ON THE HILBERT TRANSFORM I*)

By

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CONTENTS

1. Introduction. The theory of Hilbert transform has been studied by many authors. Its references will be found in the book of E. C. Titchmarsh [30]. Recently A. P. Calderón-A. Zygmund [5] show that this classical theory can be treated by another method in the n-dimensional Euclidean space. The foundation of their arguments is the interpolation of linear operation. There are several studies of J. Marcinkiewicz in Fourier series from this point of views. In particular he has presented a brief note in Comptes rendus vol. 208 (1939), 1271–1273, without proof. Recently A. Zygmund [36], a teacher of his, has completed these theorems.

In chapter 1, we shall extend one of these theorems on the totally σ-finite measure space in a sense of P. R. Halmos [11]. This may give an answer to the problem of Prof. A. Zygmund.

Using this as a main tool, we may extend the Hilbert transform to the other direction. These are treated in chapter 2.

In chapter 3 we shall prove the reciprocal formula of this operator by the complex variable methods. Setting this result as the base of arguments, we treat analytic functions in a half-plane. The ordinary case is due to R.E.A.C. Paley-N. Wiener [25] and E. Hille-J.D. Tamarkin [15, 16].

*) This contain the detailed argument of papers published in the Proc. Japan Academy, vol. 34~5 (1958~9).

Next N. I. Achiezer [1] has introduced the modified definition of Hilbert transform for the purpose to treat the more extensive class of functions. In [21, V], the author has introduced another modified one under the same idea, there we have proved that our modified transform well conserves the property having the original one. Here we only limit to the generalized Hilbert transform of order 1 and study the property of this operator in further details. This is defined by the following formula

$$\tilde{f}_1(x)=\frac{x+i}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\,\frac{dt}{x-t}. \tag{1.01}$$

This may be written formally

$$\begin{aligned}\tilde{f}_1(x)&=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}\,dt+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\,dt\\&=\tilde{f}(x)+A^f.\end{aligned} \tag{1.02}$$

Therefore if we assume the A^f to be finitely determined then $\tilde{f}$ exists a.e. and from properties of $\tilde{f}_1(x)$, those of $\tilde{f}(x)$ can be derived. We remark that the modified transform (1.01) will be equivalent that of N. I. Achiezer for certain class of functions. Our main object is functions of class W_2 to be defined later. These are argued in chapter 4.

In chapter 5, we intend to find the spectral relation between $f(x)$ and its Hilbert transform. Our main tools are N. Wiener's generalized Fourier transform and the Tauberian theorem which is called the Wiener formula [32]. These are main purpose of this paper.

Chapter 1. Interpolation of the operation

2. Let (X, R, μ) and (Y, S, ν) be two measure spaces with finite measures μ and ν respectively. We denote by L_μ^r the class of functions, which are defined on X, are μ-measurable and such that

$$\|f\|_{r,\mu}=\Big(\int_X |f|^r\,d\mu\Big)^{\frac{1}{r}}<\infty, \tag{2.01}$$

where r is any positive number.

Let us consider the operation

$$\hat{f}=Tf \tag{2.02}$$

where f and $\hat{f}$ are real or complex valued functions of μ-measurable and

ν-measurable which are defined on these measure spaces X and Y respectively.

We introduce several notations of operation after A. Zygmund [36].

Definition 1. *The operation T will be called a quasi-linear if $T(k_1f_1+k_2f_2)$ is uniquely defined whenever Tf_1 and Tf_2 are defined with constants k_1 and k_2 of any real or complex number and satisfies*

$$|T(f_1+f_2)| \leqq \kappa(|Tf_1|+|Tf_2|) \tag{2.03}$$

$$|T(kf)| = |k|\,|Tf|, \tag{2.04}$$

where κ is a constant independent on f_1 and f_2.

Definition 2. *The operation T will be called a strong type (a, b). If $\hat{f}=Tf$ is defined for every f in L^a_μ and $\hat{f}$ belongs to L^b_ν and it satisfies*

$$\|\hat{f}\|_{b,\nu} \leqq M\|f\|_{a,\mu} \tag{2.05}$$

where M is independent on f. The least value M is called the (a, b)-norm of the operation T.

Definition 3. *The operation T will be called a weak type (a, b). If $\hat{f}=Tf$ is defined for every f in L^a_μ and if we denote e_r, the set of y in which $|\hat{f}(y)|>r$, it satisfies*

$$\nu(e_r) \leqq \left(\frac{M}{r}\|f\|_{a,\mu}\right)^b, \tag{2.06}$$

where r is any positive real number and M is independent on f and r. The least value M is called the (a, b)-norm of the operation T.

We note that in these definitions, a and b are any positive real number or may be infinite. If the a is infinite, we interprete

$$\|f\|_{\infty,\mu} = \text{ess. sup.}\,|f|, \tag{2.07}$$

and similarly for the b. We observe that Definition 2 contains Definition 3. If the b is infinite in Definition 3, in this case we promise that the difference between the type of strong and that of weak vanish.

Then J. Marcinkiewicz-A. Zygmund proved the following theorem

Theorem (*J. Marcinkiewicz-A. Zygumund*) *Suppose that $\mu(X)$ and $\nu(Y)$ are both finite and that a quasi-linear operation $\hat{f}=Tf$ is of weak type (a, a) and (b, b), where $1\leqq a<b<\infty$. Suppose also that $\varphi(u)$, $u\geqq 0$, is a continuous increasing function satisfying the condition*

$$\varphi(0) = 0 \tag{2.08}$$

and

$$\varphi(2u) = O(\varphi(u)) \tag{2.09}$$

$$\int_u^\infty \frac{\varphi(t)}{t^{b+1}}\, dt = O\Big(\frac{\varphi(u)}{u^b}\Big) \tag{2.10}$$

$$\int_1^u \frac{\varphi(t)}{t^{a+1}}\, dt = O\Big(\frac{\varphi(u)}{u^a}\Big) \tag{2.11}$$

for $u\to\infty$. *Then* $\hat{f}=Tf$ *is defined for every* f *of* $\varphi(|f|)$ *to be* μ-*integrable and*

$$\int_Y \varphi(|\hat{f}|)d\nu \leqq K\int_X \varphi(|f|)d\mu+L, \tag{2.12}$$

where K *and* L *are independent on* f.

In [21, I] the author extended this theorem on spaces with infinite measures. From now on, let (X, R, μ) and (Y, S, ν) be measure spaces of totally σ-finite in a sense of P. R. Halmos [1]. In this paper it is sufficient to consider the operation under definitions 2 and 3, but in some cases such as Fourier transforms it is convenient to consider the operator under somewhat relaxed conditions. Let us suppose that the operation T is only defined on the class of simple function. Then the above definitions are relaxed as follows.

Definition 2′. *The operation* T *which is defined for every simple function is of strong type* (a, b), *if it satisfies* (2.05) *for every simple function* f *with constant* M *to be independent on* f.

Definition 3′. *The operation* T *which is defined for every simple function is of weak type* (a, b), *if it satisfies* (2.06) *for any positive real number with constant* M *to be independent on* f *and* r.

We begin with simple functions. A simple function

$$f(x) = \sum_{i=1}^n c_i\chi_{e_i}(x) \tag{2.13}$$

on measure space (X, R, μ) is integrable if $\mu(e_i)$ is finite for every index i for which $c_i \neq 0$. We limit the simple function only to the integrable one. Then the class of simple function is dense in every L_μ^r, $0<r<\infty$.

Let $\varphi(u)$ be defined for positive real arguments $u\geqq 0$ and constinuous increasing function with $\varphi(0)=0$. By L_μ^φ, we denote the class of functions which are μ-measurable and $\varphi(|f|)$ to be μ-integrable on X. We need the following two lemmas in the later arguments.

Lemma A₁. *Let* $f(x)$ *be a function of* L_μ^φ. *Then there exists a sequence of non-negative simple functions* $f_n(x)$ *such that* $f_n(x)$ *tend to*

$|f(x)|$ *increasingly for a.e. x and*

(2.14) $$\int_X \varphi(f_n)d\mu \uparrow \int_X \varphi(|f|)d\mu, \quad (n\to\infty).$$

Furthermore we have

(2.15) $$\int_X \varphi(|f_n-f_m|)d\mu \to 0, \quad (m, n\to\infty).$$

Lemma A$_2$. *Let $f_n(x)$ be a sequence of function which are not necessarily simple and belong to the class L^φ_μ. Let $f_n(x)$ be the mean fundamental with respect to $\varphi(u)$. That is*

(2.16) $$\int_X \varphi(|f_n-f_m|)d\mu \to 0, \quad (m, n\to 0).$$

Then there exists a sub-sequence (n_k) and f in L^φ_μ such that

(2.17) $$f_{n_k}(x)\to f(x), \quad a.e.\ x$$

(2.18) $$\int_X \varphi(|f-f_n|)d\mu \to 0, \quad (n\to\infty),$$

where $f(x)$ is uniqely determined for a.e. x.

Proofs of Lemmas A_1 and A_2. The Lemma A$_1$ is immediate. For the Lemma A$_2$, from (2.16) we can find that the sequence f_n form that of fundamental in measure. From this fact, (2.17) and (2.18) are obtained by the usual argument.

Next let us introduce the function $\varphi(u)$ which is defined for positive real argument. The $\varphi(u)$ is a continuous increasing function and satisfies the following properties with a, b such as $1\leqq a<b<\infty$,

(2.08) $$\varphi(0)=0$$

and

(2.09) $$\varphi(2u)=O(\varphi(u))$$

(2.10) $$\int_u^\infty \frac{\varphi(t)}{t^{b+1}}\,dt = O\left(\frac{\varphi(\mu)}{u^b}\right)$$

(2.11) $$\int_1^u \frac{\varphi(t)}{t^{a+1}}\,dt = O\left(\frac{\varphi(u)}{u^a}\right)$$

for $u\to\infty$,

(2.19) $$\varphi(u)=O(\varphi(u/2))$$

(2.20) $$\int_u^1 \frac{\varphi(t)}{t^{b+1}}\,dt = O\left(\frac{\varphi(u)}{u^b}\right)$$

$$\int_0^u \frac{\varphi(t)}{t^{a+1}}\,dt = O\Big(\frac{\varphi(u)}{u^a}\Big) \tag{2.21}$$

for $u \to 0$.

Under these assumptions we observe that

Lemma A₃. *Let $\varphi(u)$ be a continuous increasing function and satisfy the above properties* (2.08)~(2.11) *and* (2.19)~(2.21). *Then we have*

$$Au^a \leqq \varphi(u) \leqq Bu^b \tag{2.22}$$

for $u \to \infty$, and

$$A'u^a \geqq \varphi(u) \geqq B'u^b \tag{2.23}$$

for $u \to 0$ respectively with suitable constants.

Lemma A₄. *The following functions satisfy the above properties* (2.08)~(2.11) *and* (2.19)~(2.21).

$$\varphi(u) = u^r, \quad a<r<b, \tag{2.24}$$

$$\varphi(u) = u^r \log(1+u), \tag{2.25}$$

and more generally

$$\varphi(u) = u^r \psi(u), \tag{2.26}$$

where $\psi(u)$ is a slowly varying function as $u \to 0$ and $u \to \infty$ respectively.

Then we have

Theorem A. *Let (X, R, μ) and (Y, S, ν) be two measure speces of totally σ-finite. Let $\hat{f} = Tf$ be a quasi-linear operation defined for all simple function f on (X, R, μ) with $\hat{f}$ on (Y, S, ν). Suppose that T is simultaneously of the weak type (a, a) and (b, b) with $1 \leqq a < b < \infty$. Then we have*

$$\|f\|_{\varphi,\nu} \leqq A_\varphi \|f\|_{\varphi,\mu} \tag{2.27}$$

where A_φ is constant independent on f. In particular the operation T can be uniquely extended to the whole space L^φ_μ preserving the (2.27).

In [21, I], the author gave the brief proof under strong definition. The completion can be done by Lemmas A_1 and A_2.

In the (2.27) if we put $\varphi(u) = u^r$, $a<r<b$, then we get

$$\|\hat{f}\|_{r,\nu} \leqq A_r \|f\|_{r,\mu}. \tag{2.28}$$

This is the Theorem of A. P. Calderón-A. Zygmund [5], there they proved the two more interpolating theorems which are concerning to the ordinary Lebesgue measure. We can state in the following form.

Theorem B. *Let (X, R, μ) and (Y, S, ν) be two measure spaces of* *totally σ-finite. Let $\hat{f}=Tf$ be a quasi-linear operation defined for all simple functions f on (X, R, μ) with $\hat{f}$ on (Y, S, ν). Suppose that T is simultaneously of weak type $(1, 1)$ and (p, p) for some $p>1$. Then we have for any sub-set S_0 of S with finite ν-measure*

$$\int_{S_0} |\hat{f}|d\nu \leqq A\int_X |f|\{1+\log^+[\nu(S_0)^\gamma|f|]\}d\mu + B\nu(S_0)^{1-\gamma}. \tag{2.29}$$

where γ is any positive number and A, B, are absolute constants. In particular, the operation T can be uniquely extended to the whole space $L_\mu \log^+ L_\mu$ preserving (2.29).

Theorem C. *Let (X, R, μ) and (Y, S, ν) be two measure spaces of totally σ-finite. Let $\hat{f}=Tf$ be a quasi-linear operation defined for all simple functions f on (X, R, μ) with $\hat{f}$ on (Y, S, ν). Suppose that T is of weak type $(1, 1)$. Then we have for any sub-set S_0 of S with finite ν-measure*

$$\int_{S_0} |\hat{f}|^{1-\varepsilon}d\nu \leqq \frac{A}{\varepsilon}\nu(S_0)^\varepsilon\Big(\int_X |f|\,d\mu\Big)^{1-\varepsilon}, \tag{2.30}$$

where ε is any positive number such as $0<\varepsilon<1$ and A is an absolute constant. In particular the operation T can be uniquely extended to the whole space L_μ preserving (2.30).

There is no different point essentially in proofs of these theorems. For any μ-measurable function f, we shall denote by $f^*(t)$, $0<t<\infty$, a non-increasing function equi-measurable with $|f|$. We also introduce a function

$$y=\beta_f(x)=\frac{1}{y}\int_0^x f^*(t)dt, \quad x>0 \tag{2.31}$$

and the function inverse to $y=\beta_f(x)$ will be denoted by $x=\beta^f(y)$. These are used by A. P. Calderón-A. Zygmund [5]. For Theorem B, we only observe that the following lemma

Lemma B$_2$. *We have*

$$\frac{1}{y}\int_{|f|\geqq y} |f|\,d\mu \leqq \beta^f(y). \tag{2.32}$$

This is deduced by the geometrical consideration easily.

The Theorem C can be proved by the same arguments as their proof. The extension to the whole space can be obtained by Lemmas A$_1$ and A$_2$ as before.

Chapter 2. The Hilbert transform

3. Let f be a real or complex valued measurable function over $(-\infty, \infty)$. Let $\tilde{f}$ be the Hilbert transform of f, that is

$$\tilde{f}(x) = \lim_{\eta \to 0} \frac{1}{\pi} \int_{|x-t|>\eta} \frac{f(t)}{x-t} dt. \tag{3.01}$$

The property of this singular integral has been studied by many authors. In particular we feel interest in the result of L. H. Loomis [22], H. Kober [20] and A. P. Calderón-A. Zygmund [5, 6].

We consider the Hilbert transform on the totally σ-finite measure space (X, R, μ), where X is one dimensional space, R is the class of measurable sets in the ordinary Lebesgue sense and

$$\mu(e, \alpha) = \mu(e) = \int_e \frac{dx}{1+|x|^\alpha}, \quad (0 \leqq \alpha < 1) \tag{3.02}$$

for every $e \in R$. We shall write simply

$$\mu(x) = \mu\{(0, x), \alpha\} \quad x > 0, \quad \mu(-x) = \mu(x). \tag{3.03}$$

Let f be a function such that

$$\int_{-\infty}^{\infty} \frac{|f(x)|^p}{1+|x|^\alpha} dx < \infty, \quad (p \geqq 1).$$

Then we have

$$\int_{-\infty}^{\infty} \frac{|f(x)|^p}{1+|x|^\alpha} dx = \int_{-\infty}^{\infty} |f|^p d\mu. \tag{3.04}$$

The class of functions of which the integral defined by (3.04) is finite will be denoted by L^p_μ. In mathematical speaking, this measure function plays as a role of convergence factor and this enables us to treat the Hilbert transform for the more extensive class than the ordinary class L^p. On the other hand it may be interpreted that the introduce of this measure adds some varieties.

The purpose of this chapter is the systematic treatment of the Hilbert transform from the point of view of the linear operation.

We begin with the ordinary case $\alpha = 0$.

Theorem 1. *Let f belong to L^p $(p > 1)$. Then the Hilbert operator $\tilde{f}$ exists for a.e. x, and we have*

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^p dx \leqq A_p \int_{-\infty}^{\infty} |f(x)|^p dx \tag{3.05}$$

and

$$\int_{-\infty}^{\infty} |\tilde{f}(x)-\tilde{f}_\lambda(x)|^p \, dx \to 0\,, \quad \lambda \to 0, \tag{3.06}$$

where A_p is a constnat to be independent on f.

This is due to M. Riesz [28] for general p.

Theorem 2. *Let f belong to L. Then the Hilbert operator f exists for a.e. x and of weak type* (1, 1). *That is for any positive number r, if we denote by E_r the set of points at which $|\tilde{f}(x)|>r$, then we have*

$$|E_r| \leqq \frac{M}{r} \|f\|_1 \tag{3.07}$$

where M is a constant to be independent on f.

This is due to L.H. Loomis [22]. We remark that in (3.05) and so in (3.07), $\tilde{f}(x)$ may be replaced by $\tilde{f}_\lambda(x)$ and $\sup_{\lambda>0} |\tilde{f}_\lambda(x)|$ where the constants A_p and M are independent on λ and f.

4. Throughout of this section we use the term — weak or strong type of operation, linear, derivative, measurable — all in a sense concerning with μ. This section contains two fundamental theorems.

One of these is as follows:

Theorem 3. *Let f belong to L^p_μ ($p>1$, $0\leqq\alpha<1$), then the Hilbert operation is the strong type (p, p).*

Proof of Theorem 3. The existence of $\tilde{f}(x)$ is obtained by that of (3.01) for the ordinary class L^p, and the Hölder inequality. Because if we write

$$f(x) = m_n(x)+r_n(x) \tag{4.01}$$

where

$$m_n(x)=\begin{cases} f(x), & \text{if } |x-n|\leqq 1, \quad (n=0, \pm 1,\cdots) \\ 0\;, & \text{elsewhere.} \end{cases} \tag{4.02}$$

$$r_n(x) = f(x)-m_n(x) \tag{4.03}$$

Then $\tilde{m}_n(x)$ exists a.e. x in $(-\infty, \infty)$ by Theorem 1 and $\tilde{r}_n(x)$ does also for a.e. x in $|x-n|\leqq 1/2$ by the Hölder inequality. This gives the existence of $\tilde{f}(x)$ for a.e.

Now if we define

$$\tilde{f}^*(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{x-y}\,\frac{dy}{1+|y|^\beta}\,, \tag{4.04}$$

then we have

$$\tilde{f}^*(x)-\frac{\tilde{f}(x)}{1+|x|^\beta}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{1+|y|^\beta}\frac{|x|^\beta-|y|^\beta}{(x-y)(1+|x|^\beta)}dy. \tag{4.05}$$

By Theorem 1, we have

$$\|\tilde{f}^*\|_p \leqq A_p\left\|\frac{f}{1+|x|^\beta}\right\|_p. \tag{4.06}$$

Thus if we prove

$$\left\|\tilde{f}^*-\frac{\tilde{f}}{1+|x|^\beta}\right\|_p \leqq A_p\left\|\frac{f}{1+|x|^\beta}\right\|_p. \tag{4.07}$$

Then we have

$$\left\|\frac{\tilde{f}}{1+|x|^\beta}\right\|_p \leqq A_p\left\|\frac{f}{1+|x|^\beta}\right\|_p \tag{4.08}$$

and putting $\beta=\alpha/p$ we obtain Theorem 3.

Our aim is to prove the relation (4.07) but instead of this formula we prove the equivalent bilinear form (c.f. G. H. Hardy-J. E. Littlewood-G. Pólya [14, Th. 286]).

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(y)|}{1+|y|^\beta}\frac{|g(x)|}{1+|x|^\beta}K(x, y)\,dx\,dy \tag{4.09}$$

$$\leqq A\left\|\frac{f}{1+|y|^\beta}\right\|_p\left\|\frac{g}{1+|x|^\beta}\right\|_q,$$

where

$$K(x, y)=\frac{||x|^\beta-|y|^\beta|}{|x-y|(1+|x|^\beta)} \tag{4,10}$$

and

$$\frac{1}{p}+\frac{1}{q}=1. \tag{4.11}$$

We have by the Hölder inequality the absolute value of left-hand side of (4.09) does not surpass than

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{|f|}{1+|y|^\beta}(K)^{\frac{1}{p}}\frac{1+|y|^r}{1+|x|^r}\right\}\left\{\frac{|g|}{1+|x|^\beta}(K)^{\frac{1}{q}}\frac{1+|x|^r}{1+|y|^r}\right\}dx\,dy$$

$$\leqq A\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}K_1(y)\left(\frac{|f|}{1+|y|^\beta}\right)^p dy\right\}^{\frac{1}{p}}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}K_2(x)\left(\frac{|g|}{1+|x|^\beta}\right)^q dx\right\}^{\frac{1}{q}},$$

where

(4.12) $$K_1(y)=\frac{1}{\pi}\int_{-\infty}^{\infty}K(x,y)\frac{1+|y|^{p\gamma}}{1+|x|^{p\gamma}}dx$$

(4.13) $$K_2(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}K(x,y)\frac{1+|x|^{q\gamma}}{1+|y|^{q\gamma}}dy$$

and the γ is a constant to be decided later.

If we put $x/y=u$ in (4.12) we have

$$K_1(y)\leqq\frac{2}{\pi}\int_0^{\infty}\frac{1}{|1-u|}\,\frac{|y|^{\beta}|1-u^{\beta}|}{1+|uy|^{\beta}}\,\frac{1+|y|^{\gamma p}}{1+|uy|^{\gamma p}}du.$$

We estimate this integral by decomposing into three cases according to the value of y.

(i) $1/2\leqq|y|\leqq 2$. We have

$$K_1(y)\leqq A\int_0^{\infty}\frac{1}{|1-u|}\,\frac{|1-u^{\beta}|}{1+u^{\beta}}\,\frac{du}{1+u^{\gamma p}}=O(1)$$

for $\beta>0$, and $\gamma>0$.

(ii) $2<|y|$. If we decompose into the following

$$K_1(y)\leqq A\Big(\int_0^{1/2}+\int_{1/2}^{2}+\int_2^{\infty}\Big)$$
$$=A(I_1^1+I_1^2+I_1^3),\quad \text{say}.$$

Then we have

$$I_1^1\leqq A\int_0^{1/2}\frac{|y|^{\beta}}{1+|uy|^{\beta}}\,\frac{|y|^{\gamma p}}{1+|uy|^{\gamma p}}du$$
$$=A\int_0^{1/2}\frac{du}{u^{\beta+\gamma p}}=O(1),$$

for $0<\beta+\gamma p<1$,

$$I_1^2\leqq A\int_{1/2}^{2}\frac{|1-u^{\beta}|}{|1-u|}du=O(1)$$

for $\beta>0$, and

$$I_1^3\leqq A\int_2^{\infty}\frac{du}{u^{1+\gamma p}}=O(1)$$

for $\gamma>0$.

(iii) $|y|<1/2$. If we decompose similarly,

$$K_1(y)=A\Big(\int_0^{1/2}+\int_{1/2}^{2}+\int_2^{1/y}+\int_{1/y}^{\infty}\Big)$$

$$=A(I_1^4+I_1^5+I_1^6+I_1^7), \quad \text{say.}$$

Then we have

$$I_1^4 \leqq A\int_0^{1/2}(1+|y|^{\gamma p})\,|y|^{\beta}\,du=O(1)$$

for $\beta>0$, $\gamma>0$,

$$I_1^6 = O(1)$$

for $\beta>0$, $\gamma>0$, similarly,

$$I_1^6 \leqq A\int_2^{1/y}|y|^{\beta}u^{\beta-1}\,du=O(1)$$

for $\beta>0$, and

$$I_1^7 \leqq A\int_{1/y}^{\infty}\frac{du}{|y|^{\gamma p}\,u^{1+\gamma p}}=O(1)$$

for $\gamma>0$. Thus we have for all y

(4.14) $$K_1(y) = O(1)$$

Next if put $y/x=u$ in (4.13) we have

$$K_2(x)\leqq\frac{2}{\pi}\int_0^{\infty}\frac{1}{|1-u|}\,\frac{|x|^{\beta}\,|1-u^{\beta}|}{1+|x|^{\beta}}\,\frac{1+|x|^{\gamma q}}{1+|ux|^{\gamma q}}\,du\,.$$

We can estimate this integral by the similar manner.

(i) $1/2\leqq|x|\leqq 2$. We have

$$K_2(x) = O(1)$$

for $\gamma q>\beta>0$.

(ii) $2<|x|$. We have

$$K_2(x) = O(1)$$

for $0<\gamma q<1$.

(iii) $|x|<1/2$. We have

$$K_2(x) = O(1)$$

for $0<\beta<\gamma q$. Thus we have for all x

(4.15) $$K_2(x) = O(1)$$

for $\beta>0$, $\gamma>0$, and $1>\gamma q>\beta$.

From above formulas we have

(4.16) $$\left\|\frac{\tilde{f}}{1+|x|^{\beta}}\right\|_p\leqq A_p\left\|\frac{f}{1+|x|^{\beta}}\right\|_p$$

for

$$(4.17) \qquad \beta>0, \quad \gamma>0, \quad 1>\beta+\gamma p>0 \quad \text{and} \quad 1>\gamma q>\beta.$$

Hence for any $p>1$, $0<\alpha<1$, if we put $\beta=\alpha/p$, $\gamma=1/pq$ then the relation (4.17) are all satisfied. We have proved Theorem 3 completely.

Theorem 4. *Let f belong to L_μ $(0\leqq\alpha<1)$. Then the Hilbert operation is of weak type* (1, 1).

Proof of Theorem 4. The existence of $\tilde{f}(x)$ of $0\leqq\alpha\leqq 1$ is obtained by the similar manner. Proof of the weak type runs on the line of A. Zygmund [36].

Without loss of generality we may suppose that

$$(4.18) \qquad f\geqq 0, \quad \|f\|_{1,\mu}=1.$$

We may also suppose that $f=0$ in the neighbourhood of $\pm\infty$. Let us write

$$(4.19) \qquad F(x)=\int_{-\infty}^{x} f\,d\mu$$

For any fixed positive r, Q denotes the set of point x such that

$$(4.20) \qquad \frac{F(x+h)-F(x)}{\mu(x,\,x+h)}>r$$

for some $h=h_x>0$. From an absolute continuity and σ-additivity of $\mu(e,\alpha)$, if we apply the F. Riesz lemma [26] to the function $F(x)-r\,(\operatorname{sign} x)\,\mu(0,|x|)$, then the set Q, if not empty, is a family of finite or denumerable number of disjoint open intervals $I_j=(a_j,b_j)$ $(j=1,2,\cdots)$ and satisfies the following properties,

$$(4.21) \qquad \frac{F(b_j)-F(a_j)}{\mu(a_j,\,b_j)}=r, \quad \text{for all } j,$$

$$(4.22) \qquad F'_\mu(x)=f(x)\leqq r, \quad \text{a.e. in } P,$$

where the set P is the complement of set Q. Under our assumptions, the set Q is bounded. Therefore let $G(x)$ be a function which coincide with $F(x)$ in P and in each of I_j,

$$(4.23) \qquad G(x)=F(a_j)+r\mu(a_j,\,x), \quad x\in I_j,$$

and let $H(x)$ define by the equation

$$(4.24) \qquad F(x)=G(x)+H(x).$$

Then $G(x)$ and $H(x)$ have the following properties. The derivative $G_\mu'(x)=g(x)$ exists a.e. and so also does $H_\mu'(x)=h(x)$ a.e. and we have

$$(4.25) \qquad f(x)=g(x)+h(x), \quad \text{a.e.}$$

$$(4.26)\qquad g(x)=\begin{cases} f(x), & \text{a.e. in } P \\ r, & \text{in } Q \end{cases}$$

$$(4.27)\qquad 0 \leqq g(x) \leqq r, \quad \text{a.e.}$$

$$(4.28)\qquad \int_{I_j} g(x)\, d\mu = \int_{I_j} f(x)\, d\mu$$

for every j and

$$(4.29)\qquad H(x)=0 \text{ in } P, \quad h(x)=0, \text{ a.e. in } P,$$

$$(4.30)\qquad \mu(Q)=\sum_j \mu(I_j)=\frac{1}{r}\int_Q f(x)\, d\mu .$$

Since the set Q is bounded and (4.27) holds, $g(x)$ belongs to L^p_μ for every for every $p \geqq 1$ and so $h(x)$ does to L_μ. Thus we have

$$(4.31)\qquad \tilde{f}(x)=\tilde{g}(x)+\tilde{h}(x), \quad \text{a.e. } x,$$

and

$$(4.32)\qquad E_{2r}[|\tilde{f}|] \subset E_r[|\tilde{g}|]+E_r[|\tilde{h}|],$$

where $E_{2r}[|\tilde{f}|]$ denotes the set of point x such that $|\tilde{f}(x)|>2r$ and similarly as for $E_r[|\tilde{g}|]$ and $E_r[|\tilde{h}|]$ respectively.

Firstly we have by Theorem 3, (4.27) and (4.28)

$$(4.33)\qquad \begin{aligned} \mu(E_r[|\tilde{g}|]) &\leqq \frac{M}{r^p}\int_{-\infty}^{\infty} |g|^p\, d\mu \\ &\leqq \frac{M}{r}\int_{-\infty}^{\infty} f\, d\mu . \end{aligned}$$

Next we have by (4.29)

$$(4.34)\qquad \tilde{h}(x)=\int_Q \frac{h(t)}{x-t}\, dt=\sum_j \int_{I_j} \frac{h(t)}{x-t}\, dt$$

and we may prove that

$$(4.35)\qquad I^j=\int_{P^*} d\mu_x \left| \int_{I_j} \frac{h(t)}{x-t}\, dt \right| \leqq M \int_{I_j} |h(t)|\, d\mu_t$$

for every j, and the set P^* will be defined later.

For this purpose we need the following lemma:

Lemma 4_1. *For any interval $I_j=(a_j, b_j)$ and $0 \leqq \alpha < 1$, there exist two contiguous intervals $I_*=(a'_j, a_j)$ and $I^*=(b_j, b'_j)$ such that*

$$(4.36)\qquad \mu(I_*) \leqq \mu(I^1_j), \quad \mu(I^*) \leqq \mu(I^2_j)$$

$$(4.37)\qquad A_\alpha |I_j|/2 \leqq |I_*| \leqq A'_\alpha |I_j|/2$$

(4.38) $$B'_\alpha |I_j|/2 \leqq |I^*| \leqq B_\alpha |I_j|/2$$

where

(4.39) $$I_j^1=(a_j, c_j),\quad I_j^2=(c_j, b_j),\quad 2c_j=a_j+b_j.$$

and by $|I_j|$ *we denote the length of interval* I_j.

Proof of Lemma 4₄. Without loss of generality we can assume that $|a_j|\leqq|b_j|$, $b_j>0$. In this case as to I^* it is trivial since it is enough as I^* to take the same length as I_j^2. As to I_*, if $a_j\leqq 0$ then this case is also trivial. Therefore it is sufficient to prove the case $0<a_j<b_j$. We denote simply $a_j=a$, $\delta=c_j-a_j=|I_j|/2$, $\delta_*=a_j-a'_j$ and we show that the existence of δ^* such as

$$A_\alpha\delta<\delta^*<\delta$$

and

$$\mu(a'_j, a_i)\sim\mu(a_j, c_j)\ ^{1)}$$

We observe that for all x

$$\frac{1}{1+|x|^\alpha}\sim\frac{1}{(1+|x|)^\alpha}$$

(i) any $a>0$, $\delta=a$. We have

$$\int_a^{2a}\frac{dx}{1+|x|^\alpha}\sim\int_{a+1}^{2(a+1)}x^{-\alpha}\,dx=\frac{2^{1-\alpha}-1}{1-\alpha}(a+1)^{1-\alpha},$$

$$\int_{A_\alpha a}^{a}\frac{dx}{1+|x|^\alpha}\sim\int_{A_\alpha(a+1)}^{a+1}x^{-\alpha}\,dx=\frac{(1-A_\alpha)(a+1)^{-\alpha}}{1-\alpha}$$

If we put $A_\alpha=2-2^{-\alpha}$ then we have

$$\int_a^{2a}d\mu\geqq\int_{A_\alpha a}^{a}d\mu$$

and it is enough to take $\delta_*=(1-A_\alpha)a$. This estimation is also effective for $a/2\leqq\delta\leqq 2a$.

(ii) $0<a<1$, $0<\delta<a/2$. We have

$$\int_a^{a+\delta}d\mu\sim\int_{a+1}^{a+1+\delta}x^{-\alpha}\,dx>\delta\left\{\frac{3(a+1)}{2}\right\}^{-\alpha}$$

$$\int_{a-\delta_*}^{a}d\mu\sim\int_{a+1-\delta_*}^{a+1}x^{-\alpha}\,dx<\delta_*\left(\frac{a+1}{2}\right)^{-\alpha}.$$

Thus it is sufficient to put $\delta_*=3^{-\alpha}\delta$ with absolute constant multiple.

1) By $f(x)\sim g(x)$, we mean that there exist two absolute constant such that $Ag(x)\leqq f(x)\leqq Bg(x)$.

(iii) $a>1$, $0<\delta<a/2$. We have

$$\int_a^{a+\delta} d\mu \sim \int_a^{a+\delta} x^{-\alpha}\,dx \geqq \delta(a+\delta)^{-\alpha} \geqq \delta\Big(\frac{3}{2}a\Big)^{-\alpha}$$

$$\int_{a-\delta_*}^{a} d\mu \sim \int_{a-\delta_*}^{a} x^{-\alpha}\,dx \leqq \delta_*(a-\delta_*)^{-\alpha} \leqq \delta_*\Big(\frac{a}{2}\Big)^{-\alpha}$$

because $1/1+|x|^{\alpha}\sim 1/|x|^{\alpha}$ for $|x|\geqq 1$. It is sufficient to put $\delta_*=3^{-\alpha}\delta$ with absolute constant multiple.

(iv) $a>0$, $\delta=na$ $(n=2,3,\cdots)$. First we observe that if $n>3^{1/1-\alpha}$ then we have

$$\int_{-a}^{a} d\mu \leqq \int_a^{na} d\mu$$

Therefore if we put $n_0=[3^{1/1-\alpha}]+1$, then the cases $a>0$, $0<\delta<n_0 a$, $n=2,3,\cdots,n_0$ are obtained by the case (i) of $\delta_*=n_0^{-1}a$ with absolute constant multiple.

Next as for $n>n_0$, we write $\delta=kn_0 a$ $(k=2,3,\cdots)$ and we prove the case $k=2$. We have

$$\int_a^{n_0 a} d\mu \sim \int_{-a}^{a} d\mu$$

$$\int_{n_0 a}^{2n_0 a} d\mu \sim \int_{n_0(a+1)}^{2n_0(a+1)} x^{-\alpha}\,dx = \frac{2^{1-\alpha}-1}{1-\alpha}\{n_0(a+1\}^{-\alpha}$$

$$\int_{-3a}^{-a} d\mu \sim \int_{-3(a+1)}^{-(a+1)} |x|^{-\alpha}\,dx = \frac{3^{1-\alpha}-1}{1-\alpha}(a+1)^{1-\alpha}$$

and

$$\frac{2^{1-\alpha}-1}{1-\alpha}\{n_0(a+1)\}^{1-\alpha} > \frac{3^{1-\alpha}-1}{1-\alpha}(a+1)^{1-\alpha}.$$

Thus it is sufficient to put $\delta_*=5a$. As for general $k\geqq 3$ it is sufficient to put $\delta_*=(k+1)a$, because we have

$$\int_{n_0 a}^{kn_0 a} d\mu \sim \int_{n_0(a+1)}^{kn_0(a+1)} x^{-\alpha}\,dx = \frac{k^{1-\alpha}-1}{1-\alpha}\{n_0(a+1)\}^{1-\alpha}$$

$$\int_{-ka}^{-a} d\mu \sim \int_{-k(a+1)}^{-k} |x|^{-\alpha}\,dx = \frac{k^{1-\alpha}-1}{1-\alpha}(a+1)^{1-\alpha}.$$

Other case are obtained by the simple process of interpolation of the above cases.

We denote by the set Q^* the sum of family of intervals (a_j', b_j'),

$(-b'_j, -a'_j)$ $(j=1, 2, \cdots)$ and by the set P^*, the complement of the set Q^*.

Proof of (4.35). We may prove only to the following typical case

(4.40) $$0<a_j<b_j \quad \text{and} \quad a'_j=a_j-|I_*|>0\,.$$

The other case will be contained in this one. We have by (4.28) and (4.29)

(4.41) $$\begin{aligned} I^j &= \int_{P^*} d\mu_x \left| \int_{I_j} h(t)\Big(\frac{1}{x-t} - \frac{1+|c_j|^\alpha}{1+|t|^\alpha}\,\frac{1}{x-c_j}\Big) dt \right| \\ &\leqq \int_{P^*} d\mu_x \left| \int_{I_j} h(t)\Big(\frac{1}{x-t} - \frac{1}{x-c_j}\Big) dt \right| \\ &\quad + \int_{P^*} d\mu_x \left| \int_{I_j} \frac{h(t)}{1+|t|^\alpha}\, \frac{|t|^\alpha-|c_j|^\alpha}{x-c_j}\, dt \right| = I_1^j + I_2^j\,, \quad \text{say.} \end{aligned}$$

We have

(4.42) $$I_1^j \leqq \int_{I_j} \frac{|h(t)|}{1+|t|^\alpha}\, dt \int_{P^*} \frac{|t-c_j|\,(1+|t|^\alpha)}{|x-t|\,|x-c_j|}\, d\mu_x$$

and

$$\int_{P^*} \frac{|t-c_j|\,(1+|t|^\alpha)}{|x-t|\,|x-c_j|}\, d\mu_x = \int_{-\infty}^{-b'_j} + \int_{-a'_j}^{a'_j} + \int_{b'_j}^{\infty} = I_{11} + I_{12} + I_{12}\,, \quad \text{say.}$$

By the Lemma 4_1, we have for $t \in I_j$

$$I_{13} \leqq \int_{b_j+|I^*|}^{\infty} \left| \frac{t-c_j}{(x-t)(x-c_j)} \right| dx \leqq |I_j| \int_{|I_j|}^{\infty} x^{-2}\, dx = O(1)\,.$$

Similarly we have $I_{11}=O(1)$ and

$$I_{12} \leqq 2\int_0^{a_j-|I_*|} (\ )\, dx = 2\Big(\int_0^{t/2} + \int_{t/2}^{a_j-|I_*|}\Big) = I_{14} + I_{15}\,, \quad \text{say,}$$
$$(a'_j = a_j - |I_*|)\,.$$

we have

$$\begin{aligned} I_{14} &\leqq A \int_0^{t/2} \frac{|I_j|\,(1+|t|^\alpha)}{t\,|I_j|\,(1+|x|^\alpha)}\, dx \\ &\leqq \begin{cases} A\displaystyle\int_0^{t/2} t^{\alpha-1} x^{-\alpha}\, dx = O(1)\,, & \text{if } t \geqq 1, \\ A\displaystyle\int_0^{t/2} t^{-1}\, dx = O(1)\,, & \text{if } 0<t<1, \end{cases} \end{aligned}$$

and

$$I_{15} \leqq A \int_{t/2}^{a_j-|I_*|} \frac{|I_j|\, dx}{|x-c_j|\,|x-t|} \leqq A|I_j| \int_{|I_*|}^{\infty} x^{-2}\, dx = O(1)\,.$$

From the above estimation we have

$$I_1^j \leqq A \int_{I_j} |h(t)|\, d\mu_t \tag{4.43}$$

Next we have

$$I_2^j = \int_{I_j} \frac{|h|}{1+|t|^\alpha} dt \int_{P^*} \frac{||t|^\alpha - |c_j|^\alpha|}{|x-c_j|} \frac{dx}{1+|x|^\alpha} \tag{4.44}$$

and if we decompose the inner integral into three parts as before

$$\int_{P^*} \frac{||t|^\alpha - |c_j|^\alpha|}{|x-c_j|} \frac{dx}{1+|x|^\alpha} = \left(\int_{-\infty}^{-b'_j} + \int_{-a'_j}^{a'_j} + \int_{b'_j}^{\infty}\right) = I_{21} + I_{22} + I_{23},$$

and

$$I_{23} = \int_{b'_j}^{b'_j+a_j} + \int_{b'_j+a_j}^{\infty} = I_{24} + I_{25}, \quad \text{say.}$$

Since $||t|^\alpha - |c_j|^\alpha| \leqq |I_j| a_j^{\alpha-1}$ for all $t \in I_j$ under the assumption (4.40), we have

$$I_{24} \leqq A \int_{b'_j}^{b'_j+a_j} \frac{|I_j|\, a_j^{\alpha-1}}{|I_j|} \frac{1}{b_j^\alpha} dx = O(1)$$

and

$$I_{25} \leqq A \int_{b'_j+a_j}^{\infty} \frac{||t|^\alpha - |c_j|^\alpha|}{|x-c_j|} \frac{dx}{1+|x|^\alpha}\,.$$

$$\leqq A\; |b_j|^\alpha \int_{b_j}^{\infty} \frac{dx}{x^{\alpha+1}} = O(1), \quad (b'_j + a_j - c_j \geqq b_j).$$

Similarly $I_{21} = O(1)$ and if we write

$$I_{22} = 2\left(\int_0^{a_j/2} + \int_{a_j/2}^{a'_j}\right) = I_{26} + I_{27}, \quad \text{say.}$$

Then we have

$$I_{26} \leqq A \int_0^{a_j/2} \frac{|I_j|\, a_j^{\alpha-1}}{|I_j|\, x^\alpha} dx = O(1)$$

and

$$I_{27} \leqq A \int_{a_j/2}^{a_j} \frac{|I_j|\, a_j^{\alpha-1}}{|I_j|\, a_j^\alpha} dx = O(1).$$

From above estimations we have

$$I_2^j \leqq A \int_{I_j} |h(t)|\, d\mu_t\,. \tag{4.45}$$

From (4.41), (4.43) and (4.45), we obtain (4.35), and that it follows by (4.25) and (4.28)

(4.46) $$\int_{P^*} |\tilde{h}(x)|\, d\mu_x \leqq A \int_Q |h(t)|\, d\mu_t \leqq 2A \int_Q f\, d\mu_t.$$

Finally we obtain by (4.46) and (4.30)

(4.47) $$\mu(E_r[|\tilde{h}|]) \leqq \mu(Q^*) + \frac{2A}{r} \int_Q f\, d\mu \leqq \frac{M}{r} \int_Q f\, d\mu.$$

Thus we have proved Theorem 4 completely.

We remark that in Theorems 3 and 4, the operation (3.01) can be replaced by the following one

(4.48) $$T_\eta f = \tilde{f}_\eta(x) = \frac{1}{\pi} \int_{|x-t|>\eta} \frac{f(t)}{x-t}\, dt$$

and

(4.49) $$T^* f = \sup_{\eta>0} |\tilde{f}_\eta(x)|.$$

5. If we apply interpolating theorems to the result of the preceding section we have immediately

Theorem 5. *Let $f(x)$ belong to L^φ_μ $(0 \leqq \alpha < 1)$ with the $\varphi(u)$ of Theorem A with $a=1$, $b>1$. Then the Hilbert operation* (3.01) *exists for a.e. and also bolongs to the same class and we have*

(5.01) $$\int_{-\infty}^{\infty} \varphi(|\tilde{f}|)\, d\mu \leqq A \int_{-\infty}^{\infty} \varphi(|f|)\, d\mu$$

(5.02) $$\lim_{\eta\to 0} \int_{-\infty}^{\infty} \varphi(|\tilde{f} - \tilde{f}_n|)\, d\mu = 0$$

In particular we have

Corollary 5₁. *Let $f(x)$ belong to L^p_μ $(p>1,\ 0 \leqq \alpha < 1)$ then we have*

(5.03) $$\int_{-\infty}^{\infty} |\tilde{f}|^p\, d\mu \leqq A_p \int_{-\infty}^{\infty} |f|^p\, d\mu$$

and

(5.04) $$\lim_{\eta\to 0} \int_{-\infty}^{\infty} |\tilde{f} - \tilde{f}_\eta|^p\, d\mu = 0$$

Theorem 6. *Let $f(x)$ be a function such that*

(5.05) $$\int_{-\infty}^{\infty} |f|(1+\log^+ |f|)\, d\mu < \infty, \quad (0 \leqq \alpha < 1)$$

Then the Hilbert operation exists a.e. and integrable over any set S_0 *with finite* μ*-measure and*

$$\int_{S_0} |\tilde{f}|\, d\mu \leqq A \int_{-\infty}^{\infty} |f| \{1+\log^+ [\nu(S_0)^2 |f|]\}\, d\mu + B\mu(S_0)^{-1}. \tag{5.06}$$

From this theorem we get

Theorem 7. *Let* $f(x)$ *be a function such that*

$$\int_{-\infty}^{\infty} |f| \log^+ [(1+x^2)|f|]\, d\mu < \infty, \quad (0<\alpha<1). \tag{5.07}$$

Then the Hilbert operation is integrable over the whole interval $(-\infty, \infty)$ *and we have*

$$\int_{-\infty}^{\infty} |\tilde{f}|\, d\mu \leqq A \int_{-\infty}^{\infty} |f| \log^+ [(1+x^2)|f|]\, d\mu + B \tag{5.08}$$

and

$$\lim_{\eta \to 0} \int_{-\infty}^{\infty} |\tilde{f} - \tilde{f}_\eta|\, d\mu = 0 \tag{5.09}$$

Proof of Theorem 7. In the first if we put

$$\tilde{F}(x) = \tilde{f}(x) - \frac{K_1(x)}{\pi} \int_{-\infty}^{\infty} f(t)\, d\mu \tag{5.10}$$

where

$$K_1(x) = \frac{1}{x}, \quad \text{if } |x|>1, \quad =0, \quad \text{elsewhere.} \tag{5.11}$$

The same arguments of A. P. Calderón-A. Zygmund [5] lead

$$\begin{aligned} \int_{-\infty}^{\infty} |\tilde{F}|\, d\mu &\leqq A \int_{-\infty}^{\infty} |f| \{1+\log^+ |x| + \log^+ |f|\}\, d\mu + B \\ &\leqq A \int_{-\infty}^{\infty} |f| \log^+ [(1+x^2)|f|]\, d\mu + B. \end{aligned} \tag{5.12}$$

Here if $\alpha > 0$ we have

$$\int_{-\infty}^{\infty} K_1(x)\, d\mu < \infty$$

and thus we get (5.08). From the above argument we get

Theorem 8. *Let* $f(x)$ *be a function such that*

$$\int_{-\infty}^{\infty} |f| \log^+ [(1+x^2)|f|] < \infty. \tag{5.13}$$

Then we have

$$\int_{-\infty}^{\infty} |\widetilde{F}|\, dx \leqq A \int_{-\infty}^{\infty} [(1+x^2)\,|f|]\, dx + B \tag{ .14}$$

$$\lim_{\lambda,\eta\to 0} \int_{-\infty}^{\infty} |\widetilde{F}_\lambda - \widetilde{F}_\eta|\, dx = 0\,, \tag{5.15}$$

where

$$\widetilde{F}_\lambda(x) = \widetilde{f}_\lambda(x) - \frac{K_1(x)}{\pi} \int_{-\infty}^{\infty} f(t)\, dt\,. \tag{5.16}$$

Theorem 9. *Let $f(x)$ belong to L_μ $(0 \leqq \alpha < 1)$. Then we have for any sub-set S_0 with finite μ-measure*

$$\int_{S_0} |\widetilde{f}|^{1-\varepsilon}\, d\mu \leqq \frac{A}{\varepsilon} \mu(S_0)^{\varepsilon} \Big(\int_{-\infty}^{\infty} |f|\, d\mu \Big)^{1-\varepsilon}, \tag{5.17}$$

where $0 < \varepsilon < 1$ and A is an absolute constant.

From this we get

Theorem 10. *Let $f(x)$ belong to L_μ $(0 \leqq \alpha < 1)$. Then we have*

$$\int_{-\infty}^{\infty} \frac{|\widetilde{f}|^{1-\varepsilon}}{1+|x|^\beta}\, d\mu \leqq \frac{A}{\varepsilon\{\beta - \varepsilon(1-\alpha)\}} \Big(\int_{-\infty}^{\infty} |f|\, d\mu \Big)^{1-\varepsilon} \tag{5.18}$$

$$\lim_{\eta\to 0} \int_{-\infty}^{\infty} \frac{|\widetilde{f} - \widetilde{f}_\eta|^{1-\varepsilon}}{1+|x|^\beta}\, d\mu = 0 \tag{5.19}$$

where $0 < \varepsilon < 1$, $\beta > \varepsilon(1-\alpha)$ and A is an absolute constant.

Proof of Theorem 10. Since it holds that

$$\int_{2^n}^{2^{n+1}} d\mu = O(2^{n(1-\alpha)})\,,$$

we have by Theorem 9

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|f|^{1-\varepsilon}}{1+|x|^\beta}\, d\mu &= \Big\{ \int_{-1}^{1} + \sum_{n=0}^{\infty} \Big(\int_{-2^{n+1}}^{-2^n} + \int_{2^n}^{2^{n+1}} \Big) \Big\} \\ &= S_1 + S_2 + S_3\,, \quad \text{say.} \end{aligned} \tag{5.20}$$

Then we have

$$\begin{aligned} S_3 &\leqq A \sum_{n=0}^{\infty} 2^{-n\beta} \int_{2^n}^{2^{n+1}} |\widetilde{f}|^{1-\varepsilon}\, d\mu \leqq \frac{A}{\varepsilon} \sum_{n=0}^{\infty} 2^{n\{\varepsilon(1-\alpha)-\beta\}} \Big(\int_{-\infty}^{\infty} |f|\, d\mu \Big)^{1-\varepsilon} \\ &= \frac{A}{\varepsilon\{\beta - \varepsilon(1-\alpha)\}} \Big(\int_{-\infty}^{\infty} |f|\, d\mu \Big)^{1-\varepsilon}. \end{aligned} \tag{5.21}$$

Similarly we have

(5.22) $$S_2 \leqq \frac{A}{\varepsilon\{\beta-\varepsilon(1-\alpha)\}}\left(\int_{-\infty}^{\infty}|f|\,d\mu\right)^{1-\varepsilon}.$$

and

(5.23) $$S_1 \leqq \frac{A}{\varepsilon}\left(\int_{-\infty}^{\infty}|f|\,d\mu\right)^{1-\varepsilon}.$$

These give (5.18). The (5.19) is obtained by the usual way.

6. In this section we state the analogous results corresponding to the preceeding section for a sequential space.

For any sequence

(6.01) $$X=(\cdots, x_{-n},\cdots, x_{-1}, x_0, x_1,\cdots, x_n,\cdots)$$

we define the Hilbert transform by the following sequence

(6.02) $$X=(\cdots, \tilde{x}_{-n},\cdots, \tilde{x}_{-1}, \tilde{x}_0, \tilde{x}_1,\cdots, \tilde{x}_n,\cdots)$$

where

(6.03) $$\tilde{x}_n = \sum_{m=-\infty}^{\infty}{}' \frac{x_m}{n-m}$$

the prime means that the term $m=n$ is omitted from summation.

Let us denote by l_μ^φ the class of element which is a sequence such that

(6.04) $$||X||_{\varphi,\mu}=\varphi^{-1}\left\{\sum_{-\infty}^{\infty}\frac{\varphi(|x_n|)}{1+|n|^\alpha}\right\}<\infty$$

where $\varphi^{-1}(u)$ is the inverse of $\varphi(u)$. If we put

(6.05) $$f(x)=x_n\,,\quad |x-n|\leqq 1/4\,,\quad (n=0, \pm 1,\cdots)$$

then it is equivalent that $f(x)$ belongs to L_μ^φ and X belongs to l_μ^φ. In particular if $\varphi(u)=u^p$ $(p\geqq 1)$ we will denote this class by l_μ^p.

If we put

(6.06) $$f(x)=\begin{cases} x_n\,, & |n-x|\leqq 1/4\,,\quad (n=0, \pm 1,\cdots), \\ 0\,, & \text{elsewhere.}\end{cases}$$

Then we have

(6.07) $$\begin{aligned}\tilde{f}(u)&=\sum_{m=-\infty}^{\infty}{}' x_m\int_{|m-v|\leqq 1/4}\frac{dv}{u-v}+x_n\int_{|n-v|\leqq 1/4}\frac{dv}{u-v}\\ &=2^{-1}\sum_{m=-\infty}^{\infty}{}'\frac{x_m}{n-m}+x_n\tilde{\varphi}_n(u)+O(r_n)\end{aligned}$$

where $\varphi_n(u)$ is a characteristic function of the interval $|n-u|\leqq 1/4$ and

(6.08)
$$r_n = \sum_{m=-\infty}^{\infty}{}' \frac{x_m}{(n-m)^2}$$

This gives

(6.09)
$$|\tilde{x}_n| \leqq M(|\tilde{f}(u)| + |x_n \tilde{\varphi}_n(u)| + |r_n|)$$

Here if we put $x(u) = x_n$, $\tilde{x}(u) = \tilde{x}_n$, $\tilde{c}(u) = x_n \tilde{\varphi}_n(u)$, $\tilde{r}(u) = r_n$, if $|n-u| \leqq 1/2$, $(n = 0, \pm 1, \cdots)$. Then we have for all u in $(-\infty, \infty)$,

(6.10)
$$|\tilde{x}(u)| \leqq M(|\tilde{f}(u)| + |\tilde{c}(u)| + |\tilde{r}(u)|).$$

Now the correspondence from $x(u)$ to $\tilde{x}(u)$ defines a linear operation and the proposition of weak type (1, 1) and strong type (p, p) $(p>1)$ of this operation is equivalent to the following one

(6.11)
$$\sum_{n:\, |\tilde{x}_n| > r} \frac{1}{1+|n|^\alpha} \leqq \frac{M}{r} \sum_{-\infty}^{\infty} \frac{|x_n|}{1+|n|^\alpha}$$

and

(6.12)
$$\sum_{-\infty}^{\infty} \frac{|\tilde{x}_n|^p}{1+|n|^\alpha} \leqq A_p \sum_{-\infty}^{\infty} \frac{|x_n|^p}{1+|n|^\alpha}$$

respectively. For this purpose it is sufficient to prove the same matter for each of three terms of left-hand side of (6.10). That these are strong type (p, p) $(p>1)$ is obtained by the same argument as A. P. Calderón-A. Zygmund [6]. That of weak type (1, 1) is obtained by the similar manner in author [21, II].

Thus we obtain the following theorem:

Theorem 11. *Let X belong to l_μ^φ $(0 \leqq \alpha < 1)$. Then $\tilde{X}$ can be defined and belongs to the same class and we have*

(6.13)
$$\|\tilde{X}\|_{\varphi,\mu} \leqq A_\varphi \|X\|_{\varphi,\mu}$$

Corollary 11$_1$. *Let X belong to l_μ^p $(p>1,\ 0 \leqq \alpha < 1)$. Then we have*

(6.14)
$$\|\tilde{X}\|_{p,\mu} \leqq A_p \|X\|_{p,\mu}.$$

Theorem 12. *Let X be a sequence such that*

(6.15)
$$\sum_{-\infty}^{\infty} \frac{|x_n| \log^+ [(1+n^2)|x_n|]}{1+|n|^\alpha} < \infty$$

where $0 < \alpha < 1$. Then $\tilde{X}$ can be defined and we have

(6.16)
$$\sum_{-\infty}^{\infty} \frac{|\tilde{x}_n|}{1+|n|^\alpha} \leqq A \sum_{-\infty}^{\infty} \frac{|x_n| \log^+ [(1+n^2)|x_n|]}{1+|n|^\alpha} + B,$$

where A, B are absolute constants.

Theorem 13. *Let X be a sequence such that*

$$\sum_{-\infty}^{\infty}|x_n|\log^+[(1+n^2)|x_n|]<\infty \tag{6.17}$$

Then we have

$$\sum_{-\infty}^{\infty}|\tilde{x}_n^*|\leqq A\sum_{-\infty}^{\infty}|x_n|\log^+[(1+n^2)|x_n|]+B, \tag{6.18}$$

where

$$\tilde{x}_0^*=\tilde{x}_0,\quad \tilde{x}_n^*=\tilde{x}_n-\frac{1}{n}\sum_{-\infty}^{\infty}x_n,\quad (n=\pm 1,\pm 2,\cdots). \tag{6.19}$$

Theorem 14. *Let X belong to the class* l_μ $(0\leqq\alpha<1)$. *Then* $\tilde{X}$ *can be defined and we have*

$$\sum_{-\infty}^{\infty}\frac{|\tilde{x}_n|^{1-\varepsilon}}{1+|n|^{\alpha+\beta}}\leqq\frac{A}{\varepsilon\{\beta-\varepsilon(1-\alpha)\}}\left(\sum_{-\infty}^{\infty}\frac{|x_n|}{1+|n|^\alpha}\right)^{1-\varepsilon}, \tag{6.20}$$

where $0<\varepsilon<1$, $\beta>\varepsilon(1-\alpha)$ *and A is an absolute constant.*

7. As a simple application we establish some theorems concerning with the Dirichlet singular integral. This is defined as follows

$$D_\lambda(x,f)=D_\lambda(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt. \tag{7.01}$$

This can be rewritten in the following form

$$D_\lambda(x,f)=\frac{\sin\lambda x}{\pi}\int_{-\infty}^{\infty}\frac{f(t)\cos\lambda t}{x-t}\,dt-\frac{\cos\lambda x}{\pi}\int_{-\infty}^{\infty}\frac{f(t)\sin\lambda t}{x-t}\,dt. \tag{7.02}$$

Hence we have immediately.

Theorem 15. *Let* $f(x)$ *belong to* L_μ^φ $(0\leqq\alpha<1)$ *with the* $\varphi(u)$ *of Theorem* 5. *Then* $D_\lambda(x,f)$ *can be defined and we have*

$$\|D_\lambda(x,f)\|_{\varphi,\mu}\leqq A_\varphi\|f\|_{\varphi,\mu} \tag{7.03}$$

and

$$\lim_{\lambda\to\infty}\|D_\lambda(x,f)-f\|_{\varphi,\mu}=0. \tag{7.04}$$

Corollary 15. *Let* $f(x)$ *belong to* L_μ^p $(p>1,\ 0\leqq\alpha<1)$. *Then we have*

$$\|D_\lambda(x,f)\|_{p,\mu}\leqq A_p\|f\|_{p,\mu}, \tag{7.05}$$

$$\lim_{\lambda\to\infty}\|D_\lambda(x,f)-f\|_{p,\mu}=0. \tag{7.06}$$

Theorem 16. *Let* $f(x)$ *be a function of Theorem* 7. *Then we have*

$$\int_{-\infty}^{\infty}|D_\lambda(x,f)|\,d\mu\leqq A\int_{-\infty}^{\infty}|f|\log^+[(1+x^2)|f|]\,d\mu+B, \tag{7.07}$$

where $0<\alpha<1$, *and* A, B *are absolute constants.*

Furthermore we have

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty}|D_\lambda(x,f)-f|\,d\mu=0. \tag{7.08}$$

Theorem 17. *Let* $f(x)$ *be a function of Theorem* 8. *Then we have*

$$\int_{-\infty}^{\infty}|D_\lambda^*(x,f)|\,dx\leqq A\int_{-\infty}^{\infty}|f|\log^+[(1+x^2)|f|]\,dx+B \tag{7.09}$$

$$\lim_{\lambda,\eta\to\infty}\int_{-\infty}^{\infty}|D_\lambda^*(x,f)-D_\eta^*(x,f)|\,dx=0, \tag{7.10}$$

where

$$D_\lambda^*(x,f)=D_\lambda(x,f)-\frac{K_1(x)}{\pi}\int_{-\infty}^{\infty}f(t)\sin\lambda(x-t)\,dt. \tag{7.11}$$

Theorem 18. *Let* $f(x)$ *belong to* L_μ $(0\leqq\alpha<1)$. *Then we have*

$$\int_{-\infty}^{\infty}\frac{|D_\lambda(x,f)|^{1-\varepsilon}}{1+|x|^\beta}\,dt\leqq\frac{A}{\varepsilon\{\beta-\varepsilon(1-\alpha)\}}\left(\int_{-\infty}^{\infty}|f|\,d\mu\right)^{1-\varepsilon}, \tag{7.12}$$

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty}\frac{|D_\lambda(x,f)-f(x)|^{1-\varepsilon}}{1+|x|^\beta}\,d\mu=0 \tag{7.13}$$

where $0<\varepsilon<1$, $\beta>\varepsilon(1-\alpha)$ *and* A *is an absolute constant.*

8. In the final section of this chapter we show a negative example for the existence of the Hilbert transform.

Theorem 19. *For any given pair of number* $(p>1,\ \alpha\geqq1)$ *or* $(p=1,\ \alpha>1)$, *there exists a function of this class* L_μ^p *whose Hilbert transform diverges for a.e.*

Proof of Theorem 19. It is sufficient to prove cases $(p>1,\ \alpha=1)$, $(p=1,\ \alpha>1)$. If we put

$$f(x)=\begin{cases}(\log(n+1))^{-1}, & n\leqq x<n+1, \quad (n=1,2,\cdots)\\ 0 & , \text{ elsewhere.}\end{cases} \tag{8.01}$$

Then we have

$$\int_{-\infty}^{\infty}\frac{|f|^p}{1+|x|^\alpha}\,dx<A\sum_{n=1}^{\infty}\frac{1}{n(\log(n+1))^p} \tag{8.02}$$

Of course this function belongs to L_μ for $\alpha>1$. On the other hand we have

$$\tilde{f}(x)=\int_{-\infty}^{\infty}\frac{f(t)}{x-t}\,dt=\lim_{n\to\infty}\int_1^n\frac{f(t)}{x-t}\,dt \tag{8.03}$$

and for $x \in (m, m+1)$ $(m \geqq 1)$

$$0 < \int_0^{m+1} \frac{f(t)}{x-t} dt \leqq \sum_{k=1}^{m-1} \frac{1}{(m-k)\log(k+1)} < \infty \tag{8.04}$$

$$\int_{m-1}^{m+2} \frac{f(t)}{x-t} dt \quad \text{exists for a.e. } x \in (m, m+1), \tag{8.05}$$

and

$$\left| \int_{m+2}^{n} \frac{f(t)}{x-t} dt \right| \geqq \sum_{k=m+2}^{n} \frac{1}{(k-m)\log(k+1)} \to \infty, \quad n \to \infty. \tag{8.06}$$

That is $\tilde{f}(x)$ diverges for a.e. if $x \geqq 1$. Next if $x \in (m, m+1)$, $m \leqq 0$, then we have

$$\left| \int_1^{n} \frac{f(t)}{x-t} dt \right| \geqq \sum_{k=1}^{n} \frac{1}{(|m|+k)\log(k+1)} \to \infty, \quad n \to \infty. \tag{8.07}$$

These establishes Theorem 19. This example shows that the case $\alpha = 1$ is a critical case in some sense and if we wish to treat the Hilbert transform in these space we need an introduce of modified definition. These are treated in [21, IV], and we also study for the typical case in chapter 4.

Chapter 3. The analytic function in a half-plane

9. Corresponding to a rôle of conjugate function taking in a unite circle, the Hilbert transform plays the same rôle in a half-plane. Concerning to this there are studies of E. Hille-J. D. Tamarkin [15, 16]. Extension to their result for our class is the main purpose of this chapter.

Our main tools are the skew-reciprocal formula of Hilbert operator, and the two theorems of Paley-Wiener [25]. The author learned references of unicity theorem of analytic function from Prof. K. Noshiro. The author thanks to him.

We begin with the introduce of some definitions and notations. Let $f(z)$, $z = x+iy$, $y > 0$, be analytic in a half-plane $y > 0$. If the limit

$$\lim_{y \to 0} f(x+iy) = f(x) \tag{9.01}$$

exists for almot all x, $f(x)$ will be called the limit function of $f(z)$. If this limit exists in the sense of Stoltz—as an angular limit—then we shall write

$$(S)\text{-}\lim_{y \to 0} f(x+iy) = f(x). \tag{9.02}$$

By $\mathfrak{H}^p_\mu$ we denote the class of function $f(z)$ analytic in a half-plane $y>0$ and such that the integral

$$(9.03)\qquad \|f(x+iy)\|_{p,\mu}=\left(\int_{-\infty}^{\infty}|f(x+iy)|^p\,d\mu\right)^{\frac{1}{p}}\leqq M_{p,\alpha}$$

where $M_{p,\alpha}$ is a constant which depends only on p and α.

Let $g(x)$ be measurable over $(-\infty,\infty)$. We set

$$(9.04)\qquad C(z,g)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}g(t)\frac{dt}{t-z},$$

$$(9.05)\qquad P(z,g)=\frac{1}{\pi}\int_{-\infty}^{\infty}g(t)\frac{y\,dt}{(t-x)^2+y^2}$$

$$(9.06)\qquad \tilde{P}(z,g)=-\frac{1}{\pi}\int_{-\infty}^{\infty}g(t)\frac{(t-x)\,dt}{(t-x)^2+y^2}.$$

We shall call $C(z,g)$ and $P(z,g)$ integral of Cauchy type and of Poisson type associated with the function $g(t)$ respectively. We observe that, on setting $\bar{z}=x-iy$,

$$(9.07)\qquad P(z,g)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}g(t)\left\{\frac{dt}{t-z}-\frac{dt}{t-\bar{z}}\right\}$$
$$=C(z,g)-C(\bar{z},g),$$

$$(9.08)\qquad \tilde{P}(z,g)=(-i)\{C(z,g)+C(\bar{z},g)\},$$

while

$$(9.09)\qquad 2C(z,g)=P(z,g)+i\tilde{P}(z,g).$$

If $g(x)=f(x)$ is the limit function of $f(z)$ analytic for $y>0$ and such that

$$(9.10)\qquad f(z)=C(z,f)\quad\text{or}\quad f(z)=P(z,f)$$

then we shall say that $f(z)$ is represented by its proper Cauchy integral, or by its proper Poisson integral, omitting the adjective "proper" if no confusion arises.

Throughout of this chapter we assume that $g(x)$ is a real valued measurable function over $(-\infty,\infty)$.

10. In this section we show that the Hilbert operator is a skew-reciprocal. The method of proof is the so-called complex variable method which is different to the previous chapter quitely. Let us begin to prove the following two theorems.

Theorem 20. *Let $g(x)$ belong to L^p_μ $(p\geqq 1)$, then*

$$(S)\text{-}\lim_{y\to 0} P(z, g) = g(x), \quad \text{a.e.} \tag{10.01}$$

$$\lim_{y\to 0} \| P(z, g)-g(x) \|_{p,\mu} = 0 . \tag{10.02}$$

Proof of Theorem 20. (10.01) is nothing else the Fatou theorem (c.f. C. Caratheodory [7, Band I, p. 45]), and a simple application of Jessen's inequality reads (10.02).

Theorem 21. *Let $g(x)$ belong to L^p_μ $(p>1)$ or let $g(x)$ and $\tilde{g}(x)$ both belong to L_μ, then we have*

$$(S)\text{-}\lim_{y\to 0} \tilde{P}(z, g) = \tilde{g}(x), \quad a.e. \tag{10.03}$$

$$\lim_{y\to 0} \| \tilde{P}(z, g)-\tilde{g}(x) \|_{p,\mu} = 0 . \tag{10.04}$$

Proof of Theorem 21. By Theorem 20, it is sufficient to prove

$$P(z, \tilde{g}) = \tilde{P}(z, g) . \tag{10.05}$$

We state this property as a theorem.

Theorem 22. *Let $g(x)$ belong to L^p_μ $(p>1)$ or let $g(x)$ and $\tilde{g}(x)$ both belong to L_μ. Then we have*

$$P(z, \tilde{g}) = \tilde{P}(z, g) . \tag{10.05}$$

We prove this by several lemmas which are almost all well known. We introduce the Cauchy integral and the Poisson integral in a unit circle.

Lemma 22_1. *Let us write*

$$c(w, g)=\frac{1}{2\pi i}\int_{-\pi}^{\pi} g(\zeta)\frac{d\zeta}{\zeta-w}, \tag{10.06}$$

$$p(w, g)=c(w, g)-c(w^*, g) \tag{10.07}$$

$$\tilde{p}(w, g)=(-i)\{c(w, g)+c(w^*, g)\}, \tag{10.08}$$

where $w=re^{i\theta}$, $\zeta=e^{i\varphi}$, $w^*=\overline{w}^{-1}$. Then we have

$$p(w, g)=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\varphi})\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}d\varphi \tag{10.09}$$

$$\tilde{p}(w, g)=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\varphi})\frac{r\sin(\theta-\varphi)}{1-2r\cos(\theta-\varphi)+r^2}d\varphi + \frac{1}{2\pi i}\int_{-\pi}^{\pi} g(e^{i\varphi})\,d\varphi . \tag{10.10}$$

Lemma 22_2. *Let us put for $z=x+iy$, $w=re^{i\theta}$*

$$w=\frac{1+iz}{1+iz} \quad or \quad z=i\frac{1-w}{1+w}. \tag{10.11}$$

Then this linear transformation maps a half-plane $y>0$ into a unit circle $|w|<1$. Its corresponding boundary is given by

$$\zeta = \frac{1+it}{1-it} \tag{10.12}$$

and if we put $\zeta=e^{i\varphi}$ then $t=\tan\varphi/2$. Furthermore

$$w^* = \frac{1+i\bar{z}}{1-i\bar{z}}. \tag{10.13}$$

Lemma 22_3. *Using the same notation of Lemmas 22_1 and 22_2, if we substitute* (10.12) *into* (10.6) *and if we put $g(t)=\hat{g}(\zeta)$, then we have*

$$c(w, \hat{g}) = C(z, g)-C \tag{10.14}$$

$$c(w^*, \hat{g}) = C(\bar{z}, g)-C \tag{10.15}$$

where

$$\begin{aligned} C &= C(-i, g) \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} g(t)\frac{t}{1+t^2}\,dt - \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)\frac{dt}{1+t^2} \\ &= C_1+C_2, \qquad \text{say.} \end{aligned} \tag{10.16}$$

Then we have

$$2iC_1=\frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\frac{t}{1+t^2}\,dt \tag{10.17}$$

$$2iC_2=\frac{1}{2\pi i}\int_{-\pi}^{\pi} \hat{g}(e^{i\varphi})\,d\varphi \tag{10.18}$$

Lemma 22_4. *Let us put*

$$\begin{aligned} \tilde{g}^*(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{x-t}+\frac{t}{1+t^2}\right)g(t)\,dt \\ &= \tilde{g}(x)+2iC_1 \end{aligned} \tag{10.19}$$

then if we substitute $x=\tan\theta/2$, $t=\tan\varphi/2$, we have

$$\tilde{g}^*(x) = \tilde{\hat{g}}(\theta), \tag{10.20}$$

where

$$\tilde{\hat{g}}(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{g}(e^{i\varphi})\cot\frac{1}{2}(\theta-\varphi)\,d\varphi. \tag{10.21}$$

Lemma 22_4 is due to H. Kober [20].

Proof of Theorem 22. By the above lemmas and the expansion to Fourier series we have step by step

$$
\begin{aligned}
(10.22)\qquad & P(z, \tilde{g})+\Re(2iC(-i, g)) \\
&=P(z, \tilde{g}^*) \\
&=p(w, \tilde{\hat{g}}) \\
&=\tilde{p}(w, \hat{g})-i\Im(2iC(-i, g)) \\
&=\tilde{P}(z, g)+2iC(-i, g)-i\Im(2iC(-i, g)) \\
&=\tilde{P}(z, g)+\Re(2iC(-i, g)). \qquad \text{q.e.d.}
\end{aligned}
$$

Now we are in a position to establish the following fundamental theorem.

Theorem 23. *Under the same assumption of Theorem* 22, *we have*

$$
(10.23)\qquad \frac{1}{2\pi i}\int_{-\infty}^{\infty} g(t)\frac{dt}{t-z}=\frac{1}{2\pi i}\int_{-\infty}^{\infty} i\tilde{g}(t)\frac{dt}{t-z}.
$$

Proof of Theorem 23. For $w=u-iv$, $v>0$ we have by Theorems 21 and 20

$$
\begin{aligned}
(10.24)\qquad \frac{1}{2\pi i}\int_{-\infty}^{\infty} i\tilde{g}(t)\frac{dt}{t-z} &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} g(t)\frac{dt}{t-z} \\
&\quad -\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{dt}{t-z}\lim_{v\to 0}\int_{-\infty}^{\infty} g(u)\frac{du}{w-t}.
\end{aligned}
$$

Formal calculation shows that

$$
\begin{aligned}
(10.25)\qquad & \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{dt}{t-z}\lim_{v\to 0}\int_{-\infty}^{\infty} g(u)\frac{du}{w-t} \\
&= \lim_{v\to 0}\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{dt}{t-z}\int_{-\infty}^{\infty} g(u)\frac{dt}{w-t} \\
&= \lim_{v\to 0}\frac{1}{2\pi^2}\int_{-\infty}^{\infty} g(u)\,du\int_{-\infty}^{\infty}\frac{dt}{(t-z)(w-t)} \\
&= \lim_{v\to 0}-\frac{1}{\pi^2}\int_{-\infty}^{\infty} g(u)\frac{du}{w-z}=-\frac{1}{\pi i}\int_{-\infty}^{\infty} g(u)\frac{du}{u-z}.
\end{aligned}
$$

The (10.24) and (10.25) read (10.23).

We may legitimate this formal transformation. For this purpose we prove two lemmas.

Lemma 23_1. *Let $g(x)$ belong to L^p_μ $(p\geqq 1)$ then the double integral*

$$\int_{-\infty}^{\infty}\frac{dt}{t-z}\int_{-\infty}^{\infty}\frac{g(u)}{w-t}du \tag{10.26}$$

is absolutely convergent.

Proof of Lemma 23_1. It is sufficient to prove that

$$\int_{-\infty}^{\infty}\frac{dt}{|t-z||w-t|}\leqq\begin{cases}K_{y,v}, & \text{if } |u-x|\leqq 2,\\ L_{y,v}\dfrac{\log|u-x|}{|u-x|}, & \text{if } |u-x|>2.\end{cases} \tag{10.27}$$

Because we have by the Hölder inequality

$$\int_{-\infty}^{\infty}\frac{\log^+|u|}{1+|u|}|g(u)|\,du\leqq A\Big(\int_{-\infty}^{\infty}\frac{|g(u)|^p}{1+|u|^\alpha}du\Big)^{\frac{1}{p}}\Big(\int_{-\infty}^{\infty}\frac{(\log^+|u|)^q}{1+|u|^\beta}du\Big)^{\frac{1}{q}} \tag{10.28}$$

where $\beta=\Big(1-\dfrac{\alpha}{p}\Big)q>1.$

We have

$$K=\int_{-\infty}^{\infty}\frac{dt}{|t-z||w-t|}$$
$$=\int_{-\infty}^{\infty}\frac{ds}{(s^2+y^2)^{1/2}[(u-x-s)^2+v^2]^{1/2}}$$

(a) the case $|u-x|\leqq 2$. We have

$$K\leqq 2\int_0^{\infty}\frac{ds}{(s^2+y^2)^{1/2}[(u-x-s)^2+v^2]^{1/2}}$$
$$=2\Big(\int_0^3+\int_3^{\infty}\Big)=K_1+K_2,\quad\text{say.}$$

We have

$$K_1=\int_0^3\frac{1}{yv}ds=O\Big(\frac{1}{yv}\Big).$$

and

$$K_2\leqq\int_3^{\infty}\frac{ds}{s(s-2)}=O(1).$$

(b) the case $|u-x|>2$. We have

$$K\leqq 2\Big(\int_0^{|u-x|-1}+\int_{|u-x|-1}^{|u-x|+1}+\int_{|u-x|+1}^{\infty}\Big)$$
$$=K_3+K_4+K_5,\quad\text{say.}$$

We have as for K_4,

$$K_4=2\int_{1-|u-x|^{-1}}^{1+|u-x|^{-1}}\frac{dr}{r[|u-x|^2(1-r)^2+v^2]^{1/2}}\leqq\frac{2}{v}\cdot\frac{2}{|u-x|},$$

and as for K_5,

$$K_5=2\Big(\int_{|u-x|+1}^{2|u-x|}+\int_{2|u-x|}^{\infty}\Big)=K_{51}+K_{52}\,,\quad\text{say.}$$

then

$$K_{52}=2\int_2^{\infty}\frac{dr}{|u-x|\,r(1-r)}=O(|u-x|^{-1}),$$

and

$$K_{51}\leqq 2\int_{|u-x|+1}^{2|u-x|}\frac{ds}{s(s-|u-x|)}$$
$$=\frac{2}{|u-x|}\int_{|u-x|+1}^{2|u-x|}\Big(\frac{1}{s-|u-x|}-\frac{1}{s}\Big)ds=O\Big(\frac{\log|u-x|}{|u-x|}\Big).$$

In the last as for K_3, we have

$$K_3=2\Big(\int_0^{|u-x|/2}+\int_{|u-x|/2}^{|u-x|-1}\Big)=K_{31}+K_{32}\,,\quad\text{say,}$$

then

$$K_{31}\leqq 2\int_0^{|u-x|/2}\frac{dr}{[(u-x)^2r^2+y^2]^{1/2}(1-r)}$$
$$=2\int_0^{2^{-1}|u-x|^{-1}}r^{-1}\,dr=O\Big(\frac{1}{y}\cdot\frac{1}{|u-x|}\Big),$$

and

$$K_{32}\leqq 2\int_{|u-x|/2}^{|u-x|-1}\frac{ds}{(|u-x|-s)s}=O\Big(\frac{\log|u-x|}{|u-x|}\Big).$$

These above estimations give the Lemma 23_1.

Lemma 23_2. *For two complex numbers $z=x+iy$, $y>0$ and $w=u-iv$, $v>0$, we have*

(10.29) $$\int_{-\infty}^{\infty}\frac{dt}{(t-z)(w-t)}=-\frac{2\pi}{i(w-z)}$$

(10.30) $$\int_{-\infty}^{\infty}\frac{dt}{(t-\bar{z})(w-t)}=0\,.$$

Proof of Lemma 23_2. We have

(10.31) $$\frac{1}{t-z}=-\frac{1}{i}\int_{-\infty}^{0}e^{i(t-z)s}\,ds\,,$$

(10.32) $$\frac{1}{w-t}=-\frac{1}{i}\int_{-\infty}^{0}e^{i(w-t)s}\,ds\,.$$

By applying the theorem of Plancherel of Fourier transform to (10.31) and (10.32), we have

(10.33) $$\int_{-\infty}^{\infty}\frac{dt}{(t-z)(w-t)}=-2\pi\int_{-\infty}^{0}e^{i(w-z)s}\,ds=-\frac{2\pi}{i(w-z)}\,.$$

The similar arguments read to (10.30).

Now in (10.25), the first step is legitimated by Theorems 20 and 21, the second one due to Lemma 23_1 and Fubini's theorem, and the third one is deduced by Lemma 23_2. This proves Theorem 23 completely.

If we use (10.30) instead of (10.29) we have

Theorem 24. *Under the same assumption as Theorem 22, we have*

(10.34) $$\frac{1}{2\pi i}\int_{-\infty}^{\infty}g(t)\frac{dt}{t-\bar{z}}=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}i\tilde{g}(t)\frac{dt}{t-\bar{z}}\,.$$

From Theorems 20, 21, 23 and (10.08), we have immediately the reciprocity relation.

Theorem 25. *Under the same assumption as Theorem 22, we have*

(10.35) $$\tilde{\tilde{g}}(x)=-g(x)\,,\qquad \textit{for a.e. } x.$$

11. In this section we establish the representation theorem of analytic function in a half-plane under the assumption of existence of boundary function.

Theorem 26. *Under the same assumption as Theorem 22, if we put*

(10.01) $$f(z)=2C(z,g)=P(z,g)+i\tilde{P}(z,g)$$

then $f(z)$ is analytic in a half-plane $y>0$, its limit function exists as an angular limit and equals to

(11.02) $$f(x)=g(x)+i\tilde{g}(x)$$

Furthermore $f(z)$ is representable by its Cauchy integral.

Proof of Theorem 26. That $f(z)$ is analytic is trivial. That its limit function $f(x)$ is equal to $g(x)+i\tilde{g}(x)$ is obtained by Theorems 20 and 21. To prove that $f(z)$ is representable by the Cauchy integral associated with $f(x)$ we have to prove that

$$\frac{1}{\pi i}\int_{-\infty}^{\infty} g(t)\frac{dt}{t-z} = \frac{1}{2\pi i}\int_{-\infty}^{\infty}(g(t)+i\tilde{g}(t))\frac{dt}{t-z}. \tag{11.03}$$

This is nothing else Theorem 23.

Theorem 27. *If $f(z)$ is represented by its Cauchy integral with the limit function $f(x)$ of L_μ^p $(p\geqq 1)$, we have*

$$\widetilde{\Re f} = \Im f \quad \text{and} \quad \widetilde{\Im f} = -\Re f. \tag{11.04}$$

Proof of Theorem 27. If we assume that

$$f(z) = C(z, f). \tag{11.05}$$

$$\lim_{x\to 0} f(z) = f(x). \tag{11.06}$$

Let us put

$$f(x) = \Re f + i\Im f = f_1 + if_2. \tag{11.07}$$

Then by (9.09), (10.01), (10.03) and (11.06), we have

$$\begin{aligned} \lim_{y\to 0} f(z) &= \frac{1}{2}(f+i\tilde{f}) \\ &= \frac{1}{2}(f_1+if_2)+\frac{i}{2}(\tilde{f}_1+i\tilde{f}_2) \\ &= \frac{1}{2}(f_1-\tilde{f}_2)+\frac{i}{2}(f_2+\tilde{f}_1). \end{aligned} \tag{11.08}$$

From (11.06), (11.07) and (11.08) it follows that

$$f_1 = -\tilde{f}_2 \quad \text{and} \quad f_2 = \tilde{f}_1 \tag{11.09}$$

These give (11.04).

Theorem 28. *Let $f(z)$ be analytic in a half-plane $y>0$. Let $f(z)$ have limit function $f(x)$ in $L^p(p\geqq 1)$ and satisfy*

$$\widetilde{\Re f} = \Im f. \tag{11.10}$$

Furthermore this limit exists as an angular limit on the point of set x with a positive measure. Then $f(z)$ is represented by its Cauchy integral.

We need the unicity theorem of analytic function due to Lusin-Priwaloff [23, p. 164].

Theorem D. *Let $f(z)$ be analytic interior of a unit circle. Let $f(z)$ have an angular limit equal to a constant α for the point of set with a positive measure which is situated on a circumference of this circle. Then $f(z)$ is identically equal to this constant α.*

By the theorem of F. and M. Riesz [27], Theorem D is also effective as for any domain whose boundary is a rectifiable cerve, in particular in our half-plane.

Proof of Theorem 28. we have

$$\lim_{y\to 0} C(z, f)=\frac{1}{2}(f+i\tilde{f}). \tag{11.11}$$

Therefore we get by Theorem 23 and (11.10)

$$(S)\text{-}\lim_{y\to 0} C(z, f) = f(x), \quad \text{a.e. } x. \tag{11.12}$$

Thus Theorem D and (11.12) read

$$f(z) = C(z, f). \tag{11.13}$$

As for equivalency of the integral representation of the Cauchy type and that of the Poisson type, there is a study of G. Fichtenholtz [8] in a unit circle and that of E. Hille-J. D. Tamarkin [16] in a half-plane for the class $\mathfrak{H}^p$. Now we have the following.

Theorem 29. *Let $f(z)$ be analytic in a half-plane $y>0$ and have limit function in L^p_μ $(p\geqq 1)$. Then whenener $f(z)$ is representable by its Cauchy integral, it is also by its Poisson integral and vice versa.*

Proof of Theorem 29. Let $f(z)$ be represented by its Cauchy integral, then we have Theorem 27,

$$f(x) = g(x)+i\tilde{g}(x) \tag{11.14}$$

and $g(x)$, $\tilde{g}(x)$ both belong to L^p_μ $(p\geqq 1)$. Thus by Theorems 22 and 25, we have

$$\begin{aligned} C(z, f) &= \frac{1}{2}P(z, g+i\tilde{g}) + \frac{i}{2}\tilde{P}(z, g+i\tilde{g}) \\ &= \frac{1}{2}P(z, g+i\tilde{g}) + \frac{i}{2}P(z, \tilde{g}-ig) \\ &= P(z, g+i\tilde{g}) = P(z, f). \end{aligned} \tag{11.15}$$

The second half-part postpones to the end of the next section.

Remark 1. By Theorems 29, $f(z)$ in Theorems 26, 28 and 29 belong to the class $\mathfrak{H}^p_\mu$. That is

$$\|P(z, g)\|_{p,\mu} \leqq A_p \|g\|_{p,\mu} \quad (0<y<\infty). \tag{11.16}$$

Proof of (11.16). (a) $0<y\leqq 1$. Applying Jessen's inequality

$$|P(z, g)|^p \leqq \frac{1}{\pi}\int_{-\infty}^{\infty} |g(t)|^p \frac{y}{(t-x)^2+y^2}\,dt$$

$$\int_{-\infty}^{\infty} |P(z, g)|^p\, d\mu_x \leqq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\,du}{u^2+y^2}\int_{-\infty}^{\infty} \frac{|g(x+u)|^2}{1+|x|^\alpha}\,dx$$

and

$$\int_{-\infty}^{\infty}\frac{|g(x+u)|^2}{1+|x|^\alpha}dx \leqq \int_{-\infty}^{\infty}\frac{1+|x+u|^\alpha}{1+|x|^\alpha}\,\frac{|g(x+u)|^p}{1+|x+u|^\alpha}dx$$

$$\leqq \int_{-\infty}^{\infty}\frac{1+2^\alpha(|x|^\alpha+|u|^\alpha)}{1+|x|^\alpha}\,\frac{|g(x+u)|^p}{1+|x+u|^\alpha}dx$$

$$\leqq 2^\alpha\int_{-\infty}^{\infty}(1+|u|^\alpha)\frac{|g(x+u)|^p}{1+|x+u|^\alpha}dx\,.$$

Therefore

$$\int_{-\infty}^{\infty}\frac{|P(z,g)|^p}{1+|x|^\alpha}dx \leqq 2^\alpha\int_{-\infty}^{\infty}\frac{|g(v)|^p}{1+|v|^\alpha}dv\int_{-\infty}^{\infty}\frac{(1+|u|^\alpha)y}{u^2+y^2}du$$

$$\leqq 2^\alpha(1+|y|^\alpha)\int_{-\infty}^{\infty}\frac{|g(v)|^p}{1+|v|^\alpha}dv\,.$$

(b) $y>1$. Similarly

$$\int_{-\infty}^{\infty}\frac{|P(z,g)|^p}{1+|x|^\alpha}dx \leqq \int_{-\infty}^{\infty}\frac{|g(t)|^p}{1+|t|^\alpha}dt\int_{-\infty}^{\infty}\frac{(1+|t|^\alpha)\,y}{(1+|x|^\alpha)[(t-x)^2+y^2]}dt$$

Therefore it is enough to show that

$$L=\int_{-\infty}^{\infty}\frac{y\,dx}{(1+|x|^\alpha)[(t-x)^2+y^2]} \leqq \frac{A_\alpha}{1+|t|^\alpha}.$$

If $|t|<1$, then

$$L \leqq \int_{-\infty}^{\infty}\frac{y\,dx}{(t-x)^2+y^2}=\pi\sim\frac{A}{1+|t|^\alpha}$$

and if $|t|>1$ (we can assume that $t>0$), then

$$L=\int_{-\infty}^{-t}+\int_{-t}^{0}+\int_{0}^{t/2}+\int_{t/2}^{2t}+\int_{2t}^{\infty}$$

$$=L_1+L_2+L_3+L_4+L_5\,,\quad \text{say.}$$

From $(t-x)^2+y^2>2|t-x|\,y$,

$$L_5 \leqq \frac{1}{2}\int_{2t}^{\infty}\frac{dx}{x^\alpha|t-x|}$$

$$\leqq \frac{A}{2}\int_{t}^{\infty}x^{-1-\alpha}\,dx=A_\alpha t^{-\alpha}\sim\frac{A_\alpha}{1+|t|^\alpha}\,.$$

Similarly $L_1 \leqq A_\alpha/(1+|t|^\alpha)$. Secondly

$$L_3 \leqq \int_{0}^{t/2}\frac{dx}{(1+|x|^\alpha)\,|t-x|} \leqq \frac{1}{t}\int_{0}^{t/2}x^{-\alpha}\,dx \leqq \frac{A_\alpha}{1+|t|^\alpha}$$

and similarly $L_2 \leqq A_\alpha/(1+|t|^\alpha)$. In the last

$$L_4 \leqq \int_{t/2}^{2t} \frac{y\,dx}{(1+|x|^\alpha)[(t-x)^2+y^2]}$$

$$\leqq \frac{A_\alpha}{1+|t|^\alpha} \int_{-\infty}^{\infty} \frac{y\,dx}{(t-x)^2+y^2} = \frac{A_\alpha \pi}{1+|t|^\alpha}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{|P(z,g)|^p}{1+|x|^\alpha} dx \leqq A_{p,\alpha} \int_{-\infty}^{\infty} \frac{|g(x)|^p}{1+|x|^\alpha} dx.$$

These estimations give (11.16).

12. In this section we study a function of $\mathfrak{H}^p_\mu$. These are converse of the previous section. The key point is to find a limit function in each functional space. Firstly in this section we mark only the case $\alpha=0$. All theorems which we follow have been established by E. Hille-J. D. Tamarkin [15]. But our proofs are somewhat different. We need two theorems of Paley-Wiener [25] as the base of our arguments.

Theorem E. *Let $F(z)$ belong to $\mathfrak{H}^p$ $(p\geqq 1)$ in an upper half-plane then for any given $y>0$ we have*

(12.01) $$F(z+iy_0)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} F(t+iy_0)\frac{dt}{t-z},$$

and

(12.02) $$F(z+iy_0)=\frac{1}{\pi}\int_{-\infty}^{\infty} F(t+iy_0)\frac{y\,dt}{(t-x)^2+y^2},$$

for all $y>0$ $(z=x+iy)$.

They have proved originally for a case $p=2$ in the vertical strip domain. The remaing case was proved by E. Hille-J. D. Tamarkin essentinally by the same method.

Theorem F. *The two following classes of analytic functions are identical:*

(1) *the class of all functions $F(x+iy)$ analytic for $y>0$, and such that*

(12.03) $$\int_{-\infty}^{\infty} |F(x+iy)|^2\,dx < const. \quad (0<y<\infty),$$

(2) *the class of all functions defined by*

(12.04) $$F(x+iy)=\underset{A\to\infty}{\mathrm{l.i.m.}}\int_{-A}^{0} f(t)\,e^{-it(x+iy)}\,dt$$

where $f(t)$ *belongs to* L^2 *over* $(-\infty, 0)$.

Here we remark that the $f(t)$ of (12.04) is independent on y. Then we have from Theorem F by an application of the Plancherel theorem.

Theorem G. *Let* $F(z)$ *belong to* $\mathfrak{H}^2$. *Then there exists a limit function* $F(x)$ *in mean, that is*

$$\| F(x+iy) - F(x) \|_2 \to 0 \quad (y \to 0), \tag{12.05}$$

$$\| F(x+iy) \|_2 \uparrow \| F(x) \|_2 \quad (y \downarrow 0). \tag{12.06}$$

By Theorems G and E, we have

Theorem H. *Let* $F(z)$ *belong to* $\mathfrak{H}^2$. *Then there exists limit function* $F(x)$ *in mean and* $F(z)$ *is represented by its Cauchy and its Poisson integral. The* $F(z)$ *tends to this* $F(x)$ *along any non-tangential path. As for real part of* $F(x)$ *we have*

$$F(x) = 2C(z, \Re F) = P(z, \Re F) + i\tilde{P}(z, \Re F). \tag{12.07}$$

Extension to the other class $\mathfrak{H}^p$ can be done by the following way: (a) the case $p>1$. If we take y as a parameter, $f(x+iy)$ are of weakly compact. Therefore applying Theorem of F. Riesz [46] (c.f. S. Banach [38, p. 130]) to Theorem E we can prove that $f(z)$ is represented by its Cauchy and its Poisson integral respectively. The remaing part is now obtained immediately.

(b) the case $0<p\leqq 1$. There can be reduced to the case (a) by the following factorization theorem. (c.f. A. Zygmund [37, p. 162]).

Theorem I. *Let* $F(z)$ *belong to* $\mathfrak{H}^p$ $(0<p<\infty)$ *then this can be written*

$$F(z) = B_F(z) H(z), \tag{12.08}$$

where $H(z)$ *belong to the same class* $\mathfrak{H}^p$ *and non-vanish in a half-plane, and*

$$B_F(z) = \prod_{(\nu)} \frac{z - z_\nu}{z - \bar{z}_\nu} \cdot \frac{\bar{z}_\nu - i}{z_\nu + i} \tag{12.09}$$

with z_ν the *sequence of zeros of* $F(z)$ *in a half-plane* $y>0$. *The* $B_F(z)$ *will be called the Blaschke product associated with* $F(z)$ *and has following properties.*

$$| B_F(z) | < 1, \quad \textit{for all } y>0, \tag{12.10}$$

$$\lim_{y \to 0} B_F(z) = 1, \quad \textit{a.e. } x. \tag{12.11}$$

The case $p \geqq 1$ is due to E. Hille-J.D. Tamarkin and the case $0<p<1$ is due to T. Kawata [43]. All of these are special case of the results of R.M. Gabriel [9,10].

Now we extend to our class $\mathfrak{H}^p_\mu$ the result of the preceding section.

Theorem 30. *Let $f(z)$ belong to $\mathfrak{H}_\mu^p$ $(p \geqq 1)$ then the proposition of Theorem E is also true.*

Corollary 30₁. *Let $f(z)$ belong to $\mathfrak{H}_\mu^p$ $(p \geqq 1)$, then we have for any positive number η*

(12.13) $$\lim_{z \to 0} f(z) = 0, \quad \textit{unif. in } y \geqq \eta > 0.$$

By this corollary and the theorem of R. M. Gabriel we have

Theorem 31. *Let $f(z)$ belong to $\mathfrak{H}_\mu^p$ $(p \geqq 1)$ then the proposition of Theorem I is also true.*

Then we can establish the following theorem:

Theorem 32. *Let $f(z)$ belong to $\mathfrak{H}_\mu^p$ $(p \geqq 1)$ then $f(z)$ has the limit function $f(x)$ in L_μ^p along any non-tangential path.*

Proof of Theorem 32. If we put

(12.14) $$F(z) = f(z)/(z+i)$$

then $F(z)$ belong to $\mathfrak{H}^p$ $(p \geqq 1)$ respectively. If we observe that the existence of an angular limit of Blaschke product is guaranteed by another theorem of Fatou (c.f. C. Caratheodory [7, Band II, p. 40]). The Theorem 32 is a simple consequence from Theorem H.

Theorem 33. *Let $f(z)$ belong to $\mathfrak{H}_\mu^p$ $(p \geqq 1)$, then $f(z)$ is represented by its Cauchy and Poisson integral. As for real part of $f(x)$ we have also*

(12.15) $$f(z) = 2C(z, \Re f) = P(z, \Re f) + i\tilde{P}(z, \Re f).$$

Theorem 34. *Let $f(z)$ belong to $\mathfrak{H}_\mu^p$ $(0 < p < \infty)$, then we have*

(12.16) $$\| f(x+iy) - f(x) \|_{p,\mu} \to 0 \quad (y \to 0)$$

Proofs of Theorems 33 and 34. (a) the case $p > 1$. If we appeal to the idea of weakly compact too. The first half-part of Theorem 33 is an immediate consequence of Theorem E as before. Thus (12.16) is obtained by applying Jessen's inequality.

(b) the case $0 < p \leqq 1$. In the first, if we appeal to the Theorem 31, we have (12.16). Hence the first half part of Theorem 33 for $p=1$ is an immediate consequence of Theorem 30 and (12.16).

(c) By (a) and (b) and Theorem 27, we can write

(12.17) $$f(x) = g(x) + i\tilde{g}(x).$$

Then by Theorem 23, we have

(12.18) $$2C(z, g) = C(z, g) + C(z, i\tilde{g}) = C(z, f) = f(z).$$

In the last, the second half-part of Theorem 29 is obtained by Remark 1, Theorem 30 and (12.16).

Chapter 4. The generalized Hilbert transform

13.

Definition 4. *By W_2, we shall denote the class of functions to be measurable over* $(-\infty, \infty)$ *and such that*

$$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2} dx < \infty. \tag{13.01}$$

Throughout of this chapter let $g(x)$ be a real valued measurable function over $(-\infty, \infty)$ which belongs to the class W_2. We begin to state the obtained result in [21, V].

Theorem 35. *Let $g(x)$ belong to the class W_2, then its generalized Hilbert transform of order* 1, $\tilde{g}_1(x)$ *defined by* (1.01) *also does to the same class and*

$$\int_{-\infty}^{\infty} \frac{|\tilde{g}_1(x)|^2}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{|g(x)|^2}{1+x^2} dx. \tag{13.02}$$

We introduce three integrals of the Cauchy type and the Poisson type respectively.

$$C_1(z, g) = \frac{(z+i)}{2\pi i} \int_{-\infty}^{\infty} \frac{g(t)}{t+i} \frac{dt}{t-z} \tag{13.03}$$

$$P_1(z, g) = \frac{(z+i)}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{t+i} \frac{y\,dt}{(t-x)^2+y^2} \tag{13.04}$$

$$\tilde{P}_1(z, g) = -\frac{(z+i)}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{t-i} \frac{(t-x)dt}{(t-x)^2+y^2} \tag{13.05}$$

Among these formulae there is following relation.

$$2C_1(z, g) = P_1(z, g) + i\tilde{P}_1(z, g), \tag{13.06}$$

But

$$P_1(z, g) \neq C_1(z, g) - C_1(\bar{z}, g) \tag{13.07}$$

$$\tilde{P}_1(z, g) \neq (-i)(C_1(z, g) + C_1(\bar{z}, g)). \tag{13.08}$$

Then we get

Theorem 36. *Let $g(x)$ belong to the class W_2. Then*

$$\tilde{P}_1(z, g) = P_1(z, \tilde{g}_1) \tag{13.09}$$

Proof of Theorem 36. From $\tilde{P}(z, g) = P(z, \tilde{g})$,

$$P_1(z, g) = -\frac{(z+i)}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{t+i} \frac{(t-x)\,dt}{(t-x)^2+y^2}$$

$$= \frac{z+i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{s+i} \frac{ds}{t-s} \right) \frac{y\,dt}{(t-y)^2+y^2}$$

$$= \frac{z+i}{\pi} \int_{-\infty}^{\infty} \frac{\tilde{g}_1(t)}{t+i} \frac{y\,dt}{(t-x)^2+y^2} = P_1(z, \tilde{g}_1).$$

Theorem 37. *Let $g(x)$ belong to the class W_2.* Then

(13.10) $$C_1(z, g) = C_1(z, i\tilde{g}_1).$$

Then Proof of Theorem 37. From $C(z, g) = C(z, i\tilde{g})$

$$C_1(z, g) = \frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{g(t)}{t+i} \frac{dt}{t-z}$$

$$= \frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{g(s)}{s+i} \frac{ds}{t-s} \right) \frac{dt}{t-z}$$

$$= \frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{i\tilde{g}_1(t)}{t+i} \frac{dt}{t-z} = C_1(z, i\tilde{g}_1).$$

Theorem 32. *Let $g(x)$ belong to the class W_2. Then*

(13.11) $$(S)\text{-}\lim_{y\to 0} P_1(z, g) = g(x), \quad \textit{for a.e. } x,$$

(13.12) $$\lim_{y\to 0} \left\| \frac{P_1(z, g) - g(x)}{x+i} \right\|_2 = 0.$$

Proof of Theorem 38. If we write

$$P_1(z, g) = \frac{x+i}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{t+i} \frac{y\,dt}{(t-x)^2+y^2}$$

$$+ i\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{t+i} \frac{y\,dt}{(t-x)^2+y^2}$$

$$= P_1 + P_2, \quad \text{say.}$$

Then

$$P_1 \to (x+i)\frac{g(x)}{x+i} = g(x), \quad y \to 0,$$

for a.e. x as an angular limit, and

$$|P_2| \leqq y \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{g(t)}{t+i} \right|^2 dt \right)^{\frac{1}{2}} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{y}{(t-x)^2+y^2} \right)^2 dt \right)^{\frac{1}{2}}$$

$$=\sqrt{y}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|g(t)|^2}{1+t^2}dt\right)^{\frac{1}{2}}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{1+u^2}\right)^2 du\right)^{\frac{1}{2}}$$

$$=o(1), \quad y\to 0.$$

As for (13. 12), we have

$$\left\|\frac{P_1(z, g)-g(x)}{x+i}\right\|_2=\int_{-\infty}^{\infty}\frac{|P_1(z, g)-g(x)|^2}{1+x^2}dx$$

$$=\int_{-\infty}^{\infty}\frac{dx}{1+x^2}\left|\frac{z+i}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{t+i}\frac{y\,dt}{(t-x)^2+y^2}-g(x)\right|^2dx$$

$$=2^2\int_{-\infty}^{\infty}\frac{dx}{1+x^2}\left|\frac{x+i}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{t+i}\frac{y\,dt}{(t-x)^2+y^2}-g(x)\right|^2$$

$$+2^2\int_{-\infty}^{\infty}\frac{|(z+i)+(x+i)^2|}{1+x^2}\left|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{t+i}\frac{y\,dt}{(t+x)^2+y^2}\right|^2dx$$

$$=P_3+P_4, \quad \text{say.}$$

Then by Jessen's inequality

$$|P_3|\leqq 2^2y^2\int_{-\infty}^{\infty}\frac{|g(t)|^2}{1+t^2}dt\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+x^2}\frac{y\,dx}{(t-x)^2+y^2}$$

$$\leqq 2^2y^2\int_{-\infty}^{\infty}\frac{|g(t)|^2}{1+t^2}dt=o(1), \quad y\to 0.$$

and

$$|P_4|=2^2\int_{-\infty}^{\infty}\left|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{t+i}\frac{y\,dt}{(t-x)^2+y^2}-\frac{g(x)}{x+i}\right|^2dx$$

$$=o(1), \quad y\to 0$$

by Theorem 20.

Theorem 39. *Let $g(x)$ belong to the class W_2. Then*

(13.13) $$(S)\text{-}\lim_{y\to 0}\tilde{P}_1(z, g)=\tilde{g}_1(x), \quad \text{a.e. } x$$

(13.14) $$\lim_{y\to 0}\left\|\frac{\tilde{P}_1(z, g)-\tilde{g}_1(x)}{x+i}\right\|_2=0.$$

This is obtained form Theorems 36 and 38.

Theorem 40. *Let $g(x)$ belong to the class W_2. Then*

(13.15) $$(\tilde{\tilde{g}}_1)_1(x)=-g(x), \quad \textit{for a.e. } x.$$

This is obtained from Theorem 37. Next we may prove

Theorem 41. *Let $g(x)$ belong to the class W_2. Then if we put*

(13.16) $$\tilde{f}^*(x)=(x+i)\ \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{\sqrt{2\pi}}\int_{-A}^{A}(-i\ \text{sign}\ u)\,\psi(u)\,e^{ixu}\,du$$

where

(13.17) $$\psi(u)=\underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{f(t)}{t+i}e^{-iut}\,dt$$

Then

(13.18) $$\tilde{f}^*(x)=\tilde{f}_1(x),\quad for\quad a.e.\quad x.$$

The author believes that this modified definition (13.16) was first introduced by N. I. Achiezer [1].

Proof of Theorem 41. We have

$$\frac{(x+i)}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{t-z},\quad z=x+iy,\quad y>0,$$

$$=\frac{(x+i)}{2\pi i}\int_{-\infty}^{\infty}\frac{dt}{t-z}\ \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\psi(u)\,e^{itu}\,du$$

$$=\lim_{A\to\infty}\frac{(x+i)}{\sqrt{2\pi}}\int_{-A}^{A}\psi(u)\,du\ \underset{B\to\infty}{\text{l.i.m.}}\int_{-B}^{B}\frac{e^{iut}}{t-z}dt$$

$$=\lim_{A\to\infty}\int_{0}^{A}\psi(u)\,e^{i(x+iy)u}\,du.$$

We put $y\to 0$, then

$$\frac{1}{2}(f(x)+i\tilde{f}_1(x))=(x+i)\ \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{\sqrt{2\pi}}\int_{0}^{A}\psi(u)\,e^{ixu}\,du$$

$$=\frac{1}{2}(x+i)\left\{\underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\psi(u)\,e^{ixu}\,du+i\ \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{\sqrt{2\pi}}\int_{-A}^{A}(-i\ \text{sign}\ u)\,\psi(u)\,e^{ixu}\,du\right\}$$

$$=\frac{1}{2}(f(x)+i\tilde{f}^*(x)).$$

From this we have (13.18)

We also introduce another modified definition

(13.19) $$\tilde{f}_1^-(x)=\frac{(x-i)}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-i}\frac{dt}{x-t}$$

Our first definition will be written

(13.20) $$\tilde{f}_1^+(x)=\frac{x+i}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{x-t}$$

Then we get

Theorem 42. *Let $g(x)$ and $h(x)$ both belong to the class W_2. Then*

(13.21) $$\int_{-\infty}^{\infty} \frac{g(x)\widetilde{h}_1^+(x)}{1+x^2}dx = -\int_{-\infty}^{\infty} \frac{\widetilde{g}_1^-(x)h(x)}{1+x^2}dx$$

Proof of Theorem 42. We have from Theorems 35 and 38,

$$\lim_{y\to 0}\int_{-\infty}^{\infty} \frac{g(x)\widetilde{P}_1^+(z, h)}{1+x^2}dx = \int_{-\infty}^{\infty} \frac{g(x)\widetilde{h}_1^+(x)}{1+x^2}dx.$$

Therefore

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{g(x)\widetilde{P}_1^+(z, h)}{1+x^2}dx \\
&= \int_{-\infty}^{\infty} \frac{g(x)}{1+x^2}\frac{(z+i)}{\pi}\int_{-\infty}^{\infty}\frac{h(t)}{t+i}\frac{(t-x)\,dt}{(t-x)^2+y^2} \\
&= \int_{-\infty}^{\infty} dt\frac{h(t)}{1+t^2}\frac{t-i}{\pi}\int_{-\infty}^{\infty}\frac{g(x)}{x-i}\frac{z+i}{x+i}\frac{(t-x)\,dx}{(t-x)^2+y^2} \\
&= \int_{-\infty}^{\infty} \frac{h(t)\widetilde{P}_1^-(w, g)}{1+t^2}dt - iy\int_{-\infty}^{\infty} dt\frac{h(t)}{t+i}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x)}{x+i}\frac{(t-x)\,dx}{(t-x)^2+y^2} \\
&\qquad + iy\int_{-\infty}^{\infty} dt\frac{h(t)}{t+i}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x)}{1+x^2}\frac{(t-x)\,dx}{(t-x)^2+y^2} \\
&= P_5+P_6+P_7, \quad \text{say,} \quad (w=t+iy,\ y>0).
\end{aligned}$$

Then

$$\begin{aligned}
|P_6|^2 &\leqq y^2\int_{-\infty}^{\infty}\frac{|h(t)|^2}{1+t^2}dt\int_{-\infty}^{\infty}\frac{|g(x)|^2}{1+x^2}dx \\
&= o(1), \quad y\to 0.
\end{aligned}$$

and similarly

$$P_7 = o(1), \quad y\to 0.$$

and

$$\begin{aligned}
P_5 &= \int_{-\infty}^{\infty}\frac{h(t)\widetilde{P}_1^-(w, g)}{1+t^2}dt \\
&\to -\int_{-\infty}^{\infty}\frac{\widetilde{g}_1^-(t)h(t)}{1+t^2}dt, \quad \text{as} \quad y\to 0.
\end{aligned}$$

In the last we state two theorems.

Theorem 43. *Let $g(x)$ belong to the class W_2. Then if we put*

(13.22) $$f_1(z)=2C_1(z, g)$$

Then

(12.23) $$(S)\text{-}\lim_{y\to 0} f_1(z)=g(x)+i\tilde{g}_1(x), \quad a.e. \quad x.$$

(13.24) $$\left\| \frac{f_1(z)-f_1(x)}{x+i} \right\|_2 \to 0, \quad y\to 0,$$

where $f_1(x)=g(x)+i\tilde{g}_1(x)$, *and*

(13.25) $$f_1(z)=C_1(z, f_1)=P_1(z, f_1)$$

Theorem 44. *Let $f_1(z)$ be analytic in an upper half-plane and belong to the class $\mathfrak{H}_1^2$, that is*

(13.26) $$\int_{-\infty}^{\infty} \left| \frac{f_1(z)}{z+i} \right|^2 dx < \text{const.} \qquad (0<y<\infty).$$

Then there exists the limit function $f_1(x)$ such that

(13.27) $$(S)\text{-}\lim_{y\to 0} f_1(z)=f_1(x)$$

(13.28) $$\left\| \frac{f_1(z)-f_1(x)}{x+i} \right\|_2 \to 0, \quad y\to 0.$$

and if we put $\Re f_1=g$ then

(13.29) $$\tilde{g}_1(x)=\Im f_1(x).$$

We can also representate

(13.30) $$f_1(z)=C_1(z, f_1)=P_1(z, f_1)=2C_1(z, g).$$

14. In this section let us assume that the ordinal Hilbert transform exists. Then from identity

(14.01) $$\tilde{g}_1(x)=\frac{(x+i)}{\pi}\int_{-\infty}^{\infty} \frac{g(t)}{t+i}\frac{dt}{x-t}$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(t)}{x-t}dt+\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(t)}{t+i}dt,$$

it is equivalent to the constant term to be finitely determined. This constant will be called the Achiezer constant, and we write

(14.02) $$A^g=\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(t)}{t+i}dt.$$

Then from the result of previous section the property of $\tilde{g}(x)$ will be reduced.

Theorem 45. *Let $g(x)$ belong to the class W_2 and $\tilde{g}(x)$ exist for a.e. Then $\tilde{g}(x)$ also belongs to the same class and*

$$\left(\int_{-\infty}^{\infty}\frac{|\tilde{g}(x)|^2}{1+x^2}dx\right)^{\frac{1}{2}}\leqq\left(\int_{-\infty}^{\infty}\frac{|g(x)|^2}{1+x^2}dx\right)^{\frac{1}{2}}+\sqrt{\pi}A^g. \tag{14.03}$$

This is obtained from theorem 35 and (14.01).

Theorem 46. *Let $g(x)$ belong to the class W_2 and $\tilde{g}(x)$ exist for a.e. Then*

$$(S)\text{-}\lim_{y\to 0} P(z, g)=g(x), \quad a.e. \quad x \tag{14.04}$$

$$\left\|\frac{P(z, g)-g(x)}{x+i}\right\|_2\to 0, \quad y\to 0. \tag{14.05}$$

Proof of Theorem 46. From identity

$$\frac{z+i}{t+i}-1=\frac{x-t}{t+i}+\frac{iy}{t+i} \tag{14.06}$$

We have

$$\begin{aligned} P_1(z, g)&=\frac{1}{\pi}\int_{-\infty}^{\infty}g(t)\frac{y\,dt}{(t-x)^2+y^2} \\ &+\frac{(-1)}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{t+i}\,\frac{(t-x)\,y\,dt}{(t-x)^2+y^2} \\ &+i\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{t+i}\,\frac{y^2\,dt}{(t-x)^2+y^2} \\ &=P(z, g)+P_8+iP_9, \quad \text{say.} \end{aligned} \tag{14.07}$$

Then

$$P_8=-\frac{y}{z+i}\tilde{P}_1(z, g),$$

therefore

$$(S)\text{-}\lim_{y\to 0} P_8=0, \quad \text{a.e. } x. \tag{14.08}$$

$$\|P_8\|_2\to 0, \quad y\to 0, \tag{14.09}$$

and

$$P_9=i\frac{y}{z+i}P_1(z, g),$$

therefore

$$(S)\text{-}\lim_{y\to 0} P_9 = 0\,, \qquad \text{a.e. } x \tag{14.10}$$

$$\| P_9 \|_2 \to 0\,, \qquad y \to 0\,. \tag{14.11}$$

Thus our theorem is obtained from Theorem 38 and (14.07).

Theorem 47. *Under the same assumption of theorem 46, we have*

$$(S)\text{-}\lim \tilde{P}(z, g) = \tilde{g}(x)\,, \qquad a.e.\ x, \tag{14.12}$$

$$\left\| \frac{\tilde{P}(z, g) - \tilde{g}(x)}{x+i} \right\|_2 \to 0\,, \qquad y \to 0\,. \tag{14.13}$$

Proof of Theorem 47. By the similar way as Theorem 46,

$$\begin{aligned}\tilde{P}_1(z, g) = &-\frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\frac{t-x}{(t+x^2)+y^2} \\ &+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{t+i}\,\frac{(t-x)^2}{(t-x)^2+y^2}\,dt \\ &+i\frac{(-1)}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{t+i}\,\frac{(t-x)\,y}{(t-x)^2+y^2}\,dt \\ =&\ \tilde{P}(z, g) + P_{10} + P_{11}, \text{ say.}\end{aligned}$$

Then

$$\begin{aligned}P_{10} - A^g &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{t+i}\,\frac{y^2\,dt}{(t-x)^2+y^2} \\ &= \frac{y}{z+i}P_1(z, g)\,,\end{aligned}$$

and

$$(S)\text{-}\lim_{y\to 0} P_{10} = A^g\,, \qquad \text{for a.e. } x, \tag{14.14}$$

$$\left\| \frac{P_{10} - A^g}{x+i} \right\|_2 \to 0\,, \qquad y \to 0\,. \tag{14.15}$$

From

$$P_{11} = \frac{y}{z+i}\,\tilde{P}_1(z, g)$$

we have

$$(S)\text{-}\lim_{y\to 0} P_{11} = 0 \tag{14.16}$$

$$\|P_{11}\|_2 \to 0, \qquad y \to 0 \tag{14.17}$$

Therefore our theorem is obtained from Theorem 39.

We may prove

Theorem 48. *Under the same assumption of Theorem 46, if we put*

$$2C(z, g) = f(z) \tag{14.18}$$

Then

$$(S)\text{-}\lim_{y\to 0} f(z) = f(x) = g(x) + i\tilde{g}(x) \tag{14.19}$$

$$\left\| \frac{f(z)-f(x)}{x+i} \right\|_2 \to 0, \qquad y \to 0. \tag{14.20}$$

This is obtained from Theorems 46 and 47.

In the last we adds some remarks.

Definition 4′. By W_p, we will denote the class of functions to be measurable over $(-\infty, \infty)$ and such that

$$\int_{-\infty}^{\infty} \frac{|f(x)|^p}{1+|x|^p}\, dx < \infty, \qquad (p \geqq 1) \tag{14.21}$$

Then for the class W_p $(p>1)$ the analogous result of this chapter may be true. For the class W we get

Theorem 45′. *Let $f(x)$ belong to the class W. Then the ordinary Hilbert transform $\tilde{f}(x)$ exists for a.e. and for any given positive number ε such that $0<\varepsilon<1$,*

$$\int_{-\infty}^{\infty} \frac{|f(x)|^{1-\varepsilon}}{1+|x|^{1+\delta}}\, dx \leqq \frac{A}{\varepsilon(\delta-\varepsilon)} \left(\int_{-\infty}^{\infty} \frac{|f(x)|}{1+|x|}\, dx \right)^{1-\varepsilon} + B_\delta \tag{14.22}$$

where $\delta>\varepsilon$, and A is an absolute constant.

This is obtained from Theorem 4 of [21] easily.

Chapter 5. The generalized harmonic analysis of Hilbert transform

15. In chapter 4, we define the class of functions W_2. To this class the generalized Hilbert transform of order 1 is precisely corresponding. This modified one is defined by

$$\tilde{f}_1(x) = \frac{x+i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{dt}{x-t}. \tag{15.01}$$

The main purpose of this chapter is to determine the relation of spectrum between any given function $f(x)$ of W_2 and its Hilbert transform. For this purpose it is enough to remark Theorem 35. We quote the Plancherel theorem of Fourier transform repeatedly [3]. We introduce the generalized Fourier transform due to N. Wiener [31, 32]. This is defined by

$$(15.02) \qquad s^f(u)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(x)\frac{e^{-iux}-1}{-ix}\,dx$$
$$+\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right] f(x)\frac{e^{-iux}}{-ix}\,dx.$$

Let $f(x)$ belong to the class W_2. Then by the Plancherel theorem, the Fourier-Wiener transform $s^f(u)$ is defined and we have

$$(15.03) \qquad s^f(u+\varepsilon)-s^f(u-\varepsilon)=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} f(x)\,\frac{2\sin\varepsilon x}{x}\,e^{-iux}\,dx$$

$$(15.04) \qquad \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,s^f(u+\varepsilon)-s^f(u-\varepsilon)\,|^2\,du=\frac{1}{\pi\varepsilon}\int_{-\infty}^{\infty}|\,f(x)\,|^2\frac{\sin^2\varepsilon x}{x^2}\,dx.$$

If $f(x)$ belongs to the class W_2, then by Theorem 35, the Fourier-Wiener transform of $\widetilde{f}_1(x)$ is also defined. We will denote this by $\widetilde{s}_1^f(u)$. Throughout to this chapter let $g(x)$ be a real valued measurable function which belong to the class W_2. We also denote

$$(15.05) \qquad f_1(x)=g(x)+i\widetilde{g}_1(x).$$

We may prove the following funndamental theorem:

Theorem 49. *Let $g(x)$ belong to the class W_2. Then for any given positive number ε,*

(i) *if* $|\,u\,|>\varepsilon$, *then*

$$(15.06) \qquad \widetilde{s}_1^g(u+\varepsilon)-\widetilde{s}_1^g(u-\varepsilon)=(-i\ \text{sign}\ u)\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}$$

and

(ii) *if* $|\,u\,|\leqq\varepsilon$, *then*

$$(15.07) \qquad \widetilde{s}_1^g(u+\varepsilon)-\widetilde{s}_1^g(u-\varepsilon)=i\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}+2r_1^g(u+\varepsilon)+2r_2^g(u+\varepsilon),$$

where

$$(15.08) \qquad r_1^g(u)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\,\frac{e^{-ius}-1}{-is}\,ds,$$

$$(15.09) \qquad r_2^g(u)=\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\,e^{-ius}\,ds.$$

We begin with several lemmas:

Lemma 49_1. *Let $g(x)$ belong to the class W_2. Then*

$$\text{(15.10)} \qquad \underset{y\to 0}{\text{l.i.m.}}\ \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{\sqrt{2\pi}}\int_{-A}^{A} C_1(z, g)\frac{2\sin \varepsilon t}{t} e^{-iut}\,dt$$

$$=\frac{1}{2}\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}+\frac{i}{2}\{\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)\}.$$

Proof of Lemma 49_1. We have by Theorems 38 and 39

$$\text{(15.11)} \qquad \underset{y\to 0}{\text{l.i.m.}}\ C_1(z, g)\frac{2\sin \varepsilon t}{t}=\frac{1}{2}(g+i\tilde{g}_1)\frac{2\sin \varepsilon t}{t}.$$

Since the integral of square modulus of function leaves invariant under the Fourier transform, (15.10) is obtained from (15.11).

Lemma 49_2. *We have*

$$\text{(15.12)} \qquad \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin \varepsilon t}{t} e^{-itu}\,dt=\sqrt{2\pi}\,\chi_\varepsilon(u),$$

where $\chi_\varepsilon(u)$ is the characteristic function on $(-\varepsilon, \varepsilon)$.

Lemma 49_3. *We have*

$$\text{(15.13)} \qquad \underset{A\to 0}{\text{l.i.m.}}\ \frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{e^{-iut}}{s-z}\,dt=i\frac{(1+\text{sign}\,u)}{2}\sqrt{2\pi}\,e^{i(s-iy)u},$$

where $z=t+iy$, $y>0$.

From these two lemmas,

Lemma 49_4. *We have*

$$\text{(15.14)} \qquad \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin \varepsilon t}{t}\frac{e^{-iut}}{s-z}\,dt \quad (z=x+iy,\ y>0),$$

$$=\begin{cases}\sqrt{2\pi}\,i\,e^{-i(s-iy)u}\dfrac{e^{i(s-iy)\varepsilon}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}, & u>\varepsilon\\[2ex] \sqrt{2\pi}\,i\,e^{-i(s-iy)u}\dfrac{e^{i(s-iy)u}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}, & -\varepsilon\leqq u\leqq\varepsilon\\[2ex] 0, \quad u<-\varepsilon\end{cases}$$

Lemma 49_5. *We have if $o<y<1$,*

$$\text{(15.15)} \qquad \left|\frac{e^{i\varepsilon(s-iy)}-e^{-i\varepsilon(s-iy)}}{i(s-iy)}\right|=A\left\{\left|\frac{\sin \varepsilon s}{s}\right|+\frac{1}{1+|s|}\right\}.$$

Proof of Lemma 49_5. We can write

$$\frac{e^{i\varepsilon(s-iy)}-e^{-i\varepsilon(s-iy)}}{i(s-iy)}=\frac{2\sin\varepsilon s}{s-iy}+\frac{e^{i\varepsilon s}(e^{\varepsilon y}-1)}{i(s-iy)}-\frac{e^{-i\varepsilon s}(e^{-\varepsilon y}-1)}{i(s-iy)}$$

$$=K_1+K_2+K_3,\quad \text{say,}$$

then

$$|K_1|\leqq\left|\frac{2\sin\varepsilon s}{s-iy}-\frac{2\sin\varepsilon s}{s}\right|+\left|\frac{2\sin\varepsilon s}{s}\right|$$

$$\leqq\left|\frac{2\sin\varepsilon s}{s}\right|\left\{\left|\frac{s}{s-iy}-1\right|+1\right\}$$

$$\leqq\left|\frac{2\sin\varepsilon s}{s}\right|\left\{\left|\frac{y}{s-iy}\right|+1\right\}$$

$$\leqq\left|\frac{2\sin\varepsilon s}{s}\right|\left\{\frac{1+y}{1+|s|}+1\right\}\leqq 3\left|\frac{\sin\varepsilon s}{s}\right|$$

$$|K_2|\leqq\frac{A\varepsilon y}{|s-iy|}\leqq A\varepsilon\frac{1+y}{1+|s|}\leqq\frac{3A\varepsilon}{1+|s|},$$

and similarly

$$|K_3|\leqq\frac{3A\varepsilon}{1+|s|}.$$

Proof of Theorem 49. In the first we replace $g(x)$ by $g_B(x)$ which is defined as follows

(15.16) $$g_B(x)=g(x),\quad \text{if}\quad |x|\leqq B,\quad =0,\quad \text{elsewhere,}$$

Then

(15.17) $$s(z, g_B)=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}C_1(z, g_B)\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt$$

$$=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt\frac{z+i}{2\pi i}\int_{-B}^{B}\frac{g(s)}{s+i}\frac{ds}{s-z}$$

$$=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\,ds\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{z+i}{s-z}e^{-iut}\,dt.$$

From identity

(15.18) $$\frac{z+i}{s-z}=\frac{s+i}{s-z}-1$$

we have

$$(15.19)\qquad s(z, g_B)=\frac{1}{2\pi i}\int_{-B}^{B} g(s)\,ds \operatorname*{l.i.m.}_{A\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-A}^{A} \frac{2\sin\varepsilon t}{t}\,\frac{e^{-iut}}{s-z}\,dt$$

$$-\frac{1}{2\pi i}\int_{-B}^{B} \frac{g(s)}{s+i}\,ds \operatorname*{l.i.m.}_{A\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-A}^{A} \frac{2\sin\varepsilon t}{t} e^{-iut}\,dt$$

$$=s_1(u)+s_2(u),\quad \text{say.}$$

Then from Lemma 49_2,

$$(15.20)\qquad s_2(u)=\frac{i}{\sqrt{2\pi}}\int_{-B}^{B} \frac{g(s)}{s+i}\,ds\cdot \chi_\varepsilon(u).$$

We estimate $s(z, g_B)$ defined by (15.17) by decomposing into two cases.

(i) $|u|>\varepsilon$. From (15.20) and Lemma 49_4,

$$(15.21)\qquad \operatorname*{l.i.m.}_{A\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-A}^{A} C_1(z, g_B)\frac{2\sin\varepsilon t}{t} e^{-iut}\,dt$$

$$=\frac{(1+\operatorname{sign} u)}{2}\cdot\frac{1}{\sqrt{2\pi}}\int_{-B}^{B} g(s)\,\frac{e^{i\varepsilon(s-iy)}-e^{i\varepsilon(s-iy)}}{i(s-iy)}\, e^{-yu}\,e^{-ius}\,ds.$$

From (15.11) and Lemma 49_5,

$$(15.22)\qquad \operatorname*{l.i.m.}_{y\to 0} C_1(z, g_B)\,\frac{2\sin\varepsilon t}{t}=\frac{1}{2}(g_B+\tilde{g}_{B1})\frac{2\sin\varepsilon t}{t}$$

and

$$(15.23)\qquad \operatorname*{l.i.m.}_{y\to 0} g_B(s)\frac{e^{i\varepsilon(s-iy)}-e^{-i\varepsilon(s-iy)}}{i(s-iy)}=g_B(s)\,\frac{2\sin\varepsilon s}{s}.$$

Therefore

$$\operatorname*{l.i.m.}_{y\to\infty} \int_{-A}^{A} \frac{1}{2}(g_B+i\tilde{g}_{B1})\,\frac{2\sin\varepsilon t}{t}\,e^{-iut}\,dt$$

$$=\frac{(1+\operatorname{sign} u)}{2}\cdot\frac{1}{\sqrt{2\pi}}\int_{-B}^{B} g(s)\,\frac{2\sin\varepsilon s}{s}\,e^{-ius}\,ds$$

and

$$\operatorname*{l.i.m.}_{A\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-A}^{A} \frac{1}{2}(g+i\tilde{g}_1)\,\frac{2\sin\varepsilon t}{t}\,e^{-iut}\,dt$$

$$=\frac{(1+\operatorname{sign} u)}{2}\operatorname*{l.i.m.}_{B\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-B}^{B} g(s)\,\frac{2\sin\varepsilon s}{s}\,e^{-ius}\,ds.$$

That is

$$(15.24)\qquad \frac{1}{2}\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}+\frac{i}{2}\{\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)\}$$

$$=\frac{(1+\text{sign } u)}{2}\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}.$$

This gives (15.06).

(ii) if $|u|\leqq\varepsilon$. Then by the similar way but we consider now over $(-\varepsilon, \varepsilon)$, we have

$$\text{(15.25)} \qquad \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{1}{2}(g_B+i\tilde{g}_{B1})e^{-iut}\,dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}g(s)\frac{e^{-i(u+\varepsilon)s}-1}{-is}ds+\frac{i}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\,ds$$

$$=\frac{i}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\frac{e^{-i(u+\varepsilon)s}-1}{-is}\,ds+\frac{i}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}e^{-i(u+\varepsilon)s}\,ds.$$

Therefore

$$\underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{1}{2}(g+i\tilde{g}_1)\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt$$

$$=\underset{B\to\infty}{\text{l.i.m.}} \frac{i}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\frac{e^{-i(u+\varepsilon)s}-1}{-is}\,ds+\underset{B\to\infty}{\text{l.i.m.}} \frac{i}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}e^{-i(u+\varepsilon)s}\,ds.$$

That is

$$\text{(12.26)} \qquad \frac{1}{2}\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}+\frac{i}{2}\{\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)\}$$

$$=ir_1^g(u+\varepsilon)+ir_2^g(u+\varepsilon).$$

This gives (15.07).

Theorem 50. *Under the same assumption as Theorem 49, if we denote by $s_1^f(u)$ the Fourier-Wiener transform of $f_1(x)$ defined by (15.01), then we have*

(i) *if* $|u|>\varepsilon$, *then*

$$\text{(12.27)} \qquad s_1^f(u+\varepsilon)-s_1^f(u-\varepsilon)=(1+\text{sign } u)\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}$$

and

(ii) *if* $|u|\leqq\varepsilon$, *then*

$$\text{(15.28)} \qquad s_1^f(u+\varepsilon)-s_1^f(u-\varepsilon)=2ir_1^g(u+\varepsilon)+2ir_2^g(u+\varepsilon).$$

16. We introduce the following class of functions.

Definition 5. *By S_0 we denote the class of functions $f(t)$ to be measurable and such that*

$$\text{(16.01)} \qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|^2\,dt \quad \textit{exists.}$$

Then we shall prove

Theorem 51. *Let $g(x)$ be a real valued measurable function of the class S_0. Let us suppose that*

$$(K_1) \qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |\, s^g(u+\varepsilon)-s^g(u-\varepsilon)\,|^2\, du=0$$

and

(K_2) *there exists a constant a^g such that*

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{0}^{2\varepsilon}\left|\, \underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}e^{-ius}\,ds-\sqrt{\frac{\pi}{2}}a^g\right|^2 du=0.$$

Then its generalized Hilbert transform of order 1, $\tilde{g}_1(x)$ does also to the same class S_0 and

$$(16.02) \qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,\tilde{g}_1(t)\,|^2\,dt=\lim_{T\to\infty}\int_{-T}^{T}|\,g(t)\,|^2\,dt+|\,a^g\,|^2.$$

For the proof of this theorem we quote N. Wiener's Tauberian theorem which is called the Wiener formula usually (N. Wiener [32], c.f. also S. S. Bochner [3] and S. Izumi [17]).

Theorem J. *If $f(x)\geqq 0$ for $0\leqq x<\infty$, and either of the limits*

$$(16.03) \qquad \lim_{T\to\infty}\frac{1}{T}\int_{0}^{T} f(t)\,dt$$

or

$$(16.04) \qquad \lim_{\varepsilon\to 0}\frac{2}{\pi\varepsilon}\int_{0}^{\infty} f(t)\,\frac{\sin^2 \varepsilon t}{t^2}\,dt$$

exists, then the other limit exists and assumes the same value.

From this theorem, Plancherel's theorem and (15.04), we get

Theorem K. *Let $f(x)$ be a measurable function for which (16.01) is bounded in T of $1<T<\infty$. Then we have*

$$(16.05) \qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,f(t)\,|^2\,dt=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,s^f(u+\varepsilon)+s^f(u-\varepsilon)\,|^2\,du$$

in the sense that if either side of (16.05) exists, the other side does and assumes the same value.

Therefore it is enough to prove the following theorem.

Theorem 52. *Under the same assumption of Theorem 51, we have*

$$(16.06) \qquad \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)\,|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du+|a^g|^2.$$

Lemma 52_1. *Let $g(x)$ belong to the class W_2. Then for $r_1^g(u)$ defined by (15.08) we have*

(16.07) $$\lim_{\varepsilon\to 0}\frac{1}{(2\varepsilon)^2}\int_0^{2\varepsilon}|r_1^g(u)|^2\,du=0.$$

Proof of Lemma 52_1. If we denote

(16.08) $$\hat{g}_1(u)=\operatorname*{l.i.m.}_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{g(s)}{s+i}e^{-ius}\,ds,$$

then by the Schwartz inequality

$$\int_0^{2\varepsilon}|r_1^g(u)|^2\,du=\int_0^{2\varepsilon}\left|\int_0^u \hat{g}_1(v)\,dv\right|^2du$$

$$\leqq\int_0^{2\varepsilon}u\left(\int_0^u|\hat{g}_1(v)|^2\,dv\right)du=\int_0^{2\varepsilon}|\hat{g}_1(v)|^2\,dv\int_v^{2\varepsilon}u\,du$$

$$=A\varepsilon^2\int_0^{2\varepsilon}|\hat{g}_1(v)|^2\,dv.$$

Thus we have

$$\int_0^{2\varepsilon}|r_1^g(u)|^2\,du=o(\varepsilon^2),\quad \varepsilon\to 0$$

Proof of Theorem 52. From Theorem 49,

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)|^2\,du=\frac{1}{4\pi\varepsilon}\int_{|u|>\varepsilon}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du$$

$$+\frac{1}{4\pi\varepsilon}\int_{|u|\leqq\varepsilon}|\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}-2ir_1^g(u+\varepsilon)-2ir_2^g(u+\varepsilon)|^2\,du$$

$=I_1+I_2$, say,

Then by Lemma 52_1 and (K_1), (K_2)

$$I_2=\frac{1}{4\pi\varepsilon}\int_{|u|\leqq\varepsilon}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du+\frac{1}{\pi\varepsilon}\int_0^{2\varepsilon}|r_2^g(u)|^2\,du\quad+o(1)$$

$$=\frac{1}{4\pi\varepsilon}\int_{|u|\leqq\varepsilon}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du+|a^g|^2+o(1),\quad\varepsilon\to 0.$$

Thus

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s_1^g(u+\varepsilon)-s_1^g(u-\varepsilon)|^2\,du$$

$$=\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du+|a^g|^2+o(1),\quad \varepsilon\to 0.$$

Thus we have proved Theorem 52 completely. Instead of Theorem 49 if we use Theorem 50, then

Theorem 53. *Under the same assumptions as Theorem 52, $f_1(x)=g(x)+i\tilde{g}_1(x)$ also belongs to the same class S_0 and*

(16.09)
$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s_1^f(u+\varepsilon)-s_1^f(u-\varepsilon)|^2\,du$$
$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du+\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)|^2\,du$$

(16.10)
$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_1(t)|^2\,dt$$
$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2\,dt+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{g}_1(t)|^2\,dt.$$

Proof of Theorem 53. Since $g(x)$ is of real valued

(16.11)
$$s^g(-u+\varepsilon)-s^g(-u-\varepsilon)=\overline{s^g(u+\varepsilon)-s^g(u-\varepsilon)}$$

we have

(16.12)
$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du$$
$$=\frac{1}{2\pi\varepsilon}\int_{0}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du.$$

The remaining part is now obvious.

17. We attend in this section functions of classes S and S' which are introduced by N. Wiener. These are defined as follows:

Definition 6. By S, we will denote the class of functions such that

(17.01)
$$\varphi^f(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}\,dt$$

exists for every x.

Definition 7. By S', we will denote the class of functions such that $\varphi^f(x)$ exists for every x and continuous over $(-\infty,\infty)$

It is clear that

(17.02)
$$S'\subset S\subset S_0.$$

Theorem 54. *Let $g(x)$ belong to the class S. Let us suppose that*

$$(K_1)\qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du=0$$

and

$$(K_2)\qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^{2\varepsilon}\left|\underset{B\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}e^{-ius}\,ds-\sqrt{\frac{\pi}{2}}\,a^g\right|^2 du=0.$$

for certain constnat a^g. Then its generalized Hilbert transform of order 1, $\tilde{g}_1(x)$ *does also to the same class S and*

$$(17.03)\qquad \tilde{\varphi}_1^g(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{g}_1(x+t)\overline{\tilde{g}_1(t)}\,dt$$
$$=\varphi^g(x)+|a^g|^2.$$

We may prove through several lemmas which are due to N. Wiener [33].

Lemma 54_1. *Let $g(x)$ belong to the class S. Then for any real number a*

$$(17.04)\qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t+a)|^2\,dt=\lim_{T\to\infty}\int_{-T}^{T}|g(t)|^2\,dt.$$

Lemma 54_2. *We have*

$$(17.05)\qquad g(t+x)\overline{g(t)}=\frac{1}{4}\{|g(t+x)+g(t)|^2-|g(t+x)-g(t)|^2$$
$$+i|g(t+x)+ig(t)|^2-i|g(t+x)-ig(t)|^2\}.$$

Lemma 54_3. *We have for any real or complex number w such as* $|w|=1$,

$$(17.06)\qquad |g(t+x)+wg(t)|^2=|g(t+x)|^2+|g(t)|^2$$
$$+\overline{w}g(t+x)\overline{g(t)}+w\overline{g(t+x)}g(t).$$

Then

Lemma 54_4. *Let $g(x)$ belong to the class S. Then for any real or complex number w such as* $|w|=1$,

$$(17.07)\qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t+x)+wg(t)|^2\,dt$$

exists for every x.

On the other hand

Lemma 54_5. *Let $g(x)$ belong to the class W_2. Then we have*

(17.08) $$\int_{-\infty}^{\infty} |\{s_x^g(u+\varepsilon)-s_x^g(u-\varepsilon)\}-e^{iux}\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}|^2\,du$$
$$=o(\varepsilon^2),\quad \varepsilon\to 0,$$

where

(17.09) $$s_x^g(u+\varepsilon)-s_x^g(u-\varepsilon)=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} g(t+x)\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt.$$

Therefore from Lemmas 54_4 and 54_5

Lemma 54_6. *Let $g(x)$ belong to the cass S. Then for any real or complex number w such as $|w|=1$,*

(17.10) $$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |g(t+x)+wg(t)|^2\,dt$$
$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}(2+w\,e^{-iux}+\overline{w}\,e^{iux})\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du.$$

and that

Lemma 54_7. *We have*

(17.11) $$2+w\,e^{-iux}+\overline{w}\,e^{iux}=\begin{cases}2+2\cos ux, & \text{if}\quad w=1,\\ 2-2\cos ux, & \text{if}\quad w=-1,\end{cases}$$

(17.12) $$2+w\,e^{-iux}+\overline{w}\,e^{iux}=\begin{cases}2+2\sin ux, & \text{if}\quad w=i,\\ 2-2\sin ux, & \text{if}\quad w=-i.\end{cases}$$

Thus we have

Lemma 54_8. *Let $g(x)$ belong to the class S. Then we have*

(17.13) $$\varphi^g(x)=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du$$
$$=\lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_{0}^{\infty}\cos ux\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du.$$

Since $g(x)$ is of real valued, the last formnla of (17.13) is obtained from (16.11) easily.

Proof of Theorem 54. From Theorem 49 and conditions (K_1), (K_2), we have

(17.14) $$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}(2+w\,e^{-iux}+\overline{w}\,e^{iux})\,|\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)|^2\,du$$
$$=\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}(2+w\,e^{-iux}+\overline{w}\,e^{iux})\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du$$

$$+|a^g|^2\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}(2+we^{-iux}+\overline{w}e^{iux})\,du+o(1),\quad \varepsilon\to 0.$$

In Lemma 54_6, if we assume that $g(x)$ belongs to the class W_2, then (17.10) can be interpreted as follows. If either of the two limits exist then the other limit also does and assumes the same value. Therefore for any real of complex number w such as $|w|=1$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{g}_1(t+x)+w\tilde{g}_1(t)|^2\,dt$$

exists and equals to

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t+x)+wg(t)|^2\,dt$$

$$+|a^g|^2\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}(2+we^{-iux}+\overline{w}e^{iux})\,du$$

Thus we get

(17.15)
$$\begin{aligned}\tilde{\varphi}_1^g(x)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{g}_1(t+x)\overline{\tilde{g}_1(t)}\,dt\\&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}g(t+x)\overline{g(t)}\,dt+|a^g|^2\\&=\varphi^g(x)+|a^g|^2.\end{aligned}$$

Theorem 55. *Let $g(x)$ belong to the class S'. Let us suppose that conditions (K_1) and (K_2) are satisfied. Then $\tilde{g}_1(x)$ does to the same class S', and the (17.03) is true.*

This is obtained from (17.15).

As to $f_1(x)$ defined by (15.05) we shall prove

Theorem 56. *Under the same assumption as Theorem 54, the necessary and sufficient condition that $f_1(x)$ defined by (15.05) belongs to the class S, it that*

(17.16)
$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_0^{\infty}\sin ux\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du$$

exists for every x.

In this case we have

(17.17)
$$\begin{aligned}\varphi_1^f(x)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_1(t+x)\overline{f_1(t)}\,dt\\&=\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_0^{\infty}e^{iux}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du+|a^g|^2\end{aligned}$$

Proof of the necessity. From theorem 50 and conditions (K_1), (K_2), we have by the same arguments as Theorem 52

$$(17.18)\qquad \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}\,|\,s_1^f(u+\varepsilon)-s_1^f(u-\varepsilon)\,|^2\,du$$

$$=\frac{1}{\pi\varepsilon}\int_{0}^{\infty} e^{iux}\,|\,s^g(u+\varepsilon)-s^g(u+\varepsilon)\,|^2\,du$$

$$+|\,a^g\,|^2\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} e^{iux}\,du+o(1),\quad \varepsilon\to 0$$

Therefore if we assume that the limit

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}\,|\,s_1^f(u+\varepsilon)-s_1^f(u-\varepsilon)\,|^2\,du$$

exists then the following limit

$$(17.19)\qquad \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_{0}^{\infty} e^{iux}\,|\,s^g(u+\varepsilon)-s^g(u-\varepsilon)\,|^2\,du$$

also exists and we get (17.16) from (17.19) and Lemma 54_8. Thus the necessity is proved.

Proof of sufficiency. From Theorem 50, conditions (K_1), (K_2) and (17.16), for any real or complex number w such as $|\,w\,|=1$,

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}(2+w^{-iux}+\overline{w}\,e^{iux})\,|\,s_1^f(u+\varepsilon)-s_1^f(u-\varepsilon)\,|^2\,du$$

exists and equals to

$$\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_{0}^{\infty}(2+w\,e^{-iux}+\overline{w}\,e^{-iux})\,|\,s^g(u+\varepsilon)-s^g(u-\varepsilon)\,|^2\,du$$

$$+|\,a^g\,|^2\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}(2+w\,e^{-iux}+\overline{w}\,e^{iux})\,du$$

Therefore we have by Lemma 54_6,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,f_1(t+x)+w\,f_1(t)\,|^2\,dt$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}(2+w\,e^{-iux}+\overline{w}\,e^{iux})\,|\,s_1^f(u-\varepsilon)-s_1^f(u-\varepsilon)\,|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_{0}^{\infty}(2+w\,e^{-iux}+\overline{w}\,e^{iux})\,|\,s^g(u+\varepsilon)-s^g(u-\varepsilon)\,|^2\,du$$

$$+|a^g|^2 \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} (2+w\,e^{-iux}+\overline{w}\,e^{iux})\,dx.$$

Thus we get by Lemmas 54_2, 54_3, 54_4 and 54_7,

$$\varphi_1^f(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(x+t)\overline{f_1(x)}\,dt \tag{17.20}$$

exists for every x and equals to

$$\lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon} \int_0^{\infty} e^{iux}\,|\,s^g(u+\varepsilon)-s^g(u-\varepsilon)\,|^2\,du + |\,a^g\,|^2.$$

We get

Theorem 57. *Under the same assumption as Theorem 55, the necessary and sufficient condition that $f_1(x)$ defined by (15.05) belongs to the class S' is that*

$$\lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon} \int_0^{\infty} \sin ux\,|\,s^g(u+\varepsilon)-s^g(u-\varepsilon)\,|^2\,du \tag{17.21}$$

exists for every x and continuous over $(-\infty, \infty)$.

On the other hand N. Wiener [33] also proved the following two theorems:

Theorem L. *If $g(x)$ belongs to S and $\varphi^g(x)$ is coninuous at $x=0$, then it is continuous for all real arguments and $g(x)$ belongs to S'.*

Theorem M. *If $g(x)$ belongs to S, it will belong to S' when and only when*

$$\lim_{A\to\infty} \overline{\lim_{\varepsilon\to 0}} \frac{1}{4\pi\varepsilon} \left[\int_{-\infty}^{-A} + \int_{A}^{\infty} \right] |\,s^g(u+\varepsilon)-s^g(u-\varepsilon)\,|^2\,du = 0. \tag{17.22}$$

From these theorems we get immediately:

Theorem 58. *Under the same assumption as Theorem 55, if $f_1(x)$ defined by (15.05) belongs to the class S then it does to the class S'.*

Proof of Theorem 58. It is enough to prove that

$$\lim_{x\to 0} \lim_{\varepsilon\to 0} \int_0^{\infty} \sin ux\,|\,s^g(u+\varepsilon)-s^g(x-\varepsilon)\,|^2\,du \tag{17.23}$$

This is obtained from (17.22).

18. Let $g(x)$ belong to the class W_2. Let us assume that its ordinary Hilbert transform of $g(x)$ can be defined. We denote this by $\tilde{s}^g(u)$. Then by repeating the same argument as Theorem 49 we get immediately:

Theorem 59. *Let $g(x)$ belong to the class W_2. Let us assume that its ordinary Hilbert transform exist for a.e. x. Then we have*

(i) *if* $|u|>\varepsilon$, *then*

$$\tilde{s}^g(u+\varepsilon)-\tilde{s}^g(u-\varepsilon)=(-i\,\mathrm{sign}\,u)\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\} \tag{18.01}$$

and

(ii) *if* $|u|\leqq\varepsilon$, *then*

$$\begin{aligned}\tilde{s}^g(u+\varepsilon)-\tilde{s}^g(u-\varepsilon)=i\{s(u+\varepsilon)-s^g(u-\varepsilon)\}\\ +2r_1^g(u+\varepsilon)+2r_3^g(u+\varepsilon),\end{aligned} \tag{18.02}$$

Here $r_1^g(u)$ is defined by (15.08) and

$$r_3^g(u)=r_2^g(u)-\sqrt{\frac{\pi}{2}}A^g, \tag{18.03}$$

where $r_2^g(u)$ is defined by (15.09) and

$$A^g=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(s)}{s+i}ds. \tag{14.02}$$

We put

$$f(x)=g(x)+i\tilde{g}(x) \tag{18.04}$$

and denote by $s^f(u)$, the Fourier-Wiener transform of $f(x)$, then we have

Theorem 60. *Under the same assumption as Theorem 59, we have*

(i) *if* $|u|>\varepsilon$, *then*

$$s^f(u+\varepsilon)-s^f(u-\varepsilon)=(1+\mathrm{sign}\,u)\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\} \tag{18.05}$$

and

(ii) *if* $|u|\leqq\varepsilon$, *then*

$$s^f(u+\varepsilon)-s^f(u-\varepsilon)=2ir_1^g(u+\varepsilon)+2ir_3^g(u+\varepsilon). \tag{18.06}$$

From these theorems and the same arguments as the previous section we get the following theorems. We state theorems without detailed proofs.

Theorem 61. *Let $g(x)$ belong to the class S_0. Let us assume that its ordinary Hilbert transform $\tilde{g}(x)$ exists for a.e. x. Let us assume conditions (K_1) and*

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^{2\varepsilon}\left|\underset{B\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}e^{-ius}\,ds-\sqrt{\frac{\pi}{2}}A^g\right|^2du=0. \tag{K_3}$$

Then $\tilde{g}(x)$ does also to the same class and

(18.07) $$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{g}(t)|^2\,dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2\,dt$$

Theorem 62. *Under the same assumption of Theorem 61, $f(x)$ defined by (18.04) belongs to the same class and*

(18.08) $$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|^2\,dt$$
$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2\,dt+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{g}(t)|^2\,dt.$$

and

(18.09) $$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^f(u+\varepsilon)-s^f(u-\varepsilon)|^2\,du$$
$$=\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_{0}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du.$$

As to the class S and S' we have

Theorem 62. *Let $g(x)$ belong to the class S. Let us assume that its ordinary Hilbert transform exist for a.e. x. Let us also assume the conditions (K_1) and (K_3). Then $\tilde{g}(x)$ also does to the same class and*

(18.10) $$\tilde{\varphi}^g(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{g}(x+t)\,\overline{\tilde{g}(t)}\,dt$$
$$=\varphi^g(x)$$
$$=\lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_{0}^{\infty}\cos ux\,|s^g(u+\varepsilon)+s^g(u-\varepsilon)|^2\,du.$$

Theorem 63. *Let $g(x)$ belong to the class S'. Let us asume that its ordinary Hilbert transform exist for a.e. x. Let us also assume the conditions (K_1) and (K_3). Then $\tilde{g}(x)$ also does to the same class S' and (18.10) is also true.*

Theorem 64. *Under the same assumption of Theorem 62, the necessary and sufficient condition that $f(x)$ defined by (18.04) belongs to the class S is that (17.16) is true. And we have*

(18.11) $$\varphi^f(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\,\overline{f(t)}\,dt$$
$$=\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_{0}^{\infty}e^{iux}\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du$$

Theorem 65. *Under the same assumption of Theorem 63, the neces-*

sary and sufficient condition that $f(x)$ defined by (18.04) belongs to the class S' is that (17.21) is true.
and

Theorem 66. *Under the same assumption of Theorem 63, if $f(x)$ defined by (18.04) belong to the class S, then it does to the class S'.*

19. We apply the result of the preceding sections to the almost periodic function.

Theorem 67. *Let $g(x)$ be real valued measurable funciton over $(-\infty, \infty)$. Let $g(x)$ be the B_2-almost periodic function*. Let us assume that the condition (K_1) is satisfied. Then the necessary condition for the generalized Hilbert transform $\tilde{g}_1(x)$ to be B_2-almost periodic is that the condition (K_2) is satisfied for a^g—the constant term of $\tilde{g}_1(x)$.* If the *associated Fourier series with $g(x)$ is*

$$g(x) \sim \Sigma' a_n e^{i\lambda_n x} \tag{19.01}$$

then

$$\tilde{g}_1(x) \sim a^g + \Sigma' (-i \operatorname{sign} \lambda_n) a_n e^{i\lambda_n x}, \tag{19.02}$$

where the prime means that the summation does not contain the constant term.

Lemma 67_1. *Under the same assumption of Theorem 67, we have*

$$a_{-n} = \bar{a}_n, \quad \lambda_{-n} = -\lambda_n \quad (n=1,2,\cdots) \tag{19.03}$$

Proof of Lemma 67_1. If we observe that $g(x)$ is of real valued then (19.03) is obtained easily.

Lemma 67_2. *Let $g(x)$ be the B_2-almost periodic function. Let us assume that the condition (K_1) is satisfied. Then we have*

$$a_0 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)\, dt = 0 \tag{c_1}$$

Proof of Lemma 67_2. From (K_1) we have

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \{s^g(u+\varepsilon) - s^g(u-\varepsilon)\}\, du = 0, \tag{19.04}$$

and

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \{s^g(u+\varepsilon) - s^g(u-\varepsilon)\}\, du = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} g(t) \frac{\sin^2 \varepsilon t}{\varepsilon t^2} dt. \tag{19.05}$$

If we apply the theorem of Bochner-Hardy-Wiener[3, p. 30] we get (c_1).

*) We mean the almost periodic function in the sense of Besicovitch [2]

Proof of Theorem 67. In the first, from Lemmas 67_1, 67_2, the Fourier series of $g(x)$ can be written as in (19.01). Next let us assume that $\tilde{g}_1(x)$ be B_2-almost periodic. Then we get

$$(19.06)\qquad \frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)\}\,du=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\pi\varepsilon}\int_{-\infty}^{\infty}\tilde{g}_1(t)\,e^{-i\lambda t}\frac{\sin^2\varepsilon t}{t^2}\,dt$$

and

$$(19.07)\qquad \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)\}\,du=\frac{1}{\sqrt{2\pi}}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{g}_1(t)\,e^{-i\lambda t}\,dt.$$

Therefore by (i) of Theorem 49,

$$(19.08)\qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{g}_1(t)\,e^{-i\lambda t}\,dt=(-i\,\mathrm{sign}\,\lambda)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}g(t)\,e^{-i\lambda t}\,dt,$$
$$\text{if}\quad \lambda\neq 0.$$

and if we put

$$(19.09)\qquad a^g=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{g}_1(t)\,dt,$$

then (19.02) is obtained. In the last if we apply Lemma 67_1 to (19.09) we get

$$(19.10)\qquad \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)\}\,du=\frac{1}{\sqrt{2\pi}}a^g.$$

Here from the Parseval theorem of almost periodic function, we get immediately

$$(19.11)\qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{g}_1(t)|^2\,dt=|a^g|^2+\Sigma|a_n|^2$$
$$=|a^g|^2+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2\,dt,$$

and that

$$(19.12)\qquad \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(-\varepsilon)|^2\,du$$
$$=|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du.$$

From (19.12) and (K_1) we get

$$(19.13)\qquad \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)|^2\,du=|a^g|^2.$$

Substituting (19.10) to (19.13) we wet

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\{\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon\}-\sqrt{2\pi}\,a^g|^2\,du=0. \tag{19.14}$$

Applying (ii) of Theorem 49 to (19.14) and using (K_1) and Lemma 52_1 we obtain (K_2). Thus we have proved Theorem 67 completely.

Theorem 68. *Let $g(x)$ be real valued measurable function over $(-\infty, \infty)$. Let $g(x)$ be the B_2-almost periodic function. Let us assume that (K_1) is satisfied. Then the sufficient condition for the generalized Hilbert transform $\tilde{g}_1(x)$ to be B_2-almost periodic is that the condition (K_2) is satisfied. If the associated Fourier series of $g(x)$ be*

$$g(x)\sim \Sigma'\,a_n\,e^{i\lambda_n x} \tag{19.01}$$

then

$$\tilde{g}_1(x)\sim a^g+\Sigma'(-i\,\mathrm{sign}\,\lambda_n)\,e^{i\lambda_n x}. \tag{19.02}$$

Proof of Theorem 68. Since $g(x)$ is B_2-almost periodic, for any given positive number $\varepsilon>0$, there exist a trigonometrical polynomial — the Bochner-Fejér polynomial of order p — such that (c.f. S. Besicovitch [2]),

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)-\sigma_{B_p}^g(t)|^2\,dt\leqq\varepsilon, \tag{19.15}$$

where

$$\sigma_{B_p}^g(x)=\sigma_B^g(x)=\Sigma'\,d_n^B\,a_n\,e^{i\lambda_n x},\quad d_{-n}^B=d_n^B, \tag{19.16}$$

or

$$\sigma^g_{\binom{n_1,n_2,\cdots,n_p}{\beta_1,\beta_2,\cdots,\beta_p}}(x)=\Sigma'\left(1+\frac{|\nu_1|}{n_1}\right)\cdots\left(1-\frac{|\nu_p|}{n_p}\right)a_n\,e^{i\lambda_n x}, \tag{19.16$'$}$$

and

$$\lambda_n=\frac{\nu_1}{n_1}\beta_1+\frac{\nu_2}{n_2}\beta_2+\cdots+\frac{\nu_p}{n_v}\beta_p.$$

Then we put

$$\tilde{\sigma}_B^g(x)=\Sigma'(-i\,\mathrm{sign}\,\lambda_n)\,d_d^B\,a_n\,e^{i\lambda_n x} \tag{19.17}$$

and we denote by $s^\sigma(u)$ and $\tilde{s}^\sigma(u)$ the Fourier-Wiener transform of $\sigma_B^g(x)$ and $\tilde{\sigma}_B^g(x)$ respectively. If we observe (ii) of Theorem 49, (K_1) and Lemma 52_1 we get also

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\{\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)\}-\sqrt{2\pi}\,a^g|^2\,du=0 \tag{19.14}$$

Then from (i) of Theorem 49, and (19.14) we get immediately

$$(19.18)\quad \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\{\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)\}-\{\tilde{s}^\sigma(u+\varepsilon)-\tilde{s}^\sigma(u-\varepsilon)\}-\sqrt{2\pi}\,a^g\chi_\varepsilon(u)|^2du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}-\{s^\sigma(u+\varepsilon)-s^\sigma(u-\varepsilon)\}|^2\,du$$

Thus we have from (19.18) and Lemma 49_2

$$(19.19)\quad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{g}_1(t)-\tilde{\sigma}_B^g(t)-a^g|^2\,dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)-\sigma_B^g(t)|^2\,dt$$

Therefore we have

$$(19.20)\quad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{g}_1(t)-\{a^g+\tilde{\sigma}_B^g(t)\}|^2dt\leqq\varepsilon.$$

Thus the B_2-almost periodicity of $\tilde{g}_1(x)$ is proved. The remaining part is obvious.

Combining Theorems 67 and 68 we get

Theorem 69. *Let $g(x)$ be B_2-almost periodic and let us assume that the condition (K_1) is satisfied. Then the necessary and sufficient condition for $f_1(x)=g(x)+i\tilde{g}_1(x)$ to be also B_2-almost periodic is that the condition (K_2) is satisfied. If we write by (19.01) the associated Fourier series of $g(x)$, then that of $f_1(x)$ is as follows*

$$(19.21)\quad f_1(x)\sim ia^g+2\sum_{\lambda_n>0}a_n\,e^{i\lambda_n x}.$$

If we assume that the ordinary Hilbert transform $\tilde{g}(x)$ of a given function $g(x)$ exists for a.e. x. Then by repeating the same argument, we get.

Theorem 70. *Let $g(x)$ be real valued measurable function over $(-\infty, \infty)$. Let $g(x)$ be B_2-almost periodic and let us assume that (K_1) is satisfied. Let us suppose that its ordinary Hilbert transform $\tilde{g}(x)$ exist for a. e. x. Then the necessary and sufficient condition for $\tilde{g}(x)$ to be B_2-almost periodic and*

$$(19.22)\quad \tilde{a}_0=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{g}(t)\,dt=0$$

is that the condition (K_3) is satisfied. If we represent the associated Fourier series of $g(x)$ as follows

$$(19.01)\quad g(x)\sim\Sigma'\,a_n\,e^{i\lambda_n x}$$

then

$$\widetilde{g}(x) \sim \Sigma'(-i \operatorname{sign} \lambda_n)\, a_n\, e^{i\lambda_n x}. \tag{19.23}$$

Theorem 71. *Under the same assumption of Theorem 70, the necessary and sufficient condition for $f(x)=g(x)+i\widetilde{g}(x)$ to be B_2-almost periodic and*

$$c_0=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\,dt=0 \tag{19.24}$$

is that the condition (K_3) is satisfied.

We will end this chapter adding some remarks.

Remark 2. Let $g(x)$ be B_2-almost periodic. In Lemma 67_2, we proved that if (K_1) is satisfied then

$$a_0=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(t)\,dt=0 \tag{c_1}$$

Conversely, if we assume that (c_1) is satisfied, then (K_1) is deduced. Because $g(x)$ is B_2-almost periodic and therefore belongs to the class S'. By Bochner's representation theorem of the positive definite function, we can write (c.f. [3, 32]),

$$\varphi^g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}\,d\Lambda(u), \tag{19.25}$$

where $\Lambda(u)$ is a monotone increasing function of bounded variation over $(-\infty.\ \infty)$ and

$$\Lambda(u)-\Lambda(-u)=\underset{\varepsilon\to 0}{\text{l.i.m.}}\ \frac{1}{4\pi\varepsilon}\int_{-u}^{u} |\, s^g(v+\varepsilon)-s^g(v-\varepsilon)\,|^2\,dv \tag{19.26}$$

$$\Lambda(\lambda_n+0)-\Lambda(\lambda_n-0)=|\,a_n\,|^2. \tag{19.27}$$

In particular

$$\lim_{\delta\to 0}\frac{1}{2\delta}\int_{-\delta}^{\delta}\{\Lambda(u)-\Lambda(-u)\}\,du=|\,a_0\,|^2. \tag{19.28}$$

If we observe that

$$s^g(-u+\varepsilon)-s^g(-u-\varepsilon)=\overline{s^g(u+\varepsilon)-s^g(u-\varepsilon)}$$

then

$$\begin{aligned}&\lim_{\delta\to 0}\frac{1}{2\delta}\int_{-\delta}^{\delta}\{\Lambda(u)-\Lambda(-u)\}\,du\\ &=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\delta}\int_{0}^{\delta} du\,\frac{1}{\varepsilon}\int_{0}^{\varepsilon} |\, s^g(v+\varepsilon)-s^g(v-\varepsilon)\,|^2\,dv\end{aligned}$$

$$=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\delta}\left(1-\frac{v}{\delta}\right)|s^g(v+\varepsilon)-s^g(v-\varepsilon)|^2\,dv$$

$$\geqq\overline{\lim_{\varepsilon\to 0}}\frac{2}{\varepsilon}\int_0^{\varepsilon}|s^g(v+\varepsilon)-s^g(v-\varepsilon)|^2\,dv.$$

Thus we get from (c_1)

$$(K_1)\qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du=0.$$

Therefore our assumption (K_1) does not loss of generality. If we need the general case, then (K_1) is replaced by

$$(K_1)'\qquad \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}|\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}-\sqrt{2\pi}\,a_0|^2\,du=0$$

We omit in details.

Remark 3. Let $g(x)$ be B_2-almost periodic. Let associated Fourier series of $g(x)$ be

$$(19.01)\qquad g(x)\sim\Sigma' a_n e^{i\lambda_n x}$$

Let us also assume that

$$(19.29)\qquad \operatorname*{g.l.b.}_{\lambda_m,\,\lambda_n<0}|\lambda_m-\lambda_n|>0$$

or

$$(19.30)\qquad \lambda_n=-\log(|n|+1),\quad n=-1,-2,\cdots$$

then we get

$$(19.31)\qquad \sum_{\lambda_n<0}|a_n e^{\lambda_n}|<\infty.$$

If we assume that $\sigma^g_{B_p}(x)$ is the Bochner-Fejér polynomial of order p and that

$$(19.32)\qquad a^\sigma_p=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma^g_B(t)}{t+i}\,dt=-2i\sum_{\lambda_n<0}d^B_n\,a_n\,e^{\lambda_n}.$$

Then under our assumption (19.31), the condition (K_2) is equivalent to

$$(c_2)\qquad \lim_{p\to\infty}|a^g-a^\sigma_p|=0.$$

Because by Minkowski's inequality, we have

$$(19.33)\qquad \frac{\pi}{2}|a^g-a^\sigma_p|\leqq\frac{4}{\varepsilon}\int_0^{2\varepsilon}\left|\operatorname*{l.i.m.}_{B\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}e^{-ius}\,ds-\sqrt{\frac{\pi}{2}}\,a^g\right|^2du$$
$$+\frac{4}{\pi}\int_0^{2\varepsilon}\left|\operatorname*{l.i.m.}_{B\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{\sigma^g_B(s)}{s+i}e^{-ius}\,ds-\sqrt{\frac{\pi}{2}}\,a^\sigma_p\right|^2du$$

$$+\frac{4}{\varepsilon}\int_{-\infty}^{\infty}\left|\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)-\sigma_B^g(s)}{s+i}e^{-ius}\,ds\right|^2 du$$

$$=I_1+I_2+I_3, \quad \text{say.}$$

Then

$$I_2\leqq\Big|\sum_{\lambda_n+\varepsilon<0}d_n^B a_n e^{\lambda_n}\Big|^2\frac{1}{\varepsilon}\int_0^{2\varepsilon}(e^{-u}-1)^2\,du \tag{19.34}$$

$$\leqq A\varepsilon\Big(\sum_{\lambda_n<0}|a_n e^{\lambda_n}|\Big)^2=o(1), \quad \varepsilon\to 0,$$

and

$$I_3\leqq\frac{B}{\varepsilon}\int_{-\infty}^{\infty}\frac{|g(s)-\sigma_B^g(s)|^2}{1+s^2}\,ds \tag{19.35}$$

$$\leqq\frac{B}{\varepsilon}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(s)-\sigma_B^g(s)|^2\,ds.$$

Therefore

$$\lim_{p\to\infty}|a^g-a_p^\sigma|\leqq I_1+A\varepsilon\Big(\sum|a_n e^{\lambda_n}|\Big)^2$$

and tending ε to 0 we get (c_2) from (K_2). Repeating the same argument from (c_2) we get (K_2).

Remark 4. In Theorem 2, the existence of Hilbert transform for any $f\in L$ was proved by N. Lusin, Privaloff and Plessner (c.f. [30]). The real variable proof was given by A.S.Besicovitch [40] (c.f. also E.C. Titchmarch [45]). As for Fourier series the weak type (1,1) of conjugate function was proved by A.Kolmogoroff [44].

Remark 5. In Theorem 28, we find that the additional condition (11.10) is required. Therefore we should correct the two theorems. These are Theorem 8 of [21, IV] and Theorem 16 of [21, V]. And thus we change the proofs of Theorems 33 and 34.

Remark 6. In chapter 5, Theorems 57,58,65 and 66 shall be improved. These are treated in the next paper.

Remark 7. After the preperation of this paper, the author knew the paper of M.Cotlar [42] concerning with the maximal theorem of Hilbert transform in E^n. We learned this from Mr. Y.M.Chen. Author thanks to him.

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