ON THE THEORY OF A RHEONOMIC CARTAN SPACE

By

Michiaki KAWAGUCHI, Jr.

Introduction.

A CARTAN space having its origin in the CARTAN's paper "*Les espace de métriques fondés sur la notion d'aire*" was developed by many people, especially, L. BERWALD.[1] In the present paper we attempt to build up the geometry in this space from standpoint of the rheonomic theory. In this space, its $(n-1)$-dimensional area is assumed to be given *a priori* in such a way that it depends on a variation of time. The geometrical quantities of this space depend on $x^{\alpha}, t, u_{\alpha}, u_0$. If the time-area is independent of u_0, then in every moment this space reduces to a CARTAN space in ordinary sense, i.e. then this area is nothing but that of a CARTAN space. Since u_0 may be interpreted as a velocity of a small piece of hypersurface-element, we shall call u_0 *a velocity of the hypersurface-element.* As u_0 is not invariant under a rheonomic transformation, we introduce a invariant parameter v in place of u_0. This parameter v plays an important rôle in our theory. The form of fundamental function $L(x^{\alpha}, t, u_{\alpha}, u_0)$ is rewritten in $G(x^{\alpha}, t, u_{\alpha}, v)$ which is homogenous of degree one in u_{α} and lets us decide the base connection, the connection-parameters, the curvature tensors and identities of BIANCHI in our space.

§ 1. Fundamental function.

In an n-dimensional rheonomic manifold X_n with coordinates x^i, t, we consider a rheonomic hypersurface X_{n-1} given by

$$(1.1) \qquad x^{\alpha} = x^{\alpha}(v^1, v^2, \cdots, v^{n-1}, t) \qquad \alpha = 1, \cdots, n$$

and suppose that its measure of $(n-1)$-dimensional time-area in a

(1) L. BERWALD: Über die n-dimensionalen CARTANschen Räume und eine Normalform der zweiten Variation eines $(n-1)$-fachen Oberflächenintegrals, Acta Mathematica, 71 (1939), 191–248.

certain region and time interval is defined by the integral

$$(1.2)\quad 0=\int_{(n)}\cdots\int\varphi\left(x^{\alpha},\frac{\partial x^{\alpha}}{\partial v^{i}},\frac{\partial x^{\alpha}}{\partial t},t\right)dv^{1}\,dv^{2}\cdots dv^{n-1}\,dt,\ i=1,\cdots,n-1$$

extended over the region and time interval. Then we shall call the manifold a rheonomic CARTAN space. Since (1.2) must be invariant under the rheonomic transformation of parameters $\bar{v}^{i}=\bar{v}^{i}(v^{1},\cdots,v^{n-1},t)$, by using of RADON's[1] and VIVANTI's[2] theorem, φ may be written in the form $L(x^{\alpha},t,u_{\alpha},u_{0})$, which is positive and homogenous of degree one in u_{α}, u_{0}, where $u_{\alpha}=(-1)^{\alpha+1}\left|\frac{\partial x^{1}}{\partial v^{i}},\cdots,\frac{\partial x^{\alpha-1}}{\partial v^{i}},\frac{\partial x^{\alpha+1}}{\partial v^{i}},\cdots,\frac{\partial x^{n}}{\partial v^{i}}\right|$ and $u_{0}=-\frac{\partial x^{\alpha}}{\partial t}u_{\alpha}$.

Under a rheonomic transformation $\bar{x}=\bar{x}(x,t)$, u_{α} and u_{0} vary in the rule

$$(1.3)\qquad \bar{u}_{\alpha}=\left|\frac{\partial\bar{x}}{\partial x}\right|\frac{\partial x^{\beta}}{\partial\bar{x}^{\alpha}}u_{\beta},\qquad \bar{u}_{0}=\left|\frac{\partial\bar{x}}{\partial x}\right|\left(u_{0}+\frac{\partial x^{\beta}}{\partial\bar{t}}u_{\beta}\right),$$

that is, u_{α} is a vector density of weight -1 and u_{0} is not a invariant. We consider therefore the quantities u_{α} and u_{0} which depend on a rheonomic hypersurface element but not necessarily on the hypersurface (1.1) itself, and which vary in the rule (1.3) under a rheonomic transformation. L must be there a scalar under a rheonomic transformation and analytic in a certain region ϑ. L is called the *fundamental function*.

§ 2. Metric functions a^{β}.

Since under a rheonomic transformation $\frac{\partial L}{\partial u_{\alpha}}$, $\frac{\partial L}{\partial u_{0}}$ vary in the rule

$$\frac{\partial\bar{L}}{\partial\bar{u}_{\alpha}}=\left(\frac{\partial L}{\partial u_{\beta}}\frac{\partial\bar{x}^{\alpha}}{\partial x^{\beta}}+\frac{\partial L}{\partial u_{0}}\frac{\partial\bar{x}^{\alpha}}{\partial t}\right)\left|\frac{\partial x}{\partial\bar{x}}\right|,\qquad \frac{\partial\bar{L}}{\partial\bar{u}_{0}}=\left|\frac{\partial x}{\partial\bar{x}}\right|\frac{\partial L}{\partial u_{0}},$$

then

$$\bar{a}^{\gamma}=\frac{\partial\bar{x}^{\gamma}}{\partial x^{\beta}}\left(a^{\beta}+\frac{\partial x^{\beta}}{\partial t}\right),$$

(1) J. RADON: Über einige Fragen betreffend die Theorie der Maxima und Minima mehrfacher Integrale, Monatsh. f. Math. u. Phys., 22 (1911), 53-63.

(2) G. VIVANTI: Sull'equatione di Eulero per gli integrali multipli, Rend. Circ. Mat. Palermo, **33** (1912), 268 274.

putting $a^\beta = -\frac{\partial L}{\partial u_\beta} \Big/ \frac{\partial L}{\partial u_0}$.

§ 3. Parameter v and metric functions $g^{\alpha\beta}$.

Under a rheonomic transformation, the quantities

$$\mathfrak{A}^{\alpha\beta} = \frac{1}{2}\left[\frac{\partial}{\partial u_\beta}\left(\frac{\partial L^2}{\partial u_\alpha} - \frac{\partial L^2}{\partial u_0}\frac{\partial x^\alpha}{\partial t}\right) - \frac{\partial x^\beta}{\partial t}\frac{\partial}{\partial u_0}\left(\frac{\partial L^2}{\partial u_\alpha} - \frac{\partial L^2}{\partial u_0}\frac{\partial x^\alpha}{\partial t}\right)\right] \tag{3.1}$$

vary as a contravariant tensor density of weight 2. Let us put $\mathfrak{A}^{\alpha\beta} = aa^{\alpha\beta}$ where $a = |a^{\alpha\beta}|^{-1}$, from which we can find $a = \mathfrak{A}^{1/n-1}$ where $\mathfrak{A} = |\mathfrak{A}^{\alpha\beta}|$. Putting

$$v = \frac{1}{\sqrt{a}}\frac{\partial L}{\partial u_0}, \tag{3.2}$$

we have an absolute scalar which is homogenous zero in u_α, u_0. Assume that u_0 can be solved from the above equation (3.2) in the form

$$u_0 = F(x^\alpha, t, u_\alpha, v), \tag{3.3}$$

then we see $\rho u_0 = F(x^\alpha, t, \rho u_\alpha, v)$, that is, the function F is homogenous of degree one in u_α. Putting (3.3) into $L(x^\alpha, t, u_\alpha, u_0)$, the obtained function $G(x^\alpha, t, u_\alpha, v)$ is homogenous of degree one in u_α. In the same manner a^α can be brought to functions of x^α, t, u_α, v, which are homogenous of degree zero in u_α too. Afterwards we shall represent these functions by the same letters a^α.

Differentiating G with respect to u_β, we have a contravariant tensor density of weight 2: $\mathfrak{G}^{\alpha\beta} = \frac{1}{2}\frac{\partial^2 G^2}{\partial u_\alpha \partial u_\beta}$. Put $\mathfrak{G}^{\alpha\beta} = gg^{\alpha\beta}$ where $g = 1/|g^{\alpha\beta}|$, from which we can find $g = \mathfrak{G}^{\frac{1}{n-1}}$, where $\mathfrak{G} = |\mathfrak{G}^{\alpha\beta}|$, then we have a contravariant tensor

$$g^{\alpha\beta} = \mathfrak{G}^{-\frac{1}{n-1}}\mathfrak{G}^{\alpha\beta} \tag{3.4}$$

which are homogenous of degree zero in u_α and may take as the fundamental metric tensor in our space. Then $g_{\alpha\beta}$, a_α are defined by $g^{\alpha\beta} g_{\alpha\gamma} = \delta^\beta_\gamma$, $a_\alpha = g_{\alpha\beta}\, a^\beta$.

When L does not contain u_0, this space reduces to a CARTAN space for $t = \text{const.}$.

§ 4. Covariant differentiation.

We shall denote the covariant differentiation in the space with the symbol D. Since the covariant differentiation is necessary to satisfy that

1. $Dp = dp$, where p is a strong scalar,
2. $D(X+Y) = DX + DY$, where X, Y are any strong vectors,
3. $D(X \cdot Y) = X \cdot DY + DX \cdot Y$,

we define

$$DX^\lambda = dX^\lambda + \Gamma^\lambda_{\mu\nu} X^\mu dx^\nu + \Gamma^\lambda_\mu X^\mu dt + C^{\lambda\nu}_\mu X^\mu du_\nu + C^\lambda_\mu X^\mu dv. \tag{4.1}$$

In consequence of the geometrical meaning of u_α, this differentiation must be invariant under the transformation $\bar{u}_\alpha = \rho u_\alpha$. By this reason we suppose that $\Gamma^\lambda_{\mu\nu}$, Γ^λ_μ, C^λ_μ are homogenous function of degree zero in u_α and $C^{\lambda\nu}_\mu$ of degree -1 and that $C^{\lambda\nu}_\mu u_\nu$ vanish.

Let this connection be a euclidean connection, that is, when a strong vector is transported parallel to itself, the length of this vector be invariant. Then it must be that

$$\begin{aligned} \frac{\partial g_{\lambda\mu}}{\partial x^\nu} &= g_{\alpha\mu} \Gamma^\alpha_{\lambda\nu} + g_{\lambda\alpha} \Gamma^\alpha_{\mu\nu}, & \frac{\partial g_{\lambda\mu}}{\partial u_\nu} &= g_{\alpha\mu} C^{\alpha\nu}_\lambda + g_{\lambda\alpha} C^{\alpha\nu}_\mu, \\ \frac{\partial g_{\lambda\mu}}{\partial t} &= g_{\alpha\mu} \Gamma^\alpha_\lambda + g_{\lambda\alpha} \Gamma^\alpha_\mu, & \frac{\partial g_{\lambda\mu}}{\partial v} &= g_{\alpha\mu} C^\alpha_\lambda + g_{\lambda\alpha} C^\alpha_\mu. \end{aligned} \tag{4.2}$$

Supposing that $C_{\lambda\mu}{}^\nu$, $C_{\lambda\mu}$ are symmetry with respect to λ, μ, we have

$$C_{\lambda\mu}{}^\nu, = \frac{1}{2} \frac{\partial g_{\lambda\mu}}{\partial u_\nu}, \qquad C_{\lambda\mu} = \frac{1}{2} \frac{\partial g_{\lambda\mu}}{\partial v}. \tag{4.3}$$

§ 5. The normal unit vector.

From (3.4) we obtain

$$g g^{\alpha\beta} u_\alpha = \left(G \frac{\partial^2 G}{\partial u_\alpha \partial u_\beta} + \frac{\partial G}{\partial u_\alpha} \frac{\partial G}{\partial u_\beta} \right) u_\alpha = G \frac{\partial G}{\partial u_\beta}.$$

Multipling u_β and summing with respect to β, then we get

$$g g^{\alpha\beta} u_\alpha u_\beta = G^2. \tag{5.1}$$

$$(5.2)\qquad l_\lambda = \frac{u_\lambda}{\sqrt{g^{\alpha\beta}\, u_\alpha\, u_\beta}} = \frac{\sqrt{g}}{G}\, u_\lambda$$

gives therefore a normal unit vector for the hypersurface u_α. Putting

$$(5.3)\qquad A_\mu^{\lambda\nu} = \frac{G}{\sqrt{g}}\, C_\mu^{\lambda\nu} \ ,$$

which satisfy the relation

$$(5.4)\qquad A_\mu^{\lambda\nu}\, l_\nu = 0 \ ,$$

we obtain

$$(5.5)\qquad A_\mu^{\lambda\nu}\, dl_\nu = C_\mu^{\lambda\nu}\, du_\nu \ .$$

Thus the covariant differentiation (4.1) becomes

$$(5.6)\qquad DX^\lambda = dX^\lambda + \Gamma^\lambda_{\mu\nu}\, X^\mu\, dx^\nu + \Gamma^\lambda_\mu\, X^\mu\, dt + A_\mu^{\lambda\nu}\, X^\mu\, dl_\nu + C^\lambda_\mu\, X^\mu\, dv.$$

The corresponding equation for the covariant components is

$$(5.7)\qquad DX_\lambda = dX_\lambda - \Gamma^\mu_{\lambda\nu}\, X_\mu\, dx^\nu - \Gamma^\mu_\lambda\, X_\mu\, dt - A_\lambda^{\mu\nu}\, X_\mu\, dl_\nu - C^\mu_\lambda\, X_\mu\, dv$$

and for l_λ we obtain

$$(5.7')\qquad Dl_\lambda = dl_\lambda - \Gamma^0_{\lambda\nu}\, dx^\nu - \Gamma^0_\lambda\, dt - A_\lambda^{\mu\nu}\, l_\mu\, dl_\nu - C^0_\lambda\, dv \ ,$$

where $\Gamma^\alpha_{\lambda\mu}\, l_\alpha = \Gamma^0_{\lambda\mu}$, $\Gamma^\alpha_\lambda\, l_\alpha = \Gamma^0_\lambda$, etc.. Substituting (5.7′) in (5.7) and using (5.4), the covariant differentiation becomes

$$(5.8)\qquad DX_\lambda = dX_\lambda - X_\mu\, (\overset{*}{\Gamma}{}^\mu_{\lambda\nu}\, dx^\nu + A_\lambda^{\mu\nu}\, Dl_\nu + \overset{*}{\Gamma}{}^\mu_\lambda\, dt + \overset{*}{C}{}^\mu_\lambda\, dv)^{(1)}$$

where we put

$$(5.9)\qquad \overset{*}{\Gamma}{}^\mu_{\lambda\nu} = \Gamma^\mu_{\lambda\nu} + \Gamma^0_{\beta\nu}\, A_\lambda^{\mu\beta},\quad \overset{*}{\Gamma}{}^\mu_\lambda = \Gamma^\mu_\lambda + A_\lambda^{\mu\alpha}\, \Gamma^0_\alpha,\quad \overset{*}{C}{}^\mu_\lambda = C^\mu_\lambda + A_\lambda^{\mu\alpha}\, C^0_\alpha \ .$$

For $X_\lambda = l_\lambda$ we have

$$(5.10)\qquad Dl_\alpha\, (\delta^\alpha_\mu + l_\mu\, A^\alpha) = dl_\mu - \overset{*}{\Gamma}{}^0_{\mu\alpha}\, dx^\alpha - \overset{*}{\Gamma}{}^0_\mu\, dt - \overset{*}{C}{}^0_\mu\, dv \ .$$

Using $(\delta^\lambda_\mu + l_\mu\, A^\lambda)\,(\delta^\mu_\nu - l_\nu\, A^\mu) = \delta^\lambda_\nu$, Dl_α is represented by

$$(5.11)\qquad Dl_\alpha = (\delta^\lambda_\alpha - l_\alpha\, A^\lambda)\,(dl_\lambda - \overset{*}{\Gamma}{}^0_{\lambda\beta}\, dx^\beta - \overset{*}{\Gamma}{}^0_\lambda\, dt - \overset{*}{C}{}^0_\lambda\, dv) \ .$$

---

(1) Since (3.4) and (4.3) give us the relation $A_\nu{}^{0\mu} = l_\nu\, A^\mu$, where $A^\mu = A_0{}^{\mu 0}$, we see easily $A_\mu{}^{\lambda\nu}\, A_\nu{}^{0\beta} = 0$.

§ 6. Other two postulates of the covariant differentiation.

We consider the invariant differential form

$$\phi = \underset{1}{D}(g_{\lambda\mu}\underset{2}{\delta x^\lambda}\underset{3}{\delta x^\mu}) + \underset{2}{D}(g_{\lambda\mu}\underset{1}{\delta x^\lambda}\underset{3}{\delta x^\mu}) - \underset{3}{D}(g_{\lambda\mu}\underset{1}{\delta x^\lambda}\underset{2}{\delta x^\mu}) \tag{6.1}$$

where the index 1, 2, 3 under the differential symbol D, δ denote the directions of the differentiations. They are here supposed to be interchangeable, that is, $\underset{ab}{ddx^\lambda} = \underset{ba}{ddx^\lambda}$ $(a, b = 1, 2, 3)$. If we choose $\underset{2}{dt} = \underset{3}{dt} = 0$, $\underset{a}{Dl_a} = 0$ and $\underset{a}{dv} = 0$, we have

$$\begin{aligned}\phi = 2g_{\lambda\nu}\Big[&\underset{12}{ddx^\lambda} + \Big(\begin{Bmatrix}\lambda\\ \omega\mu\end{Bmatrix} + A^{\lambda a}_{\omega}\overset{*}{\Gamma}{}^{0}_{a\mu} + A^{\lambda a}_{\mu}\overset{*}{\Gamma}{}^{0}_{a\omega} - A_{\mu\omega}{}^{a}\overset{*}{\Gamma}{}^{0\lambda}_{a}\Big)\underset{2}{dx^\omega}\underset{1}{dx^\mu}\\ &+ \Big(\begin{Bmatrix}\lambda\\ \omega\end{Bmatrix} + A^{\lambda a}_{\omega}\overset{*}{\Gamma}{}^{0}_{a} + A^{\lambda a}\overset{*}{\Gamma}{}^{0}_{a\omega} - A^{a}_{\omega}\overset{*}{\Gamma}{}^{0\lambda}_{a}\Big)\underset{2}{dx^\omega}\underset{1}{dt}\Big]\underset{3}{\delta x^\nu},\end{aligned} \tag{6.2}$$

where we put

$$\begin{Bmatrix}\lambda\\ \omega\end{Bmatrix} = g^{\lambda\nu}(\partial_t g_{\nu\omega} + \partial_\omega a_\nu - \partial_\nu a_\omega), \quad A^a_h = \frac{1}{2}\frac{\partial a_h}{\partial l_a}, \quad A^{\lambda a} = g^{\lambda\nu}A^a_\nu .$$

Since $g_{\lambda\mu}$ is the strong tensor of degree 2 and $\underset{3}{\delta x^\lambda}$ is a strong vector, the term held in the bracket of above equation is a strong contravatiant vector.

In order that we desire that the covariant differentiation does not change the property "strong" and the order of tensors, we shall define

$$\overset{*}{\Gamma}{}^{\lambda}_{\omega\mu} = \begin{Bmatrix}\lambda\\ \omega\mu\end{Bmatrix} + A^{\lambda a}_{\omega}\overset{*}{\Gamma}{}^{0}_{a\mu} + A^{\lambda a}_{\mu}\overset{*}{\Gamma}{}^{0}_{a\omega} - A_{\mu\omega}{}^{a}\overset{*}{\Gamma}{}^{0\lambda}_{a}, \tag{6.3}$$

$$\overset{*}{\Gamma}{}^{\lambda}_{\omega} = \begin{Bmatrix}\lambda\\ \omega\end{Bmatrix} + A^{\lambda a}_{\omega}\overset{*}{\Gamma}{}^{0}_{a} + A^{\lambda a}\overset{*}{\Gamma}{}^{0}_{a\omega} - A^{a}_{\omega}\overset{*}{\Gamma}{}^{0\lambda}_{a}. \tag{6.4}$$

Then we can verify $Dg_{\lambda\mu}=0$. Putting $\underset{2}{\delta x^\lambda} = \underset{3}{\delta x^\lambda}$ in (6.1), we have

$$\phi = \underset{1}{D}(g_{\lambda\omega}\underset{2}{\delta x^\lambda}\underset{2}{\delta x^\omega}) = \underset{1}{D}g_{\lambda\omega}\underset{2}{\delta x^\lambda}\underset{2}{\delta x^\omega} + 2g_{\lambda\omega}\underset{1}{D}\underset{2}{\delta x^\lambda}\underset{2}{\delta x^\omega}. \tag{6.5}$$

On the other hand (6.2) becomes

$$\phi = 2g_{\lambda\omega}\underset{1}{D}\underset{2}{\delta x^\lambda}\underset{2}{\delta x^\omega}. \tag{6.6}$$

From (6.5) and (6.6) we obtain

$$\underset{1}{D} g_{\lambda\omega} \underset{2}{\delta x^{\lambda}} \underset{2}{\delta x^{\omega}} = 0 \ .$$

Hence arbitrariness of $\underset{2}{\delta x^{\lambda}}$ leads us to

$$\underset{1}{D} g_{\lambda\omega} = 0 \ . \tag{6.7}$$

§ 7. The parameters of connection $\overset{*}{\Gamma}{}^{\lambda}_{\mu\omega}$.

From (4.2) and (5.9) we have immediately

$$\overset{*}{\Gamma}_{\lambda\mu\nu} + \overset{*}{\Gamma}_{\mu\lambda\nu} = \frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} + 2A_{\lambda\mu}{}^{\omega}\, \Gamma_{\omega 0\nu} \ . \tag{7.1}$$

From this the two following equations are obtained by cyclic permutation of λ, μ, ν

$$\overset{*}{\Gamma}_{\mu\nu\lambda} + \overset{*}{\Gamma}_{\nu\mu\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + 2A_{\mu\nu}{}^{\omega}\, \Gamma_{\omega 0\lambda} \ , \tag{7.2}$$

$$-\overset{*}{\Gamma}_{\nu\lambda\mu} - \overset{*}{\Gamma}_{\lambda\nu\mu} = -\frac{\partial g_{\nu\lambda}}{\partial x^{\mu}} - 2A_{\nu\lambda}{}^{\omega}\, \Gamma_{\omega 0\mu} \ . \tag{7.3}$$

Now we shall assume that $\overset{*}{\Gamma}_{\lambda\mu\nu}$ are symmetry with respect to λ, ν. Then summing up (7.1), (7.2), and (7.3), we have

$$\overset{*}{\Gamma}_{\lambda\mu\nu} = \gamma_{\lambda\mu\nu} + A_{\lambda\mu}{}^{\omega}\, \Gamma_{\omega 0\nu} + A_{\mu\nu}{}^{\omega}\, \Gamma_{\omega 0\lambda} - A_{\nu\lambda}{}^{\omega}\, \Gamma_{\omega 0\mu} \ ,$$

where we put $\gamma_{\lambda\mu\nu} = g_{\mu\omega} \left\{ {\omega \atop \lambda\nu} \right\}$. Putting

$$H^{\lambda\mu} = g^{\lambda\mu} + A_{\omega} A^{\omega\lambda\mu} = g^{\lambda\mu} + A^{\lambda} A^{\mu} - \frac{1}{4} \frac{G}{\sqrt{g}} A_{\omega} \frac{\partial^3 G^2}{\partial u_{\lambda}\, \partial u_{\mu}\, \partial u_{\omega}} \ . \tag{7.4}$$

Under the assumption that the rank of $|H^{\lambda\mu}|$ be n, we can determine $\overset{*}{\Gamma}{}^{\omega}_{\lambda\nu}$ by the following relations

$$\begin{aligned} \overset{*}{\Gamma}{}^{\omega}_{\lambda\nu} = \left\{ {\omega \atop \lambda\nu} \right\} &+ A^{\omega\alpha}_{\lambda} (\gamma_{\alpha 0\nu} - l_{\nu}\, \gamma_{\alpha 00}) + A^{\omega\alpha}_{\nu} (\gamma_{\alpha 0\lambda} - l_{\lambda}\, \gamma_{\alpha 00}) \\ &- A_{\nu\lambda}{}^{\alpha} (\gamma^{\omega}_{\alpha 0} - l^{\omega}\, \gamma_{\alpha 00}) + (A^{\omega\alpha}_{\lambda}\, l_{\nu} + A^{\omega\alpha}_{\nu}\, l_{\lambda} - A_{\nu\lambda}{}^{\alpha}\, l^{\omega} \\ &- A^{\omega\beta}_{\lambda}\, A_{\beta\nu}{}^{\alpha} - A^{\omega\beta}_{\nu}\, A_{\beta\lambda}{}^{\alpha} + A_{\nu\lambda}{}^{\beta}\, A^{\omega\alpha}_{\beta})\, K^{\delta}_{\alpha}\, (\gamma_{\delta 00} + A^{\varepsilon}\, \gamma_{\cdot 0\delta}) \end{aligned} \tag{7.5}$$

where

$$\gamma_{\alpha 0}{}^{\omega} = g^{\beta\omega}\, \gamma_{\alpha 0\beta} \ , \qquad H^{\mu}_{\lambda}\, K^{\nu}_{\mu} = H^{\nu}_{\mu}\, K^{\mu}_{\lambda} = \delta^{\nu}_{\lambda} \ .$$

§ 8. The parameters of connection $\overset{*}{\Gamma}{}^{\lambda}_{\mu}$.

Multipling (6.4) by l_λ and summing with respect to λ, we have

$$\overset{*}{\Gamma}{}^{0}_{\alpha} = \left\{ {0 \atop \alpha} \right\} + A^{0\beta}_{\alpha} \overset{*}{\Gamma}{}^{0}_{\beta} + A^{0\beta} \overset{*}{\Gamma}{}^{0}_{\beta\alpha} - A^{\beta}_{\alpha} \overset{*}{\Gamma}{}^{00}_{\beta} \tag{8.1}$$

or

$$(\delta^{\alpha}_{\omega} - l_\omega A^{\alpha}) \overset{*}{\Gamma}{}^{0}_{\alpha} = \left\{ {0 \atop \omega} \right\} + A^{0\alpha} \overset{*}{\Gamma}{}^{0}_{\alpha\omega} - A^{\alpha}_{\omega} \overset{*}{\Gamma}{}^{00}_{\alpha} . \tag{8.2}$$

Since $(\delta^{\alpha}_{\omega} - l_\omega A^{\alpha})(\delta^{\omega}_{\beta} + l_\beta A^{\omega}) = \delta^{\alpha}_{\beta}$, we obtain

$$\overset{*}{\Gamma}{}^{0}_{\beta} = \left(\delta^{\omega}_{\beta} + l_\beta A^{\omega}\right) \left(\left\{ {0 \atop \omega} \right\} + A^{0\alpha} \overset{*}{\Gamma}{}^{0}_{\alpha\omega} - A^{\alpha}_{\omega} \overset{*}{\Gamma}{}^{00}_{\alpha}\right) . \tag{8.3}$$

In use of (8.3) and (7.5), we can determine $\overset{*}{\Gamma}{}^{\lambda}_{\mu}$ from (6.4).

§ 9. The stretch tensor W_{ij}.

We consider the invariant differential form $\phi = \bar{\delta}(g_{\lambda\omega}\, \delta x^\lambda\, \delta x^\omega)$ and choose the two (infinitesimal) variations δ, $\bar{\delta}$ in the following manner:

$$\delta x^i \neq 0 , \quad dt = 0 , \quad Dl_\lambda = 0 , \quad dv = 0 ,$$

$$\bar{\delta} x^i = 0 , \quad \bar{d}t \neq 0 , \quad \bar{D} l_\lambda = 0 , \quad \bar{d}v = 0 .$$

In words, $\bar{\delta}$ and δ mean the variation of time and that of the virtual space respectively. We suppose that δ, $\bar{\delta}$ are interchangeable. In this case ϕ shows us the stretching of the space for the time variation dt. Calculating ϕ, we have

$$\phi = 2\,[\omega_{\lambda\omega} + \overset{*}{\Gamma}{}^{0}_{\alpha} A_{\lambda\omega}{}^{\alpha} - A_{\lambda\omega}{}^{\alpha} \overset{*}{\Gamma}{}^{0}_{\alpha\beta}\, a^\beta + A_\lambda{}^{\alpha} \overset{*}{\Gamma}{}^{0}_{\alpha\omega}]\, \delta x^\lambda\, \delta x^\omega\, \bar{d}t ,$$

where we put

$$\omega_{\lambda\omega} = \frac{1}{2} (\partial_t g_{\lambda\omega} - a^\alpha \partial_\alpha g_{\lambda\omega} - 2 g_{\lambda\alpha} \partial_\omega a^\alpha) , \quad A_\lambda{}^{\alpha} = 2 (A_{\lambda\mu}{}^{\alpha} a^\mu - A^{\alpha}_{\lambda}) .$$

Using the last quantity we get a strong tensor

$$W_{\lambda\omega} = \omega_{(\lambda\omega)} + \overset{*}{\Gamma}{}^{0}_{\alpha} A_{\lambda\omega}{}^{\alpha} - A_{\lambda\omega}{}^{\alpha} \overset{*}{\Gamma}{}^{0}_{\alpha\beta}\, a^\beta + A_{(\lambda}{}^{\alpha} \overset{*}{\Gamma}{}^{0}_{|\alpha|\omega)} ,$$

which is symmetry with respect to λ, ω and shall be called the *stretch tensor of our space.*

§ 10. The curvature strong tensors.

The curvature of our space is defined by

$$(10.1)\qquad (\underset{2}{D}\underset{1}{D} - \underset{1}{D}\underset{2}{D})\, X^\lambda = \Omega^\lambda_\mu X^\mu .$$

Then this bilinear form Ω^λ_μ can be decomposed into following form

$$(10.2)\qquad \begin{aligned}\Omega^\lambda_\mu = {} & R^\lambda_{\mu\nu\omega}\, \underset{1}{dx^\nu}\, \underset{2}{dx^\omega} + P^\lambda_{\mu\nu}{}^\omega (\underset{2}{dx^\nu}\, \underset{1}{Dl_\omega} - \underset{2}{dx^\nu}\, \underset{1}{Dl_\omega}) + S^{\lambda\nu\omega}_\mu\, \underset{1}{Dl_\nu}\, \underset{2}{Dl_\omega} \\ & + \underset{13}{P}{}^\lambda_{\mu\omega}(\underset{1}{dx^\omega}\, \underset{2}{dt} - \underset{2}{dx^\omega}\, \underset{1}{dt}) + \underset{23}{P}{}^{\lambda\omega}_\mu(\underset{1}{Dl_\omega}\, \underset{2}{dt} - \underset{2}{Dl_\omega}\, \underset{1}{dt}) + \underset{34}{P}{}^\lambda_\mu(\underset{1}{dv}\, \underset{2}{dt} - \underset{2}{dv}\, \underset{1}{dt}) \\ & + \underset{14}{P}{}^\lambda_{\mu\omega}(\underset{1}{dx^\omega}\, \underset{2}{dv} - \underset{2}{dx^\omega}\, \underset{1}{dv}) + \underset{24}{P}{}^{\lambda\omega}_\mu (\underset{1}{Dl_\omega}\, \underset{2}{dv} - \underset{2}{Dl_\omega}\, \underset{1}{dv}) .\end{aligned}$$

Let f be a field in our space, we have the differential

$$(10.3)\qquad \begin{aligned} df = {} & \left(\frac{\partial f}{\partial x^\omega} + f\|^\alpha \overset{*}{\Gamma}_{\alpha 0 \omega}\right) dx^\omega + f\|^\omega Dl_\omega \\ & + \left(\frac{\partial f}{\partial t} + f\|^\alpha \overset{*}{\Gamma}{}^0_\alpha\right) dt + \left(\frac{\partial f}{\partial v} + f\|^\alpha \overset{*}{C}{}^0_\alpha\right) dv \end{aligned}$$

where $f\|^\lambda = \dfrac{G}{\sqrt{g}} \dfrac{\partial f}{\partial u_\lambda}$. In § 5 we have had the covariant differential for the contravariant vector X^ι, that is,

$$(10.4)\qquad DX^\lambda = dX^\lambda + \overset{*}{\Gamma}{}^\lambda_{\mu\nu} X^\mu dx^\nu + \overset{*}{\Gamma}{}^\lambda_\mu X^\mu dt + A^{\lambda\nu}_\mu X^\mu Dl_\nu + \overset{*}{C}{}^\lambda_\mu X^\mu dv .$$

Then we shall rewrite (10.4) in the strong tensor form

$$DX^\lambda = X^\lambda|_\nu\, \delta x^\nu + X^\lambda|^\nu Dl_\nu + X^\lambda|_t dt + X^\lambda|_v dv ,$$

where we put

$$X^\lambda|_\nu = \frac{\partial X^\lambda}{\partial x^\nu} + \overset{*}{\Gamma}{}^\lambda_{\mu\nu} X^\mu + X^\lambda\|^\alpha \overset{*}{\Gamma}{}^0_{\alpha\nu} , \qquad X^\lambda|^\nu = X^\lambda\|^\nu + A^{\lambda\nu}_\mu X^\mu,$$

$$X^\lambda|_t = \frac{\partial X^\lambda}{\partial t} + \overset{*}{\Gamma}{}^\lambda_\mu X^\mu + X^\lambda\|^\alpha \overset{*}{C}{}^0_\alpha - X^\lambda|_\nu a^\nu, \quad X^\lambda|_v = \frac{\partial X^\lambda}{\partial v} + \overset{*}{C}{}^\lambda_\mu X^\mu + X^\lambda\|^\alpha \overset{*}{C}{}^0_\alpha$$

and shall call them the *covariant differential coefficients*. We make use of (5.8) and (10.3) to calculate the left-hand member of (10.1) and compare the obtained result with (10.2), then we get

$$R^{\lambda}_{\mu\nu\omega}=\left(\frac{\partial\overset{*}{\Gamma}{}^{\lambda}_{\mu\nu}}{\partial x^{\omega}}+\overset{*}{\Gamma}{}^{\lambda}_{\mu\nu}\|^{\alpha}\,\overset{*}{\Gamma}_{\alpha 0\omega}\right)-\left(\frac{\partial\overset{*}{\Gamma}{}^{\lambda}_{\mu\omega}}{\partial x^{\nu}}+\overset{*}{\Gamma}{}^{\lambda}_{\mu\omega}\|^{\alpha}\,\overset{*}{\Gamma}_{\alpha 0\nu}\right)$$
$$+\overset{*}{\Gamma}{}^{\alpha}_{\mu\nu}\,\overset{*}{\Gamma}{}^{\lambda}_{\alpha\omega}-\overset{*}{\Gamma}{}^{\alpha}_{\mu\omega}\,\overset{*}{\Gamma}{}^{\lambda}_{\alpha\nu}-A^{\lambda\alpha}_{\mu}\left\{\left(\frac{\partial\overset{*}{\Gamma}_{\alpha 0\nu}}{\partial x^{\omega}}+\overset{*}{\Gamma}_{\alpha 0\nu}\|^{\rho}\,\overset{*}{\Gamma}_{\rho 0\omega}\right)\right.$$
$$\left.-\left(\frac{\partial\overset{*}{\Gamma}_{\alpha 0\omega}}{\partial x^{\nu}}+\overset{*}{\Gamma}_{\alpha 0\omega}\|^{\rho}\,\overset{*}{\Gamma}_{\rho 0\nu}\right)\right\},$$

$$\underset{12}{P}{}^{\lambda}_{\mu\nu}{}^{\omega}=\overset{*}{\Gamma}{}^{\lambda}_{\mu\nu}\|^{\omega}-A^{\lambda\omega}_{\mu}|_{\nu}-A^{\lambda\alpha}_{\mu}\,\overset{*}{\Gamma}{}^{\beta}_{\alpha\nu}\|^{\omega}\,l_{\beta}\,,$$

$$S^{\lambda\nu\omega}_{\mu}=A^{\alpha\lambda\nu}A_{\alpha\mu}{}^{\omega}-A^{\alpha\lambda\omega}A_{\alpha\mu}{}^{\nu}\,,$$

(10.5)
$$\underset{13}{\overset{*}{P}}{}^{\lambda}_{\mu\omega}=\frac{\partial\overset{*}{\Gamma}{}^{\lambda}_{\mu\omega}}{\partial t}-\frac{\partial\overset{*}{\Gamma}{}^{\lambda}_{\mu}}{\partial x^{\omega}}+\overset{*}{\Gamma}{}^{\lambda}_{\alpha}\,\overset{*}{\Gamma}{}^{\alpha}_{\mu\omega}-\overset{*}{\Gamma}{}^{\lambda}_{\alpha\omega}\,\overset{*}{\Gamma}{}^{\alpha}_{\mu}-\overset{*}{\Gamma}{}^{\lambda}_{\mu}\|^{\alpha}\,\overset{*}{\Gamma}_{\alpha 0\omega}+\overset{*}{\Gamma}{}^{\lambda}_{\mu\nu}\|^{\alpha}\,\overset{*}{\Gamma}{}^{0}_{\alpha}\,,$$

$$\underset{23}{\overset{*}{P}}{}^{\lambda\omega}_{\mu}=\frac{\partial A^{\lambda\omega}_{\mu}}{\partial t}-\overset{*}{\Gamma}{}^{\lambda}_{\mu}\|^{\omega}+\overset{*}{\Gamma}{}^{\lambda}_{\alpha}\,A^{\alpha\omega}_{\mu}-A^{\lambda\omega}_{\alpha}\,\overset{*}{\Gamma}{}^{\alpha}_{\mu}+A^{\lambda\omega}_{\mu}\|^{\alpha}\,\overset{*}{\Gamma}{}^{0}_{\alpha}\,,$$

$$\underset{34}{\overset{*}{P}}{}^{\lambda}_{\mu}=\frac{\partial\overset{*}{C}{}^{\lambda}_{\mu}}{\partial t}-\frac{\partial\overset{*}{\Gamma}{}^{\lambda}_{\mu}}{\partial v}+\overset{*}{C}{}^{\alpha}_{\mu}\,\overset{*}{\Gamma}{}^{\lambda}_{\alpha}-\overset{*}{C}{}^{\lambda}_{\alpha}\,\overset{*}{\Gamma}{}^{\alpha}_{\mu}+\overset{*}{C}{}^{\lambda}_{\mu}\|^{\alpha}\,\overset{*}{\Gamma}{}^{0}_{\alpha}-\overset{*}{\Gamma}{}^{\lambda}_{\mu}\|^{\alpha}\,\overset{*}{C}{}^{0}_{\alpha}\,,$$

$$\underset{14}{P}{}^{\lambda}_{\mu\omega}=\frac{\partial\overset{*}{\Gamma}{}^{\lambda}_{\mu\omega}}{\partial v}-\frac{\partial\overset{*}{C}{}^{\lambda}_{\mu}}{\partial x^{\omega}}+\overset{*}{C}{}^{\lambda}_{\alpha}\overset{*}{\Gamma}{}^{\alpha}_{\mu\omega}-\overset{*}{\Gamma}{}^{\lambda}_{\alpha\omega}\overset{*}{C}{}^{\alpha}_{\mu}-\overset{*}{C}{}^{\lambda}_{\mu}\|^{\alpha}\,\overset{*}{\Gamma}_{\alpha 0\omega}+\overset{*}{\Gamma}{}^{\lambda}_{\mu\omega}\|^{\alpha}\overset{*}{C}{}^{0}_{\alpha}\,,$$

$$\underset{24}{P}{}^{\lambda\omega}_{\mu}=\frac{\partial A^{\lambda\omega}_{\mu}}{\partial v}-\overset{*}{C}{}^{\lambda}_{\mu}\|^{\omega}+\overset{*}{C}{}^{\lambda}_{\alpha}\,A^{\alpha\omega}_{\mu}-A^{\lambda\omega}_{\alpha}\,\overset{*}{C}{}^{\alpha}_{\mu}+A^{\lambda\omega}_{\mu}\|^{\alpha}\,\overset{*}{C}{}^{0}_{\alpha}\,.$$

Since these coefficients are not all strong tensors, we deform (10.2) as follows:

$$\Omega^{\lambda}_{\mu}=R^{\lambda}_{\mu\nu\omega}\,\underset{1}{\delta x^{\nu}}\,\underset{2}{\delta x^{\omega}}+\underset{12}{P}{}^{\lambda}_{\mu\nu}{}^{\omega}\,(\underset{1}{\delta x^{\nu}}\,\underset{2}{Dl_{\omega}}-\underset{2}{\delta x^{\nu}}\,\underset{1}{Dl_{\omega}})+S^{\lambda\nu\omega}_{\mu}\,\underset{1}{Dl_{\nu}}\,\underset{2}{Dl_{\omega}}$$

(10.6)
$$+\underset{13}{P}{}^{\lambda}_{\mu\omega}(\underset{1}{\delta x^{\omega}}\,\underset{2}{dt}-\underset{2}{\delta x^{\omega}}\,\underset{1}{dt})+\underset{23}{P}{}^{\lambda\omega}_{\mu}(\underset{1}{Dl_{\omega}}\,\underset{2}{dt}-\underset{2}{Dl_{\omega}}\,\underset{1}{dt})+\underset{34}{P}{}^{\lambda}_{\mu}(\underset{1}{dv}\,\underset{2}{dt}-\underset{2}{dv}\,\underset{1}{dt})$$
$$+\underset{14}{P}{}^{\lambda}_{\mu\omega}(\underset{1}{\delta x^{\omega}}\,\underset{2}{dv}-\underset{2}{\delta x^{\omega}}\,\underset{1}{dv})+\underset{24}{P}{}^{\lambda\omega}_{\mu}(\underset{1}{Dl_{\omega}}\,\underset{2}{dv}-\underset{2}{Dl_{\omega}}\,\underset{1}{dv})\,,$$

where $\underset{13}{P}{}^{\lambda}_{\mu\omega}=\underset{13}{\overset{*}{P}}{}^{\lambda}_{\mu\omega}-\alpha^{\beta}\,R^{\lambda}_{\mu\omega\beta}$, $\underset{23}{P}{}^{\lambda\omega}_{\mu}=\underset{23}{\overset{*}{P}}{}^{\lambda\omega}_{\mu}+\alpha^{\nu}\,\underset{12}{P}{}^{\lambda}_{\mu\nu}{}^{\omega}$ and $\underset{34}{P}{}^{\lambda}_{\mu}=\underset{34}{\overset{*}{P}}{}^{\lambda}_{\mu}+\alpha^{\omega}\,\underset{14}{P}{}^{\lambda}_{\mu\omega}$.

Thus we obtain our curvature strong tensors $R^{\lambda}_{\mu\nu\omega}$, $\underset{12}{P}{}^{\lambda}_{\mu\nu}{}^{\omega}$, $S^{\lambda\nu\omega}_{\mu}$, $\underset{13}{P}{}^{\lambda}_{\mu\omega}$, $\underset{23}{P}{}^{\lambda\omega}_{\mu}$, $\underset{34}{P}{}^{\lambda}_{\mu}$, $\underset{14}{P}{}^{\lambda}_{\mu\omega}$ **and** $\underset{24}{P}{}^{\lambda\omega}_{\mu}$.

§ 11. Properties of the curvature strong tensors.

In virtue of (6. 7), we see

$$(\underset{2}{D}\underset{1}{D} - \underset{1}{D}\underset{2}{D})\, g_{\lambda\mu} = 0\,. \tag{11.1}$$

Calculate the left-hand member of (11. 1), then we have

$$\begin{aligned}
(\underset{2}{D}\underset{1}{D}-\underset{1}{D}\underset{2}{L})g_{\lambda\mu} = &-(g_{\alpha\mu}R^{\alpha}_{\lambda\nu\omega}+g_{\lambda\alpha}R^{\alpha}_{\mu\nu\omega})\underset{1}{\delta x^{\nu}}\,\underset{2}{\delta x^{\omega}}-(g_{\alpha\mu}S^{\alpha\,\nu\omega}_{\lambda}+g_{\lambda\alpha}S^{\alpha\,\nu\omega}_{\mu})\underset{1}{Dl_{\nu}}\,\underset{2}{Dl_{\omega}} \\
&-(g_{\alpha\mu}\underset{12}{R^{\alpha}_{\lambda\nu\omega}}+g_{\lambda\alpha}\underset{12}{P^{\alpha}_{\mu\nu}{}^{\omega}})(\underset{1}{\delta x^{\nu}}\,\underset{2}{Dl_{\omega}}-\underset{2}{\delta x^{\nu}}\,\underset{1}{Dl_{\omega}}) \\
&-(g_{\alpha\mu}\underset{13}{P^{\alpha}_{\lambda\omega}}+g_{\lambda\alpha}\underset{13}{P^{\alpha}_{\mu\omega}})(\underset{1}{\delta x^{\omega}}\,\underset{2}{dt}-\underset{2}{\delta x^{\omega}}\,\underset{1}{dt}) \\
&-(g_{\alpha\mu}\underset{23}{P^{\alpha\omega}_{\lambda}}+g_{\lambda\alpha}\underset{23}{P^{\alpha\omega}_{\mu}})(\underset{1}{Dl_{\omega}}\,\underset{2}{dt}-\underset{2}{Dl_{\omega}}\,\underset{1}{dt}) \\
&-(g_{\alpha\mu}\underset{34}{P^{\alpha}_{\lambda}}+g_{\lambda\alpha}\underset{34}{P^{\alpha}_{\mu}})(\underset{1}{dv}\,\underset{2}{dt}-\underset{2}{dv}\,\underset{1}{dt}) \\
&-(g_{\alpha\mu}\underset{14}{P^{\alpha}_{\lambda\omega}}+g_{\lambda\alpha}\underset{14}{P^{\alpha}_{\mu\omega}})(\underset{1}{\delta x^{\omega}}\,\underset{2}{dv}-\underset{2}{\delta x^{\omega}}\,\underset{1}{dv}) \\
&-(g_{\alpha\mu}\underset{24}{P^{\alpha\omega}_{\lambda}}+g_{\lambda\alpha}\underset{24}{P^{\alpha\omega}_{\mu}})(\underset{1}{Dl_{\omega}}\,\underset{2}{dv}-\underset{2}{Dl_{\omega}}\,\underset{1}{dv})\,,
\end{aligned} \tag{11.2}$$

consequently,

$$\begin{aligned}
&R_{\lambda\mu\nu\omega}+R_{\mu\lambda\nu\omega}=0\,, && \underset{12}{P_{\lambda\mu\nu}{}^{\omega}}+\underset{12}{P_{\mu\lambda\nu}{}^{\omega}}=0\,, && S^{\nu}_{\lambda\mu}{}^{\omega}+S^{\nu}_{\mu\lambda}{}^{\omega}=0\,, \\
&\underset{13}{P_{\lambda\mu\omega}}+\underset{13}{P_{\mu\lambda\omega}}=0\,, && \underset{23}{P_{\lambda\mu}{}^{\omega}}+\underset{23}{P_{\mu\lambda}{}^{\omega}}=0\,, && \underset{34}{P_{\mu\lambda}}+\underset{34}{P_{\lambda\mu}}=0\,, \\
&\underset{14}{P_{\lambda\mu\omega}}+\underset{14}{P_{\mu\lambda\omega}}=0\,, && \underset{24}{P_{\lambda\mu\omega}}+\underset{24}{P_{\mu\lambda\omega}}=0\,.
\end{aligned} \tag{11.3}$$

That is, they are skew-symmetry with respect to λ, μ.

From definition (10. 5), we have evidently

$$R^{\lambda}_{\mu\nu\omega} = -R^{\lambda}_{\mu\omega\nu}\,, \qquad S^{\lambda\nu\omega}_{\mu} = -S^{\lambda\omega\nu}_{\mu}\,. \tag{11.4}$$

§ 12. The identities of RICCI and BIANCHI.

Now we shall proceed to find the identities between the torsion strong tensors, the curvature strong tensors and their derivatives, which correspond to the so-called identities of RICCI and BIANCHI. The PFAFFian forms

$$\Pi^{\lambda} = (\varpi^{\lambda})' + [\omega^{\lambda}_{\mu}\, \varpi^{\mu}] \tag{12.1}$$

are components of a covariant strong vector, where ω, ϖ are CARTAN's symbols. Consider the external derivative of (12.1), then we have the relation

$$(\Pi^{\lambda})' + [\omega^{\lambda}_{\mu}\, \Pi^{\mu}] = [\Omega^{\lambda}_{\mu}\, \varpi^{\mu}]. \tag{12.2}$$

By help of (11.4), this relation offers us the required identities

$$R^{\lambda}_{[\mu\nu\omega]} + A^{\lambda\alpha}_{[\mu} R_{0|\alpha|\nu\omega]} = 0, \qquad \underset{12}{P}{}^{\lambda}_{[\mu\nu]}{}^{\omega} - A^{\lambda\omega}_{[\mu}|_{\nu]} + A^{\lambda\alpha}_{[\mu} \underset{12}{P}{}_{0|\alpha|\nu]}{}^{\omega} = 0,$$

$$2\underset{13}{P}{}^{\lambda}_{[\mu\nu]} - 2a^{\lambda}_{(x)[\mu}|_{\nu]} + 2A^{\lambda\alpha}_{[\mu} \underset{13}{P}{}_{0|\alpha|\nu]} - a^{\lambda\alpha}_{(e)} R_{0\alpha[\mu\nu]} = 0,$$

$$\underset{14}{P}{}^{\lambda}_{[\mu\nu]} - \overset{*}{C}{}^{\lambda}_{[\mu}|_{\nu]} + A^{\lambda\alpha}_{[\mu} \underset{14}{P}{}_{0|\alpha|\nu]} = 0,$$

$$S^{\lambda}_{\mu}{}^{\nu\omega} + A^{\lambda[\nu}_{\mu}|^{\omega]} + A^{\lambda[\nu}_{\beta} A^{|\beta|\omega]}_{\mu} + A^{\lambda\alpha}_{\mu} S_{0\alpha}{}^{\nu\omega} = 0,$$

$$\begin{aligned} &\underset{23}{P}{}^{\lambda\omega}_{\mu} + A^{\lambda\omega}_{\mu}|_{t} - a^{\lambda}_{(x)\mu}|^{\omega} + A^{\lambda\omega}_{\alpha} a^{\alpha}_{(x)\mu} - a^{\lambda}_{(x)\alpha} A^{\alpha\omega}_{\mu} + A^{\lambda\alpha}_{\mu} \underset{23}{P}{}_{0\alpha}{}^{\omega} + a^{\lambda\omega}_{(e)}|_{\mu} \\ &\qquad - a^{\lambda\alpha}_{(e)} \underset{12}{P}{}_{0\alpha\mu}{}^{\omega} = 0, \end{aligned} \tag{12.3}$$

$$\underset{24}{P}{}^{\lambda\omega}_{\mu} + A^{\lambda\omega}_{\mu}|_{v} - \overset{*}{C}{}^{\lambda}_{\mu}|^{\omega} + A^{\lambda\omega}_{\alpha} \overset{*}{C}{}^{\alpha}_{\mu} - \overset{*}{C}{}^{\lambda}_{\alpha} A^{\alpha\omega}_{\mu} + A^{\lambda\alpha}_{\mu} \underset{24}{P}{}_{0\alpha}{}^{\omega} = 0,$$

$$\begin{aligned} &-\underset{34}{P}{}^{\lambda}_{\mu} + a^{\lambda}_{(x)\mu}|_{v} - \overset{*}{C}{}^{\lambda}_{\mu}|_{t} + a^{\lambda}_{(x)\alpha} \overset{*}{C}{}^{\alpha}_{\mu} - \overset{*}{C}{}^{\lambda}_{\alpha} a^{\alpha}_{(x)\mu} + A^{\lambda\alpha}_{\mu} \underset{34}{P}{}_{0\alpha} - a^{\lambda}_{(v)}|_{\mu} \\ &\qquad + a^{\lambda\alpha}_{(e)} \underset{14}{P}{}_{0\alpha\mu} = 0, \end{aligned}$$

$$2a^{\lambda\mu}_{(e)}|^{\omega} - 2A^{\lambda\omega}_{\alpha} a^{\alpha\mu}_{(e)} + a^{\lambda\alpha}_{(e)} S_{0\alpha}{}^{\mu\omega} = 0,$$

$$a^{\lambda\mu}_{(e)}|_{v} - a^{\lambda}_{(v)}|^{\mu} + A^{\lambda\mu}_{\alpha} a^{\alpha}_{(v)} - \overset{*}{C}{}^{\lambda}_{\alpha} a^{\alpha\mu}_{(e)} + a^{\lambda\alpha}_{(e)} \underset{24}{P}{}_{0\alpha}{}^{\mu} = 0,$$

where we put

$$a^{\lambda}_{(x)\alpha} = \overset{*}{\Gamma}{}^{\lambda}_{\alpha} - \frac{\partial a^{\lambda}}{\partial x^{\alpha}} - \frac{\partial a^{\lambda}}{\partial l_{\mu}} \overset{*}{\Gamma}{}^{0}_{\mu\alpha}, \qquad a^{\lambda\alpha}_{(e)} = -\frac{\partial a^{\lambda}}{\partial l_{\alpha}} - \frac{\partial a^{\lambda}}{\partial l_{\beta}} A^{0\alpha}_{\beta},$$

$$a^{\lambda}_{(v)} = -\frac{\partial a^{\lambda}}{\partial l_{\alpha}} \overset{*}{C}{}^{0}_{\alpha} - \frac{\partial a^{\lambda}}{\partial v}.$$

The identities corresponding to the identities of BIANCHI, are obtained from the coefficients of crotchets $[\underset{1}{\delta x^\nu}\,\underset{2}{\delta x^\omega}\,\underset{3}{\delta x^\alpha}]$, $[\underset{1}{Dl_\nu}\,\underset{2}{Dl_\omega}\,\underset{3}{Dl_\alpha}]$, $[\underset{1}{\delta x^\nu}\,\underset{2}{\delta x^\omega}\,\underset{3}{Dl_\alpha}]$, $[\underset{1}{Dl_\nu}\,\underset{2}{Dl_\omega}\,\underset{3}{\delta x^\alpha}]$, $[\underset{1}{\delta x^\nu}\,\underset{2}{\delta x^\omega}\,\underset{3}{dt}]$, etc. in the relation of PFAFFian forms

$$(\Omega^{\lambda}_{\mu})' - [\omega^{\nu}_{\mu}\,\Omega^{\lambda}_{\nu}] + [\Omega^{\nu}_{\mu}\,\omega^{\lambda}_{\nu}] = 0 \tag{12.4}$$

and have the forms:

$$\begin{aligned}
&R^{\lambda}_{\mu[\nu\omega}|_{\alpha]} + \underset{12}{P}{}^{\lambda}_{\mu[\nu}{}^{h}\,R_{0|h|\omega\alpha]} = 0\,, \qquad S^{\lambda[\nu\omega}_{\mu}|^{\alpha]} + S^{\lambda\beta[\omega}_{\mu}\,S_{0\beta}{}^{\nu\alpha]} = 0\,,\\
&R^{\lambda}_{\mu\nu\omega}|^{\alpha} + R^{\lambda}_{\mu\delta[\omega}\,A^{\delta\alpha}_{\nu]} + \underset{12}{P}{}^{\lambda}_{\mu[\nu}{}^{\beta}\,\underset{12}{P}{}_{0|\beta|\omega]}{}^{\alpha} - S^{\lambda\beta\alpha}_{\mu}\,R_{0\beta\nu\omega} + \underset{12}{P}{}^{\lambda}_{\mu[\nu|\alpha|}|_{\omega]} = 0\,,\\
&S^{\lambda\nu\omega}_{\mu}|_{\alpha} - S^{\lambda\beta[\omega}_{\mu}\,\underset{12}{P}{}_{0\beta\alpha}{}^{\nu]} + \underset{12}{P}{}^{\lambda}_{\mu\alpha}{}^{\beta}\,S_{0\beta}{}^{\nu\omega} + \underset{12}{P}{}^{\lambda[\nu}_{\mu\alpha}|^{\omega]} - \underset{12}{P}{}^{\lambda[\omega}_{\mu\beta}\,A^{|\beta|\nu]}_{\alpha} = 0\,,\\
&R^{\lambda}_{\mu\nu\omega}|_{\mu} + R^{\lambda}_{\mu\beta[\omega}\,a^{\beta}_{(x)\nu]} + \underset{12}{P}{}^{\lambda}_{\mu[\nu}{}^{\alpha}\,\underset{13}{P}{}_{0|\alpha|\omega]} - \underset{23}{P}{}^{\lambda\alpha}_{\mu}\,R_{0\alpha\nu\omega} - \underset{13}{P}{}^{\lambda}_{\mu[\nu}|_{\omega]} = 0\,,\\
&R^{\lambda}_{\mu\nu\omega}|_{v} + R^{\lambda}_{\mu\beta[\omega}\,\overset{*}{C}{}^{\beta}_{\nu]} + \underset{12}{P}{}^{\lambda}_{\mu[\nu}{}^{\alpha}\,\underset{14}{P}{}_{0|\alpha|\omega]} - \underset{24}{P}{}^{\lambda\alpha}_{\mu}\,R_{0\alpha\nu\omega} - \underset{14}{P}{}^{\lambda}_{\mu[\nu}|_{\omega]} = 0\,,\\
&S^{\lambda\nu\omega}_{\mu}|_{t} + S^{\lambda\beta[\omega}_{\mu}\,\underset{23}{P}{}_{0\beta}{}^{\nu]} - \underset{23}{P}{}^{\lambda[\nu}_{\mu}|^{\omega]} - \underset{23}{P}{}^{\lambda\alpha}_{\mu}\,S_{0\alpha}{}^{\nu\omega} + \underset{12}{P}{}^{\lambda[\omega}_{\mu\beta}\,a^{|\beta|\nu]}_{(e)} = 0\,,\\
&S^{\lambda\nu\omega}_{\mu}|_{v} + S^{\lambda\beta[\omega}_{\mu}\,\underset{24}{P}{}_{0\beta}{}^{\nu]} - \underset{24}{P}{}^{\lambda[\nu}_{\mu}|^{\omega]} - \underset{24}{P}{}^{\lambda\alpha}_{\mu}\,S_{0\alpha}{}^{\nu\omega} = 0\,,
\end{aligned}$$

$$\begin{aligned}
&S^{\lambda\beta\omega}_{\mu}\,\underset{13}{P}{}_{0\beta\alpha} + \underset{12}{P}{}^{\lambda}_{\mu\beta}{}^{\omega}\,a^{\beta}_{(x)\alpha} + \underset{12}{P}{}^{\lambda}_{\mu\alpha}{}^{\beta}\,\underset{23}{P}{}_{0\beta}{}^{\omega} - \underset{23}{P}{}^{\lambda\beta}_{\mu}\,\underset{12}{P}{}_{0\beta\alpha}{}^{\omega} - \underset{13}{P}{}^{\lambda}_{\mu\beta}\,A^{\beta\omega}_{\alpha}\\
&\qquad + \underset{12}{P}{}^{\lambda}_{\mu\alpha}{}^{\omega}|_{t} - \underset{13}{P}{}^{\lambda}_{\mu\alpha}|^{\omega} + \underset{23}{P}{}^{\lambda\omega}_{\mu}|_{\alpha} - R^{\lambda}_{\mu\beta\alpha}\,a^{\beta\omega}_{(e)} = 0\,,\\
&S^{\lambda\beta\omega}_{\mu}\,\underset{14}{P}{}_{0\beta\alpha} + \underset{12}{P}{}^{\lambda}_{\mu\beta}{}^{\omega}\,\overset{*}{C}{}^{\beta}_{\alpha} + \underset{12}{P}{}^{\lambda}_{\mu\alpha}{}^{\beta}\,\underset{24}{P}{}_{0\beta}{}^{\omega} - \underset{24}{P}{}^{\lambda\beta}_{\mu}\,\underset{12}{P}{}_{0\beta\alpha}{}^{\omega} - \underset{14}{P}{}^{\lambda}_{\mu\beta}\,A^{\beta\omega}_{\alpha}\\
&\qquad + \underset{12}{P}{}^{\lambda}_{\mu\alpha}{}^{\omega}|_{v} - \underset{14}{P}{}^{\lambda}_{\mu\alpha}|^{\omega} + \underset{24}{P}{}^{\lambda\omega}_{\mu}|_{\alpha} = 0\,,\\
&- S^{\lambda\beta\omega}_{\mu}\,\underset{34}{P}{}_{0\beta} + \underset{23}{P}{}^{\lambda\beta}_{\mu}\,\underset{24}{P}{}_{0\beta}{}^{\omega} - \underset{24}{P}{}^{\lambda\beta}_{\mu}\,\underset{23}{P}{}_{0\beta}{}^{\omega} + \underset{23}{P}{}^{\lambda\omega}_{\mu}|_{v} + \underset{34}{P}{}^{\lambda}_{\mu}|^{\omega} - \underset{24}{P}{}^{\lambda\omega}_{\mu}|_{t}\\
&\qquad + \underset{12}{P}{}^{\lambda}_{\mu\beta}{}^{\omega}\,a^{\beta}_{(v)} + \underset{14}{P}{}^{\lambda}_{\mu\beta}\,a^{\beta\omega}_{(e)} = 0\,,\\
&\underset{12}{P}{}^{\lambda}_{\mu\omega}{}^{\beta}\,\underset{34}{P}{}_{0\beta} + \underset{13}{P}{}^{\lambda}_{\mu\beta}\,\overset{*}{C}{}^{\beta}_{\omega} + \underset{23}{P}{}^{\lambda\beta}_{\mu}\,\underset{14}{P}{}_{0\beta\omega} - \underset{14}{P}{}^{\lambda}_{\mu\beta}\,a^{\beta}_{(x)\omega} - \underset{24}{P}{}^{\lambda\beta}_{\mu}\,\underset{13}{P}{}_{0\beta\omega} + \underset{13}{P}{}^{\lambda}_{\mu\omega}|_{v}\\
&\qquad + \underset{34}{P}{}^{\lambda}_{\mu}|_{\omega} - \underset{14}{P}{}^{\lambda}_{\mu\omega}|_{t} - R^{\lambda}_{\mu\beta\omega}\,a^{\beta}_{(v)} = 0\;.
\end{aligned} \tag{12.5}$$