MONOTONY AND FLATNESS OF THE NORMS BY MODULARS

By

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§ 1. Introduction

Let R be a modulared semi-ordered linear space with a modular m[1]. We define the two norms on R by the formulas:

$$\text{(a)} \qquad \|x\| = \inf_{\xi>0} \frac{1+m(\xi x)}{\xi},$$

$$\text{(b)} \qquad |||x||| = \inf_{m(\xi x)\leqq 1} \frac{1}{|\xi|}.$$

The norms $\|x\|$ and $|||x|||$ are called the first norm and the second norm by m respectively.

$\tilde{R}$ denotes the totality of all linear functionals which are bounded under the norm $\|x\|$. The associated modular $\tilde{m}$ of m is defined on $\tilde{R}$ by the formula:

$$\text{(c)} \qquad \tilde{m}(\tilde{x}) = \sup_{x\in R} \{\tilde{x}(x) - m(x)\} \qquad \text{for all } \tilde{x}\in\tilde{R}.$$

The functional $\tilde{m}$ satisfies all the modular conditions (cf. [2, §38]).

We know (cf. [3, §§80–83])

$$\text{(d)} \qquad m(a) = \sup_{\tilde{x}\in\tilde{R}} \{\tilde{x}(a) - \tilde{m}(\tilde{x})\} \qquad \text{for all } a\in R,$$

and

$$\text{(e)} \qquad \|a\| = \sup_{\tilde{m}(\tilde{x})\leqq 1} |\tilde{x}(a)| \qquad \text{for all } a\in R.$$

The first and second norms satisfy always

$$\text{(f)} \qquad |||x||| \leqq \|x\| \leqq 2|||x||| \qquad \text{for all } x\in R,$$

and a fortiori, they are equivalent (cf. [3, §83]). The first and second norms by the associated modular $\tilde{m}$ on $\tilde{R}$ are denoted by $\|\tilde{x}\|$ and $|||\tilde{x}|||$ respectively. Then we know ([3, §83]),

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1) We use definitions, notation and terminology of [2,3].

(g) $$\|\tilde{a}\| = \sup_{|||x|||\leqq 1} |\tilde{a}(x)| \quad \text{and} \quad |||\tilde{a}||| = \sup_{\|x\|\leqq 1} |\tilde{a}(x)| \qquad \text{for all } \tilde{a}\in\tilde{R}.$$

One of the important problems in the theory of modulared semi-ordered linear spaces is to characterize the properties of the norms by those of the modulars. In this paper we shall give such characterizations for (uniform) monotony and (uniform) flatness[2] of the norms.

§2. Preliminaries

A norm $\|x\|$ on a semi-ordered linear space S is said to be *monotone*[3], if $a\frown b=0$, $\|a\|=1$, $\|b\|\geqq\varepsilon>0$ implies

(h) $$\|a+b\|\geqq 1+\delta \qquad \text{for some } \delta=\delta(\varepsilon, a, b)>0,$$

and is said to be *flat*, if $a\frown b=0$, $\|a\|=\|b\|=1$ implies

(i) $$\lim_{\varepsilon\to 0}\frac{\|a+\varepsilon b\|-1}{\varepsilon}=0.$$

A norm is said to be *uniformly monotone*, if in (h) δ depends only on ε, and is said to be *uniformly flat*, if in (i) the convergence is uniform with respect to both a and b.

A modular m is said to be *simple*, if

(j) $$m(a)=0 \quad \text{implies} \quad a=0,$$

and is said to be *uniformly simple*, if

(k) $$\inf_{|||x|||=1} m(\xi x)>0 \qquad \text{for all } \xi>0.$$

m is said to be *monotone*[4], if

(l) $$\lim_{\xi\to 0}\frac{m(\xi x)}{\xi}=0 \qquad \text{for all } x\in R,$$

and is said to be *uniformly monotone*, if

(m) $$\lim_{\xi\to 0}\sup_{|||x|||=1}\frac{m(\xi x)}{\xi}=0.$$

We use frequently the following properties of modulars:

(n) $m(\xi x)$ is a convex function of $\xi\geqq 0$,

(n′) $\dfrac{m(\xi x)}{\xi}$ is an increasing function of $\xi\geqq 0$,

(o) $a\frown b=0$ implies $m(a+b)=m(a)+m(b)$,

2) For the definitions, see §2.

3) We may define monotony as follows: $0\leqq a<b$ implies $\|a\|<\|b\|$.

4) Do not confuse with the monotony of a norm.

(p) $0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda} a$ implies $\sup_{\lambda \in \Lambda} m(a_\lambda) = m(a)$.

In the remainder of this section we state some known facts used later.

Lemma A ([2, §31]). *A norm on a semi-ordered linear space is uniformly monotone, if and only if its associated norm is uniformly flat.*

Lemma B ([3, §85]). *A modular is uniformly simple, if and only if its associated modular is uniformly monotone.*

Lemma C ([3, §84]). *A modular is monotone, if and only if its associated modular is simple.*

Lemma D ([4]). *If the norm by a simple, monotone complete*[5] *modular is continuous,*[5] *the modular is uniformly simple.*

Lemma E ([3, §79]). *If for fixed $a \in R$, $\alpha \geqq 0$, $\beta \geqq 0$*

$$\alpha\xi \leqq \beta + m(\xi a) \qquad \text{for all } \xi \geqq 0,$$

then there exists $\tilde{a} \in \tilde{R}$ such that $\tilde{a}(a) = \alpha$ and $\tilde{m}(\tilde{a}) \leqq \beta$. If in addition $\beta = \sup_{\xi \geqq 0} \{\alpha\xi - m(\xi a)\}$, then $\tilde{m}(\tilde{a}) = \beta$.

§3. Monotony

In the sequel R denotes a modulared semi-ordered linear space the dimension of which is greater than 2.

In this section, we give conditions for the norms by modulars to be (uniformly) monotone. We begin with some elementary but important lemmas concerning the norms by modulars.

Lemma 3.1. (1) *If $m(a) < |||a|||$, then $|||a||| \leqq 1$.* (2) *For $a \in R$, $m(a) < |||a||| = 1$, if and only if $m(a) < 1$ and $m(\xi a) = +\infty$ for all $\xi > 1$.* (3) *If $m(a) < |||a||| = 1$ and $a = a_1 + a_2$, $a_1 \frown a_2 = 0$, then $|||a_1||| = 1$ or $|||a_2||| = 1$.*

Proof. (1) and (2) are direct consequences of the definition (b). (3) If $|||a_i||| < 1$ $(i = 1, 2)$, then by (b)

$$m(\xi_0 a_i) < 1 \quad (i = 1, 2) \quad \text{for some} \quad \xi_0 > 1,$$

hence by (0) $\qquad m(\xi_0 a) = m(\xi_0 a_1) + m(\xi_0 a_2) < +\infty.$

This is a contradiction, because of (2). Q.E.D.

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5) A modular is said to be monotone complete, if $\sup_{\lambda \in \Lambda} m(a_\lambda) < +\infty$, $0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda}$ implies existence of $\bigcup_{\lambda \in \Lambda} a_\lambda$. A norm $\|\cdot\|$ is said to be continuous, if $a_\nu \downarrow_{\nu=1}^{\infty} 0$ implies $\lim_{\nu \to \infty} \|a_\nu\| = 0$.

Lemma 3.2. (1) *For any* $a \in R$, $\sup_{\xi>0} \frac{m(\xi a)}{\xi} = \|a\|$, or $\xi_0 \|a\| = 1 + m(\xi_0 a)$ *for some* $\xi_0 > 0$. (2) *If* $a \in R$ $\tilde{a} \in \tilde{R}$, $|||\tilde{a}||| = 1$ *and* $\tilde{a}(a) = 1 + m(a)$, *then* $\|a\| = \tilde{a}(a)$.

Proof. (1) By the definition (a) and (n′), we have

$$\sup_{\xi>0} \frac{m(\xi a)}{\xi} \geqq \|a\|.$$

If

$$\sup_{\xi>0} \frac{m(\xi a)}{\xi} > \|a\|,$$

by the continuity of the convex real function $m(\xi a)$, there exists $\xi_0 > 0$ such that

$$\|a\| = \frac{1 + m(\xi_0 a)}{\xi_0}.$$

(2) Since

$$1 = \frac{1+m(a)}{\tilde{a}(a)} = \frac{1 + m\left(\tilde{a}(a) \cdot \frac{a}{\tilde{a}(a)}\right)}{\tilde{a}(a)},$$

by the definition (a) we have $\|a\| \leqq \tilde{a}(a)$. On the other hand, by (g) $\tilde{a}(a) \leqq \|a\| \cdot |||\tilde{a}||| \leqq \|a\|$. Q.E.D.

Lemma 3.3. *If* $m(\xi a) \leqq \xi \|a\|$ *for all* $\xi \geqq 0$, *then the first norm is of* L^1*-type on* $[a]R$[6], *i.e.*

$$\|x+y\| = \|x\| + \|y\| \quad \text{for all} \quad 0 \leqq x, y \in [a]R.$$

Proof. From Lemma 3.2 it follows

$$\sup_{\xi>0} \frac{m(\xi a)}{\xi} = \|a\|.$$

Since by (n′) for any $p \in R$,

$$\begin{aligned} &\|[p]a\| + \|(1-[p]a\| \\ &\leqq \sup_{\xi>0} \frac{m(\xi[p]a)}{\xi} + \sup_{\xi>0} \frac{m(\xi(1-[p])a)}{\xi} \\ &= \sup_{\xi>0} \frac{m(\xi a)}{\xi} = \|a\|, \end{aligned}$$

we have

6) For $a \in R$, $[a]$ denotes the projection operator defined by $[a]x = \bigcup_{\nu=1}^{\infty} (x \frown \nu |a|)$ for all $x \geqq 0$.

$$\| [p]a \| = \sup_{\xi>0} \frac{m(\xi[p]a)}{\xi} \qquad \text{for all } p \in R.$$

From this, we can conclude that for any orthogonal system $\{p_\nu\}_{\nu=1}^{\kappa}$ and for any real numbers $\{\alpha_\nu\}_{\nu=1}^{\kappa}$,

$$\left\| \sum_{\nu=1}^{\kappa} \alpha_\nu [p_\nu] a \right\| = \sup_{\xi>0} \frac{m\left(\xi \sum_{\nu=1}^{\kappa} \alpha_\nu [p_\nu] a\right)}{\xi},$$

consequently
$$\left\| \sum_{\nu=1}^{\kappa} \alpha_\nu [p_\nu] a \right\| = \sum_{\nu=1}^{\kappa} | \alpha_\nu | \cdot \| [p_\nu] a \|.$$

It follows that $|x| \frown |y| = 0$, $x, y \in [a]R$ implies $\| x+y \| = \| x \| + \| y \|$, and in turn this is sufficient for the assertion. Q.E.D.

As is well-known (cf. [2, §42]), there exists the decomposition of R: $R = R_1 \oplus R_2$, where R_i $(i=1,2)$ are mutually orthogonal normal manifolds such that m is simple on R_1 and semi-singular[7] on R_2.

Theorem 3.1. *In order that the first norm be monotone, it is necessary and sufficient that one of the following conditions is satisfied:*

(1) $R_2 = \{0\}$ *i.e.* m *is simple,*

(2) $m(x) \leqq \| x \|$ *for all* $x \in R$,

(3) $0 < \dim (R_2) < +\infty$ *and* $\| x \| < m(x) + 1$ *for all* x *with* $(1 - [x])R_2 \neq \{0\}$.

Proof. Necessity. Suppose that $R_2 \neq \{0\}$. Let $0 < b \in R_2$, $m(b) = 0$ and $a \frown b = 0$. If $1 + m(a) = \| a \|$,

$$\| a+b \| \leqq 1 + m(a+b) = 1 + m(a) = \| a \|,$$

contradicting monotony of the first norm. Thus by Lemma 3.2

$$m(a) \leqq \| a \| < m(a) + 1.$$

If furthermore $\dim (R_2) = +\infty$, for any $x \in R$ there exists a family of projectors $[p_\lambda]_{\lambda \in \Lambda}$ such that $[p_\lambda] \uparrow_{\lambda \in \Lambda} [x]$ and $(1 - [p_\lambda])R_2 \neq \{0\}$.

From the above and (p) we obtain

$$m(x) = \sup_{\lambda \in \Lambda} m([p_\lambda]x) \leqq \sup_{\lambda \in \Lambda} \| [p_\lambda]x \| = \| x \|.$$

Sufficiency. Let $a \frown b = 0$, $b > 0$. If

$$\sup_{\xi>0} \frac{m(\xi(a+b))}{\xi} = \| a+b \|,$$

(in particular if (2) holds), by Lemma 3.3 we obtain

$$\| a+b \| = \| a \| + \| b \| > \| a \|.$$

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7) R is said to be semi-singular, if for any $0 < a \in R$ there exists $0 < b \in R$ such that $b \leqq a$, $m(b) = 0$.

If $m(\xi_0(a+b))+1=\xi_0\|a+b\|$ for some $\xi_0>0$, then the condition (1) implies

$$\|a\|<\frac{1+m(\xi_0 a)}{\xi_0}+\frac{m(\xi_0 b)}{\xi_0}=\|a+b\|,$$

and the condition (3) implies

$$(1-[a])R_2=\{0\} \quad \text{or} \quad (1-[b])R_2=\{0\},$$

namely $a\in R_1$ or $b\in R_1$. In case $a\in R_1$, $\frac{1+m(\xi_0 a)}{\xi_0}>\|a\|$ by (3), and if $b\in R_1$, $m(\xi_0 b)>0$. Thus the first norm is monotone. Q.E.D.

Now let the first norm be uniformly monotone. Then, considering $\tilde{\tilde{R}}$ with the second-associated modular $\tilde{\tilde{m}}$[8] and using (d) and Lemma A, we may assume that R is monotone complete. If (1) in Theorem 3.1 occurs, by Lemma D m is uniformly simple. If (3) occurs, let $[R_2]=[a_1, a_2,\cdots, a_n]$ where $a_i>0$ $(i=1,2,\cdots,n)$ be mutually orthogonal discrete elements with $m(a_i)=0$. Put.

$$\alpha=\min_{i=1,2,\cdots,n}\sup\{\xi: m(\xi a_i)=0\}.$$

For any $\gamma>0$ and $a\in R$, $\|a\|=1$ with $a\frown a_i=0$ (for some i)

$$\begin{aligned}\left\|a+\frac{\alpha}{\gamma}a_i\right\| &\leqq \frac{1+m(\xi a)}{\xi}+\frac{m\left(\frac{\xi\alpha}{\gamma}a_i\right)}{\xi}\\ &=\frac{1+m(\xi a)}{\xi} \qquad \text{for all } 0<\xi\leqq\gamma.\end{aligned}$$

Since uniform monotony of the first norm implies

$$\min_{i=1,2,\cdots,n}\ \inf_{\substack{x\frown a_i=0\\ \|x\|=1}}\left\|x+\frac{\alpha}{\gamma}a_i\right\|\geqq 1+\delta \qquad \text{for some } \delta>0,$$

we obtain

$$\xi(1+\delta)-m(\xi a)\leqq 1 \qquad \text{for all } 0\leqq\xi\leqq\gamma.$$

On the other hand, as in case (1), m is uniformly simple on R_1.

Thus we have proved the "necessity" part of the following theorem.

Theorem 3.2. *In order that the first norm be uniformly monotone, it is necessary and sufficient that one of the following conditions is satisfied:*

(1) *m is uniformly simple,*

(2) *$m(x)\leqq\|x\|$ for all $x\in R$,*

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8) $\tilde{\tilde{R}}$ and $\tilde{\tilde{m}}$ denote the associated space of $\tilde{R}$ and the associated modular of $\tilde{m}$ respectively.

(3) $0<\dim(R_2)<+\infty$ *and* m *is uniformly simple on* R_1, *and for any* $\gamma>0$ *there exists* $\delta>0$ *such that* $\sup_{0\leqq\xi\leqq\gamma}\{\xi(1+\delta)-m(\xi x)\}\leqq 1$ *for all* x *with* $\|x\|=1$, $(1-[x])R_2\neq\{0\}$.

Proof of sufficiency. The proof for (1) is a slight modification of [2, Theorem 48.8]. If (2) holds, by Lemma 3.3 the first norm is of L^1-type, a fortiori uniformly monotone. Now suppose that (3) holds. Let $\|a\|=1$, $\|b\|\geqq\varepsilon>0$, $a\frown b=0$. As in the proof of Theorem 3.1 we may restrict ourselves to the case that $a\in R_1$ (or $b\in R_1$) and

$$\|a+b\|=\frac{1+m(\xi_0 a)}{\xi_0}+\frac{m(\xi_0 b)}{\xi_0}\qquad\text{for some }\xi_0\geqq\frac{1}{2}.$$

Since m is uniformly simple on R_1, there exists $\delta'>0$ such that

$$\inf_{\|x\|=\frac{\varepsilon}{2},\,x\in R_1} m(x)\geqq\delta'>0.$$

If $b\in R_1$, $\|a+b\|=\dfrac{1+m(\xi_0 a)}{\xi_0}+\dfrac{m(\xi_0 b)}{\xi_0}\geqq 1+\dfrac{m\left(\frac{b}{2}\right)}{\frac{1}{2}}\geqq 1+2\delta'$, because of (n′).

If $a\in R_1$ and $\gamma\leqq\xi$, where $\gamma=\sup_{\|x\|=\varepsilon}\dfrac{2}{|||x|||}$, then

$$\|a+b\|\geqq 1+\frac{m(\gamma b)}{\gamma}\geqq 1+\frac{m\left(\frac{2b}{|||b|||}\right)}{2}|||b|||\geqq 1+|||b|||\geqq 1+\frac{\varepsilon}{2},$$

because of (f) and (n′). Finally, if $a\in R_1$ and $\xi_0\leqq\gamma$, then by (3)

$$\|a+b\|\geqq\frac{1+m(\xi_0 a)}{\xi_0}\geqq 1+\delta.$$

Thus the first norm is uniformly monotone. Q.E.D.

Remark 3.1. *If* R *is non-atomic*[9], *in Theorems 3.1-2 the conditions (3) disappear.*

Remark 3.2. *The conditions* (2) *and* (3) *in Theorem 3.1 can be written in terms of modulars as follows*:

(2′) $\xi m(x)\leqq m(\xi x)+1$ *for all* $x\in R$ *and* $\xi\geqq 0$,

(3′) $0<\dim(R_2)<+\infty$ *and* $\sup_{\xi>0}\dfrac{m(\xi x)}{\xi}<m(x)+1$ *for all* x *with* $(1-[x])R_2\neq\{0\}$.

9) This means that for any $0<a\in R$ there exist $b, c\in R$ such that $b+c\leqq a$, $b\frown c=0$, $b>0$, $c>0$.

Now we shall consider the second norm.

Theorem 3.3. *In order that the second norm be monotone, it is necessary and sufficient that the following two conditions are satisfied:*

(1) *m is simple,*

(2) *"$|||x|||=1$" is equivalent to "$m(x)=1$".*

Proof. Necessity. (1) If $m(a)=0$, we may assume that $[a]<\mathrm{I}$. Let $a\frown b=0$ and $|||b|||=1$, then by (o) and Lemma 3.1 we have

$$1=|||b|||\leqq|||a+b|||\leqq m(a+b)=m(b)\leqq 1.$$

Thus monotony of the second norm implies $a=0$. (2) If for some $a>0$, $m(a)<|||a|||=1$, by Lemma 3.1 we may assume that $[a]<\mathrm{I}$. Since there exists $b>0$ such that $a\frown b=0$, $m(a+b)\leqq 1$, we have

$$1\leqq|||a|||\leqq|||a+b|||\leqq 1,$$

contradicting monotony of the second norm.

Sufficiency. If $0<b$, $a\frown b=0$ and $|||a|||=1$, then by (1) and (2)

$$1=|||a|||=m(a)<m(a+b),$$

hence $|||a+b|||>1=|||a|||$ by the definition (b). Q.E.D.

Theorem 3.4. *In order that the second norm be uniformly monotone, it is necessary and sufficient that the following two conditions are satisfied:*

(1) *m is simple,*

(2) $\sup\limits_{0<\xi<1,}\ \inf\limits_{|||x|||=1} m(\xi x)=1.$

Proof of sufficiency is a slight modification of [2, Theorem 48.9].

Necessity. We prove firstly that for any $0<p\in R$

$$(*)\qquad \sup_{0<\xi<1,}\ \inf_{\substack{x\frown p=0\\ |||x|||=1}} m(\xi x)=1.$$

Otherwise there exists a sequence of positive elements $\{a_\nu\}_{\nu=1}^{\infty}\subset R$, and $\varepsilon>0$ such that $a_\nu\frown p=0$, $1>|||a_\nu|||\geqq 1-\dfrac{1}{\nu}$, $m(a_\nu)\leqq 1-\varepsilon$ $(\nu=1,2,\cdots)$. Then chosing $0<m(\alpha p)\leqq\varepsilon$, we have by (o)

$$m(x_\nu+\alpha p)=m(x_\nu)+m(\alpha p)\leqq 1\quad(\nu=1,2,\cdots),$$

hence by Lemma 3.1

$$\sup_{\nu\geqq 1}|||x_\nu+\alpha p|||\leqq 1.$$

On the other hand, uniform monotony of the norm implies that

$$|||x_\nu+\alpha p|||\geqq|||x_\nu|||+\delta\ (\nu=1,2,\cdots)\text{ for some }\delta>0,$$

consequently

$$\sup_{\nu\geqq 1} |||x_\nu+\alpha p||| \geqq \sup_{\nu\geqq 1} |||x_\nu|||+\delta\geqq 1+\delta,$$

contradicting the above. Thus we have proved (*).

Now choose $p\in R$, $0<[p]<1$. From (*) it follows that for any $0<\xi<1$ there exists $\gamma>1$ such that $m(x)\leqq\xi$ implies $|||[p]x|||\leqq\frac{1}{\gamma}$ and $|||(I-[p])x|||\leqq\frac{1}{\gamma}$, hence by (o) $m(\gamma x)=m(\gamma[p]x)+m(\gamma(1-[p])x)\leqq 2$. Since by Theorem 3.3 $m\left(\frac{a}{|||a|||}\right)=1$ for all $a\neq 0$, using (n) we obtain that, if $|||x|||\geqq\frac{1}{\gamma}$,

$$\gamma-1=(\gamma-1)m\left(\frac{x}{|||x|||}\right)\leqq\left(\gamma-\frac{1}{|||x|||}\right)m(x)+\left(\frac{1}{|||x|||}-1\right)m(\gamma x)$$
$$\leqq\left(\gamma-\frac{1}{|||x|||}\right)\xi+2\left(\frac{1}{|||x|||}-1\right),$$

hence

$$|||x|||\leqq\frac{2-\xi}{\gamma(1-\xi)+1}\equiv\rho<1.$$

This means that $|||x|||\geqq\rho$ implies $m(x)\geqq\xi$. Since $0<\xi<1$ is arbitrary, we obtain (2). Q.E.D.

Corollary. *If the second norm is (uniformly) monotone, then the first norm is also (uniformly) monotone.*

Remark 3.3. *If R is non-atomic,* (2)[10] *in Theorem 3.3 may be replaced by finiteness of the modular, and in Theorem 3.4* (1) *and* (2) *together may be replaced by uniform simplicity of the modular by Lemma D.*

Remark 3.4. *The condition* (2) *in Therem 3.3 can be written in terms of modulars as follows:*

(2′) $$\sup_{\xi}\{m(\xi x): m(\xi x)<+\infty\}\geqq 1 \textit{ for all } x\neq 0.$$

§4. Flatness

From the definition (i), to prove flatness, sometimes we may restrict ourselves to the two-dimensional subspace spanned by two (fixed) orthogonal elements.

Using Hahn-Banach's theorem (cf. [3, §44]) we can prove:

10) We know that (2) in Theorem 3.3 implies the continuity of the norm. The argument in [1] shows that in a non-atomic space continuity of the norm is equivalent to finiteness of the modular.

Lemma 4.1. *Let S be a normed semi-ordered linear space. In order that for (fixed) $a \in S$, $\| a \| = 1$*

$$\lim_{\varepsilon \to 0} \frac{\| a + \varepsilon b \| - 1}{\varepsilon} = 0 \qquad \text{for all } b \in S,\ a \frown b = 0,$$

it is necessary and sufficient that for any $\tilde{a} \in \tilde{S}$ with $\tilde{a}(a) = \| \tilde{a} \| = 1$, $\tilde{a}(b) = 0$ for all $b \in S$, $a \frown b = 0$.

Lemma 4.2. *A norm on a semi-ordered linear space S is flat (or monotone), if the associated norm on $\tilde{S}$ is monotone (or flat resp.).*

Theorem 4.1. *In order that the first norm be flat, it is necessary and sufficient that the following two conditions are satisfied:*

(1) *m is monotone,*

(2) $\sup_{\xi > 0} \{ \xi - m(\xi a) \} = 1$ *for all a with $\| a \| = 1$.*

Proof. Necessity. We may assume that R is two-dimensional. Since R is reflexive, by Lemma 4.2 the second norm $||| \tilde{x} |||$ by the associated modular is monotone. Hence, by Theorem 3.3 the associated modular $\tilde{m}$ on $\tilde{R}$ is simple and "$\tilde{m}(\tilde{a}) = 1$" is equivalent to "$||| \tilde{a} ||| = 1$". By Lemma C m is monotone. For any $a \in R$, $[a] < \mathrm{I}$, $\| a \| = 1$ by Hahn-Banach's theorem there exists $\tilde{a} \in \tilde{R}$ such that $\tilde{a}(a) = ||| \tilde{a} ||| = 1$, $\tilde{a} = \tilde{a}[a]$. Since R is two-dimensional, by the definition (c) we obtain

$$1 = \tilde{m}(\tilde{a}) = \sup_{\xi > 0} \{ \xi - m(\xi a) \}.$$

If $[a] = \mathrm{I}$, $||| a ||| = 1$ and $\sup_{\xi > 0} \{ \xi - m(\xi a) \} < 1$, then we have

$$\sup_{\xi > 0} \frac{m(\xi a)}{\xi} = \| a \|.$$

hence by Lemma 3.3 the first norm on R is of L^1-type and is not flat. Thus (2) holds. The proof of sufficiency proceeds along the same idea. Q.E.D.

Theorem 4.2. *In order that the first norm be uniformly flat, it is necessary and sufficient that the following two conditions are satisfied:*

(1) *m is monotone,*

(2) $\sup_{0 < \eta < 1} \inf_{\| a \| = 1} \sup_{\xi > 0} \{ \xi \eta - m(\xi a) \} = 1.$

Proof. Necessity. It is sufficient to prove (2). The second norm $||| \tilde{x} |||$ by the associated modular on $\tilde{R}$ is uniformly monotone by Lemma A, therefore by Theorem 3.4

$$(\sharp) \qquad \sup_{0 < \xi < 1} \inf_{||| \tilde{x} ||| = 1} \tilde{m}(\xi \tilde{x}) = 1.$$

Since for any $a\in R$, $\|a\|=1$ and for $1>\eta>0$ by Hahn-Banach's theorem there exists $\tilde{b}\in\tilde{R}$ such that $|||\tilde{b}|||=\tilde{b}(a)=1$, we have $\xi\eta-m(\xi a)\leqq\xi-m(\xi a)\leqq\tilde{m}(\tilde{b})\leqq 1$ for all $\xi\geqq 0$. Hence by Lemma E there exists $\tilde{a}\in\tilde{R}$ such that $|||\tilde{a}|||\geqq\tilde{a}(a)=1$,

$$\tilde{m}(\eta\tilde{a})=\sup_{\xi>0}\{\xi\eta-m(\xi a)\}.$$

Thus the condition (♯) implies

$$\sup_{0<\eta<1}\ \inf_{\|a\|=1}\ \sup_{\xi>0}\{\xi\eta-m(\xi a)\}\geqq 1.$$

The converse inequality is trivial.

Sufficiency. It is not difficult to see that the second associated modular $\tilde{\tilde{m}}$ on $\tilde{\tilde{R}}$ also satisfies

$$\sup_{0<\eta<1}\ \inf_{\|\tilde{\tilde{x}}\|=1}\ \sup_{\xi>0}\{\xi\eta-\tilde{\tilde{m}}(\xi\tilde{\tilde{x}})\}\geqq 1.$$

For any $\tilde{a}\in\tilde{R}$, $|||\tilde{a}|||=1$ by Hahn-Banach's theorem there exists $\tilde{\tilde{a}}\in\tilde{\tilde{R}}$ such that $\|\tilde{\tilde{a}}\|=\tilde{\tilde{a}}(\tilde{a})=1$. Since for any $\xi,\eta>0$, $\tilde{m}(\eta\tilde{a})\geqq\xi\eta-\tilde{\tilde{m}}(\xi\tilde{\tilde{a}})$ by (c), we obtain from the above

$$\sup_{0<\eta<1}\ \inf_{|||\tilde{a}|||=1}\tilde{m}(\eta\tilde{a})=\sup_{0<\eta<1}\ \inf_{\|\tilde{\tilde{a}}\|=1}\ \sup_{\xi>0}\{\xi\eta-\tilde{\tilde{m}}(\xi\tilde{\tilde{a}})\}=1.$$

Since the associated modular $\tilde{m}$ on $\tilde{R}$ is simple by Lemma C, Theorem 3.4 tells us that the second norm on $\tilde{R}$ is uniformly monotone, consequently the first norm on R is uniformly flat by Lemma *A*. Q.E.D.

Remark 4.1. *If R is non-atomic, in Theorem 4.2* (1) *and* (2) *together may be replaced by the single condition that m is uniformly monotone, because of the similar reason as in Remark 3.3.*

Remark 4.2. *The condition* (2) *in Theorem 4.1 can be written in terms of modulars as follows*:

$$(2')\qquad \sup_{\xi\geqq 0}\ \inf_{\eta>0}\frac{\xi m(\eta a)-\eta m(\xi a)+\xi}{\eta}\geqq 1 \qquad \text{for all } a\in R.$$

Now we turn our attention to the second norm.

Lemma 4.3. *Let $a,b\in R$, $|||a|||=1$ and $a\frown b=0$.*

(1) *If*
$$\sup_{0<\xi<1}\frac{1-m(\xi a)}{1-\xi}\equiv\gamma<+\infty,$$
there exists $\tilde{a}\in\tilde{R}$ such that $\tilde{a}(a)=\|\tilde{a}\|$ and $\tilde{a}(b)=\inf_{\xi>0}\frac{m(\xi b)}{\xi}$.

(2) *If* $\sup_{0<\xi<1}\frac{1-m(\xi a)}{1-\xi}=+\infty$, then $\lim_{\varepsilon\to 0}\frac{|||a+\varepsilon b|||-1}{\varepsilon}=0$.

Proof. (1) First we remark $m(a)=1$. By Lemma E there exists $0\leqq\tilde{a}_0\in\tilde{R}$ such that $\tilde{a}_0(a)=\gamma=1+\tilde{m}(\tilde{a}_0)$ and $\tilde{a}_0[a]=\tilde{a}_0$.
Put

$$\tilde{a}(x)=\begin{cases}\inf\limits_{\xi>0}\dfrac{m(\xi x)}{\xi} & \text{for all } 0\leqq x\in(1-[a])R,\\ \tilde{a}_0(x) & \text{for all } x\in[a]R,\end{cases}$$

and extend $\tilde{a}$ as a linear functional over all R. Then we have $\tilde{a}(a)=1+\tilde{m}(\tilde{a})$, because $\tilde{a}((\mathrm{I}-[a])x)-m(x)\leqq 0$ for all $x\in R$ implies $\tilde{m}(\tilde{a}(1-[a]))=0$ by the definition (c). Thus by Lemma 3.2 we have

$$\|\tilde{a}\|=\tilde{a}(a) \text{ and } \tilde{a}(b)=\inf_{\xi>0}\frac{m(\xi b)}{\xi}.$$

(2) Suppose that $\tilde{a}\in\tilde{R}$, $\tilde{a}(a)=\|\tilde{a}\|=1$ and $\tilde{a}(b)\neq 0$. If $\xi_0\tilde{a}(a)=\xi_0\|\tilde{a}\|=1+\tilde{m}(\xi_0\tilde{a})$ for some $\xi_0>0$, then by (c) we have $m(a)=1$ and

$$m(a)-m(\xi a)\leqq\xi_0(1-\xi)\tilde{a}(a) \text{ for all } 1\geqq\xi>0,$$

consequently
$$\sup_{0<\xi<1}\frac{1-m(\xi a)}{1-\xi}\leqq\xi_0\tilde{a}(a)<+\infty,$$

contradicting the assumption. Thus by Lemma 3.2 we have

$$\sup_{\xi>0}\frac{\tilde{m}(\xi\tilde{a})}{\xi}=\|\tilde{a}\|,$$

hence by Lemma 3.3

$$1=\|\tilde{a}\|=\|\tilde{a}[a]\|+\|\tilde{a}(1-[a])\|>\|\tilde{a}[a]\|=1,$$

because
$$\|\tilde{a}(1-[a])\|\geqq\frac{\tilde{a}(b)}{|||b|||}>0.$$

This is a contradiction. By Lemma 4.1 we obtain

$$\lim_{\varepsilon\to 0}\frac{|||a+\varepsilon b|||-1}{\varepsilon}=0. \qquad \text{Q.E.D.}$$

As is well-known, there exists the decomposition of R:

$$R=R_3\oplus R_4,$$

where R_i ($i=3,4$) are mutually orthogonal normal manifold such that m is monotone on R_3 and is ascending[11] on R_4.

Now suppose that the second norm on R is flat and $R_4\neq\{0\}$. If $a\in R$ and $(\mathrm{I}-[a])R_4\neq\{0\}$, there exists $b>0$ such that

$$a\frown b=0 \text{ and } \inf_{\xi>0}\frac{m(\xi b)}{\xi}>0.$$

11) R is said to be ascending, if for any $0<a\in R$ there exists $0<b\in R$ such that $b\leqq a$ and $\inf\limits_{\xi>0}\dfrac{m(\xi b)}{\xi}>0$.

If furthermore $|||a|||=1$ and

$$\sup_{0<\xi<1} \frac{1-m(\xi a)}{1-\xi} < +\infty,$$

then by Lemma 4.3 there exists $\tilde{a} \in \tilde{R}$ such that

$$\tilde{a}(a) = ||\tilde{a}|| = 1 \quad \text{and} \quad \tilde{a}(b) = \inf_{\xi>0} \frac{m(\xi b)}{\xi} > 0,$$

contradicting flatness of the second norm by Lemma 4.1. Thus we have proved that

$$\sup_{0<\xi<1} \frac{1-m(\xi a)}{1-\xi} = +\infty,$$

consequently by virtue of convexity of the real function $m(\xi x)$, $(1-[x])R_4 \neq \{0\}$ $m(x) < +\infty$ implies $m(x) \leqq 1$. If R_4 is infinite-dimensional, for any $x \in R$, $m(x) < +\infty$, there exists a family of projectors $[p_\lambda]_{\lambda \in \Lambda}$ such that $[p_\lambda] \uparrow_{\lambda \in \Lambda} [x]$ and $(1-[p_\lambda])R_4 \neq \{0\}$, consequently by (p)

$$m(x) = \sup_{\lambda \in \Lambda} m([p_\lambda]x) \leqq 1.$$

Thus we have proved the "necessity" part of the following theorem.

Theorem 4.3. *In order that the second norm be flat, it is necessary and sufficient that one of the following conditions is satisfied*:

(1) $R_4 = \{0\}$ *i.e.* m *is monotone,*

(2) $\sup_{m(x)<+\infty} m(x) \leqq 1$,

(3) $0 < \dim(R_4) < +\infty$ *and* $\sup_{0<\xi<1} \dfrac{1-m(\xi x)}{1-\xi} = +\infty$ *for all* x *with* $(1-[x])R_4 \neq \{0\}$.

Proof of sufficiency. Let $a \frown b = 0$ and $|||a||| = |||b||| = 1$. Suppose (1) to be valid. Then for any $0 < \varepsilon < 1$ by Lemma 3.1 we have $|||a+\varepsilon b||| \leqq 1$ or $|||a+\varepsilon b||| \leqq m(a+\varepsilon b)$, consequently $\lim_{\varepsilon \to 0} \dfrac{|||a+\varepsilon b|||-1}{\varepsilon} \leqq \inf_{\xi>0} \dfrac{m(\xi b)}{\xi} = 0$. If (2) is valid, it is easy to see that the second norm is of L^∞-type, i.e. $|||a \smile b||| = \operatorname{Max}\{|||a|||, |||b|||\}$ for all $0 \leqq a, b \in R$, a fortiori uniformly flat. Finally suppose (3) to be valid. If $(1-[a])R_4 \neq \{0\}$, then by Lemma 4.3

$$\lim_{\varepsilon \to 0} \frac{|||a+\varepsilon b|||-1}{\varepsilon} = 0.$$

If $(1-[a])R_4 = \{0\}$, then $b \in R_3$ and as in case (1)

$$\lim_{\varepsilon \to 0} \frac{|||a+\varepsilon b|||-1}{\varepsilon} = 0. \qquad \text{Q.E.D.}$$

Now suppose that the second norm is uniformly flat. If (1) in Theorem 4.3 halds, m is uniformly monotone by Lemmas A, B and D. If (2) halds, as above the second norm is of L^{∞}-type. Since the first norm $\|\tilde{x}\|$ by the associated modular $\tilde{m}$ on $\tilde{R}$ is uniformly monotone, by Theorem 3.2, in case (3) for any $\gamma>2$ there exists $1>\delta>0$ such that

$$(**) \qquad \sup_{0\leqq\xi\leqq\gamma} \sup_{\substack{\|\tilde{x}\|=1 \\ (I-[\tilde{x}])[R_4]\neq 0}} \{\xi(1+\delta)-\tilde{m}(\xi\tilde{a})\}\leqq 1.$$

If $|||a|||=1$, $(I-[a])R_4\neq\{0\}$, by Lemma E for any $\frac{1}{2}>\varepsilon>0$ there exists $\tilde{a}\in\tilde{R}$ such that $\|\tilde{a}\|=1$ and $m((1-\varepsilon)a)+\tilde{m}(\xi_0\tilde{a})=\xi_0(1-\varepsilon)\tilde{a}(a)$ for some $\xi_0>0$, consequently $\quad \dfrac{1-m((1-\varepsilon)a)}{\varepsilon}\geqq\dfrac{1+\tilde{m}(\xi_0\tilde{a})-\xi_0(1-\varepsilon)}{\varepsilon}$.

Then, if $0\leqq\xi_0\leqq\gamma$, by $(**)$ we have $\quad \dfrac{1-m((1-\varepsilon)a)}{\varepsilon}\geqq\dfrac{\delta}{\varepsilon}$, and

if $\xi_0\geqq\gamma$, we have $\quad \dfrac{1-m((1-\varepsilon)a)}{\varepsilon}\geqq\xi_0\xi_0\tilde{a}(a)\geqq\dfrac{\gamma}{2}$.

Since $\gamma>2$ and $\frac{1}{2}>\varepsilon>0$ are arbitrary, we can conclude

$$\sup_{0<\xi<1} \inf_{\substack{|||x|||=1 \\ (I-[x])R_4\neq\{0\}}} \frac{1-m(\xi x)}{1-\xi}=+\infty.$$

Thus we have proved the "necessity" part of the following theorem.

Theorem 4.4. *In order that the second norm be uniformly flat, it is necessary and sufficient that one of the following conditions is satisfied:*

(1) m *is uniformly monotone,*

(2) $\sup_{m(x)<+\infty} m(x)\leqq 1$,

(3) $0<\dim(R_4)<+\infty$ *and* m *is uniformly monotone on* R_3, *and*

$$\sup_{0<\xi<1} \inf_{\substack{|||x|||=1 \\ (I-[x])R_4\neq\{0\}}} \frac{1-m(\xi x)}{1-\xi}=+\infty.$$

Proof of sufficiency. The proof for the case (1) was given in [2, §48]. If (2) holds, the second norm is of L^{∞}-type, a fortiori uniformly flat. Finally, suppose (3) holds. It is sufficient to prove uniform monotony of the first norm by the associoted modular on $\tilde{R}$ by virtue of Lemma *A*. For this purpose it is sufficient to prove $(**)$, because the associated modular $\tilde{m}$ is uniformly simple on $[R_3]\tilde{R}$ by Lemma B and (3). Considering $\tilde{\tilde{R}}$, we can easily

prove that for any $\gamma>0$ there exists $0<\eta_0<1$ such that $1-\tilde{\tilde{m}}(\eta_0\tilde{\tilde{x}})\geqq 2\gamma(1-\eta_0)$ for all $\tilde{\tilde{x}}\in\tilde{\tilde{R}}$ with $(\mathrm{I}-[\tilde{\tilde{x}}])R_4\neq\{0\}$, $|||\tilde{\tilde{x}}|||=1$. By Hahn-Banach's theorem for any $\tilde{a}\in\tilde{R}$ with $(\mathrm{I}-[\tilde{a}])[R_4]\neq 0$ $\|\tilde{a}\|=1$, there exists $\tilde{\tilde{a}}\in\tilde{\tilde{R}}$ such that $(\mathrm{I}-[\tilde{\tilde{a}}])[R_4]\neq 0$, $|||\tilde{\tilde{a}}|||=\tilde{\tilde{a}}(\tilde{a})=1$, consequently by the definition (c) we have $\eta_0\xi-\tilde{m}(\xi\tilde{a})\leqq\tilde{\tilde{m}}(\eta_0\tilde{\tilde{a}})\leqq 1-2\gamma(1-\eta_0)$ for all $\xi\geqq 0$.
Finally, we obtain

$$\begin{aligned} 1 &\geqq \xi\left(\eta_0+\frac{2\gamma}{\xi}(1-\eta_0)\right)-\tilde{m}(\xi\tilde{a}) \\ &\geqq \xi(2-\eta_0)-\tilde{m}(\xi\tilde{a}) \qquad \text{for all } 0\leqq\xi\leqq\gamma. \end{aligned}$$

Putting $\delta=1-\eta_0$, we obtain (**). Q.E.D.

Corollary. *If the first norm is (uniformly) flat, then the second norm is also (uniformly) flat.*

Remark 4.3. *If R is non-atomic, the conditions (3) in Theorems 4.3–4 disappear.*

§5. Some Comments on Uniformly Simple Modulars

As is seen in the foregoing sections, monotony of the norms is closely connected with simplicity of the modular. On the other hand, Yamamuro's theorem (cf. Lemma D) shows that if a modular is simple and the norm is continuous on the monotone completion[12] of R, then it is uniformly simple. In this section we shall give some conditions which (together with simplicity) imply uniform simplicity.

Lemma 5.1. *Each of the following conditions implies the continuity of the norm on the monotone completion $\bar{\bar{R}}$ of R:*

(A) $$\inf_{|||x|||=1} m(\xi x)=\varepsilon>0 \qquad \textit{for some } 0<\xi<1,$$

(B) $$\sup_{\|x\|=1} m(\xi x)=\gamma<+\infty \qquad \textit{for some } \xi>1.$$

Proof. First we shall show that (A) implies continuity of the norm on R. Suppose that $a_\nu\downarrow_{\nu=1}^\infty 0$ and $\inf_{\nu\geqq 1}|||a_\nu|||=\alpha$. If $\alpha>0$, choose $\delta>0$ such that

$$1>\frac{\alpha}{\alpha+\delta}\geqq\xi.$$

Since $$\frac{a_\nu}{\alpha+\delta}\downarrow_{\nu=1}^{\infty} 0 \text{ and } 1>\inf_{\nu\geqq 1}\left|\left|\left|\frac{a_\nu}{\alpha+\delta}\right|\right|\right|=\frac{\alpha}{\alpha+\delta}\geqq\xi,$$

12) In this case the monotone completion coincide withs $\bar{\bar{R}}$.

by (A) we have for some μ

$$1 > \left\| \frac{a_\nu}{\alpha+\delta} \right\| \geqq m\left(\frac{a_\nu}{\alpha+\delta}\right) \geqq \varepsilon \qquad \text{for all } \nu \geqq \mu.$$

On the other hand, by (p) we have

$$\lim_{\nu\to\infty} m\left(\frac{a_\nu}{\alpha+\delta}\right) = 0.$$

Thus α must be equal to 0, i.e. the norm is continuous. Next we shall show that (A) holds on the monotone completion $\bar{\bar{R}}$ of R with a slightly changed ξ'. For any $0 \leqq \bar{\bar{a}} \in \bar{\bar{R}}$, $|||\bar{\bar{a}}||| = \xi'$ (where $\xi < \xi' < 1$), there exists $\{a_\lambda\} \subset R$ such that $0 \leqq a_\lambda \uparrow_{\lambda\in\Lambda} \bar{\bar{a}}$, hence $\sup_{\lambda\in\Lambda} |||a_\lambda||| = |||\bar{\bar{a}}||| = \xi'$, consequently by (A) and (p) we have $\bar{\bar{m}}(\bar{\bar{a}}) = \sup_{\lambda\in\Lambda} m(a_\lambda) \geqq \varepsilon$.

The proof for the condition (B) is similar. Q.E.D.

Lemma 5.2. *In order that m satisfy the condition*

(A) $$\inf_{|||x|||=1} m(\xi x) = \varepsilon > 0 \qquad \textit{for some } 0 < \xi < 1,$$

it is necessry and sufficient that $\tilde{m}$ satisfies the condition

(A*) $$\sup_{\|\tilde{x}\|=1} \tilde{m}(\tilde{x}) = \xi' < 1.$$

In order that m satisfy the condition

(B) $$\sup_{\|x\|=1} m(\xi x) = \gamma < +\infty \qquad \textit{for some } \xi > 1,$$

it is necessary and sufficient that $\tilde{m}$ satisfies the condition

(B*) $$\inf_{|||\tilde{x}|||=1} \frac{\tilde{m}(\gamma\tilde{x})}{\gamma} = \xi' > 1 \qquad \text{for some } \gamma > 0.$$

Proof. Suppose that m on R satisfies (A).
Let $\tilde{a} \in \tilde{R}$ $\|\tilde{a}\| = 1$, then

$$\tilde{a}(x) - m(x) \leqq \begin{cases} 0 & \text{for } |||x||| > 1, \\ 1-\varepsilon & \text{for } 1 \geqq |||x||| \geqq \xi, \\ \xi & \text{for } |||x||| \leqq \xi, \end{cases}$$

consequently we have

$$\tilde{m}(\tilde{a}) \leqq \max\{1-\varepsilon, \xi\} < 1.$$

Thus the associated modular $\tilde{m}$ satisfies (A*).

Conversely suppose that m satisfies (A*). If $\tilde{a} \in \tilde{R}$, $|||\tilde{a}||| = 1$, then by (g) there exists $a \in R$ such that $\|a\| = 1$ and $\tilde{a}(a) \geqq \frac{1+\xi'}{2}$.

If $1>\alpha>\dfrac{2\xi'}{1+\xi'}$, then we have

$$\alpha\tilde{a}(a)-m(a)\geqq\frac{\alpha(1+\xi')-2\xi'}{2}>0,$$

consequently $\inf_{|||\tilde{a}|||=1}\tilde{m}(\alpha\tilde{a})\geqq\dfrac{\alpha(1+\xi')-2\xi'}{2}>0$. Thus $\tilde{m}$ satisfies (A).

The proof of the relation between (B) and (B*) is similar. Q.E.D.

Theorem 5.1. *If m is simple, the following conditions are mutually equivalent*:

(1) *The first norm is uniformly monotone,*

(2) *m is uniformly simple,*

(3) *m satisfies* (A),

(4) *m satisfies* (B).

Proof. (1)$\leftrightarrow$(2) follows from Theorem 3.2 and Lemma D. (2)$\to$(3) is trivial. (3) or (4)$\to$(2) follows from Lemma 5.1 and Lemma D. (2)$\to$(4): By (f) (2) implies

$$\inf_{\|x\|=1}m(x)\geqq\inf_{|||x|||=1}m\left(\frac{1}{2}x\right)=\delta>0.$$

Let $\|a\|=1$. If $1+m(\eta a)=\eta$ for some $\eta>1$, then $\eta>1+m(a)\geqq1+\delta$ by the above, consequently

$$\frac{m((1+\delta)a)}{1+\delta}\leqq\frac{m(\eta a)+1}{\eta}=\|a\|=1.$$

If such η does not exists, by Lemma 3.2 we have

$$1=\|a\|=\sup_{\xi>0}\frac{m(\xi a)}{\xi}.$$

Thus in any case we have $\sup_{\|x\|=1}m((1+\delta)x)\leqq1+\delta$. Q.E.D.

Using Lemmas A, B and Lemma 5.2 we obtain:

Theorem 5.2. *If m is monotone, the following conditions are mutually equivalent*:

(1) *the second norm is uniformly flat,*

(2) *m is uniformly monotone,*

(3) *m satisfies* (A*),

(4) *m satisfies* (B*).

Finally we shall state a remark for the result of T. Shimogaki [5].

Theorem 5.3. *The following conditions are mutually equivalent*:

(1) $$\inf_{0 \neq x \in R} \frac{\|x\|}{|||x|||} = \gamma > 1,$$

(2) *m satisfies* (B) *and* (A*),

(3) *m satisfies* (B*) *and* (A).

Proof. (1)$\rightarrow$(2). (1) is equivalent to $\sup_{\|x\|=1} m(\gamma x) \leqq 1$. Hence this implies $\sup_{\|x\|=1} m(x) \leqq \frac{1}{\gamma} < 1$. Thus (1) implies (2). (2)$\rightarrow$(1). Let $\varphi(\xi) = \sup_{\|x\|=1} m(\xi x)$ for all $\xi \geqq 0$. Then $\varphi(\xi)$ is a convex function of $\xi \geqq 0$. (2) implies that $\varphi(1) < 1$ and $\varphi(\xi_0) < +\infty$ for some $\xi_0 > 1$. Convexity of φ implies $\varphi(\gamma) \leqq 1$ for some $\gamma > 1$. As above this is equivalent to (1).

The proof of the equivalence between (1) and (3) is similar. Q.E.D.

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