_{\mu'(N)/\nu'} P_{\mu(N)\alpha(N)} U^{\alpha(N)}_{\alpha'(N)} \\ &\quad + U^{\alpha(N)}_{\alpha'(N)/\nu'} P_{\mu(N)\alpha(N)} U^{\mu(N)}_{\mu'(N)}). \end{aligned}$$

Putting $\lambda'_1 = \mu'_1$, $\lambda'_2 = \mu'_2$, $\cdots\cdots$, $\lambda'_{N-1} = \mu'_{N-1}$, $\lambda'_N = \lambda'$, $\mu'_N = \mu'$ and contracting over the indices $\mu'_1, \mu'_2, \cdots\cdots, \mu'_{N-1}$ we have

$$\begin{aligned} Q^{\lambda'}_{\mu'\nu'} &= Q^{\lambda}_{\mu\nu} U^{\lambda'}_{\lambda} U^{\mu}_{\mu'} U^{\nu}_{\nu'} + NP^{(\mu'_1 \cdots\cdots \mu'_{N-1}\sigma')(\alpha'_1 \cdots\cdots \alpha'_{N-1}\lambda')} \\ &\quad \times P_{(\mu'_1 \cdots\cdots \mu'_{N-1}\mu')(\alpha'_1 \cdots\cdots \alpha'_{N-1}\rho')} U^{\rho'}_{\alpha} U^{\alpha}_{\sigma'\nu'} \\ &\quad + U^{(\mu'_1}_{\mu_1} U^{\mu'_2}_{\mu_2} \cdots\cdots U^{\mu'_{N-1}}_{\mu_{N-1}} U^{\lambda')}_{\mu} (U^{\mu_1}_{\mu'_1} \cdots\cdots U^{\mu_{N-1}}_{\mu'_{N-1}} U^{\mu}_{\mu'})_{/\nu'} \end{aligned}$$

or

$$\begin{aligned} (5.1) \qquad Q^{\lambda'}_{\mu'\nu'} &= Q^{\lambda}_{\mu\nu} U^{\lambda'}_{\lambda} U^{\mu}_{\mu'} U^{\nu}_{\nu'} + K^{\sigma'\lambda'}_{\mu'\rho'} U^{\rho'}_{\alpha} U^{\alpha}_{\sigma'\nu'} \\ &\quad + pU^{\lambda'}_{\mu} U^{\mu}_{\mu'\nu'} + q\delta^{\lambda'}_{\mu'} \partial_{\nu'} \log \varDelta, \end{aligned}$$

where we put $\qquad Q^{\lambda'}_{\mu'\nu'} = Q^{\mu'(N-1)\lambda'}_{\mu'(N-1)\mu'\nu'}$,

$$K^{\sigma'\lambda'}_{\mu'\rho'} = NP^{(\mu'_1 \cdots\cdots \mu'_{N-1}\sigma')(\alpha'_1 \cdots\cdots \alpha'_{N-1})\lambda'} P_{(\mu'_1 \cdots \cdot \mu'_{N-1}\mu')(\alpha'_1 \cdots\cdots \alpha'_{N-1}\rho')}$$

$$p = \frac{(K+2)\cdots(K+N)}{N!} \text{ and } q = \frac{(N-1)(K+2)\cdots(K+N-1)}{N!} \ (N \neq 2),$$

$$q = \frac{1}{2} \quad (N=2).$$

Moreover from (5.1) we have

$$(5.2) \qquad Q_{\mu'} = Q_{\mu} U^{\mu}_{\mu'} + K^{\sigma'\lambda'}_{\mu'\rho'} U^{\rho'}_{\alpha} U^{\alpha}_{\sigma'\lambda'} + (p+q)\, \partial_{\mu'} \log \varDelta,$$

putting $Q_{\mu'} = Q^{\lambda'}_{\mu'\lambda'}$.

By eliminating $\partial_{\nu'} \log \Delta$ from (5.1) and (5.2) we get

$$(5.3)\qquad (p+q)\, Q^{\lambda'}_{\mu'\nu'} - q\delta^{\lambda'}_{\mu'} Q_{\nu'} = [(p+q)\, Q^{\lambda}_{\mu\nu} - q\delta^{\lambda}_{\mu} Q_{\nu}]\, U^{\lambda'}_{\lambda} U^{\mu}_{\mu'} U^{\nu}_{\nu'} + [(p+q)\, \delta^{\omega'}_{\nu'} K^{\sigma'\lambda'}_{\mu'\rho'} + p\,(p+q)\, \delta^{\lambda'}_{\rho'} \delta^{\sigma'}_{\mu'} \delta^{\omega'}_{\nu'} - q\delta^{\lambda'}_{\mu'} K^{\sigma'\omega'}_{\nu'\rho'}]\, U^{\rho'}_{\alpha} U^{\alpha}_{\sigma'\omega'}.$$

Now we put

$$(p+q)\, K^{\sigma'\lambda'}_{\mu'\rho'} \delta^{\omega'}_{\nu'} + p(p+q)\delta^{\lambda'}_{\rho'} \delta^{\sigma'}_{\mu'} \delta^{\omega'}_{\nu'} - q\delta^{\lambda'}_{\mu'} K^{\sigma'\omega'}_{\nu'\rho'} = N^{\lambda'}_{\mu'\nu',\rho'}{}^{\sigma'\omega'}$$

and assume that the K^3-rowed determinant $|N^{\lambda'}_{\mu'\nu',\rho'}{}^{\sigma'\omega'}|$ is different from zero, then we can obtain the quantities $n^{\mu'\nu'}_{\lambda'},{}^{\alpha'\tau'}_{\beta'}$ such that

$$N^{\lambda'}_{\mu'\nu',\rho'}{}^{\sigma'\omega'} n^{\mu'\nu'}_{\lambda'},{}^{\alpha'}_{\beta'\tau'} = \delta^{\alpha'}_{\rho'} \delta^{\sigma'}_{\beta'} \delta^{\omega'}_{\tau'}.$$

Since $K^{\sigma'\lambda'}_{\mu'\rho'}$ has the tensor character with respect to its indices under the transformations (1.1) and (1.2), the quantities $N^{\lambda'}_{\mu'\nu',\rho'}{}^{\sigma'\omega'}$ and $n^{\mu'\nu'}_{\lambda'}{}^{\alpha'}_{\beta'\tau'}$ are tensors. Hence, if we put

$$n^{\mu'\nu'}_{\lambda'},{}^{\alpha'}_{(\beta'\tau')} ((p+q)\, Q^{\lambda'}_{\mu'\nu'} - q\delta^{\lambda'}_{\mu'} Q_{\nu'}) = G^{\alpha'}_{\beta'\tau'},$$

it is easily seen from (5.3) that the quantity $G^{\alpha'}_{\beta'\tau'}$ obeys the law of transformation:

$$G^{\alpha'}_{\beta'\tau'} = G^{\alpha}_{\beta\tau} U^{\alpha'}_{\alpha} U^{\beta}_{\beta'} U^{\tau}_{\tau'} + U^{\alpha'}_{\alpha} U^{\alpha}_{\beta'\tau'}.$$

Thus obtained $G^{\alpha'}_{\beta'\tau'}$ will play an important rôle in our theories.

Let us next consider the EULER vector which is concerned with the first variation of the integral (4.1):

$$(5.4)\qquad E_i(F) = \sum_{r=0}^{m} (-1)^r (F;{}^{\alpha(r)}_{i})_{/\alpha(r)},$$

It is seen from the first variation of (4.1) that the EULER vector E_i is transformed by the parameter transformations as follows:

$$(5.5)\qquad E_i(F') = \Delta E_i(F).$$

If in (5.4) F be substituted by $F^* = F\phi$, ϕ being any function of u's, we have

$$E_i(F\phi) = \sum_{r=0}^{m} E^{\alpha(r)}_{i} \phi_{/\alpha(r)},$$

where $E^{\alpha(r)}_{i}$ $(r = 0, 1, \cdots\cdots, m)$ are vectors of the form

$$E^{\alpha(r)}_{i} = \sum_{s=r}^{m} (-1)^s \binom{s}{r} (F;{}^{\alpha(r)}{}_{i}^{\beta(s-r)})_{/\beta(s-r)} \qquad (r = 0, 1, \cdots\cdots, m),$$

and called the SYNGE vectors. If we effect the parameter transformations, it follows that

$$E_i(F'\phi')=\sum_{s=0}^{m}E_i^{a'(s)}\phi_{/a'(s)}=\sum_{s=0}^{m}\sum_{r=0}^{s}E_i^{a(s)}A_{a'(s)}^{a(r)}\phi_{/a(r)}$$
$$=\sum_{r=0}^{m}\sum_{s=r}^{m}E_i^{a'(s)}A_{a'(s)}^{a(r)}\phi_{/a(r)}\,.$$

On the other hand by (5.5) we have

$$E_i(F'\phi')=\Delta E_i(F\phi)=\Delta\sum_{r=0}^{m}E_i^{a(r)}\phi_{/a(r)}\,.$$

Hence, the SYNGE vectors are transformed by the parameter transformations in the manners

$$E_i^{a(r)}=\Delta^{-1}\sum_{s=r}^{m}E_i^{a'(s)}A_{a'(s)}^{a(r)}\qquad(r=0,1,\cdots\cdots,m),$$

so that we can derive from the SYNGE vectors a system of the intrinsic vectors $\mathfrak{G}_i^{\beta(l)}\ (l=0,1,\cdots\cdots,m)$ as similar manner as theorem 5, that is

$$\mathfrak{G}_i^{\beta(l)}=-\frac{1}{F}\sum_{r=l}^{m}E_i^{a(r)}K_{a(r)}^{\beta(l)}\qquad(l=1,2,\cdots\cdots,m)\,.$$

By the definitions of $E_i^{a(r)}$ and $K_{a(r)}^{\beta(l)}$ it is seen that $\mathfrak{G}_i^{\beta(l)}$ is a quantity of $F_n^{(2m-1)}$ and the relations

$$\begin{aligned}&\mathfrak{G}_i^{\beta}p_\gamma^i=\delta_\gamma^\beta\,,\quad \mathfrak{G}_i^{\beta(l)}p_\gamma^i=0\qquad(l=2,3,\cdots\cdots,m)\ [5]\,,\\ (5.6)\qquad&(\delta_j^i-p_a^i\mathfrak{G}_j^a)\,p_\gamma^j=0,\quad(\delta_j^i-p_a^i\mathfrak{G}_j^a)\,E_i^\gamma=0\,,\\ &(\delta_j^i-p_a^i\mathfrak{G}_j^a)(\delta_k^j-p_\beta^j\mathfrak{G}_k^\beta)=(\delta_k^i-p_a^i\mathfrak{G}_k^a)\end{aligned}$$

hold.

We shall now go on to derive the quantity $\Gamma_{j\gamma}^i$ which obeys the transformation law as mentioned in § 3. When we put $T^A=F;_j^{\gamma(m)}$ and $l=m-1$ in theorem 6, one gets the intrinsic quantity

$$\frac{1}{F}\,(mK_{a(m-1)}^{\beta(m-1)}F;_j^{\gamma(m)};_i^{a(m-1)\beta}v^i{}_{/\beta}+K_{a(m-1)}^{\beta(m-1)}F;_j^{\gamma(m)};_i^{a(m-1)}v^i$$
$$+\,K_{a(m)}^{\beta(m-1)}F;_j^{\gamma(m)};_i^{a(m)}v^i)\,,$$

or by virtue of (3.8) we have

$$\begin{aligned}(5.7)\qquad&\frac{m}{F}\,F;_j^{\gamma(m)};_i^{\beta(m-1)\beta}v^i{}_{/\beta}+\frac{1}{F}\,F;_j^{\gamma(m)};_i^{\beta(m-1)}\\ &+\frac{m\,(m-1)}{2}\,F;_j^{\gamma(m)};_i^{\omega_1\omega_2(\beta_1\cdots\cdots\beta_{m-2}}G_{\omega_1\omega_2}^{\beta_{m-1})})\,v^i\,,\end{aligned}$$

where v^i is a vector of $F_n^{(m)}$. The $n\binom{K+m-1}{m}$-rowed matrix $(F;_j^{\gamma(m)};_i^{\beta(m)})$

has the rank $\binom{K+m-1}{m}(n-K)$ at most because of $F;^{\gamma(m)}_{j};^{\beta(m)}_{i}p^i_\gamma = 0$ $(\gamma = \dot{1}, \dot{2}, \cdots\cdots, \dot{K})$. Suppose that the matrix $(F;^{\gamma(m)}_{j};^{\beta(m)}_{i})$ is of rank $\binom{K+m-1}{m}(n-K)$, then by virtue of (5.6) it is seen that we can find the quantities $G^k_{\alpha(m)}{}^j_{\gamma(m)}$ such that

$$(5.8) \qquad G^k_{\alpha(m)}{}^j_{\gamma(m)}F;^{\gamma(m)}_{j};^{\beta(m)}_{i} = F(\delta^k_i - p^k_\alpha \mathfrak{G}^\alpha_i)\,\delta^{\beta(m)}_{\alpha(m)}\,.$$

We see that thus obtained $G^k_{\alpha(m)}{}^j_{\gamma(m)}$ obey the transformation law of the form

$$(5.9) \qquad G^k_{\alpha(m)}{}^j_{\gamma(m)} = G^{k'}_{\alpha'(m)}{}^{j'}_{\gamma'(m)}X^k_{k'}X^j_{j'}U^{\alpha'(m)}_{\alpha(m)}U^{\gamma'(m)}_{\gamma(m)} + R^a p^j_a\,.$$

We may write all system of the solutions of (5.8) in the forms

$$(5.10) \qquad G^k_{\alpha(m)}{}^j_{\gamma(m)} = g^k_{\alpha(m)}{}^j_{\gamma(m)} + \varphi^i_{\alpha(m)}{}^j_{\gamma(m)}\mathfrak{G}^a_i p^k_a + \psi^k_{\alpha(m)}{}^a_{\gamma(m)}p^j_a\,,$$

where $g^k_{\alpha(m)}{}^j_{\gamma(m)}$ and $\varphi^i_{\alpha(m)}{}^j_{\gamma(m)}$ are quantities of $F^{(m)}_n$ and $\psi^k_{\alpha(m)}{}^a_{\gamma(m)}$ are any quntities. Accordingly, it is known that the quantity $G^k_{\alpha(m)}{}^j_{\gamma(m)}T^A_j$ is intrinsic and is the same for all system of the solutions of (5.8), when T^A_j is an intrinsic quantity satisfying the relations $p^j_\gamma T^A_j = 0$ $(\gamma = \dot{1}, \dot{2}, \cdots\cdots, \dot{K})$, that is to say, the intrinsic quantity $G^k_{\alpha(m)}{}^j_{\gamma(m)}T^A_j$ is uniquely determined by the equations (5.8).

If (5.7) be multiplied by $G^k_{\alpha(m)}{}^j_{\gamma(m)}$ and summed for $\gamma_1, \gamma_2, \cdots\cdots, \gamma_m$ and j, we have the intrinsic quantity

$$m\,(\delta^k_i - p^k_a \mathfrak{G}^a_i)\,\delta^{\beta(m-1)\beta}_{\alpha(m)}v^i{}_{/\beta} + \frac{1}{F}\,G^k_{\alpha(m)}{}^j_{\gamma(m)}\,(F;^{\gamma(m)}_{j};^{\beta(m-1)}_{i}$$
$$+\frac{m\,(m-1)}{2}\,F;^{\gamma(m)}_{j};^{\omega_1\omega_2(\beta_1\cdots\cdots\beta_{m-2}}_{i}G^{\beta_m}_{\omega_1}{}^{1)}_{\omega_2})\,v^i\,.$$

Putting $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, $\cdots\cdots$, $\beta_{m-1} = \alpha_{m-1}$ and contracting over these indices, we get

$$(\delta^k_i - p^k_\beta \mathfrak{G}^\beta_i)\,\frac{(K+1)\cdots(K+m-1)}{(m-1)\,!}\,v^i{}_{/\alpha_m} + \frac{1}{F}\,G^k_{\alpha(m)}{}^j_{\gamma(m)}\,(F;^{\gamma(m)}_{j};^{\alpha(m-1)}_{i}$$
$$+\frac{m\,(m-1)}{2}\,F;^{\gamma(m)}_{j};^{\omega_1\omega_2(\alpha_1\cdots\cdots\alpha_{m-2}}_{i}G^{\alpha_m}_{\omega_1}{}^{1)}_{\omega_2})\,v^i\,.$$

Accordingly, if we put

$$\Lambda^k_{ia} = \frac{(m-1)\,!}{(K+1)\cdots(K+m-1)}\,\frac{1}{F}\,G^k_{\alpha(m-1)a}{}^j_{\gamma(m)}\,(F;^{\gamma(m)}_{j};^{\alpha(m-1)}_{i}$$
$$+\frac{m\,(m-1)}{2}\,F;^{\gamma(m)}_{j};^{\omega_1\omega_2(\alpha_1\cdots\cdots\alpha_{m-2}}_{i}G^{\alpha_m}_{\omega_1}{}^{1)}_{\omega_2}) \qquad (m>2)\,,$$

$$A^k_{ia} = \frac{1}{K+1}\,\frac{1}{F}\,G^k_{\alpha_1 a}{}^j_{\gamma(2)}\,F_{;}{}^{\gamma(2)}_{j}{}_{;}{}^{\alpha_1}_{i} + (F_{;}{}^{\gamma(2)}_{j}{}_{;}{}^{\omega_1\omega_2}_{i}G^{\alpha_1}_{\omega_1\omega_2}$$
$$+ 2\mathfrak{G}^{(\gamma_1}_{i}F_{;}{}^{\gamma_2)\alpha_1}_{j} - 2\mathfrak{G}_{[i}^{\alpha_1}F_{;}{}^{\gamma_1\gamma_2}_{j]}) \qquad (m=2),$$

we have the intrinsic quantity

$$(5.11) \qquad \Delta_a v^k = (\delta^k_i - p^k_\beta \mathfrak{G}^\beta_i)\, v^i{}_{/a} + A^k_{ia} v^i$$

which is the same for all systems of the solutions of (5.8). If w_i be any covariant vector of $F^{(m)}_n$ such that $w_i p^i_\gamma = 0$ $(\gamma = \dot{1}, \dot{2}, \cdots\cdots, \dot{K})$, one obtains from (5.11) the intrinsic derivative of w_i:

$$\Delta_a w_i = w_{i/a} - A^k_{ia} w_k$$

by means of $\Delta_a (v^i w_i) = (v^i w_i)_{/a}$. Accordingly, it is easily seen that the quantity

$$(5.12) \qquad \Delta_a F_{;}{}^{\alpha(m)}_{i} = \frac{1}{F}(F_{;}{}^{\alpha(m)}_{i}{}_{/a} - A^k_{ia}F_{;}{}^{\alpha(m)}_{k} + mG^{(\alpha_1}_{\beta}{}_{a}F_{;}{}^{\alpha_2\cdots\cdots\alpha_m)\beta}_{i}$$
$$- G^\beta_{\beta a}F_{;}{}^{\alpha(m)}_{i})$$

is intrinsic. Moreover, we see from (5.10) and the definition of A^k_{ia} that $A^k_{ia}F_{;}{}^{\alpha(m)}_{k}$ does not depend on $\mathfrak{G}^\alpha_{i'}$, so that (5.12) can be writen in the form

$$(5.13) \qquad \Delta_{(a}F_{;}{}^{\alpha(m))}_{i} = H^{\alpha(m)}_{a}{}_{i}{}^{\beta(m+1)}_{j}p^j_{\beta(m+1)} + P^{\alpha(m)}_{a}{}_{i}(x^j, p^j_{\beta(1)}, \cdots\cdots, p^j_{\beta(m)}).$$

When $m > 2$, we have from (4.3b) an identity

$$\sum_{s=m-1}^{m} \binom{s}{m-1} p^i_{a\gamma(s-m+1)}F_{;}{}^{\beta(m-1)\gamma(s-m+1)}_{i} = 0.$$

Differentiating this with respect to $p^j_{\alpha(m)}$ one obtains

$$F_{;}{}^{\alpha(m)}_{j}{}_{;}{}^{\beta(m-1)}_{i}p^i_a = -mF_{;}{}^{\alpha(m)}_{j}{}_{;}{}^{\beta(m-1)\gamma}_{i}p^i_{a\gamma},$$

and consequently it follows that

$$p^i_\gamma \Delta_a F_{;}{}^{\alpha(m)}_{i} = p^i_\gamma\,(F_{;}{}^{\alpha(m)}_{i}{}_{/a} - A^k_{ia}F_{;}{}^{\alpha(m)}_{k})$$
$$= -F_{;}{}^{\alpha(m)}_{i}p^i_{a\gamma} - \frac{(m-1)!}{(K+1)\cdots(K+m-1)}\,\frac{1}{F}\,G^k_{\beta(m-1)a}{}^j_{\gamma(m)}F_{;}{}^{\gamma(m)}_{i}{}_{;}{}^{\beta(m-1)}_{j}F_{;}{}^{\alpha(m)}_{k}p^i_\gamma$$
$$= -F_{;}{}^{\alpha(m)}_{i}p^i_{a\gamma} + \frac{m!}{(K+1)\cdots(K+m-1)}\,\frac{1}{F}\,G^k_{\beta(m-1)a}{}^j_{\gamma(m)}F_{;}{}^{\gamma(m)\beta(m-1)\beta}_{j}{}_{;}{}_{i}p^i_{\beta\gamma}F_{;}{}^{\alpha(m)}_{k}$$
$$= -F_{;}{}^{\alpha(m)}_{i}p^i_{a\gamma} + \frac{m!}{(K+1)\cdots(K+m-1)}(\delta^k_i - p^k_\lambda E^\lambda_i)\delta^{\beta(m-1)\beta}_{\beta(m-1)a}p^i_{\beta\gamma}F_{;}{}^{\alpha(m)}_{k}$$
$$= -F_{;}{}^{\alpha(m)}_{i}p^i_{a\gamma} + F_{;}{}^{\alpha(m)}_{i}p^i_{a\gamma} = 0.$$

When $m = 2$, it follows that

$$\begin{aligned} p^i_\gamma \varDelta_\alpha F_{;}{}^{\alpha(2)}_{i} &= p^i_\gamma \left(F_{;}{}^{\alpha(2)}_{i}{}_{|\alpha} - \varLambda^k_{i\alpha} F_{;}{}^{\alpha(2)}_{k}\right) \\ &= -F_{;}{}^{\alpha(2)}_{i} p^i_{\alpha\gamma} - \frac{1}{K+1}\frac{1}{F} G^k_{\beta_1 \alpha}{}^{j}_{\gamma(2)} \left(F_{;}{}^{\lambda(2)}_{j}{}_{;}{}^{\beta_1}_{i} p^i_\gamma + 2\delta^{(\gamma_1}_{\gamma} F_{;}{}^{\gamma_2)\beta_1}_{j} \right. \\ &\quad \left. - \delta^{\beta_1}_{\gamma} F_{;}{}^{\gamma_1\gamma_2}_{j}\right) F_{;}{}^{\alpha(2)}_{k}. \end{aligned}$$

On the other hand, differentiating the identity

$$F_{;}{}^{\beta_1}_{i} p^i_\gamma + 2F_{;}{}^{\beta_1\beta}_{i} p^i_{\beta\gamma} = \delta^{\beta_1}_{\gamma} F$$

with respect to $p^j_{\gamma(2)}$, we have

$$F_{;}{}^{\gamma(2)}_{j}{}_{;}{}^{\beta_1}_{i} p^i_\gamma + 2F_{;}{}^{\beta_1(\gamma_1}_{j} \delta^{\gamma_2)}_{\gamma} + 2F_{;}{}^{\gamma(2)}_{j}{}_{;}{}^{\beta_1\beta}_{i} p^i_{\beta\gamma} - \delta^{\beta_1}_{\gamma} F_{;}{}^{\gamma(2)}_{j} = 0$$

and consequently

$$\begin{aligned} p^i_\gamma \varDelta_\alpha F_{;}{}^{\alpha(2)}_{i} &= -F_{;}{}^{\alpha(2)}_{i} p^i_{\alpha\gamma} + \frac{2}{K+1}\frac{1}{F} G^k_{\beta_1 \alpha}{}^{j}_{\gamma(2)} F_{;}{}^{\gamma(2)}_{j}{}_{;}{}^{\beta_1\beta}_{i} p^i_{\beta\gamma} F_{;}{}^{\alpha(2)}_{k} \\ &= -F_{;}{}^{\alpha(2)}_{i} p^i_{\alpha\gamma} + \frac{2}{K+1}\frac{1}{F} \left(\delta^k_i - p^k_\lambda E^\lambda_i\right) \delta^{(\beta}_{\beta_1}\delta^{\beta)}_{\alpha} p^i_{\alpha\gamma} F_{;}{}^{\alpha(2)}_{k} \\ &= 0. \end{aligned}$$

Hence, we have from (5.13) the relations

$$p^i_\gamma P^{\alpha(m)}_{\alpha}{}_{i} = 0 \qquad (\gamma = \dot{1}, \dot{2}, \cdots\cdots, \dot{K}). \tag{5.14}$$

Suppose now that the rank of the $n\binom{K+m}{m+1}$-rowed matrix $(H^{\alpha(m)}_{\alpha}{}_{i}{}^{\beta(m+1)}_{j})$ is $(n-K)\binom{K+m}{m+1}$, then we can find the quantities ${}'H^{\alpha}_{\alpha(m)}{}^{i}{}^{k}_{\gamma(m+1)}$ such that

$${}'H^{\alpha}_{\alpha(m)}{}^{i}{}^{k}_{\gamma(m+1)} H^{\alpha(m)}_{\alpha}{}_{i}{}^{\beta(m+1)}_{j} = \left(\delta^k_j - p^k_\lambda \mathfrak{G}^\lambda_j\right) \delta^{\beta(m+1)}_{\gamma(m+1)}. \tag{5.15}$$

We may write all systems of the solutions of (5.15) in the form

$${}'H^{\alpha}_{\alpha(m)}{}^{i}{}^{k}_{\gamma(m+1)} = h^{\alpha}_{\alpha(m)}{}^{i}{}^{k}_{\gamma(m+1)} + \varphi^{\alpha}_{\alpha(m)}{}^{i}{}^{j}_{\gamma(m+1)} \mathfrak{G}^\lambda_j p^k_\lambda + \psi^{\alpha}_{\alpha(m)}{}^{\beta}{}^{k}_{\gamma(m+1)} p^i_\beta, \tag{5.16}$$

where $h^{\alpha(m)i}_{\alpha}{}^{k}_{\gamma(m+1)}$ and $\varphi^{\alpha}_{\alpha(m)}{}^{i}{}^{j}_{\gamma(m+1)}$ are quantities of $F^{(m)}_{n}$, and $\psi^{\alpha}_{\alpha(m)}{}^{\beta}{}^{k}_{\gamma(m+1)}$ are any quantities.

If (5.13) be multiplied by ${}'H^{\alpha}_{\alpha(m)}{}^{i}{}^{k}_{\gamma(m+1)}$ and contracted over the indices $i, \alpha, \alpha_1, \cdots\cdots, \alpha_m$, it is obtained the intrinsic quantity

$$\left(\delta^k_j - p^k_\lambda \mathfrak{G}^\lambda_j\right) p^j_{\gamma(m+1)} + {}'H^{\alpha}_{\alpha(m)}{}^{i}{}^{k}_{\gamma(m+1)} P^{\alpha(m)}_{\alpha}{}_{i} \tag{5.17}$$

and by (5.14) it is seen that the quantity ${}'H^{\alpha}_{\alpha(m)}{}^{i}{}^{k}_{\gamma(m+1)} P^{\alpha(m)}_{\alpha}{}_{i}$ is the same for all solutions of (5.15).

If (5.17) be multiplied by $(\delta^l_k - p^l_\mu \mathfrak{G}^\mu_k)$ and summed for k, we have, in consequence of (5.6) and (5.16),

$$(\delta^l_j - p^l_\lambda \mathfrak{G}^\lambda_j)[p^j_{\gamma(m+1)} + H^j_{\gamma(m+1)}(x^i, p^i_{\alpha(1)}, \cdots\cdots, p^i_{\alpha(m)})]$$

or

$$T^l_{\gamma(m+1)} = p^l_{\gamma(m+1)} + H^l_{\gamma(m+1)} - p^l_\lambda \chi^\lambda_{\gamma(m+1)}, \tag{5.18}$$

putting $H^k_{\gamma(m+1)} = h^\alpha_{\alpha(m)}{}^i{}^k_{\gamma(m+1)} P^{\alpha(m)}_\alpha{}_i$, $\chi^\lambda_{\gamma(m+1)} = \mathfrak{G}^\lambda_j(p^j_{\gamma(m+1)} + H^j_{\gamma(m+1)})$. It should be observed that $H^l_{\gamma(m+1)}$ is a quantity of $F^{(m)}_n$.

Let us now observe the following facts. If Q^A be any quantity of $F^{(m)}_n$ which is transformed by arbitrary transformations of parameters in the manner

$$Q^{A'}(x^i, p^i_{\alpha'(1)}, \cdots\cdots, p^i_{\alpha'(m)}) = R^{A'}(Q^A, U^\alpha_{\alpha'(1)}, \cdots\cdots, U^\alpha_{\alpha'(s)}),$$

$R^{A'}$ being a rational function of its arguments, then the relations

$$Q^A{}_;{}^{\alpha(m)}_i p^i_\gamma = 0 \qquad (\gamma = \dot{1}, \dot{2}, \cdots\cdots, \dot{K})$$

hold when $s<m$. On the other hand, since $A^{\alpha(t)}_{\alpha'(s)}$ $(t \leq s)$ are polynomials of $U^\alpha_{\alpha'(1)}$, $U^\alpha_{\alpha'(2)}$, $\cdots\cdots$, $U^\alpha_{\alpha'(s-t+1)}$, the equation of transformation of the quantity

$$F_;{}^{\alpha(r)}_i{}^{\beta(s-r)}{}_{/\beta(s-r-1)} \qquad (1 \leqq r < s \leqq m)$$

appearing in the course of formation of the Synge vector contains the derivatives of u^α's not more than the $(m-r)$-th order. And the quantity $K^{\beta(s)}_{\gamma(t)}$ $(1 \leqq s \leqq t \leqq m-1)$ determined from (3.5) and (3.7) has the transformation law:

$$K^{\beta'(s)}_{\alpha'(t)} = U^{\beta'(s)}_{\beta(s)} \sum_{r=s}^{t} A^{\gamma(r)}_{\alpha'(t)} K^{\beta(s)}_{\gamma(r)}$$

in which $U^\gamma_{\alpha(t-s+1)}$'s are contained as the highest derivatives of u^γ.

Let us now define the differential operator D_γ applied to the quantity Q^A as follows:

$$D_\gamma Q^A = Q^A{}_{;\gamma} + \sum_{s=0}^{m-1} Q^A{}_;{}^{\alpha(s)}_i p^i_{\alpha(s)\gamma} - Q^A{}_;{}^{\alpha(m)}_i (H^i_{\alpha(m)\gamma} - p^i_\lambda \chi^\lambda_{\alpha(m)\gamma})$$

or

$$D_\gamma Q^A = Q^A{}_{;\gamma} + \sum_{s=0}^{m-1} Q^A{}_;{}^{\alpha(s)}_i p^i_{\alpha(s)\gamma} - Q^A{}_;{}^{\alpha(m)}_i H^i_{\alpha(m)\gamma},$$

then it is also a quantity of $F^{(m)}_n$ and its transformation law under (1.1) and (1.2) is the same as that of $Q^A{}_{/\gamma}$. Consequently, if the quantities $K^{\beta(s)}_{\gamma(t)}$ and $\mathfrak{G}^{\alpha(r)}_i$ in which the operator D_γ is applied instead of the operator $/\gamma$ are denoted by $\mathring{K}^{\beta(s)}_{\gamma(t)}$ and $\mathring{\mathfrak{G}}^{\alpha(r)}_i$ respectively, these are quantities of $F^{(m)}_n$, and the transformation laws of $\mathring{K}^{\beta(s)}_{\gamma(t)}$ and $\mathring{\mathfrak{G}}^{\alpha(r)}_i$ are the same as that of $K^{\beta(s)}_{\gamma(t)}$ and $\mathfrak{G}^{\alpha(r)}_i$ respectively. Moreover the same relations as (5.6) hold in this case.

We can now find one and only one system of the quantities $\mathring{G}^k_{\alpha(m)}{}^j_{\gamma(m)}$ satisfying the equations

$$\mathring{G}^k_{\alpha(m)}{}^j_{\gamma(m)} F_{;}{}^{\gamma(m)}_{j}{}_{;}{}^{\beta(m)}_{i} = F\,(\delta^k_i - p^k_\alpha \mathring{\mathfrak{G}}^\alpha_i)\,\delta^{\beta(m)}_{\alpha(m)}\,,$$

$$\mathring{G}^k_{\alpha(m)}{}^j_{\gamma(m)} \mathring{\mathfrak{G}}^\alpha_j = 0 \qquad (\alpha = \dot{1}, \dot{2}, \cdots\cdots, \dot{K}),$$

since we have supposed that the rank of the matrix $(F_{;}{}^{\gamma(m)}_{j}{}_{;}{}^{\beta(m)}_{i})$ is $(n-K)\binom{K+m-1}{m}$. And it is easily seen that thus obtained $\mathring{G}^k_{\alpha(m)}{}^j_{\gamma(m)}$ is intrinsic and symmetric with respect to the rows $\underset{\alpha(m)}{k}$ and $\underset{\gamma(m)}{j}$, so that

$$\text{(5.19)} \qquad \mathring{G}^k_{\alpha(m)}{}^j_{\gamma(m)} \mathring{\mathfrak{G}}^\alpha_k = 0 \qquad (\alpha = \dot{1}, \dot{2}, \cdots\cdots, \dot{K}).$$

If we put

$$\mathring{\Lambda}^k_{i\alpha} = \frac{(m-1)}{(K+1)\cdots(K+m-1)}\frac{1}{F}\mathring{G}^k_{\alpha(m-1)\,\alpha}{}^j_{\gamma(m)}\,(F_{;}{}^{\gamma(m)}_{j}{}_{;}{}^{\alpha(m-1)}_{i}$$
$$+\frac{m\,(m-1)}{2}F_{;}{}^{\gamma(m)}_{j}{}_{;}{}^{\omega_1\omega_2(\alpha_1\cdots\cdots\alpha_{m-1}}_{i}\mathring{G}^{\alpha_m}_{\omega_1}{}^{)}_{\omega_2}) \qquad (m \geqq 2),$$

then

$$(\delta^k_i - p^k_\beta \mathring{\mathfrak{G}}^\beta_i)\,v^i{}_{/\alpha} + \mathring{\Lambda}^k_{i\alpha} v^i$$

is an intrinsic quantity as it is seen from (5.11).

When we put $v^i = p^i_\gamma v^\gamma$, by virtue of (5.6) it follows that $\mathring{\mathfrak{G}}^\gamma_i v^i = v^\gamma$, and consequently we have

$$\text{(5.20)} \qquad (\delta^k_i - p^k_\beta \mathring{\mathfrak{G}}^\beta_i)\,v^i{}_{/\alpha} + \mathring{\Lambda}^k_{i\alpha} v^i = v^k{}_{/\alpha} - p^k_\beta\,(v^\beta{}_{/\alpha} + \mathring{G}^\beta_{\alpha\gamma} v^\gamma)$$
$$+\,(p^k_\beta G^\beta_{\alpha\gamma}\mathring{\mathfrak{G}}^\gamma_i + p^k_\beta \mathring{\mathfrak{G}}^\beta_{i\,/\alpha} + \mathring{\Lambda}^k_{i\alpha})\,v^i$$

from which we see that

$$\text{(5.21)} \qquad \Delta_\alpha v^k = v^k{}_{/\alpha} + \Gamma^k_{i\alpha} v^i$$

defines an intrinsic derivative of the vector v^k, where we put

$$\Gamma^k_{i\alpha} = \mathring{\Lambda}^k_{i\alpha} + p^k_\beta \mathring{\mathfrak{G}}^\beta_{i/\alpha} + \mathring{G}^\beta_{\alpha\gamma} p^k_\beta \mathring{\mathfrak{G}}^\gamma_i\,.$$

From (5.5), (5.19) and (5.20) we have

$$\mathring{\mathfrak{G}}^\beta_k\,(v^k{}_{/\alpha} + \Gamma^k_{i\alpha} v^i) = v^\beta{}_{/\alpha} + G^\beta_{\alpha\gamma} v^\gamma\,,$$

Consequently, if one defines the intrinsic derivative along the K-dimensional surface $x^i = x^i(u^\alpha)$ by (5.21), putting

$p^i_{\alpha(s)} = \dfrac{\partial^s x^i}{\partial u^{\alpha_1}\cdots\partial u^{\alpha_s}}$ $(s = 0, 1, \cdots\cdots, m)$, then

$$\Delta_\alpha v^\beta \equiv \mathfrak{G}^\beta_k \Delta_\alpha v^k = v^\beta{}_{/\alpha} + \mathring{G}^\beta_{\alpha\gamma} v^\gamma$$

may be regarded as the intrinsic derivative induced on the K-dimensional surface. Therefore, the covariant differential of the vector v^k along a K-dimensional surface is given by

$$\delta_S v^k = dv^k + \Gamma^k_{j\gamma} v^j du^\gamma$$

and the induced covariant differential is given by

$$\delta_S v^\beta = du^\beta + \mathring{G}^\beta_{\alpha\gamma} v^\alpha du^\gamma ,$$

when we put $p^i_{\alpha(s)} = \dfrac{\partial^s x^i}{\partial u^{\alpha_1}\cdots\partial u^{\alpha_s}}$ $(s = 0, 1, \cdots\cdots, m)$.

In order to determine the base connections and the connections in $F^{(m)}_n$ we put

$$\mathring{\Gamma}^i_{j\gamma} = \Lambda^i_{j\gamma} + p^i_\beta D_\gamma \mathfrak{G}^\beta_j + \mathring{G}^\beta_{\alpha\gamma} p^i_\beta \mathfrak{G}^\alpha_j \quad [14][15],$$

then it is a quantity of $F^{(m)}_n$ and has the same transformation law as that of $\Gamma^i_{j\gamma}$ under the transformations (1.1) and (1.2), that is,

$$\mathring{\Gamma}^i_{j\gamma} = \mathring{\Gamma}^{i'}_{j'\gamma'} X^i_{i'} X^{j'}_j U^{\gamma'}_\gamma - X^i_{j'k'} X^{j'}_j p^{k'}_{\gamma'} U^{\gamma'}_\gamma , \tag{5.22}$$

as it is seen from the tensor character of (5.21).

§ 6. **Base connections in $F^{(m)}_n$ and covariant differentials.** In order to define the base connections in $F^{(m)}_n$, we shall introduce the intrinsic PFAFFian forms by means of theorem 8.

Since dx^i may be regarded as an intrinsic PFAFFian form,

$$\omega^i_{\beta(1)}(d) = P^i_{\beta(1)}{}^{\alpha(1)}_k dp^k_{\alpha(1)} + P^i_{\beta(1)k} dx^k$$

is an intrinsic PFAFFian form, where the coefficients $P^i_{\beta(1)}{}^{\alpha(1)}_k$ and $P^i_{\beta(1)k}$ are determined from the recurring formulae (3.11) as follows:

$$P^i_{\beta(1)}{}^{\alpha(1)}_k = \delta^i_k \delta^\alpha_{\beta_1} , \quad P^i_{\beta(1)k} = \mathring{\Gamma}^i_{k\beta_1} ,$$

so that

$$\omega^i_{\beta(1)}(d) = dp^i_{\beta(1)} + \mathring{\Gamma}^i_{k\beta_1} dx^k$$

is an intrinsic PFAFFian form of $F^{(m)}_n$. In general, we obtain the intrinsic PFAFFian forms

$$\omega^i_{\beta(s)}(d) = \sum_{r=0}^{s} P^i_{\beta(s)}{}^{\alpha(r)}_k dp^k_{\alpha(r)} \qquad (s = 0, 1, \cdots\cdots, m), \tag{6.1}$$

where the coefficients $P^{i}_{\beta(s)}{}^{\alpha(r)}_{k}$ are determined from the recurring formulae

$$P^{i}_{\beta(s)}{}^{\alpha(r)}_{k} = P^{i}_{(\beta(s-1)}{}^{\alpha(r-1)}_{|k|}\delta^{\alpha r}_{\beta_s)} + D_{(\beta_s}P^{i}_{\beta(s-1))}{}^{\alpha(r)}_{k}$$
$$+\Gamma^{i}_{j(\beta_s}P^{j}_{\beta(s-1))}{}^{\alpha(r)}_{k} - (s-1)\,G^{\gamma}_{(\beta_s\beta_{s-1}}P^{i}_{\beta_{s-2}\cdots\cdots\beta_1)\gamma}{}^{\alpha(r)}_{k}$$
$$(0 \leqq r \leqq s \leqq m),$$

putting $P^{i}_{\beta(0)}{}^{\alpha(0)}_{k} = \delta^{i}_{k}$.

It is evident from the above recurring formulae that the coefficients of the differential $dp^{k}_{\alpha(s)}$ in the Pfaffian form $\omega^{i}_{\beta(s)}(d)$, say $P^{i}_{\beta(s)}{}^{\alpha(s)}_{k}$, is of the form $\delta^{i}_{k}\delta^{\alpha(s)}_{\beta(s)}$, so that (6.1) becomes

$$\omega^{i}_{\beta(s)}(d) = dp^{i}_{\beta(s)} + \sum_{r=0}^{s-1} P^{i}_{\beta(s)}{}^{\alpha(r)}_{k}dp^{k}_{\alpha(r)} \qquad (s = 0, 1, \cdots\cdots, m). \tag{6.2}$$

Hence, we can define the base connetcions in $F^{(m)}_{n}$ by the equations

$$\omega^{i}_{\beta(s)}(d) = 0 \qquad (s = 0, 1, \cdots\cdots, m).$$

We shall next introduce a covariant differential of vector of $F^{(m)}_{n}$ by means of theorem 5.

We have seen in theorem 5 that the transformation law of the quantity $\mathfrak{P}^{\beta}(L)$ defined by

$$\mathfrak{P}^{\beta}(L) = \sum_{t=1}^{m} K^{\beta}_{\gamma(t)} \sum_{s=t}^{m} \binom{s}{t} L_{;}{}^{\gamma(t)\alpha(s-t)}_{k} dp^{k}_{\alpha(s-t)}$$

is

$$\mathfrak{P}^{\beta}(L) = U^{\beta}_{\beta'}\mathfrak{P}^{\beta'}(L),$$

when L is a quantity of $F^{(m)}_{n}$. If we put $L = \mathring{\Gamma}^{i}_{j\beta}$ into $\mathfrak{P}^{\beta}(L)$ and contract over the indix β, it follows from (5.22) that the quantity $\mathfrak{P}^{\beta}(\Gamma^{i}_{j\beta})$ is transformed by the transformations (1.1) and (1.2) in the manner

$$\mathfrak{P}^{\beta}(\mathring{\Gamma}^{i}_{j\beta}) = \mathfrak{P}^{\beta'}(\mathring{\Gamma}^{i'}_{j'\beta'})X^{i}_{i'}X^{j'}_{j} - KX^{i}_{j'k'}X^{j'}_{j}dx^{k'},$$

so that the Pfaffian form

$$\Gamma^{i}_{j} = \frac{1}{K}\mathfrak{P}^{\beta}(\mathring{\Gamma}^{i}_{j\beta}) \tag{6.3}$$

obeys the transformation law

$$\Gamma^{i}_{j} = X^{i}_{i'}X^{j'}_{j}\Gamma^{i'}_{j'} - X^{i}_{j'k'}X^{j'}_{j}dx^{k'}.$$

Consequently, if v^{i} be a vector of $F^{(m)}_{n}$,

$$\delta v^{i} = dv^{i} + \Gamma^{i}_{j}v^{j} \tag{6.4}$$

defines a covariant differential of the vector v^{i}.

We may write (6.4) in the form

$$(6.5) \qquad \delta v^i = dv^i + \sum_{s=0}^{m-1} C_j^i{}^{\beta(s)}_k v^j dp^k_{\beta(s)},$$

when we put

$$C_j^i{}^{\beta(s)}_k = \frac{1}{K}\sum_{t=1}^{m-s}\binom{t+s}{t}\mathring{K}^{\beta}_{\gamma(t)}\Gamma^i_{j\beta;}{}^{\gamma(t)\beta(s)}_k \qquad (s=0,1,\cdots\cdots,m-1).$$

On the other hand, we may also introduce another intrinsic differential of vector by means of theorem 7.

From (6.2) we have the intrinsic PFAFFian form

$$\omega^j_{\beta(m)}(d) = dp^j_{\beta(m)} + \sum_{s=0}^{m} P^j_{\beta(m)}{}^{\alpha(s)}_i dp^i_{\alpha(s)}.$$

Applying theorem 7 one obtains the intrinsic quantity

$$\delta^j_i \delta^{\alpha(m)}_{\beta(m)} dv^i + \sum_{s=0}^{m-1} P^j_{\beta(m)}{}^{\gamma(s)}_i{}_;{}^{\alpha(m)}_k v^k dp^i_{\gamma(s)}$$

from which we obtain the intrinsic differential of the vector v^i:

$$'\delta v^j = dv^j + \frac{m\,!}{K(K+1)\cdots(K+m-1)}\sum_{s=0}^{m-1} P^j_{\beta(m)}{}^{\gamma(s)}_i{}_;{}^{\beta(m)}_k v^k dp^i_{\gamma(s)}.$$

§ 7. Covariant derivatives, torsion tensors and curvature tensors.

When we put

$$\delta v^i = \sum_{s=0}^{m} (\nabla^{\beta(s)}_k v^i)\,\omega^k_{\beta(s)}(d),$$

it is obtained from (6.2) and (6.5) the recurring formulae for the covariant derivatives $\nabla^{\beta(s)}_k v^i\ (s=0,1,\cdots\cdots,m)$, that is

$$(7.1) \qquad \begin{aligned} &\nabla^{\beta(m)}_k v^i = v^i{}_;{}^{\beta(m)}_k, \\ &\nabla^{\alpha(t)}_j v^i = v^i{}_;{}^{\alpha(t)}_j + C^i_k{}^{\alpha(t)}_j v^k - \sum_{s=t+1}^{m} P^k_{\beta(s)}{}^{\alpha(t)}_j \nabla^{\beta(s)}_k v^i. \end{aligned}$$

We shall next determine the torsion tensors and the curvature tensors of $F^{(m)}_n$.

If δ_1 and δ_2 denote the intrinsic differential operators corresponding to the increments d_1 and d_2 respectively, we can find the torsion tensors $A^i_{\alpha(r)}{}^{\beta(p)}_j{}^{\gamma(q)}_k$ and $A^i_j{}^{\omega(q)}_k$ by means of the equations

$$\delta_1\omega^i_{\alpha(r)}(d_2) - \delta_2\omega^i_{\alpha(r)}(d_1) = \sum_{p=0}^{r}\sum_{q=0}^{r} A^i_{\alpha(r)}{}^{\beta(p)}_j{}^{\gamma(q)}_k \omega^j_{\beta(p)}(d_2)\omega^k_{\gamma(q)}(d_1)$$
$$+\sum_{p=0}^{r}\sum_{q=r+1}^{m} A^i_{\alpha(r)}{}^{\beta(p)}_j{}^{\gamma(q)}_k[\omega^j_{\beta(p)}(d_2)\,\omega^k_{\gamma(q)}(d_1)]^* \qquad (r=0,1,\cdots\cdots,m).$$

Indeed, we have

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*The symbol $[\omega^j_{\beta(p)}(d_2)\,\omega^k_{\gamma(q)}(d_1)]$ means $\omega^j_{\beta(p)}(d_2)\omega^k_{\gamma(q)}(d_1) - \omega^j_{\beta(p)}(d_1)\,\omega^k_{\gamma(q)}(d_2)$.

$$A^{i}_{\alpha(r)}{}^{\beta(r)\gamma(q)}_{j\;\;k} = C^{i\;\gamma(q)}_{j\;k}\delta^{\beta(r)}_{\alpha(r)} - C^{i\;\beta(r)}_{k\;j}\delta^{\gamma(q)}_{\alpha(r)} - \sum_{s=q+1}^{m-1} A^{i}_{\alpha(r)}{}^{\beta(r)\omega(s)}_{j\;\;l} P^{l}_{\omega(s)}{}^{\gamma(q)}_{k}$$

$$(r=m,\ m-1, \cdots, 0;\ q=m-1, \cdots, 0),$$

$$\begin{aligned} A^{i}_{\alpha(r)}{}^{\beta(p)\gamma(q)}_{j\;\;k} = {} & P^{i}_{\alpha(r)}{}^{\beta(p)}_{j}{}_{;}{}^{\gamma(q)}_{k} - P^{i}_{\alpha(r)}{}^{\gamma(q)}_{k}{}_{;}{}^{\beta(p)}_{j} \\ & + C^{i\;\gamma(q)}_{l\;k} P^{l}_{\alpha(r)}{}^{\beta(p)}_{j} - C^{i\;\beta(p)}_{l\;j} P^{l}_{\alpha(r)}{}^{\gamma(q)}_{k} \\ & - \sum_{t=p+1}^{r-1} A^{i}_{\alpha(r)}{}^{\delta(t)\gamma(q)}_{l\;\;k} P^{l}_{\delta(t)}{}^{\beta(p)}_{j} - \sum_{s=q+1}^{m-1} A^{i}_{\alpha(r)}{}^{\beta(p)\omega(s)}_{j\;\;l} P^{l}_{\omega(s)}{}^{\gamma(q)}_{k} \\ & - \sum_{t=p+1}^{r-1}\sum_{s=q+1}^{m-1} A^{i}_{\alpha(r)}{}^{\delta(t)\omega(s)}_{h\;\;l} P^{h}_{\delta(t)}{}^{\beta(p)}_{j} P^{l}_{\omega(s)}{}^{\gamma(q)}_{k} \\ & - A^{i\;\gamma(q)}_{l\;k} P^{l}_{\alpha(r)}{}^{\beta(p)}_{j} - \sum_{s=q+1}^{m-1} A^{i\;\omega(s)}_{l\;h} P^{l}_{\alpha(r)}{}^{\beta(p)}_{j} P^{h}_{\omega(s)}{}^{\gamma(q)}_{k} \end{aligned}$$

$$(r=m, \cdots, 1;\ p=r-1, \cdots, 0;\ q=m-1, \cdots, 0),$$

where we put $P^{l}_{\omega(s)}{}^{\gamma(q)}_{k} = 0$ $(s \leqq q)$ and $A^{i\;\gamma(q)}_{l\;k} = A^{i}_{\alpha(0)}{}^{\beta(0)\gamma(q)}_{l\;\;k}$.

Next, by means of the equation

$$\begin{aligned} [\delta_1\delta_2 - \delta_2\delta_1]v^i = {} & \sum_{t=0}^{m-1} R^{i\;\alpha(t)\beta(m)}_{l\;h\;\;k} [\omega^{k}_{\beta(m)}(d_1)\omega^{h}_{\alpha(t)}(d_2)]\, v^l \\ & + \sum_{s=0}^{m-1}\sum_{t=0}^{m-1} R^{i\;\alpha(t)\beta(s)}_{l\;h\;\;k}\omega^{k}_{\beta(s)}(d_1)\omega^{h}_{\alpha(t)}(d_2)\, v^l, \end{aligned}$$

we have the recurring formulae for the curvature tensors:

$$R^{i\;\alpha(m-1)\beta(m)}_{l\;h\;\;\;\;k} = C^{i\;\alpha(m-1)}_{l\;h}{}_{;}{}^{\beta(m)}_{k},$$

$$R^{i\;\alpha(t)\beta(m)}_{l\;h\;\;k} = C^{i\;\alpha(t)}_{l\;h}{}_{;}{}^{\beta(m)}_{k} - \sum_{s=t+1}^{m-1} R^{i\;\gamma(s)\beta(m)}_{l\;g\;\;k} P^{g}_{\gamma(s)}{}^{\alpha(t)}_{h}$$

$$(t=m-1,\ m-2, \cdots\cdots, 1, 0),$$

$$\begin{aligned} R^{i\;\beta(t)\alpha(s)}_{l\;h\;\;f} = {} & C^{i\;\beta(t)}_{j\;h}{}_{;}{}^{\alpha(s)}_{f} - C^{i\;\alpha(s)}_{j\;f}{}_{;}{}^{\beta(t)}_{h} + C^{i\;\alpha(s)}_{j\;f} C^{j\;\beta(t)}_{l\;h} \\ & - C^{i\;\beta(t)}_{j\;h} C^{j\;\alpha(s)}_{l\;f} + R^{i\;\alpha(s)\gamma(m)}_{l\;f\;\;k} P^{k}_{\gamma(m)}{}^{\beta(t)}_{h} \\ & - R^{i\;\beta(t)\gamma(m)}_{l\;h\;\;k} P^{k}_{\gamma(m)}{}^{\alpha(s)}_{f} + \sum_{r=s+1}^{m-1} R^{i\;\delta(r)\mu(m)}_{l\;g\;\;k} P^{k}_{\mu(m)}{}^{\beta(t)}_{h} P^{g}_{\delta(r)}{}^{\alpha(s)}_{f} \\ & - \sum_{r=t+1}^{m-1} R^{i\;\delta(r)\mu(m)}_{l\;g\;\;k} P^{k}_{\mu(m)}{}^{\alpha(s)}_{f} P^{g}_{\delta(r)}{}^{\beta(t)}_{h} \\ & + \sum_{r=s+1}^{m-1} R^{i\;\beta(t)\delta(r)}_{l\;h\;\;k} P^{k}_{\delta(r)}{}^{\alpha(s)}_{f} - \sum_{r=t+1}^{m-1} R^{i\;\delta(r)\alpha(s)}_{l\;k\;\;f} P^{k}_{\delta(r)}{}^{\beta(t)}_{h} \\ & - \sum_{r=s+1}^{m-1}\sum_{p=t+1}^{m-1} R^{i\;\gamma(p)\delta(r)}_{l\;g\;\;k} P^{k}_{\delta(r)}{}^{\alpha(s)}_{f} P^{g}_{\gamma(p)}{}^{\beta(t)}_{h} \end{aligned}$$

$$(t, s = m-1, m-2, \cdots\cdots, 1, 0).$$

§ 8. **Method of A. KAWAGUCHI.** In the case of one parameter

Prof. A. KAWAGUCHI [3] has introduced a base connection in the manifold of line-elements of higher order. We shall generalize this method to the manifold of surface-elements of higher order.

By theorem 5 the quantities

$$\frac{1}{F}\mathfrak{P}^{\gamma(s)}(F_{;}{}^{\alpha(m)}_{i}) = \frac{1}{F}\sum_{l=s}^{m}\overset{\circ}{K}{}^{\gamma(s)}_{\beta(l)}\sum_{t=l}^{m}\binom{t}{l}F_{;}{}^{\alpha(m)}_{i}{}_{;}{}^{\beta(l)\mu(t-l)}_{j}dp^{j}_{\mu(t-l)}$$

$$(s=1,2,\cdots\cdots,m)$$

are intrinsic. Putting $t-l=r$, we have the intrinsic quantities

$$(8.1)\qquad \frac{1}{F}\binom{m}{s}F_{;}{}^{\alpha(m)}_{i}{}_{;}{}^{\gamma(s)\mu(m-s)}_{j}dp^{j}_{\mu(m-s)}+\frac{1}{F}\sum_{r=0}^{m-s-1}M^{\alpha(m)}_{i}{}^{\gamma(s)\mu(r)}_{j}dp^{j}_{\mu(r)}$$

$$(s=1,2,\cdots\cdots,m),$$

where

$$M^{\alpha(m)}_{i}{}^{\gamma(s)\mu(r)}_{j}=\sum_{l=s}^{m-r}\overset{\circ}{K}{}^{\gamma(s)}_{\beta(l)}\binom{l+r}{l}F_{;}{}^{\alpha(m)}_{i}{}_{;}{}^{\beta(l)\mu(r)}_{j}.$$

Hence, we can derive from (8.1) the intrinsic PFAFFian forms

$$(8.2)\qquad \delta p^{k}_{\gamma(m-s)}=(\delta^{k}_{i}-p^{k}_{\alpha}\overset{\circ}{\mathfrak{G}}{}^{\alpha}_{j})dp^{j}_{\gamma(m-s)}+\sum_{r=0}^{m-s-1}N^{k}_{\gamma(m-s)}{}^{\beta(r)}_{j}dp^{j}_{\beta(r)}$$

$$(s=1,2,\cdots\cdots,m),$$

where

$$N^{k}_{\gamma(m-s)}{}^{\beta(r)}_{j}=\frac{1}{F}\binom{m+K-1}{s}^{-1}\overset{\circ}{G}{}^{k}_{\gamma(m-s)\nu(s)}{}^{i}_{\alpha(m)}M^{\alpha(m)}_{i}{}^{\nu(s)\beta(r)}_{j}.$$

Moreover we have the intrinsic quantity

$$\frac{1}{F}\delta F_{;}{}^{\alpha(m)}_{i}=\frac{1}{F}(dF_{;}{}^{\alpha(m)}_{i}-\Gamma^{j}_{i}F_{;}{}^{\alpha(m)}_{j})$$

from which one gets the intrinsic PFAFFian form

$$(8.3)\qquad \delta p^{k}_{\gamma(m)}=(\delta^{k}_{j}-p^{k}_{\alpha}\overset{\circ}{\mathfrak{G}}{}^{\alpha}_{j})dp^{j}_{\gamma(m)}+\sum_{t=0}^{m-1}N^{k}_{\gamma(m)}{}^{\beta(t)}_{j}dp^{j}_{\beta(t)},$$

where

$$N^{k}_{\gamma(m)}{}^{\beta(t)}_{j}=\frac{1}{F}G^{i}_{\alpha(m)}{}^{k}_{\gamma(m)}(F_{;}{}^{\alpha(m)}_{i}{}_{;}{}^{\beta(t)}_{j}-C^{l}_{i}{}^{\beta(t)}_{j}F_{;}{}^{\alpha(m)}_{l}).$$

We may difine the base connections in $F^{(m)}_{n}$ by the equations

$$\delta p^{k}_{\gamma(t)}=0\qquad\qquad (t=0,1,\cdots\cdots,m).$$

Using of (6.5), (8.2) and (8.3) we can determine the covariant derivatives $'\nabla^{\alpha(t)}_{j}v^{i}$ under the conditions $'\nabla^{\alpha(t)}_{j}v^{i}p^{j}_{\gamma}=0$ $(\gamma=\dot{1},\dot{2},\cdots\cdots,\dot{K})$, that is,

$$'\nabla^{\alpha(m)}_{j} v^i = v^i{}_{;}{}^{\alpha(m)}_{j} ,$$

$$'\nabla^{\alpha(t)}_{j} v^i = v^i{}_{;}{}^{\alpha(t)}_{j} + C^{i\alpha(t)}_{j\ j} v^k - \sum_{s=t+1}^{m} P^k_{\beta(s)}{}^{\alpha(t)}_{j} \nabla^{\beta(s)}_{k} v^i .$$

§ 9. **Method of D. D. KOSAMBI.** D. D. KOSAMBI has introduced a system of covariant derivatives in his work [2] on the path space of higher order by using of the special method. We shall now generalize this method in our metric space.

If v^j be a vector of $F^{(m)}_n$, the quantity $\overset{K}{\nabla}{}^{\alpha(m)}_{i} v^j = v^j{}_{;}{}^{\alpha(m)}_{i}$ is an intrinsic derivative.

Now we see that when $m > 1$,

$$\overset{K}{\nabla}{}^{\alpha(m-1)}_{i} v^j = \frac{m}{K+m-1} (\overset{K}{\nabla}{}^{\alpha(m-1)\sigma}_{i} \Delta_\sigma v^j - \Delta_\sigma \overset{K}{\nabla}{}^{\alpha(m-1)\sigma}_{i} v^j)$$
$$- \frac{m}{K+m-1} (\overset{K}{\nabla}{}^{\alpha(m-1)\sigma}_{i} \Gamma'^{j}_{l\sigma}) v^l$$

is an intrinsic quantity, because $\overset{K}{\nabla}{}^{\alpha(m-1)\sigma}_{i} \Gamma'^{j}_{l\sigma}$ is a tensor. Moreover we see that $\overset{K}{\nabla}{}^{\alpha(m-1)}_{i} v^j$ may be written in the form

$$\overset{K}{\nabla}{}^{\alpha(m-1)}_{i} v^j = v^j{}_{;}{}^{\alpha(m-1)}_{i} + S^{l\alpha(m-1)}_{i\ \beta(m)} \overset{K}{\nabla}{}^{\beta(m)}_{l} v^j \qquad (m>1), \tag{9.1}$$

where

$$S^{l\beta(m-1)}_{i\beta(m)} = \frac{m}{K+m-1} (\Gamma'^{l}_{i\beta_m} \delta^{\alpha(m-1)}_{\beta(m-1)} - G_{\beta_m} \delta^{\alpha(m-1)}_{\beta(m-1)} \delta^l_i$$
$$- (m-1)\, G^{(\alpha_{m-1}}_{\beta_m \beta_{m-1}} \delta^{\alpha(m-2))}_{\beta(m-2)} \delta^l_i) , \tag{9.2}$$

putting $G_{\beta_m} = G^{\lambda}_{\beta_m \lambda}$.

We shall prove that in general the representations

$$\overset{K}{\nabla}{}^{\alpha(r)}_{i} v^j = v^j{}_{;}{}^{\alpha(r)}_{i} + \sum_{s=r+1}^{m} S^{l\alpha(r)}_{i\beta(s)} \overset{K}{\nabla}{}^{\beta(s)}_{l} v^j \qquad (r = 1, 2, \cdots\cdots, m-1)$$

and

$$\overset{K}{\nabla}_i v^j = v^j{}_{;i} + \sum_{s=1}^{m} S^l_{i\beta(s)} \overset{K}{\nabla}{}^{\beta(s)}_{l} v^j + \frac{1}{K} \overset{K}{\nabla}{}^{\beta}_{i} \Gamma'^{j}_{l\beta} v^l$$

is true when we put

$$\overset{K}{\nabla}{}^{\alpha(r)}_{i} v^j = \frac{r+1}{K+r} \left(\overset{K}{\nabla}{}^{\alpha(r)\beta}_{i} \Delta_\beta - \Delta_\beta \overset{K}{\nabla}{}^{\alpha(r)\beta}_{i}\right) v^j$$
$$- (1-\delta^0_r) \frac{r+1}{K+r} \left(\overset{K}{\nabla}{}^{\alpha(r)\beta}_{i} \Gamma'^{j}_{l\beta}\right) v^l \quad (r = 0, 1, \cdots, m-1).$$

First of all we have

$$\overset{K}{\nabla}{}^{\alpha(m)}_{i}(D_\gamma v^j) - D_\gamma(\overset{K}{\nabla}{}^{\alpha(m)}_{i} v^j)$$
$$= v^j{}_{;}{}^{(\alpha(m-1)}_{i}\delta^{\alpha}_{\gamma}{}^{m)} - H^l_{\beta(m)\gamma;}{}^{\alpha(m)}_{i}\overset{K}{\nabla}{}^{\beta(m)}_{l} v^j$$

or by (9.1)

$$(9.3)\qquad \overset{K}{\nabla}{}^{\alpha(m)}_{i}(D_\gamma v^j) - D_\gamma(\overset{K}{\nabla}{}^{\alpha(m)}_{i} v^j)$$
$$= \delta^{(\alpha}_{\gamma}{}^{m}\overset{K}{\nabla}{}^{\alpha(m-1))}_{i} v^j + U^l_{\beta(m)\gamma}{}^{\alpha(m)}_{i}\overset{K}{\nabla}{}^{\beta(m)}_{l} v^j ,$$

where

$$(9.4)\qquad U^l_{\beta(m)\gamma}{}^{\alpha(m)}_{i} = -S^{l(\alpha(m-1)}_{i\,\beta(m)}\delta^{\alpha}_{\gamma}{}^{m)} - H^l_{\beta(m)\gamma;}{}^{\alpha(m)}_{i} ,$$

$H^l_{\beta(m)\gamma}$ being that of § 5.

Let us now assume that the representations of two kinds:

$$(9.5)\qquad \overset{K}{\nabla}{}^{\alpha(r)}_{i} v^j = v^j{}_{;}{}^{\alpha(r)}_{i} + \sum_{s=r+1}^{m} S^{k\alpha(r)}_{i\beta(s)}\overset{K}{\nabla}{}^{\beta(s)}_{k} v^j \quad (r=m, m-1, \cdots\cdots, t)$$

and

$$(9.6)\qquad \overset{K}{\nabla}{}^{\alpha(r+1)}_{i} D_\gamma v^j - D_\gamma \overset{K}{\nabla}{}^{\alpha(r+1)}_{i} v^j$$
$$= \delta^{(\alpha}_{\gamma}{}^{r+1}\overset{K}{\nabla}{}^{\alpha(r))}_{i} v^j + \sum_{s=r+1}^{m} U^l_{\beta(s)\gamma}{}^{\alpha(r+1)}_{i}\overset{K}{\nabla}{}^{\beta(s)}_{l} v^j$$
$$(r=m-1, m-2, \cdots\cdots, t)$$

are true, then, after some calculation we see that the equalities

$$\overset{K}{\nabla}{}^{\alpha(r-1)}_{i} v^j = v^j{}_{;}{}^{\alpha(r-1)}_{i} + \sum_{s=r}^{m} S^{l\alpha(r-1)}_{i\beta(s)}\nabla^{\beta(s)}_{l} v^j \qquad (r>1),$$
$$\overset{K}{\nabla}_i v^j = v^j{}_{;i} + \sum_{s=1}^{m} S^k_{i\beta(s)}\nabla^{\beta(s)}_{k} v^j + \frac{1}{K}(\nabla^{\beta}_{i}\Gamma^j_{l\beta})\, v^l$$

and

$$\overset{K}{\nabla}{}^{\alpha(r)}_{i} D_\gamma v^j - D_\gamma \overset{K}{\nabla}{}^{\alpha(r)}_{i} v^j = \delta^{(\alpha}_{\gamma}{}^{r}\nabla^{\alpha(r-1))}_{i} v^j + \sum_{s=r}^{m} U^l_{\beta(s)\gamma}{}^{\alpha(r)}_{i}\overset{K}{\nabla}{}^{\beta(s)}_{l} v^j$$

hold good, where the coefficients $S^{l\alpha(r-1)}_{i\beta(s)}$, $S^k_{i\beta(s)}$ and $U^l_{\beta(s)\gamma}{}^{\alpha(r)}_{i}$ are determined from (9.2), (9.4) and the recurring formulae

$$S^{l\alpha(r-1)}_{i\beta(m)} = \frac{r+1}{K+r}\Big(\sum_{t=r+1}^{m} S^{j\alpha(r-1)\gamma}_{i\omega(t)} U^l_{\beta(m)\gamma}{}^{\omega(t)}_{j}$$
$$- D_\gamma S^{l\alpha(r-1)\gamma}_{i\beta(m)} - H^l_{\beta(m)\gamma;}{}^{\alpha(r-1)\gamma}_{i}\Big) \qquad (r<m),$$

$$S^{l\alpha(r-1)}_{i\beta(r)} = \frac{r+1}{K+r}(S^{l\alpha(r-1)\gamma}_{i\beta(r)}{}_{\gamma} + \Gamma^l_{i\beta_r}\delta^{\alpha(r-1)}_{\beta(r-1)})$$

$$-(r-1)\,G^{(\alpha_r}{}_{\beta_r\beta_{r-1}}{}^{1}\delta^{\alpha(r-2)}_{\beta(r-2)}\delta^{l}_{i}-G_{\beta_r}\delta^{\alpha(r-1)}_{\beta(r-1)}\delta^{l}_{i})\,,$$

$$S^{l\alpha(r-1)}_{i\beta(s)}=\frac{r+1}{K+r}\,(S^{l\alpha(r-1)\gamma}_{i\beta(s)}{}_{\gamma}-D_{\gamma}S^{l\alpha(r-1)\gamma}_{i\beta(s)}+\sum_{t=r+1}^{s}S^{j\alpha(r-1)\gamma}_{i\omega(t)}U^{l}_{\beta(s)\gamma}{}^{\omega(t)}_{j})$$

$$(s=r+1,r+2,\cdots\cdots,m-1)$$

and

$$U^{l}_{\beta(m)\gamma}{}^{\alpha(r)}_{i}=\sum_{t=r+1}^{m}\delta^{j\alpha(r)}_{i\omega(t)}U^{l}_{\beta(m)\gamma}{}^{\omega(t)}_{j}-S^{l(\alpha(r-1)}_{i\ \beta(m)}\delta^{\alpha_r)}_{\gamma}$$
$$-H^{l}_{\beta(m)\gamma;}{}^{\alpha(r)}_{i}-D_{\gamma}S^{l\alpha(r)}_{i\beta(m-1)}\,,$$

$$U^{l}_{\beta(r)\gamma}{}^{\alpha(r)}_{i}=\Gamma^{l}_{i\gamma}\delta^{\alpha(r)}_{\beta(r)}-rG^{(\alpha_r}_{\beta_r\gamma}\delta^{\alpha(r-1))}_{\beta(r-1)}\delta^{l}_{i}+S^{l\alpha(r)}_{i\beta(r)\gamma}-S^{l(\alpha(r-1)}_{i\beta(r)}\delta^{\alpha_r)}_{\gamma}\,,$$

$$U^{l}_{\beta(s)\gamma}{}^{\alpha(r)}_{i}=\sum_{t=r+1}^{s}S^{j\alpha(r)}_{i\omega(t)}U^{l}_{\beta(s)\gamma}{}^{\omega(t)}_{j}-S^{l(\alpha(r-1)}_{i\ \beta(s)}\delta^{\alpha_r)}_{\gamma}$$
$$+S^{l\alpha(r)}_{i\beta(s)\gamma}-D_{\gamma}S^{l\alpha(r)}_{i\beta(s)}\qquad(s=r+1,\cdots\cdots,m-1)\,.$$

Consequently, we have the intrinsic PFAFFian forms of the second kind

$$\overset{K}{\delta}{}^{i}_{\beta(s)}=dp^{i}_{\beta(s)}-\sum_{r=0}^{s-1}S^{i\beta(r)}_{l\alpha(s)}dp^{l}_{\beta(r)}\qquad(s=0,1,\cdots\cdots,m)\,.$$

§ 10. **Metric tensors and metric connection.** If we put

$$G^{i}_{\alpha(m-1)\alpha}{}^{j}_{\beta(m-1)\beta}\overset{\circ}{\mathfrak{E}}{}^{\alpha(m-1)}_{i}\overset{\circ}{\mathfrak{E}}{}^{\beta(m-1)}_{j}=G_{\alpha\beta}$$

it is easily seen from (5.6), (5.9) and (5.10) that when $m>2$, $G_{\alpha\beta}$ is an intrinsic quantity of $F^{(m)}_{n}$ and is the same for all the solutions $G^{i}_{\alpha(m-1)\alpha}{}^{j}_{\beta(m-1)\beta}$ of the equations (5.8). When $m=2$, we put

$$\frac{1}{F}\,G^{i}_{\alpha(2)}{}^{j}_{\beta(2)}F_{;}{}^{\alpha(2)}_{i}F_{;}{}^{\beta(2)}_{j}=f$$

and derive the intrinsic vectors $\overset{\circ}{H}{}^{\alpha}_{i}$ from the scalar f as if we derive the intrinsic vectors $\overset{\circ}{E}{}^{\alpha}_{i}$ from the scalar F. If we put

$$(\delta^{i}_{a}-p^{i}_{\lambda}\overset{\circ}{\mathfrak{E}}{}^{\lambda}_{a})\,(\delta^{j}_{b}-p^{j}_{\mu}\overset{\circ}{\mathfrak{E}}{}^{\mu}_{b})\,G^{a}_{\alpha_1\alpha}{}^{b}_{\beta_1\beta}\overset{\circ}{H}{}^{\alpha_1}_{i}\overset{\circ}{H}{}^{\beta_1}_{j}=G_{\alpha\beta}\,,$$

this is an intrinsic quantity of $F^{(m)}_{n}$ and is the same for all the solutions $G^{i}_{\alpha(2)}{}^{j}_{\beta(2)}$.

Moreover, if we put $F^{\frac{2}{K}}G^{-\frac{1}{K}}G_{\alpha\beta}=g_{\alpha\beta}$ assuming that $G=|G_{\alpha\beta}|\neq 0$, then the measure of K-dimensional surface is given by

$$\int_{(K)}|g_{\alpha\beta}|^{\frac{1}{2}}du^{1}\cdots\cdots du^{K}\,.$$

Hence, it is adequate to take $g_{\alpha\beta}$ as the metric tensor on the K-dimensional surface, when we put $p^i_{\alpha(s)} = \partial^s x^i / \partial u^{\alpha_1} \cdots \partial u^{\alpha_s}$ $(s=0, \cdots, m)$.

Now we put

$$\frac{1}{F} g_{\alpha_1\beta_1} g_{\alpha_2\beta_2} \cdots\cdots g_{\alpha_m\beta_m} F_{;\,i}^{\,\alpha(m)}{}_{;\,j}^{\,\beta(m)} + g_{\alpha\beta} \mathfrak{F}^{\alpha}_{i} \mathfrak{F}^{\beta}_{j} = g_{ij} \ ,$$

and assume that the determinant $|g_{ij}|$ does not vanish, then g_{ij} is a tensor of $F^{(m)}_n$ and the relation

$$g_{ij} p^i_\alpha p^j_\beta = g_{\alpha\beta}$$

holds good, so that we may take g_{ij} as the metric tensor of $F^{(m)}_n$. If $g^{\alpha\beta}$ and g^{ij} be the inverses of $g_{\alpha\beta}$ and g_{ij} respectively, it is easily seen that

$$g^{ij} g_{\alpha\beta} \mathfrak{F}^{\beta}_{j} = p^i_\alpha, \quad g^{ij} \mathfrak{F}^{\alpha}_{i} \mathfrak{F}^{\beta}_{j} = g^{\alpha\beta} .$$

By the method of Prof. A. KAWAGUCHI [4] we obtain the metric connection:

$$dv^i + \frac{1}{2} (\Gamma^i_j - g^{ik} \Gamma^l_k g_{lj} + g^{ik} dg_{jk}) v^j = 0 .$$

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