NON-HOLONOMIC SYSTEM IN A SPACE OF HIGHER ORDER II. ON THE THEORY OF EXTENSORS ON THE SUBSPACE

By

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Introduction The concept of the non-holonomic system in the higher order space has been already given by the present author [1][1], and many operations in the system has been studied too [2]. It is purpose of the present paper to treat the theory of extensors in a subspace of the higher order space under such the concept of the non-holonomic system. That is, we study, in §2 the operations introduced by A. KAWAGUCHI [3] in the exsurface and the expseudonormal defined in §1 and give the D-symbols of these operations. The same discussion is made for the excovariant differentiation in the space of the connection in §§3–4. In this paper we use certain of the ideas, notations and results given in the previous paper [1] without explanation.

The present author wishes to offer to Prof. A. KAWAGUCHI her thanks for his guidance.

§1 The exsurface and the expseudonormal. Let us give an m-dimensional subspace in the n-dimensional space by the parameter form

$$x^i = x^i(u^{j'}) \qquad i=1,\cdots,n\,;\ j'=1,\cdots,m;\ m\leqq n \tag{1.1}$$

and differentiate (1.1) in succession along parameterized arc of class P in the subspace, then we have the following results:[2]

$$\begin{cases} x'^i = \dfrac{\partial x^i}{\partial u^{j'}} u'^{j'} \\ x^{(2)i} = \dfrac{\partial x^i}{\partial u^{j'}} u^{(2)j'} + \dfrac{\partial^2 x^i}{\partial u^{i'} \partial u^{j'}} u^{(1)i'} u^{(1)j'} \\ \vdots \end{cases} \tag{1.2}$$

(1) Numbers in brackets refer to the references at the end of the paper.

(2) Throughout this paper, repeated lower case Latin indices call for summation 1 to n, while the summations indicated by repeated lower case Latin indices with prime are from 1 to m.

$$x^{(M)i} = \frac{\partial x^i}{\partial u^{j'}} u^{(M)j'} + (M-1)\frac{\partial^2 x^i}{\partial u^{i'}\partial u^{j'}} u^{(1)i'} u^{(M-1)j'} + \cdots \quad \text{for } M \leqq P.$$

Indicating any integer not exceeding P with a Greek letter, (1.2) give rise to the following relations:(3)

$$\frac{\partial x^{(\tau)i}}{\partial u^{i'}} = \left(\frac{\partial x^{(\tau-\delta)i}}{\partial u^{i'}}\right)^{(\delta)} = \left(\frac{\partial x^i}{\partial u^{i'}}\right)^{(\tau)}, \tag{1.3}$$

$$\frac{\partial x^{(\alpha)i'}}{\partial u^{(\beta)i'}} = \binom{\alpha}{\beta}\frac{\partial x^{(\alpha-\beta)i}}{\partial u^{i'}} \quad \text{for } \alpha \geqq \beta \geqq 0, \tag{1.4}$$

$$\frac{\partial x^{(\alpha)i}}{\partial u^{(\beta)i'}} = \frac{\partial x^i}{\partial u^{i'}} \quad \text{for } \alpha = \beta, \tag{1.5}$$

$$\frac{\partial x^{(\alpha)i}}{\partial u^{(\beta)i'}} = \frac{\alpha!\,(\beta-\gamma)!}{\beta!\,(\alpha-\gamma)!}\frac{\partial x^{(\alpha-\gamma)i}}{\partial u^{(\beta-\gamma)i'}} \quad \text{for } \alpha \geqq \beta \geqq \gamma \geqq 0, \tag{1.6}$$

$$\frac{\partial x^{(\alpha)i}}{\partial u^{(\beta)i'}} = 0 \quad \text{for } \beta > \alpha. \tag{1.7}$$

Remark. The rank of the matrix: $(B^{\alpha i}_{\beta j'})$ is $(M+1)m$, putting $B^{\alpha i}_{\beta j'} (\equiv B^{(\alpha)i}_{(\beta)j'}) \equiv \frac{\partial x^{(\alpha)i}}{\partial u^{(\beta)j'}}$ $(\alpha, \beta = 0, \cdots, M;\ i = 1, \cdots, n;\ j' = 1, \cdots, m)$.

Let us consider the space of line-elements of order M: $K^{(M)}_n$, then the set of expoints expressed by the right members of (1.1) and (1.2) will be called the parameterized $(M+1)m$-*dimensional subspace* in $K^{(M)}_n$ or briefly *exsurface* of order M and $u^{i'}, u'^{i'}, \cdots, u^{(M)i'}$ may be considered as the *extended parameters* (or *exparameters*). Then we have the theorem:

Theorem 1.1. *The quantities $B^{\alpha i}_{\beta j'}$ are components of the extensor of the type indicated by the indices, that is, excontravariant in $K^{(M)}_n$ and excovariant in $K^{(M)}_m$.*

Proof. By the reason that the quantities $B^{\alpha i}_{\beta j'}$ are changed as follows:

$$\bar{B}^{\lambda a}_{\mu b'} = \sum_{\alpha=\mu}^{\lambda}\sum_{\beta=\mu}^{\alpha} X^{\lambda a}_{\alpha i} U^{\beta j'}_{\mu b'} B^{\alpha i}_{\beta j'}$$

under any extended coordinate and exparameter transformation: $\bar{x}^a = \bar{x}^a(x^i)$, $\bar{x}^{(\lambda+1)a} = \sum_{\alpha=0}^{\lambda} X^{\lambda a}_{\alpha i} x^{(\alpha+1)i}$ and $\bar{u}^{b'} = \bar{u}^{b'}(u^{j'})$, $\bar{u}^{(\mu+1)b'} = \sum_{\beta=0}^{\mu} U^{\mu b'}_{\beta j'} u^{(\beta+1)j'}$, putting $U^{\mu b'}_{\beta j'} \equiv \frac{\partial \bar{u}^{(\mu)b'}}{\partial u^{(\beta)j'}}$, the theorem follows.

Further let us associate with each expoint on the exsurface: $K^{(M)}_m$ $(M+1)(n+m)$ **excontravariant extensors**: $\lambda^{\alpha i}_{\beta p'}$, $(\beta = 0, \cdots, M;\ p' = m+1,$

(3) These relations can be verified by the same manner as that of KAWAGUCHI ([3], pp. 17–19).

$\cdots, n$) of the range M in $K_n^{(M)}$, then at each expoint on the exsurface $(M+1)n$ excontravariant extensors, that is, $\lambda^{\alpha i}_{\beta p''}$ and $B^{\alpha i}_{\beta j'}$ are endowed. In addition, we take over the assumptions that $(M+1)n$ extensors are mutually independent (i.e., the determinant constructed from $\lambda^{\alpha i}_{\beta p''}$ and $B^{\alpha i}_{\beta j'}$ does not vanish) and $\lambda^{\alpha i}_{\beta p''}=0$ for $\beta>\alpha$. A field of $(M+1)(n-m)$ extensors $\lambda^{\alpha i}_{\beta j'}$ having such the properties is called the *expseudonormal* and indicated by $X_n^{(M)(n-m)}$. Of course, it will be understood that $\lambda^{\alpha i}_{\beta p''}$ and $B^{\alpha i}_{\beta j'}$ are the functions of exparameters $u^{j'}, u'^{j'}, \cdots, u^{(M)j'}$. Hence we can introduce the reciprocal excovariant extensors $B^{\gamma j'}_{\alpha i}$ and $\lambda^{\beta p''}_{\alpha i}$ satisfying the following equations

$$(1.8)\quad \begin{cases} \sum_{\alpha=0}^{M} B^{\alpha i}_{\beta i'} B^{\gamma j'}_{\alpha i} = \delta^{\gamma}_{\beta}\delta^{j'}_{i'}, & \sum_{\alpha=0}^{M} \lambda^{\alpha i}_{\beta p''} B^{\gamma j'}_{\alpha i} = 0; \\ \sum_{\alpha=0}^{M} B^{\alpha i}_{\beta i'} \lambda^{\gamma p''}_{\alpha i} = 0, & \sum_{\alpha=0}^{M} \lambda^{\alpha i}_{\beta p''} \lambda^{\gamma q''}_{\alpha i} = \delta^{\gamma}_{\beta}\delta^{q''}_{p''}. \end{cases}$$

Consequently, it will be found that such the excovariant extensors are the functions of $u^{j'}, \cdots, u^{(M)j'}$. From (1.8), we may state without proof the following theorem:

Theorem 1.2. *The following relation consists*

$$\sum_{\alpha=\delta}^{\Upsilon} (B^{i}_{\alpha j'} B^{\alpha j'}_{\delta j} + \lambda^{\Upsilon i}_{\alpha p''} \lambda^{\alpha p''}_{\delta j}) = \delta^{\Upsilon}_{\delta}\delta^{i}_{j}.^{(4)}$$

Similarly as the case of the ordinary point space, an extensor: $T'^{\alpha_1 i'_1 \cdots \alpha_A i'_A}{}_{\beta_1 j'_1 \cdots \beta_B j'_B}$ of characteristic $(A+B, 0, R, D)$ $(R \leqq M, M \leqq D \leqq P)$ in $K_m^{(D)}$ and an extensor: $T'^{\alpha_1 p''_1 \cdots \alpha_A p''_A}{}_{\beta_1 q''_1 \cdots \beta_B q''_B}$ of the same characteristic in $K_n^{(D)(n-m)}$ have the components referred to the x coordinate system in $K_n^{(D)}$:

$$T^{\Upsilon_1 i_1 \cdots \Upsilon_A i_A}{}_{\delta_1 j_1 \cdots \delta_B j_B} = T'^{\alpha_1 i'_1 \cdots \alpha_A i'_A}{}_{\beta_1 j'_1 \cdots \beta_B j'_B} \prod_{\lambda=1}^{A} \prod_{\mu=1}^{B} B^{\Upsilon}_{\alpha}{}^{\lambda}_{\lambda}{}^{i}_{i'}{}^{\lambda}_{\lambda} B^{\beta}_{\delta}{}^{\mu}_{\mu}{}^{j'}_{j}{}^{\mu}_{\mu} \quad (5)$$

and

$$T^{\Upsilon_1 i_1 \cdots \Upsilon_A i_A}{}_{\delta_1 j_1 \cdots \delta_B j_B} = T'^{\alpha_1 p''_1 \cdots \alpha_A p''_A}{}_{\beta_1 q''_1 \cdots \beta_B q''_B} \prod_{\lambda=1}^{A} \prod_{\mu=1}^{B} \lambda^{\Upsilon}_{\alpha}{}^{\lambda}_{\lambda}{}^{i}_{p''}{}^{\lambda}_{\lambda} \lambda^{\beta}_{\delta}{}^{\mu}_{\mu}{}^{q''}_{j}{}^{\mu}_{\mu}$$

respectively. Conversely, if an extensor in $K_m^{(D)}$ resp. $K_n^{(D)(n-m)}$ have the components: $T^{\Upsilon_1 i_1 \cdots \Upsilon_A i_A}{}_{\delta_1 j_1 \cdots \delta_B j_B}$ referred to the x coordinate system in $K_n^{(D)}$, its components referred to the u coordinate system in $K_m^{(D)}$ resp. to the expseudonormal should be expressed by

$$T'^{\alpha_1 i'_1 \cdots \alpha_A i'_A}{}_{\beta_1 j'_1 \cdots \beta_B j'_B} = T^{\Upsilon_1 i_1 \cdots \Upsilon_A i_A}{}_{\delta_1 j_1 \cdots \delta_B j_B} \prod_{\lambda=1}^{A} \prod_{\mu=1}^{B} B^{\alpha}_{\Upsilon}{}^{\lambda}_{\lambda}{}^{i'}_{i}{}^{\lambda}_{\lambda} B^{\beta}_{\delta}{}^{\mu}_{\mu}{}^{j}_{j'}{}^{\mu}_{\mu}$$

(4) Repeated lower case Latin indices with doublet primes call for summation $m+1$ to n.

(5) The summations indicated by repeated lower case Greek indices are from zero to R (range).

resp.

$$T'^{\alpha_1 p''_1 \cdots \alpha_A p''_A}{}_{\beta_1 q''_1 \cdots \beta_B q''_B} = T^{\tau_1 i_1 \cdots \tau_A i_A}{}_{\delta_1 j_1 \cdots \delta_B j_B} \prod_{\lambda=1}^{A} \prod_{\mu=1}^{B} \lambda^{\alpha_\lambda}_{\tau_\lambda} \lambda^{p''_\lambda}_{i_\lambda} \lambda^{\delta_\mu}_{\beta_\mu} \lambda^{j_\mu}_{q''_\mu} ;$$

For example, if we adopt $u^{(\alpha)j'}$ $(\alpha = 1, \cdots, M)$ as an extensor in $K_m^{(M)}$, then its components referred to the coordinate system in the $K_n^{(M)}$ will be given by

$$x^{(\alpha+1)i} = \sum_{\beta=0}^{\alpha} B^{\alpha i}_{\beta j'} u^{(\beta+1)j'} \qquad \alpha = 0, \cdots, M-1.$$

If we consider a general extensor in $K_n^{(D)}$ associated with an expoint on the exsurface, for example, if we take up an excontravariant extensor $v^{\alpha i}$ of order one, then it is possible to express it in the form

$$v^{\alpha i} = \sum_{\beta=0}^{\alpha} B^{\alpha i}_{\beta j'} v^{\beta j'} + \sum_{\tau=0}^{\alpha} \lambda^{\alpha\ i}_{\tau p''} v^{\tau p''} ,$$

consequently it follows that

$$v^{\beta j'} = \sum_{\alpha=0}^{\beta} B^{\beta j'}_{\alpha i} v^{\alpha i} , \qquad v^{\tau p''} = \sum_{\alpha=0}^{\tau} \lambda^{\tau p''}_{\alpha i} v^{\alpha i} ,$$

that mean the projections of the extensor $v^{\alpha i}$ upon $K_m^{(D)}$ and $K_n^{(D)(n-m)}$ respectively. Such the representation is likely possible for any general extensor of higher order.

§2 The $\mathfrak{C}^H$-operation. We shall state the following theorem as the first step to the study on the $\mathfrak{C}^H$-operation of excontravariant extensors $v^{\alpha i}$ belonging to $K_m^{(D)}$.

Theorem 2.1. *Let*

$$v^{\tau i} = \sum_{\beta=0}^{\tau} B^{\tau i}_{\beta i'} v^{\beta i'} \tag{2.1}$$

be an excontravariant extensor of characteristic $(1, 0, R, M)$ *in* $K_m^{(M)}$, *then the extensor* $\mathfrak{C}^H v^{\tau i}$ ([3], p. 29) *belongs to* $K_m^{(M+H)}$ (6) *and its components referred to the u exparameters are* $\mathfrak{C}^H v^{\tau i'} = \sum_{\lambda=0}^{H} (-1)^{H-\lambda} \binom{H}{\lambda} v^{\tau+\lambda i'(H-\lambda)}$ $(\tau = 0, \cdots, R-H)$.

Proof. The method of proof is same as that of KAWAGUCHI ([3], p. 29). That is, from (2.1) it follows that

$$\begin{aligned} \mathfrak{C}^H v^{\tau i} &= \sum_{\lambda=0}^{H} (-1)^{H-\lambda} \binom{H}{\lambda} (B^{\tau+\lambda i}_{\beta\ i'} v^{\beta i'})^{(H-\lambda)} \\ &= \sum_{\lambda=0}^{H} (-1)^{H-\lambda} \binom{H}{\lambda} \sum_{\mu=0}^{H-\lambda} \binom{H-\lambda}{\mu} B^{\tau+\lambda i^{(\mu)}}_{\beta\ i'} v^{\beta i'(H-\lambda-\mu)} \end{aligned}$$

(6) Afterward $M+H$ denotes an integer not exceeding P.

$$= \sum_{\mu=0}^{H}\sum_{\lambda=0}^{H-\mu}\sum_{\beta=0}^{\gamma+\lambda}(-1)^{H-\lambda}\binom{H}{\mu}\binom{H-\mu}{\lambda}\binom{\gamma+\lambda}{\beta}\left(\frac{\partial x^i}{\partial u^{i'}}\right)^{(\gamma+\lambda-\beta+\mu)} v^{\beta i'(H-\lambda-\mu)}$$

$$= \sum_{\nu=0}^{H}(-1)^{H-\nu}\binom{H}{\nu}\sum_{\beta=0}^{\gamma+\nu}\left\{\sum_{\mu=0}^{\nu}(-1)^{\mu}\binom{\nu}{\mu}\binom{\gamma+\nu-\mu}{\beta}\right\}\left(\frac{\partial x^i}{\partial u^{i'}}\right)^{(\gamma-\beta+\nu)} v^{\beta i'(H-\nu)}$$

(putting: $\nu = \lambda + \mu$)

$$= \sum_{\nu=0}^{H}\sum_{\delta=0}^{\gamma}\binom{\gamma}{\delta}\left(\frac{\partial x^i}{\partial u^{i'}}\right)^{(\gamma-\delta)}(-1)^{H-\nu}\binom{H}{\nu} v^{\delta+\nu i'(H-\nu)} = B^{\gamma i}_{\delta i'}\,\mathfrak{C}^H v^{\delta i'}$$

(putting: $\delta = \beta - \nu$).

Multiplying the last result by $B^{\beta j'}_{\gamma i}$ and summing with respect to γ and i, we have $\mathfrak{C}^H v^{\beta j'} = \sum_{\gamma=0}^{\beta} B^{\beta j'}_{\gamma i}\,\mathfrak{C}^H v^{\gamma i}$.

Next let us consider the $\mathfrak{C}^H$-operation of an excontravariant extensor belonging to $K_n^{(M)}$ associated with each expoint on a curve in the exsuface, and use the symbol $\mathfrak{C}^H v^{\alpha q''}$ for the extensor in $K_n^{(M+H)(n-m)}$ which is obtained by applying the $\mathfrak{C}^H$-operation to an extensor $v^{\alpha q''}$ and has the structure $\sum_{\rho=0}^{H}\sum_{\delta=0}^{\gamma+\rho}(-1)^{H-\rho}\binom{H}{\rho} v^{\delta p''(H-\rho)} C^{\rho\,\alpha q''}_{\delta\,p''}(\mathfrak{C})$ ([1], p.197) and the symbol ${}^{\blacktriangle}\mathfrak{C}^H v^{\alpha i'}$ for $\sum_{\rho=0}^{H}\sum_{\delta=0}^{\gamma+\rho}(-1)^{H-\rho}\binom{H}{\rho} v^{\delta p''(H-\rho)} C^{\rho\,\alpha i'}_{\delta\,p''}(\mathfrak{C})$, where $C^{\rho\,\alpha i'}_{\delta\,p''}(\mathfrak{C})$, $C^{\rho\,\delta p''}_{\alpha\,i'}(\mathfrak{C})$, $C^{\rho\,\alpha q''}_{\beta\,\rho''}(\mathfrak{C})$ and $C^{\rho\,\alpha i'}_{\beta\,j'}(\equiv \delta_\beta{}^\alpha{}_\rho\,\delta^{i'}_{j'})$ are the $\mathfrak{C}$-operation coefficients of excontravariant extensor in the non-holonomic system: $B^{\alpha i}_{\beta i'}$ and $\lambda^{\alpha i}_{\beta p''}$ ([1], p. 197), then we get the following statement.

Theorem 2.2. *When we write an extensor of characteristic* $(1, 0, R, M)$ *in the form*

$$v^{\gamma i} = \sum_{\alpha=0}^{\gamma} B^{\gamma i}_{\alpha i'} v^{\alpha i'} + \sum_{\alpha=0}^{\dot{\gamma}} \lambda^{\gamma i}_{\alpha p''} v^{\alpha p''} \qquad \gamma = 0, \cdots, R, \tag{2.2}$$

the following relations consist

$$\mathfrak{C}^H v^{\gamma i} = \sum_{\beta=0}^{\dot{\gamma}} B^{\gamma i}_{\beta i'}\,\mathfrak{C}^H v^{\beta i'} + \sum_{\rho=0}^{H}\sum_{\delta=0}^{\gamma+H-\rho}(-1)^{\rho}\binom{H}{\rho}\,\mathfrak{C}^{H-\rho}\,\lambda^{\gamma i}_{\delta p''} v^{\delta p''(\rho)},$$

$$\sum_{\gamma=0}^{\alpha} \dot{B}^{\alpha i'}_{\gamma i}\,\mathfrak{C}^H v^{\gamma i} = \mathfrak{C}^H v^{\alpha i'} - {}^{\blacktriangle}\mathfrak{C}^H v^{\alpha i'}$$

and

$$\sum_{\gamma=0}^{\alpha} \lambda^{\alpha q''}_{\gamma i}\,\mathfrak{C}^H v^{\gamma i} = \mathfrak{C}^H v^{\alpha q''} \qquad \alpha = 0, \cdots, R-H.$$

The proof of this statement is essentially the same to the explanation of the $\mathfrak{C}^H$-operation of an excontravariant extensor in the non-holonomic system ([1], p. 197). That is, from the definition of $\mathfrak{C}^H v^{\gamma i}$ and Theorem 2.1, the first relation in this theorem follows and multi-

plying the both-hand members of this relation by $B^{\alpha i'}_{\gamma i}$ or $\lambda^{\alpha p''}_{\gamma i}$ and summing with respect to γ and i, the second relation or the third relation is obtained.

Corollary. *If an extensor of characteristic* $(1, 0, R, M)$ *is given by*

$$v^{\gamma i} = \sum_{\alpha=0}^{\gamma} B^{\gamma i}_{\alpha i'} v^{\alpha i'} + \sum_{\alpha=0}^{\gamma} \lambda^{\gamma i}_{\alpha p''} v^{\alpha p''} \qquad \gamma = 0, \cdots, R,$$

then the difference between the projection of the extensor $\mathfrak{C}^H v^{\gamma i}$ *upon* $K_m^{(M+H)}$ *and the extensor* $\mathfrak{C}^H v^{\alpha i'}$ *is equal to* ${}^{\blacktriangle}\mathfrak{C}^H v^{\alpha i'} \left(\equiv \sum_{\rho=0}^{H} \sum_{\delta=0}^{\gamma+\rho} (-1)^{H-\rho} \binom{H}{\rho} v^{\delta p''(H-\rho)} \times C^{\rho\, \alpha i'}_{\delta\, p''}(\mathfrak{C}) \right)$, *and the projection of the extensor* $\mathfrak{C}^H v^{\gamma i}$ *upon* $K_m^{(M+H)(n-m)}$ *is nothing but* $\mathfrak{C}^H v^{\alpha q''}$.

We get the following theorems for excovariant extensors.

Theorem 2.3. *Let*

$$w_{\gamma i} = \sum_{\alpha=0}^{\gamma} B^{\alpha i'}_{\gamma i} w_{\alpha i'} \tag{2.3}$$

be an excovariant extensor of characteristic $(1, 0, R, M)$ *belonging to* $K_m^{(M)}$, *then the extensor in* $K_m^{(M+H)}$ *obtained by applying* $\mathfrak{C}^H$*-operation to the extensor* $w_{\alpha i'}$,

$$\mathfrak{C}^H w_{\alpha i'} \equiv H! \sum_{\nu=0}^{H} (-1)^{\nu} \binom{\alpha+\nu}{\alpha} \binom{R-\alpha-\nu}{R-H-\alpha} w_{\alpha+\nu i'}{}^{(\nu)} \qquad \alpha = 0, \cdots, R-H,$$

is equal to the projection of the extensor $\mathfrak{C}^H w_{\gamma i}$ ([3], p. 31) *upon* $K_m^{(M+H)}$; *and the projection of the extensor* $\mathfrak{C}^H w_{\gamma i}$ *upon* $K_n^{(M+H)(n-m)}$ *is* ${}^{\blacktriangle}\mathfrak{C} w_{\alpha p''}$, *where* ${}^{\blacktriangle}\mathfrak{C}^H w_{\alpha p''} = \sum_{\rho=0}^{H} \sum_{\delta=0}^{R} H! (-1)^{\rho} \binom{\delta}{\rho} \binom{R-\delta}{H-\rho} w_{\delta j'}{}^{(\rho)} C^{\rho\, \delta j'}_{\alpha\, p''}(\mathfrak{C}^H)$ *and* $C^{\rho\, \delta j'}_{\alpha\, p''}(\mathfrak{C}^H)$ $(j' = 0, \cdots, m;\ p'' = m+1, \cdots, n)$ *are a part of* $\mathfrak{C}^H$*-operation coefficients of excontravariant extensor in the non-holonomic system* ([1], p. 197).

Proof. By virtue of (2.3), we have $w_{\alpha i'} = \sum_{\gamma=\alpha}^{R} B^{\gamma i}_{\alpha i'} w_{\gamma i}$; then it follows that

$$\begin{aligned}
\mathfrak{C}^H w_{\alpha i'} &= H! \sum_{\nu=0}^{H} (-1)^{\nu} \binom{\alpha+\nu}{\alpha} \binom{R-\alpha-\nu}{R-H-\alpha} w_{\alpha+\nu i}{}^{(\nu)} \\
&= H! \sum_{\nu=0}^{H} (-1)^{\nu} \binom{\alpha+\nu}{\alpha} \binom{R-\alpha-\nu}{R-H-\alpha} \sum_{\gamma=\alpha+\nu}^{R} \binom{\gamma}{\alpha+\nu} \\
&\qquad \times \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \left(\frac{\partial x^i}{\partial u^{i'}} \right)^{(\gamma-\alpha-\nu+\mu)} w_{\gamma i}{}^{(\nu-\mu)} \\
&= H! \sum_{\nu=0}^{H} (-1)^{\nu} \binom{\alpha+\nu}{\alpha} \binom{R-\alpha-\nu}{R-H-\alpha} \sum_{\lambda=0}^{R} \binom{\nu}{\lambda} \sum_{\gamma=\alpha+\nu}^{R} \binom{\lambda}{\alpha+\nu} \\
&\qquad \times \left(\frac{\partial x^i}{\partial u^{i'}} \right)^{(\gamma-\alpha-\lambda)} w_{\gamma i}{}^{(\lambda)}
\end{aligned}$$

(putting: $\lambda = \nu - \mu$)

$$= H! \sum_{\lambda=0}^{H} \sum_{\nu=\lambda}^{H} (-1)^{\nu} \binom{\alpha+\nu}{\nu} \binom{R-\alpha-\nu}{R-H-\alpha} \times \binom{\nu}{\lambda} \sum_{\beta=\alpha+\nu-\lambda}^{R-\lambda} \binom{\beta+\lambda}{\alpha+\nu} \left(\frac{\partial x^{i}}{\partial u^{i'}}\right)^{(\beta-\alpha)} w_{\beta+\lambda i}^{(\lambda)} .$$

(putting: $\beta = \gamma - \lambda$)

$$= H! \sum_{\lambda=0}^{H} \sum_{\beta=0}^{R-H} (-1)^{\lambda} \binom{\beta+\lambda}{\beta} \binom{R-\beta-\lambda}{R-H-\beta} \left\{ \sum_{\mu=0}^{H-\lambda} (-1)^{H-\mu-\lambda} \binom{H-\lambda}{\mu} \times \binom{R-H-\alpha+\mu}{R-\beta-\lambda} \binom{R-H-\gamma}{\beta-\gamma}^{-1} \right\} \binom{\beta}{\alpha} \left(\frac{\partial x^{i}}{\partial u^{i'}}\right)^{(\beta-\alpha)} w_{\beta+\lambda i}^{(\lambda)}$$

(putting: $\mu = H - \nu$)

$$= \sum_{\beta=0}^{R-H} \frac{\partial x^{(\beta)i}}{\partial u^{(\alpha)i'}} H! \sum_{\lambda=0}^{H} (-1)^{\lambda} \binom{\beta+\lambda}{\beta} \binom{R-\beta-\lambda}{R-H-\beta} w_{\beta+\lambda i}^{(\lambda)} = \sum_{\beta=0}^{R-H} \frac{\partial x^{(\beta)i}}{\partial u^{(\alpha)i'}} \mathfrak{S}^{H} w_{\beta i} .$$

Using the same method as that of the proof of Theorem 11 in the previous paper ([1], p. 200–201), we get the result

$$\sum_{\gamma=\alpha}^{R-H} \lambda_{\alpha p''}^{\gamma i} \mathfrak{S}^{H} w_{\gamma i} = \sum_{\rho=0}^{H} \sum_{\delta=0}^{R} H! (-1)^{\rho} \binom{\delta}{\rho} \binom{R-\delta}{H-\rho} w_{\delta j'}^{(\rho)} C_{\alpha p''}^{\rho \delta j'}(\mathfrak{S}^{H}) ,$$

where

$$C_{\alpha p''}^{\rho \delta \nu}(\mathfrak{S}^{H}) = \left\{ (H-\rho)! \binom{\delta}{\rho} \binom{R-\delta}{H-\rho} \right\}^{-1} \sum_{\gamma=\alpha}^{R-H} \binom{\gamma+\rho}{\rho} \lambda_{\alpha p'}^{\gamma i} \mathfrak{S}^{H-\rho} B_{\gamma+\rho i}^{\delta j'} \text{ ([1], p. 201).}$$

Consider the $\mathfrak{S}^{H}$-operation of an excovariant extensor in $K_n^{(M)}$ associated with each expoint on a curve in the exsurface, then we can see the following theorem, where the symbol $\mathfrak{S}^{H} w_{\alpha p''}$ denotes the extensor in $K_n^{(M+H(n-m))}$ which is obtained by applying the $\mathfrak{S}^{H}$-operation to an extensor $w_{\gamma p''}$ and has the structure $\sum_{\rho=0}^{H} \sum_{\delta=0}^{R} H! (-1)^{\rho} \binom{\delta}{\rho} \binom{R-\delta}{H-\rho} w_{\delta q'}^{(\rho)} C_{\alpha p''}^{\rho \delta q''}(\mathfrak{S}^{H})$, and $C_{\alpha p''}^{\rho \delta i'}(\mathfrak{S}^{H})$, $C_{\alpha i'}^{\rho \delta p''}(\mathfrak{S}^{H})$, $C_{\alpha q''}^{\rho \delta p''}(\mathfrak{S}^{H})$ and $C_{\alpha j'}^{\rho \delta i'}(\mathfrak{S}^{H})$ form a complete set of the $\mathfrak{S}^{H}$-operation coefficients of excovariant extensor in the non-holonomic system: $B_{\beta i'}^{\alpha i}$, $\lambda_{\beta q''}^{\alpha i}$ ([1], p. 201).

Theorem 2.4. *If an excovariant extensor of characteristic* $(1, 0, R, M)$ *in* $K_n^{(M)}$ *associated with each expoint on a curve in the exsurface is given by*

$$w_{\gamma i} = \sum_{\alpha=\gamma}^{R} (B_{\gamma i}^{\alpha i'} w_{\alpha i'} + \lambda_{\gamma i}^{\alpha p''} w_{\alpha p''}) , \tag{2.4}$$

then the following relations hold good

$$\mathfrak{S}^{H} w_{\gamma i} = \sum_{\alpha=\gamma}^{R-H} \left\{ B_{\gamma i}^{\alpha i'} \mathfrak{S}^{H} w_{\alpha i'} + \lambda_{\gamma i}^{\alpha p''} (\mathfrak{S}^{H} w_{\alpha p''} + \mathfrak{S}^{H} w_{\alpha p''}) \right\} ,$$

$$\sum_{\gamma=\alpha}^{R-H} B_{\alpha i'}^{\gamma i} \mathfrak{S}^{H} w_{\gamma i} = \mathfrak{S}^{H} w_{\alpha i'}$$

and

$$\sum_{\tau=\alpha}^{R-H} \lambda^{\tau i}_{\alpha p''} \mathfrak{C}^H w_{\tau i} = {}^{\blacktriangle}\mathfrak{C}^H w_{\alpha p''} + \mathfrak{C}^H w_{\alpha p''} .$$

Proof. By consequence of (2.4), we have the relation:

$$\mathfrak{C}^H w_{\tau i} = \mathfrak{C}^H \left(\sum_{\alpha=\tau}^{R} B^{\alpha i'}_{\tau i} w_{\alpha i'} \right) + \mathfrak{C}^H \left(\sum_{\alpha=\tau}^{R} \lambda^{\alpha p''}_{\tau i} w_{\alpha p''} \right),$$

the first term of the right hand member of the above expression is equal to $\sum_{\alpha=\tau}^{R-H} (B^{\alpha i'}_{\tau i} \mathfrak{C}^H w_{\alpha i'} + \lambda^{\alpha p''}_{\tau i} {}^{\blacktriangle}\mathfrak{C}^H w_{\alpha p''})$ according to Theorem 2.3, and the second term becomes $\sum_{\alpha=\tau}^{R-H} \lambda^{\alpha p''}_{\tau i} \mathfrak{C}^H w_{\alpha p''}$ as we see in the same way as the explanation of the $\mathfrak{C}^H$-operation of an excovariant extensor in the non-holonomic system ([1], p. 201). Further multiplying the both-hand members of the resulting expression by $B^{\beta j'}_{\tau i}$ or $\lambda^{\beta q''}_{\tau i}$ and summing with respect to τ and i, the second expression or the third expression in the theorem is obtained.

Corollary. *When an excovariant extensor of characteristic* $(1, 0, R, M)$ *in* $K^{(M)}_n$ *associated with each expoint of a curve on the exsurface is expressed by* (2.4), *the projection of the extensor* $\mathfrak{C}^H w_{\tau i}$ *upon* $K^{(M+H)}_m$ *coincides with* $\mathfrak{C}^H w_{\alpha i'}$, *and the difference between the projection of* $\mathfrak{C}^H w_{\tau i}$ *upon* $K^{(M+H)(n-m)}_n$ *and the extensor* $\mathfrak{C}^H w_{\alpha p''}$ *is* ${}^{\blacktriangle}\mathfrak{C}^H w_{\alpha p''}$.

There are the corresponding theorems for the $\mathfrak{Z}^H$, $\mathfrak{Y}^H$-operations of extensors, and these theorems are given by the same statement as those for the $\mathfrak{C}^H$-operation.

At last, let us express any one of $\mathfrak{C}$, $\mathfrak{Z}$ and $\mathfrak{Y}$-operations by a symbol $\mathfrak{R}$, then we may define D^H-symbols referred to the $\mathfrak{R}^H$-operation of an excontravariant extensor and of an excovariant extensor of characteristic $(1, 0, R, M)$ by

$$D^H T^{\tau i} = \mathfrak{R}^H T^{\tau i} \qquad \text{for } T^{\tau i} \text{ in } K^{(M)}_n ,$$

$$D^H T^{\tau i'} = \sum_{\rho=0}^{\tau} B^{\tau i'}_{\rho i} \mathfrak{R}^H T^{\rho i} \; (= \mathfrak{R}^H T^{\tau i'}) \qquad \text{for } T^{\tau i'} \text{ in } K^{(M)}_m ,$$

$$D^H T^{\tau p''} = \sum_{\rho=0}^{\tau} \lambda^{\tau p''}_{\rho i} \mathfrak{R}^H T^{\rho i} \; (= \mathfrak{R}^H T^{\tau p''}) \qquad \text{for } T^{\tau p''} \text{ in } K^{(M)(n-m)}_n$$

and

$$D^H T_{\tau i} = \mathfrak{R}^H T_{\tau i} \qquad \text{for } T_{\tau i} \text{ in } K^{(M)}_n ,$$

$$D^H T_{\tau i'} = \sum_{\rho=\tau}^{G} B^{\rho i}_{\tau i'} \mathfrak{R}^H T_{\rho i} \; (= \mathfrak{R}^H T_{\tau i'}) \qquad \text{for } T_{\tau i'} \text{ in } K^{(M)}_m ,$$

$$D^H T_{\tau p''} = \sum_{\rho=\tau}^{G} \lambda^{\rho i}_{\tau p''} \mathfrak{R}^H T_{\rho i} \; (= R^H T_{\tau p''}) \qquad \text{for } T_{\tau p''} \text{ in } K^{(M)(n-m)}_n$$

respectively, where G is the range of $\mathfrak{R}^{H}T'_{\rho i}$.

§ 3. **The exsurface in a space $C_n^{(M)}$ with an affine connection.** We shall suppose such a continuous family of curves that one and only one curve of it passes through every one point in the considered domain of the space and confine ourselves to consider only the space $C_n^{(M)}$ with affine connection, of which parameters $\Gamma_{\alpha i\ \beta j}^{\tau k}$ are defined on every curve in the space. Then, for example, the excovariant differential of an excontravariant extensor field $v^{ik}(x^{(\alpha)i})$ of characteristic $(1, 0, R, M)$ $R \leqq M$ is introduced in the form

$$\delta v^{ik} = dv^{ik} + \sum_{\tau=0}^{\tau} \sum_{\delta=0}^{M} \Gamma_{\tau i\ \delta j}^{ik}\, v^{\tau i} dx^{(\delta)j} \qquad \tau = 0, \cdots, R$$

where the displacement $dx^{(\delta)j}$ means difference of the line element at any two infinitesimally near points lying on any two infinitesimally near curves of the familly respectively. Then, supposing a similar continuous family of curves in the exsurface $C_m^{(M)}$, the connection parameter ${}'\Gamma_{\theta j'\ \delta k'}^{\tau i'}$ induced upon $C_m^{(M)}$ from such the connection parameter $\Gamma_{\alpha i\ \beta j}^{\rho k}$ at the expoint on a curve in $C_m^{(M)}$ is given by

$$(5.1) \qquad {}'\Gamma_{\delta j'\ \theta k'}^{\tau i'} = \sum_{\alpha=\delta}^{\tau} \sum_{\beta=\theta}^{M} - B_{\delta j'}^{\alpha i} B_{\theta k'}^{\beta j} \left\{ \frac{\partial B_{\alpha i}^{\tau i'}}{\partial x^{(\beta)j}} - \sum_{\rho=0}^{\tau} \Gamma_{\alpha i\ \beta j}^{\rho k} B_{\rho k}^{\tau i'} \right\}$$

$$= \sum_{\alpha=\delta}^{\tau} \sum_{\beta=\theta}^{M} - B_{\delta j'}^{\alpha i} B_{\theta k'}^{\beta j} B_{\alpha i,\ \beta j}^{\tau i'}$$

where $B_{\alpha i,\ \beta j}^{\tau i'}$ means the excovariant derivative of $B_{\alpha i}^{\tau i'}$ regarded as an excovariant extensor of $C_n^{(M)}$, and the excovariant differential and the excovariant derivative of an extensor field $v^{\tau i'}$ $(u^{(\alpha)i'})$ of characteristic $(1, 0, R, M)$ in $C_m^{(M)}$, along the exsurface: $C_m^{(M)}$, are defined by

$$\delta v^{\tau i'} = dv^{\tau i'} + \sum_{\rho=0}^{\tau} \sum_{\theta=0}^{M} {}'\Gamma_{\delta j'\ \theta k'}^{\tau i'}\, v^{\delta j'} du^{(\theta)k'} \qquad \tau = 0, \cdots, R,$$

and

$$v^{\tau i'}{}_{,\theta k'} = \frac{\partial v^{\tau i'}}{\partial u^{(\theta)k'}} + \sum_{\rho=0}^{\tau} {}'\Gamma_{\delta j'\ \theta k'}^{\tau i'}\, v^{\delta j'}$$

respectively. Further the connectin parameter ${}''\Gamma_{\theta q''\ \delta i'}^{\tau p''}$ that offers the excovariant differential (or derivative) in $K_n^{(M)(n-m)}$ along the exsurface is given by

$$ {}''\Gamma_{\theta q''\ \delta i'}^{\tau p''} = \sum_{\alpha=\delta}^{M} \sum_{\beta=\theta}^{\tau} - B_{\delta i'}^{\alpha i} \lambda_{\theta q''}^{\beta j} \left\{ \frac{\partial \lambda_{\beta j}^{\tau p''}}{\partial x^{(\alpha)i}} - \sum_{\rho=0}^{\tau} \Gamma_{\beta j\ \alpha i}^{\rho k} \lambda_{\rho k}^{\tau p''} \right\}$$

$$= \sum_{\alpha=\delta}^{M} \sum_{\beta=\theta}^{\tau} - B_{\delta i'}^{\alpha i} \lambda_{\theta q''}^{\beta j} \lambda_{\beta j,\ \alpha i}^{\tau p''},$$

such as the connection parameter in the non-holonomic system of the previous paper ([2], p. 64).

We can now define D-symols referred to the excovariant differentiation of an extensor of characteristic $(1, 0, R, M)$ as follows:

$$
(5.3)\quad
\begin{cases}
D_{\alpha i}P = \nabla_{\alpha i}P = \dfrac{\partial P}{\partial x^{(\alpha)i}} & \text{for a scalar } P \\
D_{\alpha i}q^{\beta j} = \nabla_{\alpha i}q^{\beta j} = q^{\beta j}{}_{,\alpha i} & \text{for an extensor } q^{\beta j} \text{ in } C_n^{(M)} \\
D_{\alpha i}v^{\gamma i'} = \sum_{\delta=0}^{\gamma} B_{\delta j}^{\gamma i'}\nabla_{\alpha i}v^{\delta j} = \sum_{\delta=0}^{\gamma} B_{\delta j}^{\gamma i'}v^{\delta j}{}_{,\alpha i} & \text{for an extensor } v^{\gamma i'} \text{ in } C_m^{(M)} \\
D_{\alpha i}w^{\gamma p''} = \sum_{\delta=0}^{\gamma} \lambda_{\delta j}^{\gamma p''}\nabla_{\alpha i}w^{\delta j} = \sum_{\delta=0}^{\gamma} \lambda_{\delta j}^{\gamma p''}w^{\delta j}{}_{,\alpha i} & \\
\qquad\qquad \text{for an extensor } w^{\gamma p''} \text{ in } C_n^{(M)(n-m)} & \\
D_{\alpha i}x^{(\beta)j} = \delta_\alpha^\beta\,\delta_i^j & ,
\end{cases}
$$

$$
(5.4)\quad
\begin{cases}
D_{\alpha i'}P = \sum_{\gamma=\alpha}^{M} B_{\alpha i'}^{\gamma i}P_{;\gamma i} = \dfrac{\partial P}{\partial u^{(\alpha)i'}} & \text{for a scalar } P \\
D_{\alpha i'}q^{\beta j} = \sum_{\gamma=\alpha}^{M} B_{\alpha i'}^{\gamma i}\nabla_{\gamma i}q^{\beta j} = \sum_{\gamma=\alpha}^{M} B_{\alpha i'}^{\gamma i}q^{\beta j}{}_{,\gamma i} & \text{for an extensor } q^{\beta i} \text{ in } C_n^{(M)} \\
D_{\alpha i'}v^{\beta j'} = \sum_{\gamma=\alpha}^{M}\sum_{\delta=0}^{\beta} B_{\alpha i'}^{\gamma i}B_{\delta j}^{\beta j'}\nabla_{\gamma i}v^{\delta j} = \sum_{\gamma=\alpha}^{M}\sum_{\delta=0}^{\beta} B_{\alpha i'}^{\gamma i}B_{\delta j}^{\beta j'}v^{\delta j}{}_{,\gamma i} = v^{\beta j'}{}_{,\alpha i'} & \\
\qquad\qquad \text{for an extensor } v^{\beta j'} \text{ in } C_m^{(M)} & \\
D_{\alpha i'}w^{\beta p''} = \sum_{\gamma=\alpha}^{M}\sum_{\delta=0}^{\beta} B_{\alpha i'}^{\gamma i}\lambda_{\delta j}^{\beta p''}\nabla_{\gamma i}w^{\delta j} = \sum_{\gamma=\alpha}^{M}\sum_{\delta=0}^{\beta} B_{\alpha i'}^{\gamma i}\lambda_{\delta j}^{\beta p''}w^{\delta j}{}_{,\gamma i} = w^{\gamma p''}{}_{,\alpha i'} & \\
\qquad\qquad \text{for an extensor } w^{\gamma p''} \text{ in } C_n^{(M)(n-m)} & \\
D_{\alpha i'}x^{(\beta)j} = \sum_{\gamma=\alpha}^{\beta} B_{\alpha i'}^{\gamma k}\delta_\gamma^\beta\delta_k^j = B_{\alpha i'}^{\beta j} & ,
\end{cases}
$$

and

$$
(5.5)\quad
\begin{cases}
D_{\alpha p''}P = \sum_{\beta=\alpha}^{M} \lambda_{\alpha p''}^{\beta j}P_{;\beta j} & \text{for a scalar } P \\
D_{\alpha p''}q^{\gamma j} = \sum_{\beta=\alpha}^{M} \lambda_{\alpha p''}^{\beta k}q^{\gamma j}{}_{,\beta k} & \text{for an extensor } q^{\gamma j} \text{ in } C_n^{(M)} \\
D_{\alpha p''}v^{\beta i'} = \sum_{\gamma=\alpha}^{M}\sum_{\delta=0}^{\beta} \lambda_{\alpha p''}^{\gamma i}B_{\delta j}^{\beta i'}v^{\delta j}{}_{,\gamma i} & \text{for an extensor } v^{\alpha i'} \text{ in } C_m^{(M)} \\
D_{\alpha p''}w^{\beta q''} = \sum_{\gamma=\alpha}^{M}\sum_{\delta=0}^{\beta} \lambda_{\alpha p''}^{\gamma i}\lambda_{\delta j}^{\beta q''}v^{\delta j}{}_{,\gamma i} & \text{for an extensor } w^{\beta q''} \text{ in } C_n^{(M)(n-m)} \\
D_{\alpha p''}x^{\beta j} = \lambda_{\alpha p''}^{\beta i} & .
\end{cases}
$$

We can now introduce the *extended Euler-Schouten curvature extensor* of $C_m^{(M)}$:

$$\bar{D}_{\beta j'}B^{\gamma j}_{\alpha i'} = \frac{\partial B^{\gamma j}_{\alpha i'}}{\partial u^{(\beta)j'}} + \sum_{\delta=\alpha}^{\gamma}\sum_{\theta=\beta}^{M} \Gamma_{\delta i\ \theta k}^{\gamma j}B^{\delta i}_{\alpha i'}B^{\theta k}_{\beta j'} - \sum_{\theta=0}^{\gamma}{}'\Gamma_{\alpha i'\ \beta j'}^{\theta k'}B^{\gamma j}_{\theta k'} \equiv H_{\beta j'\alpha i'}^{\gamma j}.$$

Since it follows that $\sum_{\gamma=0}^{t} B^{\gamma l'}_{\gamma j}H_{\beta j'\alpha i'}^{\gamma j} = 0$ according to (5.1), the quantities $H_{\beta j'\alpha i'}^{\gamma j}$ are regarded as the components of an excontravariant extensor in $K_n^{(M)(n-m)}$, and are expressed

$$D_{\beta j'}B^{\gamma j}_{\alpha i'} = H_{\beta j'\alpha i'}^{\gamma j} = \sum_{\theta=0}^{\gamma} H_{\beta j'\alpha i'}^{\theta q''}\lambda^{\gamma j}_{\theta q''} \qquad \gamma = 0, \cdots, M. \tag{5.6}$$

Further if we consider

$$D_{\beta j'}\lambda^{\gamma j}_{\alpha p''} = \frac{\partial \lambda^{\gamma j}_{\alpha p''}}{\partial u^{(\beta)j'}} + \sum_{\theta=0}^{\gamma}\sum_{\delta=0}^{M} \Gamma_{\theta k\ \delta l}^{\gamma j}\lambda^{\theta k}_{\alpha p''}B^{\delta l}_{\beta j'} = \sum_{\delta=0}^{\gamma}{}''\Gamma_{\alpha p''\ \beta j'}^{\delta q'}\lambda^{\gamma j}_{\delta q''} \equiv L_{\beta j'\alpha p''}^{\gamma j},$$

then the quantities $L_{\beta j'\alpha p''}^{\gamma j}$ are rewritten in the form

$$L_{\beta j'\alpha p''}^{\gamma j} = \sum_{\rho=0}^{\gamma} L_{\beta j'\alpha p''}^{\rho j'}B^{\gamma i}_{\rho j'} \qquad \gamma = 0, \cdots, M \tag{5.7}$$

in consequence of (5.2).

Theorem 5.1. *If* $v^{\alpha i} = \sum_{\beta=0}^{\alpha} B^{\alpha i}_{\beta i'}v^{\beta i'}$ *is an excontravariant extensor of range* M *in* $C_m^{(M)}$, *then the projections of the excovariant differential* $\delta v^{\alpha i}$ *along* $C_m^{(M)}$ *upon* $C_m^{(M)}$ *and upon* $C_n^{(M)(n-m)}$ *are equal to* $\delta v^{\beta j'}$ *and* $\sum_{\theta=0}^{\beta}\sum_{\rho=0}^{M} H_{\rho k'\theta h''}^{\beta j''} \times v^{\theta h'}du^{(\rho)k'}$ *respectively.*

Proof. Differentiating $v^{\alpha i} = \sum_{\beta=0}^{\alpha} B^{\alpha i}_{\beta i'}v^{\beta i'}$ $(\alpha = 0, \cdots, M)$ excovariantly along $C_m^{(M)}$, we have

$$\begin{aligned}\delta v^{\alpha i} &= \sum_{\beta=0}^{\alpha}\left\{\delta B^{\alpha i}_{\beta j'}v^{\beta j'} + B^{\alpha i}_{\beta j'}\delta v^{\beta j'}\right\} \\ &= \sum_{\beta=0}^{\alpha}\sum_{\delta=0}^{M} D_{\delta k'}B^{\alpha i}_{\beta j'}v^{\beta j'}du^{(\delta)k'} + \sum_{\beta=0}^{\alpha} B^{\alpha i}_{\beta j'}\delta v^{\beta j'} \\ &= \sum_{\beta=0}^{\alpha}\sum_{\delta=0}^{M} H_{\delta k'\beta j'}^{\alpha i}v^{\beta j'}du^{(\delta)k'} + \sum_{\beta=0}^{\alpha} B^{\alpha i}_{\beta j'}\delta v^{\beta j'}.\end{aligned}$$

Multiplying the both-hand members of the above equation by $B^{\gamma l'}_{\alpha i}$ or $\lambda^{\gamma l''}_{\alpha i}$ and summing with respect to α and i, we get

$$\sum_{\alpha=0}^{\gamma} B^{\gamma l'}_{\alpha i}\delta v^{\alpha i} = \delta v^{\gamma l}$$

or

$$\sum_{\alpha=0}^{\gamma} \lambda^{\gamma l''}_{\alpha i}\delta v^{\alpha i} = \sum_{\beta=0}^{\gamma}\sum_{\delta=0}^{M} H_{\delta k'\beta j'}^{\gamma l''}v^{\beta j'}du^{(\delta)k'}$$

in consequence of (5.6).

Theorem 5.2. *If $v^{\gamma i}$ is an excontravariant extensor of range M in $C_n^{(M)}$ associated with each expoint on each curve of the curve family in the exsurface, that is denoted by* (2.2), *then the projections of the excovariant differential $\delta v^{\alpha i}$ along $C_m^{(M)}$ upon $C_m^{(M)}$ and upon $C_n^{(M)(n-m)}$ are $\delta v^{\beta j'} + L_{\rho k' \theta h''}^{\beta j'} \times v^{\theta h''} du^{(\rho)k'}$ and $\delta v^{\beta j''} + H_{\rho k' \theta h'}^{\beta j''} v^{\theta h'} du^{(\rho)k'}$ respectively.*

Proof. Differentiating $v^{\gamma i} = \sum_{\alpha=0}^{\gamma} \left\{ B_{\alpha i'}^{\gamma i} v^{\alpha i'} + \lambda_{\alpha p''}^{\gamma i} v^{\alpha p''} \right\}$ excovariantly,

$$\delta v^{\gamma i} = \delta \left\{ \sum_{\alpha=0}^{\gamma} B_{\alpha i'}^{\gamma i} v^{\alpha i'} \right\} + \delta \left\{ \sum_{\alpha=0}^{\gamma} \lambda_{\alpha p''}^{\gamma i} v^{\alpha p''} \right\}$$

$$= \sum_{\alpha=0}^{\gamma} \sum_{\delta=0}^{M} H_{\delta k' \alpha i'}^{\gamma i} v^{\alpha i'} du^{(\delta)k'} + \sum_{\alpha=0}^{\gamma} B_{\alpha i'}^{\gamma i} \delta v^{\alpha i'}$$

$$+ \sum_{\alpha=0}^{\gamma} \lambda_{\alpha p''}^{\gamma i} \delta v^{\alpha p''} + \sum_{\alpha=0}^{\gamma} \sum_{\delta=0}^{M} L_{\delta k' \alpha p''}^{\gamma i} v^{\alpha p''} du^{(\delta)k'} .$$

Multiplying the both-hand members of the resulting equation by $B_{\gamma i}^{\beta j'}$ or $\lambda_{\gamma i}^{\beta j''}$ and summing with respect to γ and i

$$\sum_{\gamma=0}^{\beta} B_{\gamma i}^{\beta j'} \delta v^{\gamma i} = \delta v^{\beta j'} + \sum_{\alpha=0}^{\beta} \sum_{\delta=0}^{M} L_{\delta k' \alpha p''}^{\beta j'} v^{\alpha p''} du^{(\delta)k'}$$

or

$$\sum_{\gamma=0}^{\beta} \lambda_{\gamma i}^{\beta j''} \delta v^{\gamma i} = \sum_{\alpha=0}^{\beta} \sum_{\delta=0}^{M} H_{\delta k' \alpha i'}^{\beta j''} v^{\alpha i'} du^{(\delta)k'} + \delta v^{\beta j''} .$$

Corollary. *If $v^{\gamma i}$ is an excontravariant extensor of range M in $C_n^{(M)}$ associated with each expoint on each curve of the curve family in the exsurface, that is denoted by* (2.2), *then the difference between the projections of the excovariant differential $\delta v^{\alpha i}$ upon $C_m^{(M)}$ and $\delta v^{\beta j'}$ is equal to $\sum_{\theta=0}^{\beta} \sum_{\rho=0}^{M} L_{\rho k' \theta h''}^{\beta j'} \times v^{\theta h''} du^{(\rho)k'}$ and the difference between the projection of $\delta v^{\alpha i}$ upon $C_n^{(M)(n-m)}$ and $\delta v^{\beta j''}$ is $\sum_{\theta=0}^{\beta} \sum_{\rho=0}^{M} H_{\rho k' \theta h'}^{\beta j''} v^{\theta h'} du^{(\rho)k'}$.*

§ 4. The exsurface in a space $\mathbf{R}_n^{(M)}$ with a metric extensor. Let us assume that our space $K_n^{(M)}$ is a space $R_n^{(M)}$ with a symmetric excovariant extensor $g_{\alpha i \beta j}$ of characteristic $(2, 0, M, M)$ which is so called metric extensor, at each expoint in the considered domain of $R_n^{(M)}$, where $(M+1) \times n$-rowed determinant g constructed of $g_{\alpha i \beta j}$ with respect to the doublet indices αi and βj does not vanish in the domain, and suppose a continuous family of curves as stated in the previous section, in $R_n^{(M)}$ too. Then following Craig ([4], p. 797), the connection parameter of an extensor in $R_n^{(M)}$ is given by Christoffel symbol of the second kind by means of $g_{\alpha i \beta j}$ and $g^{\alpha i \gamma j}$, i. e.,

$$\Gamma_{\beta j\ \gamma k}^{\alpha i} = \frac{1}{2} g^{\alpha i \theta l} \left\{ \frac{\partial g_{\theta l \gamma k}}{\partial x^{(\beta) j}} + \frac{\partial g_{\beta j \theta l}}{\partial x^{(\gamma) k}} - \frac{\partial g_{\beta j \gamma k}}{\partial x^{(\theta) l}} \right\}.$$

In this case, the metric extensor of the exsurface: $R_m^{(M)}$ is offered by

$$g_{\gamma i' \delta j'} = g_{\alpha i \beta j} B_{\gamma i'}^{\alpha i} B_{\delta j'}^{\beta j}$$

and the induced connection parameter of $R_m^{(M)}$ coincides with CHRISTOFFEL symbol of the second kind $\Gamma_{\beta j'\ \gamma k'}^{\alpha i'}$ by means of $g_{\gamma i' \delta j'}$ and $g^{\gamma i' \delta j'}$. The connection theory of extensors referred to such the exsurface will be established parallelly that of tensors in the subspace of RIEMANNian space. (August, 1951).

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