ON INFINITESIMAL HOLOMORPHICALLY PROJECTIVE TRANSFORMATION

By

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§0. Introduction. Recently, T. Ôtsuki and Y. Tashiro [1][1] have studied holomorphically projective correspondences of Kählerian manifolds.

On the other hand, K. Yano and T. Nagano [2] and T. Sumitomo [3] have studied infinitesimal projective transformations in a Riemannian manifold and obtained valuable results. Further, S. Tachibana and S. Ishihara [4] have considered analogous problems concerning the infinitesimal holomorphically projective transformations, which will be briefly called an *HP-transformation*, and obtained that a Kählerian manifold satisfying $R_{ij;k}=0$, which admits a non-trivial analytic HP-transformation reduces to an Einstein one.

The purpose of the present paper is to generalize more the above result of S. Tachibana and S. Ishihara, that is, we shall give a theorem about a Ricci-recurrent Kählerian manifold in §1 and one in a Ricci-recurrent K-space in §2, which is one of the generalization of the theorem in §1.

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§1. An analytic HP-transformation on a Kählerian manifold.

A vector field v^i is called an HP-transformation, if it satisfies

$$\underset{v}{\mathfrak{L}}\left\{{i \atop jk}\right\}=P_h(\delta_j^h\delta_k^i-\varphi_j^h\varphi_k^i)+P_h(\delta_k^h\delta_j^i-\varphi_k^h\varphi_j^i), \tag{1.1}$$

where P_h is a certain vector and φ_j^i is the complex structure, and semi-colon and $\underset{v}{\mathfrak{L}}$ denote the covariant differentiation with respect to v^i and Lie differentiation with respect to v^i, respectively. We shall call P_h in (1.1) the associated vector of the HP-transformation. Contracting (1.1)

1) Numbers in brackets refer to the references at the end of the paper.

with respect to i and k, we get $P_h=\frac{1}{n+2}v^i{}_{;i;h}$, which shows that P_h is gradient.

An infinitesimal affine transformation v^i is defined by

$$\underset{v}{\mathfrak{L}}\{{}^{i}_{jk}\}\equiv v^i{}_{;j;k}+R^i_{jkl}v^l=0.$$

If $P_h=0$, then the HP-transformation reduces to an affine one.

A vector field v^i is called analytic on a Kählerian manifold, if it satisfies

$$\underset{v}{\mathfrak{L}}\varphi^i_j=-\varphi^k_j v^i{}_{;k}+\varphi^i_k v^k{}_{;j}=0. \tag{1.2}$$

We shall give here preliminary formulas on Kählerian manifold. Let our manifold be a real $n(=2m>2)$ dimensional Kählerian manifold with local coordinates $\{x^i\}$. Then the Riemannian metric g_{ij} and the complex structure φ^j_i satisfy

$$\varphi^k_i\varphi^j_k=-\delta^j_i, \qquad g_{hk}\varphi^h_i\varphi^k_j=g_{ij},$$
$$\varphi^j_{i;k}=0, \qquad g_{ij;k}=0.$$

Then the following equation holds:

$$R_{hk}\varphi^h_i\varphi^k_j=R_{ij}, \tag{1.3}$$

where R^i_{jkl} is the Riemannian curvature tensor, and

$$R^i_{jki}=R_{jk}, \qquad R^h_{jkl}g_{hi}=R_{ijkl}.$$

If P_h is the associated vector of an analytic HP-transformation, then we get

$$P_{h;k}\varphi^h_i\varphi^k_j=P_{i;j}. \tag{1.4}$$

Moreover, if v^i be an analytic HP-transformation, then we have

Lemma. *Let v^i be an analytic HP-transformation, then the following relation holds:*

$$(\underset{v}{\mathfrak{L}}g_{ik})R^k_j=(\underset{v}{\mathfrak{L}}g_{jk})R^k_i. \tag{1.5}$$

Proof. From the assumptions, it follows that

$$\underset{v}{\mathfrak{L}}\varphi^j_i=0,$$

$$\underset{v}{\mathfrak{L}}\{{}^{i}_{jk}\}\equiv v^i{}_{;j;k}+R^i_{jkl}v^l=P_h(\delta^h_j\delta^i_k-\varphi^h_j\varphi^i_k)+P_h(\delta^h_k\delta^i_j-\varphi^h_k\varphi^i_j).$$

Since R_{ijkl} is anti-symmetric with respect to i and j, we get

$$(\underset{v}{\mathfrak{L}}g_{ij})_{,k}\equiv(v_{i;j}+v_{j;i})_{;k}=2P_kg_{ij}+P_jg_{ik}+P_ig_{jk}-P_a\varphi^a_j\varphi_{ki}-P_a\varphi^a_i\varphi_{kj}. \tag{1.6}$$

The integrable condition of the above equation is that

$$(\underset{v}{\mathfrak{L}}g_{aj})R^a_{ikl}+(\underset{v}{\mathfrak{L}}g_{ia})R^a_{jkl}=P_{j;k}g_{il}+P_{i;k}g_{jl}-P_{j;l}g_{ik}-P_{i;l}g_{jk}$$
$$+\varphi^a_j(P_{a;l}\varphi_{ki}-P_{a;k}\varphi_{li})+\varphi^a_i(P_{a;l}\varphi_{kj}-P_{a;k}\varphi_{lj}).$$

If we contract g^{jl} to this equation and take account of (1.4), then we have

$$(\underset{v}{\mathfrak{L}} g_{aj})R_{ik}^{a\ j} - (\underset{v}{\mathfrak{L}} g_{ia})R_{k}^{a} = nP_{i;k} - P_{a}{}^{;a} g_{ik}.$$

Since P_h is gradient and $(\underset{v}{\mathfrak{L}} g_{aj})R_{ik}^{a\ j}$ is symmetric with respect to i and k, we obtain the conclusion.

Recently S. Tachibana and S. Ishihara [4] obtained the following

Theorem. *If a Kählerian manifold satisfying* $R_{ij;k}=0$ *admits an analytic non-affine HP-transformation, it is a Kähler-Einstein manifold.*

We shall now consider a Ricci-recurrent Kählerian manifold, i.e., a Kählerian manifold such that $R_{ij;k}=R_{ij}v_k$, and we obtain the following

Theorem. *If a Kählerian manifold satisfying* $R_{ij;k}=R_{ij}v_k$ *admits an analytic non-affine HP-transformation, it is a Kähler-Einstein manifold.*

Proof. Covariantly differentiating (1.5) with respect to x^l and making use of (1.5), we find

$$(\underset{v}{\mathfrak{L}} g_{ia})_{;l} R_{k}^{a} = (\underset{v}{\mathfrak{L}} g_{ka})_{;l} R_{i}^{a}.$$

Substituting (1.6) into the last equation, we have easily

$$\begin{aligned}(P_a g_{il} + P_i g_{al} + 2P_l g_{ia} - \varphi_a^b \varphi_{li} P_b - \varphi_i^b \varphi_{la} P_b) R_k^a \\ = (P_a g_{kl} + P_k g_{al} + 2P_l g_{ka} - \varphi_a^b \varphi_{lk} P_b - \varphi_k^b \varphi_{la} P_b) R_i^a.\end{aligned}$$

Contracting this equation with g^{il} and R^{il}, and taking account of (1.3) and (1.4), we have

$$nP_a R_k^a = RP_k,$$

$$RR_k^a P_a = R_{ij} R^{ij} P_k.$$

From the above equations, we get

$$\left(R_{ij}R^{ij} - \frac{R^2}{n}\right) P_k = 0.$$

Since $P_k \neq 0$, we must have

$$R_{ij}R^{ij} - \frac{R^2}{n} = 0.$$

On the other hand, according to the theorem obtained by T. Sumitomo [3], a Riemannian manifold satisfying the relation $R_{ij}R^{ij} = \dfrac{R^2}{n}$ is an Einstein manifold. Therefore, we get the conclusion.

§2. An analytic HP-transformation in a K-space.

In this section, we shall consider only a K-space, which is another generalization of a Kählerian manifold.

If φ_{ij} $(\varphi_{ij}\overset{\text{def}}{=}\varphi_i^k g_{kj})$ is a Killing tensor, i.e., it satisfies the equation

$$\varphi_{ij;k}+\varphi_{ik;j}=0,$$

an almost-Hermitian space is called a K-space. After some calculations we get also the following identities in a K-space:

$$R_{hk}\varphi_i^h\varphi_j^k=R_{ij}, \tag{2.1}$$

$$P_{h;k}\varphi_i^h\varphi_j^k=P_{i;j}. \tag{2.2}$$

Thus, by virtue of (2.1), (2.2), and Lemma, we have the following

Theorem. *If a K-space satisfying $R_{ij;k}=R_{ij}v_k$ admits an analytic non-affine HP-transformation, it is an Einstein K-space.*

The method of the proof is analogous to that in Kählerian manifold.

References

[1] OTSUKI and Y. YASHIRO: On curves in Kählerian spaces, Math. Jour. Okayama Univ., 4 (1954), 57–78.

[2] K. YANO and T. NAGANO: Some theorems on projective and conformal transformations, Indag. Math., 14 (1957), 451–458.

[3] T. SUMITOMO: Projective and conformal transformations in compact Riemannian manifolds, Tensor, New Series, 9 (1959), 113–135.

[4] S. TACHIBANA and S. ISHIHARA: On infinitesimal holomorphically projective transformations in Kählerian manifolds, Tôhoku Math. Jour., 12 (1960), 77–101.

[5] T. SUMITOMO: On infinitesimal projective transformation on a Riemannian manifold, To appear.

[6] S. TACHIBANA: On almost analytic vectors in certain almost Hermitian manifolds, Tôhoku Math. Jour., 11 (1959), 351–363.

[7] K. YANO: The theory of Lie derivatives and its applications, Amsterdam, (1957).

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