ON THE EXTENDED CONNECTION PARAMETERS IN A SPACE WITH AFFINE CONNECTION AND IN A RIEMANNIAN SPACE

By

Yoshie KATSURADA

Introduction. The extensor introduced first by H. V. CRAIG [1][(1)] appears as an extension of the tensor concept and holds tensor member as a part of its components, for example, let $T^{\alpha a}_{\beta b}$ be the components of a second order mixed extensor of range G, then the quantities T^{0a}_{Gb} are components of its tensor member. From this fact it has seemed to the present author to be desirable to find such an operation of an extensor that it applied to tensor member of the extensor just coincides with the similar well-known tensor operation. As the first step towards the desire, the principal purpose of the present paper is to establish an excovariant derivation, in the above stated sense, of an extensor in a space with an affine connection as well as in a RIEMANNian space, using the extended connection parameter $\Gamma_{\beta}{}^{\alpha}_{b}{}^{a}_{\gamma c}$ formed by the affine connection parameter Γ^{a}_{bc} and CHRISTOFFEL symbol by means of the metric extensor $g_{\alpha a \beta b}$ induced by the metric tensor g_{ab} respectively.

In the present paper we use certain of the ideas, notations and results given in the previous paper [2] without explanation. The present author wishes to express to Prof. A. KAWAGUCHI her appreciation for his helpful guidance.

§ 1. **The connection of extensor in the space with an affine connection.** We shall proceed to find what is so called the extended connection parameter and to build up such the excovariant differential that contains the covariant differential of a tensor as a part of it, in the space with a symmetric affine connection.

At each point on a parameterized arc of class M: $x^a = x^a(t)$, where t is a fixed essential parameter, the quantities defined by differentiating the connection parameter $\Gamma^a_{bc}(t)$ successively by t:

(1) Numbers in brackets refer to the references at the end of the paper.

$$\Gamma_{\beta b \gamma c}^{\alpha a} \equiv \binom{\alpha}{\beta\gamma} \Gamma_{bc}^{a\,(\alpha-\beta-\gamma)} \qquad \text{for} \quad \beta+\gamma \leqq \alpha \tag{1.1}$$
$$\equiv 0 \qquad \text{for} \quad \beta+\gamma > \alpha$$

are considered along the arc, where α is an integer not exceeding M, and

$$\binom{\alpha}{\beta\gamma} \equiv \frac{\alpha\,!}{\beta!\,\gamma!\,(\alpha-\beta-\gamma)\,!} \qquad \text{for} \quad \beta+\gamma \leqq \alpha,$$
$$\equiv 0 \qquad \text{for} \quad \beta+\gamma > \alpha.$$

We can then state the following theorems:

Theorem 1.1. *The quantities* $\Gamma_{\beta b \gamma c}^{\alpha a}$ are changed by

$$\Gamma_{\beta b \gamma c}^{\alpha a} = \sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\varphi=\beta}^{\alpha-\theta} \sum_{\psi=\alpha}^{\theta+\varphi} \Gamma_{\varphi j \theta k}^{\psi i} X_{\psi i}^{\alpha a} X_{\beta b}^{\varphi j} X_{\gamma c}^{\theta k} + \sum_{\theta=\alpha}^{\beta+\gamma} X_{\beta b \gamma c}^{\theta i} X_{\theta i}^{\alpha a} \tag{1.2}$$

under a transformation of the expoint ([1], *p.* 765), *where* $X_{\beta b \gamma c}^{\theta i} \equiv \dfrac{\partial^2 x^{(\theta)i}}{\partial x^{(\beta)b}\, \partial x^{(\gamma)c}}$.

Proof. Differentiating the transformation equation of the affine connection parameter:

$$\Gamma_{bc}^{a} = \Gamma_{jk}^{i} \bar{X}_{i}^{a} X_{b}^{j} X_{c}^{k} + X_{bc}^{i} \bar{X}_{i}^{a} \qquad \left(X_{bc}^{i} \equiv \frac{\partial^2 x^i}{\partial x^b\, \partial x^c}\right)$$

$(\alpha-\beta-\gamma)$ times by LEIBNITZ rule and multiplying the generalized binomial coefficient $\binom{\alpha}{\beta\gamma}$, the left-hand member is equal to $\Gamma_{\beta b \gamma c}^{\alpha a}$, and the first and second terms of the right-hand member become as follows:

$$\binom{\alpha}{\beta\gamma} (\Gamma_{jk}^{i} X_{i}^{a} X_{b}^{j} X_{c}^{k})^{(\alpha-\beta-\gamma)} = \sum_{\theta=\gamma}^{\alpha-\beta} \binom{\alpha}{\beta\gamma} \binom{\alpha-\beta-\gamma}{\theta-\gamma} (\Gamma_{jk}^{i} X_{i}^{a} X_{b}^{j})^{(\alpha-\beta-\theta)} X_{c}^{k(\theta-\gamma)}$$
$$= \sum_{\theta=\gamma}^{\alpha-\beta} \binom{\alpha-\beta}{\theta} \binom{\alpha}{\beta} (\Gamma_{jk}^{i} X_{i}^{a} X_{b}^{j})^{(\alpha-\beta-\theta)} X_{\gamma c}^{\theta k}$$
$$= \sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\varphi=\beta}^{\alpha-\theta} \binom{\alpha-\beta}{\theta} \binom{\alpha}{\beta} \binom{\alpha-\beta-\theta}{\varphi-\beta} (\Gamma_{jk}^{i} X_{i}^{a})^{(\alpha-\theta-\varphi)} X_{b}^{j(\varphi-\beta)} X_{\gamma c}^{\theta k}.$$
$$= \sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\varphi=\beta}^{\alpha-\theta} \binom{\alpha}{\theta\varphi} (\Gamma_{jk}^{i} X_{i}^{a})^{(\alpha-\theta-\varphi)} X_{\beta b}^{\varphi j} X_{\gamma c}^{\theta k}$$
$$= \sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\varphi=\beta}^{\alpha-\theta} \sum_{\psi=\alpha}^{\theta+\varphi} \binom{\alpha}{\theta\varphi} \binom{\alpha-\theta-\varphi}{\alpha-\psi} \Gamma_{jk}^{i(\psi-\theta-\varphi)} X_{i}^{a(\alpha-\varphi)} X_{\beta b}^{\psi j} X_{\gamma c}^{\theta k}$$
$$= \sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\varphi=\beta}^{\alpha-\theta} \sum_{\psi=\alpha}^{\theta+\varphi} \binom{\psi}{\theta\varphi} \Gamma_{jk}^{i(\psi-\theta-\varphi)} X_{\psi i}^{\alpha a} X_{\beta b}^{\varphi i} X_{\gamma c}^{\theta k}$$
$$= \sum_{\theta=\gamma}^{\alpha-\beta} \sum_{\varphi=\beta}^{\alpha-\theta} \sum_{\psi=\alpha}^{\theta+\varphi} \Gamma_{\varphi j \theta k}^{\psi i} X_{\psi i}^{\alpha a} X_{\beta b}^{\varphi j} X_{\gamma c}^{\theta k}$$

and

$$\binom{\alpha}{\beta\gamma}(X^i_{bc}X^a_i)^{(\alpha-\beta-\gamma)} = \sum_{\theta=\alpha}^{\beta+\gamma}\binom{\alpha}{\beta\gamma}\binom{\alpha-\beta-\gamma}{\alpha-\theta}X^{i\,(\theta-\beta-\gamma)}_{bc}X^{a\,(\alpha-\theta)}_{i}$$

$$= \sum_{\theta=\alpha}^{\beta+\gamma}\binom{\theta}{\beta\gamma}X^{i\,(\theta-\beta-\gamma)}_{bc}X^{\alpha a}_{\theta i} = \sum_{\theta=\alpha}^{\beta+\gamma}\binom{\theta-\beta}{\gamma}\binom{\theta}{\beta}X^{i\,(\theta-\beta-\gamma)}_{bc}X^{\alpha a}_{\theta i}$$

$$= \sum_{\theta=\alpha}^{\beta+\gamma}X_{\beta b \gamma c}^{\theta i}X^{\alpha a}_{\theta i}$$

respectively, accordingly (1.2) is obtained.

Theorem 1.2. *$\Gamma_{\beta b \gamma c}^{\alpha a}$ is symmetric with respect to the doublet indices βb and γc.*

Theorem 1.3. *When the index α of $\Gamma_{\beta b \gamma c}^{\alpha a}$ is equal to zero, both the indices β and γ are also equal to zero and $\Gamma_{\beta b \gamma c}^{\alpha a}$ coincides with Γ^a_{bc}.*

The two theorems are self-evident from (1.1).

Such the quantities $\Gamma_{\beta b \gamma c}^{\alpha a}$ are called the *extended connection parameter* in the space with an affine connection.

Now we shall suppose such a continuous family of curves that one and only one curve belonging to it passes through every one point in the considered domain of the space. In order to define the excovariant differential of an extensor field defined on each curve belonging to it, for example, let us consider an extensor $T'^{\alpha a}{}_{\beta b}{}^{c}$ of the type indicated by the indices (range with respect to α and β being G_1 and G_2 respectively, and of functional order M; $G_1, G_2 \leqq M$), then the quantities given by

$$(1.3)\qquad \delta^* T'^{\alpha a}{}_{\beta b}{}^{c} = dT'^{\alpha a}{}_{\beta b}{}^{c} + \sum_{\gamma,\delta=0}^{G_1}\Gamma_{\gamma d\,\delta e}^{\alpha a}T'^{\gamma d}{}_{\beta b}{}^{c}\,dx^{(\delta)e}$$
$$- \sum_{\gamma,\delta=0}^{G_2}\Gamma_{\beta b\,\delta e}^{\gamma d}T'^{\alpha a}{}_{\gamma d}{}^{c}\,dx^{(\delta)e} + \Gamma^{c}_{ef}T'^{\alpha a}{}_{\beta b}{}^{e}\,dx^{f}$$

are the components of an extensor of the same kind as the original extensor $T'^{\alpha a}{}_{\beta b}{}^{c}$, where the displacement $dx^{(\delta)e}$ means difference of the line elements at any two infinitesimally near points lying on any two infinitesimally near curves belonging to the family respectively. Such the quantities $\delta^* T'^{\alpha a}{}_{\beta b}{}^{c}$ and the quantities $T'^{\alpha a}{}_{\beta b}{}^{c}{}_{,\gamma d}$ defined by

$$(1.4)\qquad T'^{\alpha a}{}_{\beta b}{}^{c}{}_{,\xi d} = T'^{\alpha a}{}_{\beta b}{}^{c}{}_{;(\delta)d} + \sum_{\iota=1}^{G_1}\Gamma_{\gamma e\,\delta d}^{\alpha a}T'^{\gamma e}{}_{\beta b}{}^{c}$$
$$- \sum_{\gamma=0}^{G_2}\Gamma_{\beta b\,\delta d}^{\gamma e}T'^{\alpha a}{}_{\gamma e}{}^{c} + \Gamma^{c}_{ek}T'^{\alpha a}{}_{\beta b}{}^{e}\,\delta^0_\delta$$

(δ^0_δ : KRONECKER delta)

are called the *excovariant differential* and the *excovariant derivative* of the extensor $T'^{\alpha a}{}_{\beta b}{}^{c}$ respectively and when $\delta^* T'^{\alpha a}{}_{\beta b}{}^{c} = 0$, we say that the *extensor* $T^{\alpha a}{}_{\beta b}{}^{c}$ *is displaced parallelly.* From the definition we can get the following theorems.

Theorem 1.4. *The excovariant derivative of* $T^{\alpha a}{}_{\beta b}{}^{c}$: $T^{\alpha a}{}_{\beta b}{}^{c}{}_{,\delta d}$ *is an extensor of the type indicated by the indices, of ranges with respect to* α, β *and* δ *being* G_1, G_2 *and* M *respectively, and of functional order* M.

Theorem 1.5. *Let* $v^{\alpha a}$ *be an excontravariant extensor of characteristic* $(1+0, 0, R, R)$ $R \leqq M$, *then the following relations are observed*

$$\delta^* v^{0a} = dv^{0a} + \Gamma_{b}{}^{a}{}_{c} v^{0b} dx^{c} \ (\equiv \delta v^{0a})$$

for the tensor member of $v^{\alpha a}$, *and* S *being an integer not exceeding* R,

$$\delta^* v^{\bar{\alpha} a} = dv^{\bar{\alpha} a} + \sum_{\beta, \bar{\gamma}=0}^{\bar{\alpha}} \Gamma_{\beta b}{}^{\bar{\alpha} a}{}_{\bar{\gamma} c} v^{\beta b} dx^{(\bar{\gamma})c} \ (\equiv \delta^S v^{\bar{\alpha} a})$$

for the member $v^{\bar{\alpha} a}$ $(\bar{\alpha} = 0, \cdots, S)$ *of range* S *which is a part of the extensor* $v^{\alpha a}$ $(\alpha = 0, \cdots, R)$.

The theorems are verified without difficulty by virtue of (1.1) and (1.3).

Theorem 1.6. *If* v^a *is a contravariant vector, then it follows that*

$$(\delta^* v^a)^{(\alpha)} = \delta^* v^{(\alpha)a}.$$

Proof. From Theorem 1.5, there exists the relation

$$\delta^* v^a = \delta v^a \equiv dv^a + \Gamma_{b}{}^{a}{}_{c} v^b dx^c .$$

Differentiating the above equation α times, we have

$$\begin{aligned}(\delta^* v^a)^{(\alpha)} &= dv^{(\alpha)a} + (\Gamma_{b}{}^{a}{}_{c} v^b dx^c)^{(\alpha)} \\ &= dv^{(\alpha)a} + \sum_{\beta, \gamma=0}^{\alpha} \binom{\alpha}{\beta\ \gamma} \Gamma_{b}{}^{a}{}_{c}{}^{(\alpha-\beta-\gamma)} v^{(\beta)b} dx^{(\gamma)k}\end{aligned}$$

and

$$(\delta^* v^a)^{(\alpha)} = \delta^* v^{(\alpha)a}$$

in consequence of (1.1) and (1.3).

Corollary. *If a vector* v^a *is displaced parallelly, that is,* $\delta v^a = 0$, *then the extensor* $v^{(\alpha)a}$ *induced by differentiating* v^a *by* t *is displaced parallelly too.*

These facts state that the components of the excovariant differential $\delta^* v^{\alpha a} (\alpha = 0, \cdots, R)$ hold as a part those of the excovariant differential $\delta^S v^{\bar{\alpha} a}$ $(\bar{\alpha} = 0, \cdots, S)$ of the subextensor $v^{\bar{\alpha} a}$ whose components are a part of those of the extensor $v^{\alpha a}$, particularly the excovariant differential of the tensor member becomes the covariant differential of the tensor in the space with an affine connection, and when an extensor $v^{\alpha a}$ is displaced parallelly, that is, $\delta^* v^{\alpha a} = 0$, the tensor member v^{0a} is displaced.

parallelly by means of usual sense. It should be specially noticed that the connection theory of extensors by means of the above-stated excovariant differential contains the theory referred to tensors in the space of the affine connection as a part.

The similar statements for general extensors of higher order will be indicated also.

§ 2. **The metric extensor of a RIEMANNian space.** Let us suppose a one-parameter system of the metric tensors $g_{ab}(t)$ along a parameterized arc of class M: $x^a = x^a(t)$ in an n-dimensional RIEMANNian space, differentiate successively by t, and constract the quantities $\overset{M}{g}_{\alpha a \beta b}$

$$\overset{M}{g}_{\alpha a \beta b} \equiv \binom{M}{\alpha\ \beta} g_{ab}^{(M-\alpha-\beta)} \qquad \alpha, \beta = 0, \cdots, M, \tag{2.1}$$

then, as proved by CRAIG ([3], p. 336), the quantities $\overset{M}{g}_{\alpha a \beta b}$ are the components of an extensor of characteristic $(0+2, 0, M, M)$. We shall call such the quantities $\overset{M}{g}_{\alpha a \beta b}$ the components of the *metric extensor* in the RIEMANNian space of M-th order line-elements: $R_n^{(M)}$. The properties of $\overset{M}{g}_{\alpha a \beta b}$ are stated by the following theorems.

Theorem 2.1. *The metric extensor $\overset{M}{g}_{\alpha a \beta b}$ is symmetric.*

Proof. From $g_{ab} = g_{ba}$ and (2.1), we have $\overset{M}{g}_{\alpha a \beta b} = \overset{M}{g}_{\beta b \alpha a}$.

Theorem 2.2. *If G is $(M+1)\,n$-rowed determinant constructed of $\overset{M}{g}_{\alpha a \beta b}$ with respect to the pairs (αa) and (βb), then*

$$G \equiv |\overset{M}{g}_{\alpha a \beta b}| = |g_{ab}|^{M+1} \neq 0 .$$

Proof. The statement follows from the fact that $\overset{M}{g}_{\alpha a \beta b} = 0$ for $\beta > M-\alpha$ and $\overset{M}{g}_{M-\beta a \beta b} = g_{ab}$ for $\beta = 0, 1, \cdots, M$ in consequence of (2.1).

Here we can define such the unique *reciprocal excontravariant metric extensor* $\underset{M}{g}^{\alpha a \beta b}$ of the excovariant metric extensor $\overset{M}{g}_{\alpha a \beta b}$ that satisfies the following equation:

$$\overset{M}{g}_{\alpha a \beta b}\, \underset{M}{g}^{\alpha a \gamma c} = \delta_\beta^\gamma\, \delta_b^c , \tag{2.2}$$

where symbol δ denotes KRONECKER delta. The structure of $\underset{M}{g}^{\alpha a \beta b}$ is given by the next theorem:

Theorem 2.3. *The quantities $\begin{bmatrix} \alpha\ \beta \\ M \end{bmatrix} g^{ab(\alpha+\beta-M)}$ that are the components* *of an extensor* ([3], *p.* 335) *are equal to* $\underset{M}{g}^{\alpha a\,\beta b}$, *where*

$$\begin{bmatrix}\alpha\,\beta\\ M\end{bmatrix}=\binom{M}{M-\alpha\;M-\beta}\binom{M}{\alpha}^{-1}\binom{M}{\beta}^{-1}\qquad \text{for } \alpha+\beta-M\geqq 0$$
$$=0\qquad\qquad \text{for } \alpha+\beta-M<0.$$

Proof. Putting $\underset{M}{\tilde g}^{\alpha a\,\beta b}\equiv\begin{bmatrix}\alpha\,\beta\\ M\end{bmatrix}g^{ab(\alpha+\beta-M)}$, we get

$$\sum_{\alpha=1}^{M}\underset{M}{\tilde g}^{\alpha a\,\beta b}\overset{M}{g}_{\alpha a\,\gamma c}=\sum_{\alpha=0}^{M}\binom{\beta-\gamma}{\alpha+\beta-M}\binom{\beta}{\gamma}g^{ab(\alpha+\beta-M)}\,g_{ac}{}^{(M-\alpha-\gamma)}$$
$$=\binom{\beta}{\gamma}\sum_{\mu=0}^{\beta-\gamma}\binom{\beta-\gamma}{\mu}g^{ab(\mu)}\,g_{ac}{}^{(\beta-\gamma-\mu)}$$
$$(\text{putting}\quad \mu=\alpha+\beta-M)$$
$$=\binom{\beta}{\gamma}(g^{ab}g_{ac})^{(\beta-\gamma)}=\binom{\beta}{\gamma}\delta_c^{b(\beta-\gamma)}=\delta_c^b\,\delta_\gamma^\beta\,,$$

consequently $\underset{M}{\tilde g}^{\alpha a\,\beta b}=\underset{M}{g}^{\alpha a\,\beta b}$, since $\underset{M}{g}^{\alpha a\,\beta b}$ is determined uniquely by equation (2.2).

If we look for the relations between the metric extensor $\overset{R}{g}_{\alpha a\,\beta b}$ (or $\underset{R}{g}^{\alpha a\,\beta b}$) defined by (2.1) in the space of R-th order line-elements and $\overset{M}{g}_{\alpha a\,\beta b}$ (or $\underset{M}{g}^{\alpha a\,\beta b}$) of $R_n^{(M)}$, R being any fixed integer not exceeding M, we can find the following theorem:

Theorem 2.4. *The following relation holds good*:

$$\binom{M+\sigma}{M}\overset{M}{g}_{\lambda a\,M-R+\sigma b}=\binom{M+\sigma}{R}\overset{R}{g}_{\lambda a\sigma b}\qquad \sigma,\lambda=0,1,\cdots,R\,.$$

Proof. From (2.1), we obtain that $\overset{M}{g}_{\lambda a\,M-R+\sigma b}=\binom{M+\sigma}{R}\binom{M+\sigma}{M}^{-1}\overset{R}{g}_{\lambda a\sigma b}$, from which the statement follows at once.

Theorem 2.5. *There exists the relation*

$$\frac{M\,!}{M-R+\sigma\,!}\underset{M}{g}^{\alpha a\,M-R+\sigma b}=\frac{R\,!}{\sigma\,!}\underset{R}{g}^{\alpha a\sigma b}\qquad \alpha,\sigma=0,1,\cdots,R\,.$$

We define the quantities $v_{\beta b}\equiv\overset{M}{g}_{\alpha a\beta b}\,v^{\alpha a}$ determined by components of an extensor $v^{\alpha a}$ of characteristic $(1+0,0,M,G)$ as the *excovariant components* of the extensor, then we have

Theorem 2.6. *If v_b are covariant components of a contravariant vector v^a, i.e., $v_b=g_{ba}v^a$, then the excovariant components of the extensor $v^{(\alpha)a}$ are equal to the quantities $\binom{M}{\beta}v_b{}^{(M-\beta)}$ that are the components of an excovariant* *extensor in virtue of the theorem of Kawaguchi* ([4], *p.* 22), *that is,*

$$v_{\beta b} \equiv \underset{M}{g}_{\alpha a \beta b}\, v^{(\alpha)a} = \binom{M}{\beta} v_b^{(M-\beta)}$$

Proof. Differentiating the equation: $g_{ba} v^a = v_b$ $(M-\beta)$ times by LEIBNITZ rule and multiplying the binomial coefficient $\binom{M}{\beta}$ to the both members of the resulting equation, we get $g_{\alpha a \beta b} v^{(\alpha)a} = \binom{M}{\beta} v_b^{(M-\beta)}$.

Corollary 1. *The following relation consists*: $v_{\beta b} \underset{M}{g}^{\alpha a \beta b} = v^{\alpha a}$.

Corollary 2. *It follows that* $\sum_{\beta=0}^{M} \binom{M}{\beta} v_b^{(M-\beta)} \underset{M}{g}^{\alpha a \beta b} = v^{(\alpha)a}$.

Let $v^{\alpha a}$ be the components of an excontravariant extensor in $R_n^{(M)}$ and consider the quantity $\underset{M}{g}_{\alpha a \beta b} v^{\alpha a} v^{\beta b}$, then $\underset{M}{g}_{\alpha a \beta b} v^{\alpha a} v^{\beta b}$ is a scalar. The square root of the scalar $(\underset{M}{g}_{\alpha a \beta b} v^{\alpha a} v^{\beta b})^{\frac{1}{2}}$ is called the *length* of the excontravariant extensor $v^{\alpha a}$, and the scalar: $\underset{M}{g}_{\alpha a \beta b} v_1^{\alpha a} v_2^{\beta b} (\underset{M}{g}_{\alpha a \beta b} v_1^{\alpha a} v_1^{\beta b})^{-\frac{1}{2}} \times (\underset{M}{g}_{\alpha a \beta b} v_2^{\alpha a} v_2^{\beta b})^{-\frac{1}{2}}$ formed by any two excontravariant extensors $v_1^{\alpha a}$ and $v_2^{\alpha a}$ of the same kind is called *the cosine of the angle* between the two extensors. Further we define the *length* T' of a general extensor $T'^{\alpha_1 a_1 \cdots \alpha_A a_A}{}_{\beta_1 b_1 \cdots \beta_B b_B}$ with the different ranges G_r and G'_S for the indices α_r and β_S and the *angle* θ between two extensors $T'^{\alpha_1 a_1 \cdots \alpha_A a_A}_1{}_{\beta_1 b_1 \cdots \beta_B b_B}$ and $T'^{\alpha_1 a_1 \cdots \alpha_A a_A}_2{}_{\beta_1 b_1 \cdots \beta_B b_B}$ of the same kind as follows:

$$T'^2 = \underset{G_1}{g}_{\alpha_1 a_1 \gamma_1 c_1} \cdots \underset{G_A}{g}_{\alpha_A a_A \gamma_A c_A} \underset{G'_1}{g}^{\beta_1 b_1 \delta_1 d_1} \cdots \underset{G'_B}{g}^{\beta_B b_B \delta_B d_B} \times T'^{\alpha_1 a_1 \cdots \alpha_A a_A}{}_{\beta_1 b_1 \cdots \beta_B b_B} T'^{\gamma_1 c_1 \cdots \gamma_A c_A}{}_{\delta_1 d_1 \cdots \delta_B d_B} \tag{2.4}$$

and

$$\cos\theta = \frac{1}{T'_1 T'_2} \underset{G_1}{g}_{\alpha_1 a_1 \gamma_1 c_1} \cdots \underset{G_A}{g}_{\alpha_A a_A \gamma_A c_A} \underset{G'_1}{g}^{\beta_1 b_1 \delta_1 d_1} \cdots \underset{G'_B}{g}^{\beta_B b_B \delta_B d_B} \times T'_1{}^{\alpha_1 a_1 \cdots \alpha_A a_A}{}_{\beta_1 b_1 \cdots \beta_B b_B} T'_2{}^{\gamma_1 c_1 \cdots \gamma_A c_A}{}_{\delta_1 d_1 \cdots \delta_B d_B} \tag{2.5}$$

respectively. By means of this definition, the following theorem is stated:

Theorem 2.7. *If* v *is the length of an excontravariant extensor* $v^{\alpha a}$, *then we get*: $v^2 = \underset{M}{g}^{\alpha a \beta b} v_{\alpha a} v_{\beta b}$.

Theorem 2.8. *When* v *is the length of a contravariant vector* v^a, *the relation:*

$$\frac{d^M v^2}{dt^M} = \overset{M}{g}_{\alpha a \beta b} v^{(\alpha) a} v^{(\beta) b}$$

holds good.

Proof. Differentiating the equation $v^2 = g_{ab} v^a v^b$ M times by LEIBNITZ rule, the above equation is obtained.

§ 3. **Extended CHRISTOFFEL symbol in the RIEMANNian space.** Denoting CHRISTOFFEL symbol of the first kind and of the second kind by means of $\overset{M}{g}_{\alpha a \beta b}$ and $\underset{M}{g}^{\alpha a \gamma c}$ by

$$\overset{M}{\Gamma}_{\beta b \gamma c, \alpha a} = \frac{1}{2} (\overset{M}{g}_{\alpha a \gamma c;(\beta) b} + \overset{M}{g}_{\beta b \alpha a;(\gamma) c} - \overset{M}{g}_{\beta b \gamma c;(\alpha) a}) \tag{3.1}$$

and

$$\overset{M}{\Gamma}{}_{\beta b}{}^{\alpha a}{}_{\gamma c} = \sum_{\delta=0} \underset{M}{g}^{\alpha a \delta e} \overset{M}{\Gamma}_{\beta b \gamma c, \delta e} \tag{3.2}$$

respectively, we have the following theorems:

Theorem 3. 1. *CHRISTOFFEL symbol of the second kind* $\overset{M}{\Gamma}{}_{\beta b}{}^{\alpha a}{}_{\gamma c}$ *coincides just with the extended connection parameter:* $\binom{\alpha}{\beta \gamma} \overset{0}{\Gamma}{}_{b}{}^{a}{}_{c}{}^{(\alpha-\beta-\gamma)}$, *where* $\overset{0}{\Gamma}{}_{b}{}^{a}{}_{c}$ *indicates CHRISTOFFEL symbol of the second kind by means of* g_{ab} *and* g^{ac}.

Proof. At first let us calculate the first term in the right-hand member of (3.2), that is,

$$\overset{M}{\Gamma}{}_{\beta b}{}^{\alpha a}{}_{\gamma c} = \frac{1}{2} \sum_{\delta=0}^{M} \underset{M}{g}^{\alpha a \delta e} (\overset{M}{g}_{\delta e \gamma c;(\beta) b} + \overset{M}{g}_{\beta b \delta e;(\gamma) c} - \overset{M}{g}_{\beta b \gamma c;(\delta) e}),$$

then we have

$$\sum_{\delta=0}^{M} \underset{M}{g}^{\alpha a \delta e} \overset{M}{g}_{\delta e \gamma c;(\beta) b} = \sum_{\delta=0}^{M} \begin{bmatrix} \alpha\delta \\ M \end{bmatrix} g^{ae(\alpha+\delta-M)} \binom{M}{\delta\gamma} g_{ec;(\beta) b}{}^{(M-\delta-\gamma)}$$
$$= \sum_{\delta=M-\gamma-\beta}^{M-\alpha} \begin{bmatrix} \alpha\delta \\ M \end{bmatrix} \binom{M}{\delta\gamma} \binom{M-\delta-\gamma}{\beta} g^{ae(\alpha+\delta-M)} g_{ec;b}{}^{(M-\delta-\gamma-\beta)}$$

and putting $\mu = \alpha + \delta - M$,

$$\sum_{\delta=0}^{M} \underset{M}{g}^{\alpha a \delta e} \overset{M}{g}_{\delta e \gamma c;(\beta) b} = \sum_{\mu=0}^{\alpha-\gamma-\beta} \binom{\alpha}{\beta\gamma} \binom{\alpha-\beta-\gamma}{\mu} g^{ae(\mu)} g_{ec;b}{}^{(\alpha-\gamma-\beta-\mu)}$$
$$= \binom{\alpha}{\beta\gamma} (g^{ae} g_{ec;b})^{(\alpha-\beta-\gamma)}$$

from the reason that $\begin{bmatrix} \alpha\delta \\ M \end{bmatrix} \binom{M}{\delta\gamma} \binom{M-\delta-\gamma}{\beta} = \binom{\alpha}{\beta\gamma} \binom{\alpha-\beta-\gamma}{\mu}$. Similarly the second and third terms in the right-hand member of (3.2) become as follows:

$$\sum_{\delta=0}^{M} \overset{M}{g}{}^{\alpha a\,\delta e} \overset{M}{g}_{\beta b \delta e;(\gamma)c} = \binom{\alpha}{\gamma\beta} (g^{ae}\, g_{be;c})^{(\alpha-\gamma-\beta)}$$

and

$$\sum_{\delta=0}^{M} \overset{M}{g}{}^{\alpha a\,\zeta e} \overset{M}{g}_{\beta b \gamma c;(\delta)e} = \binom{\alpha}{\beta\gamma} (g^{ae}\, g_{bc;e})^{(\alpha-\beta-\gamma)}$$

respectively. Consequently we get

$$\overset{M}{\Gamma}{}_{\beta b}{}^{\alpha a}{}_{\gamma c} = \binom{\alpha}{\beta\gamma} (\overset{0}{\Gamma}{}^{a}_{bc})^{(\alpha-\beta-\gamma)} .$$

Theorem 3.2. *If R is an integer not exceeding M and if $\overset{R}{\Gamma}_{\alpha a\beta b,\sigma c}$ is* CHRISTOFFEL *symbol of the first kind by means of $\overset{R}{g}_{\alpha a\beta b}$, then we see that*

$$\binom{M+\sigma}{R} \overset{R}{\Gamma}_{\alpha a\beta b,\sigma c} = \binom{M+\sigma}{M} \overset{M}{\Gamma}_{\alpha a\beta b,M-R+\sigma c} \qquad \alpha,\beta,\sigma = 0,\cdots,R.$$

Proof. In virtue of Theorem 2.4, we can find

$$\begin{aligned}
\overset{M}{g}_{\alpha a\, M-R+\sigma c;(\beta)b} &= \binom{M+\sigma}{R}\binom{M+\sigma}{M}^{-1} \overset{R}{g}_{\alpha a\sigma c;(\beta)b} && \text{for } \beta \leqq R \\
&= 0 && \text{for } \beta > R, \\
\overset{M}{g}_{M-R+\sigma c\beta b;(\alpha)a} &= \binom{M+\sigma}{R}\binom{M+\sigma}{M}^{-1} \overset{R}{g}_{\sigma c\beta b;(\alpha)a} && \text{for } \alpha \leqq R \\
&= 0 && \text{for } \alpha > R
\end{aligned}$$

and

$$\begin{aligned}
\overset{M}{g}_{\alpha a\beta b;(M-R+\sigma)c} &= \binom{M}{\alpha\beta} g_{ab}{}^{(M-\alpha-\beta)}{}_{;(M-R+\sigma)c} \\
&= \binom{M}{\alpha\beta}\binom{M-\alpha-\beta}{M-R+\sigma} g_{ab}{}^{(M-\alpha-\beta-\overline{M-R+\sigma})}{}_{;c} \\
&= \binom{M}{\alpha\beta}\binom{M-\alpha-\beta}{M-R+\sigma}\binom{R-\alpha-\beta}{\sigma}^{-1} g_{ab}{}^{(R-\alpha-\beta)}{}_{;(\sigma)c} \\
&= \binom{M+\sigma}{R}\binom{M+\sigma}{M}^{-1}\binom{R}{\alpha\beta} g_{ab}{}^{(R-\alpha-\beta)}{}_{;(\sigma)c} \\
&= \binom{M+\sigma}{R}\binom{M+\sigma}{M}^{-1} \overset{R}{g}_{\alpha a\beta b;(\sigma)c} && \text{for } \sigma \leqq R \\
&= 0 && \text{for } \sigma > R,
\end{aligned}$$

consequently, it follows that

$$\binom{M+\sigma}{R} \overset{R}{\Gamma}_{\alpha a\beta b,\sigma c} = \binom{M+\sigma}{M} \overset{M}{\Gamma}_{\alpha a\beta b,\,M-R+\sigma c} .$$

Theorem 3.3. *There exists the relation*

$$\overset{R}{\Gamma}{}_{\alpha a}{}^{\gamma c}{}_{\beta b} = \sum_{\sigma=M-R}^{M} \overset{M}{g}{}^{\gamma c\,\sigma e} \overset{M}{\Gamma}_{\alpha a\beta b,\sigma e} \qquad \alpha,\beta,\gamma = 0,\cdots,R,$$

$\overset{R}{\Gamma}{}_{\alpha a}{}^{\gamma c}{}_{\beta b}$ *being CHRISTOFFEL symbol by means of* $\overset{R}{g}_{\alpha a \beta b}$ *and* $\underset{R}{g}^{\alpha a \gamma c}$.

The theorem is verified without difficulty from Theorem 2.5 and Theorem 3.2.

Corollary 1. *If* $\overset{0}{\Gamma}{}_{a}{}^{c}{}_{b}$ *is CHRISTOFFEL symbol by means of the metric tensor* g_{ab}, *then*

$$\overset{0}{\Gamma}{}_{a}{}^{c}{}_{b} = \overset{M}{g}{}^{\gamma c M e}\, \overset{M}{\Gamma}{}_{\alpha a \beta b M e} \qquad \alpha, \beta, \gamma = 0, \ (M\text{: not summing}).$$

In virtue of the theorem, we know that the quantities $\sum_{\sigma=M-R}^{M} \underset{M}{g}^{\gamma c \sigma e} \times \overset{M}{\Gamma}{}_{\alpha a \beta b, \sigma e}$ being a part of CHRISTOFFEL symbol by means of $\overset{M}{g}_{\alpha a \beta b}$ and $\underset{M}{g}^{\alpha a \gamma c}$ may be adopted as connection parameter for a member of range R of an extensor and especially $\underset{M}{g}^{\gamma c M e} \overset{M}{\Gamma}{}_{\alpha a \beta b, M e}$ (M: not summing) as that for a tensor member. Such a set of $(M+1)$ CHRISTOFFEL symbols in $R_n^{(M)}$, i.e., $\sum_{\sigma=M-R}^{M} \underset{M}{g}^{\gamma c \sigma e} \overset{M}{\Gamma}{}_{\alpha a \beta b, \sigma e} \left(= \overset{R}{\Gamma}{}_{\alpha a}{}^{\gamma c}{}_{\beta b}\right)$ $R=0, \cdots, M$ is called *extended CHRISTOFFEL symbol*.

§ 4. The excovariant differentiation of the RIEMANNian space.

We shall adopt the definition of excovariant differential of an extensor given by (1.1) in §1. Then we have the following theorems:

Theorem 4.1. *The extensor* $\overset{R}{g}_{\alpha a \beta b}$, $\underset{R}{g}^{\alpha a \beta b}$ $(R = 0, \cdots, M)$ *and* $\delta^{\alpha}_{\beta} \delta^{a}_{b}$ *behave as constants in excovariant differentiation.*

Theorem 4.2. *If* $T^{\alpha_1 a_1 \cdots \alpha_A a_A}{}_{\beta_1 b_1 \cdots \beta_B b_B}$ *is an extensor of range R, its excovariant differential* $\delta^R T^{\alpha_1 a_1 \cdots \alpha_A a_A}{}_{\beta_1 b_1 \cdots \beta_B b_B}$ *by means of* $\overset{R}{\Gamma}{}_{\beta b}{}^{\alpha a}{}_{\gamma c}$ *is equal to* $\delta^* T^{\alpha_1 a_1 \cdots \alpha_A a_A}{}_{\beta_1 b_1 \cdots \beta_B b_B}$.

Corollary. *When* $T^{a_1 \cdots a_A}{}_{b_1 \cdots b_B}$ *is a tensor, the well known covariant differential* $\delta T^{a_1 \cdots a_A}{}_{b_1 \cdots b_B}$ *in the RIEMANNian space is nothing but* $\delta^* T^{a_1 \cdots a_A}{}_{b_1 \cdots b_B}$.

Theorem 4.3. *If* $v_1^{\alpha a}$ *and* $v_2^{\alpha a}$ *are two extensors of range R satisfying* $\delta^* v_1^{\alpha a} = 0$ *and* $\delta^* v_2^{\alpha a} = 0$, *that is, the extensors are displaced parallelly to themselves, then it follows that* $\delta^* v = 0$ and $\delta^* \theta = 0$, where v and θ *mean the length of* $v_1^{\alpha a}$ *and the angle between* $v_1^{\alpha a}$ *and* $v_2^{\alpha a}$ *respectively.*

Theorem 4.4. *If* v *is the length of a vector* v^a *and if* v^a *is displaced parallelly, then the extensor* $v^{(\alpha) a}$ $(\alpha = 0, \cdots, R)$ *formed by differentiating the vector* v^a *R times is displaced parallelly and the scalar* $\dfrac{d^{\alpha} v^2}{dt^{\alpha}}$ $(\alpha = 0, \cdots, R)$ *does not change.*

The theorem is proved by virtue of Theorem 1.6, Theorem 2.8 and Theorem 3.1.

Theorem 4.5. *If* $T'^{\alpha_1 a_1 \cdots \alpha_A a_A}{}_{\beta_1 b_1 \cdots \beta_B b_B}$ *is any general extensor with different range for each Greek index,* $\delta^* T'^2 = 0$ *follows from* $\delta^* T'^{\alpha_1 a_1 \cdots \alpha_A a_A}{}_{\beta_1 b_1 \cdots \beta_B b_B} = 0$.

Corollary. *The length of a vector and the angle between two vectors are invariant under the parallel displacement of the vectors.*

In consequence of the above-mentioned result, our excovariant differential is the operation with the geometrical meaning such that the scalar $\frac{d^R v^2}{dt^R}$ resp. $\frac{d^R \cos\theta}{dt^R}$ $(R = 0, \cdots, M)$ obtained by differentiating the square of the length of a vector v^a resp. cosine of the angle between two unit vectors v_1^a and v_2^a R times becomes invariant under the parallel displacement of the extensors $v_1^{(\alpha)a}$ and $v_2^{(\alpha)a}$ of range R and that the operation coincides with the ordinary covariant differential in the RIEMANNian space for tensor members. Thus it will be very interesting that the connection theory of extensor which holds the already known tensor theory in the RIEMANNian space as a part of it has been established.

§ 5. The connection of extensor in an EUCLIDean space. Consider an EUCLIDean space that is a special one of the RIEMANNian space, then the space has the rectangular cartesian coordinate system x^a in which the metric tensor g_{ab} becomes equal to δ_{ab}, and the metric extensor $\overset{M}{g}_{\alpha a \beta b}$ and $\underset{M}{g}^{\alpha a \beta b}$ are equal to $\delta_{ab} \overset{M}{\delta}_{\alpha\beta}$ and $\delta^{ab} \underset{M}{\delta}^{\alpha\beta}$ respectively, putting $\overset{M}{\delta}_{\alpha\beta} = \binom{M}{\alpha} \delta_{M-\alpha\,\beta}$ and $\underset{M}{\delta}^{\alpha\beta} = \binom{M}{\alpha}^{-1} \delta^{M-\alpha\,\beta}$. Evidently, we have the following theorems:

Theorem 5.1. *The following relation holds good:*

$$\sum_{\alpha=0}^{M} \overset{M}{\delta}_{\alpha\beta} \underset{M}{\delta}^{\alpha\gamma} = \delta_\beta^\gamma .$$

Theorem 5.2. *In the cartesian coordinate system, the extended* CHRISTOFFEL *symbol* $\overset{R}{\Gamma}{}_{\beta b}{}^{\alpha a}{}_{\gamma c}$ $(R = 0, \cdots, M)$ *vanishes identically and the excovariant differential becomes the ordinary differential.*

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