CONVEXITY AND EVENNESS IN MODULARED SEMI-ORDERED LINEAR SPACES

By

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CONTENTS

Introduction

Let R be a universally continuous semi-ordered linear space[1]. A functional m on R is called a *modular*, if it satisfies the following modular conditions:

(1) $\quad 0 \leqq m(a) \leqq \infty \quad$ for all $\quad a \in R$;

(2) $\quad$ if $\quad m(\xi a)=0 \quad$ for all $\quad \xi \geqq 0, \quad$ then $\quad a=0$;

(3) $\quad$ for any $\quad a \in R \quad$ there exists $\quad \alpha>0 \quad$ such that $\quad m(\alpha a)<\infty$;

(4) $\quad$ for every $a \in R, \quad m(\xi a) \quad$ is a *convex* function of ξ;

(5) $\quad |a| \leqq |b| \quad$ implies $\quad m(a) \leqq m(b)$;

(6) $\quad a \frown b=0 \quad$ implies $\quad m(a+b)=m(a)+m(b)$;

(7) $\quad 0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda} a \quad$ implies $\quad \sup_{\lambda \in \Lambda} m(a_\lambda)=m(a)$.

When a modular m is defined on R, R is called a *modulared semi-ordered linear space* with the modular m and is denoted by (R, m), if necessary. We can define two kinds of norms on R by the formulas:

1) We use mainly notation and terminology of [12], [13].

$$(8) \qquad \|a\| = \inf_{\xi>0} \frac{1+m(\xi a)}{\xi},$$

$$(9) \qquad |||a||| = \inf_{m(\xi a)\leq 1} \frac{1}{|\xi|}.$$

The former norm is called the *first* one by m and the latter the *second* (or sometimes *modular*-) one by m. Among the important problems in the theory of modulared semi-ordered linear spaces are to investigate relations between properties of a modular and those of its norms and to investigate order-structure of the space. In this paper we shall turn our attention mainly to *convexity* and *evenness* (for the definitions, see below).

In Chapter I we shall concern ourselves with the question whether (uniform) convexity of the norm implies (uniform) convexity of the modular. We can give satisfactory answers in the case R is non-atomic (Theorems 2.1, 2.2 and 3.4). In connection with convexity we shall treat also evenness. Recently H. W. Milnes [10] investigated similar problems in the case of Orlicz spaces[2]. Since Orlicz spaces are special concrete examples of modulared semi-ordered linear spaces, our results in Chapter I are considered as an extension of H. W. Milnes' ones to modulared semi-ordered linear spaces, and are more complete, because he did not treat the second norm and evenness.

In Chapter II we shall investigate conditions under which a given modular can be converted to a convenient one (for example, strictly convex, uniformly convex etc.) and give some necessary and sufficient conditions in terms of topological or order structure of the space. Our standpoint is similar to that of M. M. Day [8], and we shall solve completely questions raised by him, so far as modulared semi-ordered linear spaces are considered. In fact, we shall give topological conditions under which the modular norm is equivalent to a strictly convex (even, uniformly convex etc.) one (Theorems 6.4, 6.6 and 7.4). Among the important results is that if a modulared semi-ordered linear space is reflexive as a Banach space, then its modular norm is equivalent to a uniformly convex one (Theorem 7.4).

In connection with the above result, a conjecture arises that a semi-ordered linear space with a uniformly convex norm is *modularable* i.e. its norm is equivalent to a modular norm by some modular. In Appendix, we shall give a negative answer to this conjecture.

In the remainder of this Introduction, we shall state definitions and results used later from the theory of modulared semi-ordered linear spaces

2) For Orlicz spaces, see [17].

developed by H. Nakano in [12, 13].

$\tilde{R}$ and $\overline{R}$ denote the totality of all linear functionals and that of all universally continuous linear functionals on R respectively, which are bounded under the norm. On $\tilde{R}$ the *associated modular* $\tilde{m}$ of m is defined by the formula:

$$(10) \qquad \tilde{m}(\tilde{a})=\sup_{x\in R}\{\tilde{a}(x)-m(x)\} \qquad \text{for } \tilde{a}\in\tilde{R}.$$

$\tilde{m}$ satisfies all the modular conditions (see [12; §38]), and

$$(11) \qquad \tilde{a}(a)\leqq m(a)+\tilde{m}(\tilde{a}) \qquad \text{for all } a\in R,\ \tilde{a}\in\tilde{R}.$$

When we consider the associated modular only on $\overline{R}$, we call it the *conjugate* modular of m and denote it by $\overline{m}$. In this paper *projectors* $[p]$ and *projection operators* $[N]$ are frequently used (for the definition see [12; §§ 5~6]

In this paper we always assume *semi-regularity* of R, i.e. for any $0\neq a\in R$ there exists $\overline{a}\in\overline{R}$ such that $\overline{a}(a)\neq 0$. By semi-regularity the following formulas are valid (see [11], [12; §39–40] and [13; §83])

$$(12) \qquad m(a)=\sup_{\overline{x}\in\overline{R}}\{\overline{x}(a)-\overline{m}(\overline{x})\} \qquad \text{for all } a\in R;$$

$$(13) \qquad \|a\|=\sup_{\overline{m}(\overline{x})\leqq 1}|\overline{x}(a)| \qquad \text{for all } a\in R.$$

Two norms satisfy always (see [12; §40])

$$(14) \qquad |||a|||\leqq\|a\|\leqq 2|||a||| \qquad \text{for all } a\in R,$$

hence they are equivalent. The first norm and the second one by the conjugate modular $\overline{m}$ are denoted by $\|\cdot\|$ and $|||\cdot|||$ respectively. Then we have

$$(15) \qquad \|\overline{a}\|=\sup_{|||x|||\leqq 1}|\overline{a}(x)|, \quad |||\overline{a}|||=\sup_{\|x\|\leqq 1}|\overline{a}(x)| \qquad \text{for all } \overline{a}\in\overline{R}.$$

An element $a\in R$ is said to be *finite*, if $m(\xi a)<\infty$ for all $\xi\geqq 0$. A modular m is said to be *almost finite* or *finite*, according as the totality of all finite elements constitutes a complete semi-normal manifold of R or is identical with R itself. An element $a\in R$ is said to be *domestic*, if $m(\gamma a)<\infty$ for some $\gamma>1$. An element $a\in R$ is said to be *simple*, if $m([p]a)=0$ implies $[p]a=0$. If all elements of R are simple, m is said to be *simple*. m is said to be *increasing* or *monotone*, according as $\sup_{\xi>0}\frac{m(\xi a)}{\xi}=\infty$ or $\inf_{\xi>0}\frac{m(\xi a)}{\xi}=0$ for all $0\neq a\in R$. m is said to be *continuous*, if for any $a\in R$ with $m(a)=\infty$ $\sup_{\substack{|x|\leqq|a|\\ m(x)<\infty}} m(x)=\infty$. m is said to be *infinitely increasing*, if for every $0\neq a\in R$,

$$\sup_{\xi>0}\frac{m(\xi a)}{\xi}=\infty \quad \text{or} \quad \sup_{\xi>0}\{\xi\gamma-m(\xi a)\}=\infty \quad \text{where} \quad \gamma=\sup_{\xi>0}\frac{m(\xi a)}{\xi}.$$

As relations between these properties, we know (see [12; §§42–46], [13; §84]).

Theorem A.

(A_1) *Almost finiteness and increasingness are conjugate*[3].

(A_2) *Simplicity and monotony are conjugate, and further if m is monotone, the associated modular $\tilde{m}$ is simple.*

(A_3) *Continuity and infinite increasingness are conjugate.*

The properties defined above are concerned with *individual* convex functions $m(\xi x)$. When we require some *uniformity*, we use the following *modular functions:*

$$\omega(\xi\,|\,a)=m\left(\frac{\xi a}{|||a|||}\right) \qquad \text{for } 0\neq a\in R. \tag{16}$$

Then m is said to be *uniformly finite* or *uniformly simple*, according as

$$\sup_{x\neq 0}\omega(\xi\,|\,x)<\infty \quad \text{or} \quad \inf_{x\neq 0}\omega(\xi\,|\,x)>0 \qquad \text{for all } \xi>0.$$

m is said to be *uniformly increasing* or *uniformly monotone*, according as

$$\sup_{\xi>0}\inf_{x\neq 0}\frac{\omega(\xi\,|\,x)}{\xi}=\infty \quad \text{or} \quad \inf_{\xi>0}\sup_{x\neq 0}\frac{\omega(\xi\,|\,x)}{\xi}=0.$$

m is said to be *upper bounded*, if there exists $\gamma>0$ such that

$$m(2a)\leqq\gamma m(a) \qquad \text{for all } a\in R,$$

and to be *lower bounded*, if there exist $\alpha>\beta>1$ such that

$$m(\beta a)\geqq\alpha m(a) \qquad \text{for all } a\in R.$$

m is said to be *bounded*, if it is upper bounded and lower bounded at the same time. Upper boundedness implies uniform simplicity and uniform finiteness, similarly lower boundedness implies uniform increasingness and uniform monotony (see [12; §49]). As relations between these uniform properties, we known (see [12; §49] and [13; §§84–85]):

Theorem B.

(B_1) *Uniform finiteness and uniform increasingness are associated*[3].

(B_2) *Uniform simplicity and uniform monotony are associated.*

(B_3) *Upper boundedness and lower boundedness are associated.*

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3) General properties P and Q are said to be *conjugate* or *associated*, according as "R possesses P" is equivalent to "$\bar{R}$ possesses Q" or to "$\tilde{R}$ does Q".

When R is semi-regular, by (12) we can imbed R into $\overline{\overline{R}}$ preserving the order structure and the modular structure. If there is no confusion, elements of R, considered in $\overline{\overline{R}}$, are denoted by the same symbols.

Let S be a universally continuous semi-ordered linear space. $S \ni a > 0$ is called an *atom*, if $a \geqq b \geqq 0$ implies $b = \alpha a$ for some $\alpha \geqq 0$. If the totality D fo all atoms constitutes a complete manifold, then S is said to be *atomic*. If S contains no atom, then it is called *non-atomic*. We call $S_d (\equiv [D]S)$ the *atomic part* and $S_c (\equiv S_d^{\perp})$ the *non-atomic part* of S respectively.

When we call $(S, \|.\|)$ a normed semi-ordered linear space, the norm is assumed to satisfy the condition:

$$(\bigstar) \qquad |a| \leqq |b| \quad \text{implies} \quad \|a\| \leqq \|b\| .$$

$(S, \|.\|)$ is said to be *monotone complete*, if for any $0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda}$ with $\sup_{\lambda \in \Lambda} \|a_\lambda\| < \infty$ there exists $S \ni a = \bigcup_{\lambda \in \Lambda} a_\lambda$. A norm is said to be *continuous*, if $a_\nu \downarrow_{\nu=1}^{\infty} 0$ implies $\lim_{\nu \to \infty} \|a_\nu\| = 0$. Concerning a normed semi-ordered linear space we know (see [12; §§ 30–31]):

Theorem C.

(C_1) *The conjugate space* $(\overline{S}, \|\bar{.}\|)$ *and the associated space* $(\widetilde{S}, \|\tilde{.}\|)$ *are always monotone complete.*

(C_2) *A norm on* S *is continuous, if and only if* $\overline{S} = \widetilde{S}$.

(C_3) *If* S *is complete under both two norms, then they are equivalent.*

Chapter I. Convexity and Evenness of the Norms

§ 1. Some preliminary lemmas

Throughout this Chapter R denotes a modulared semi-ordered linear space with a modular m. Since $m(\xi a)$ is a convex function of $\xi \geqq 0$, it is natural to define strict convexity as follows: m is said to be *strictly convex*, if

$$(\mathrm{s}) \quad a \neq 0 \quad \alpha \geqq \beta \geqq 0 \quad m(\alpha a) < \infty, \quad m\left(\frac{\alpha+\beta}{2} a\right) = \frac{m(\alpha a) + m(\beta a)}{2}$$

$$\text{implies} \qquad \alpha = \beta .$$

We can easily prove that (s) is equivalent to

$$(\mathrm{s}') \quad m(a) < \infty \quad m(b) < \infty \quad m\left(\frac{a+b}{2}\right) = \frac{m(a) + m(b)}{2} \quad \text{implies} \quad a = b .$$

From the definition, strict convexity implies simplicity.

Before defining a conjugate type of strict convexity, we introduce the following notations:

(17) $$\pi_-(a)=\begin{cases}\lim\limits_{\varepsilon\downarrow 0}\dfrac{m(a)-m((1-\varepsilon)a)}{\varepsilon} & \text{if } m(a)<\infty,\\ \infty & \text{if } m(a)=\infty,\end{cases}$$

$$\pi_+(a)=\begin{cases}\lim\limits_{\varepsilon\downarrow 0}\dfrac{m((1+\varepsilon)a)-m(a)}{\varepsilon} & \text{if } m(a)<\infty,\\ \infty & \text{if } m(a)=\infty.\end{cases}$$

These limits exist because of convexity of $m(\xi a)$. Now m is said to be *even*, if it is monotone and satisfies the following conditions:

(e) $$\begin{cases}\pi_+(a)<\infty & \text{implies} \quad \pi_+(a)=\pi_-(a);\\ \pi_-(a)<\infty & \text{implies} \quad \pi_+([p]a)<\infty \quad \text{for some } 0\neq[p]\leqq[a].\end{cases}$$

In order to obtain the conjugate type, the following formulas play an important rôle:

(18) $$m(a)+\overline{m}(\overline{a})=\overline{a}(a) \quad \text{implies} \quad m([p]a)+\overline{m}([p]\overline{a})=\overline{a}([p]a) \quad \text{for all } p\in R;$$

(19) $$m\left(\frac{a+b}{2}\right)=\frac{m(a)+m(b)}{2} \quad \text{implies}$$
$$m\left(\frac{[p]a+[p]b}{2}\right)=\frac{m([p]a)+m([p]b)}{2} \quad \text{for all } p\in R.$$

These are immediate consequences of (4), (6) and (11).

H. Nakano [12; Theorem 39.1] proved that for any domestic $0\leqq a\in R$ there exists $\overline{a}\in\overline{R}$ such that

$$m(a)+\overline{m}(\overline{a})=\overline{a}(a)$$

and $$\overline{a}([p]a)=\pi_+([p]a) \quad \text{for all } p\in R.$$

We use the following generalized form:

Lemma 1.1. *For any* $0\leqq a\in R$ $\pi_-(a)<\infty$ *there exists* $\overline{a}\in\overline{R}$ *such that*

$$\overline{a}([p]a)=\pi_-([p]a) \quad \textit{for all } p\in R$$

and

$$m(a)+\overline{m}(\overline{a})=\overline{a}(a).$$

Proof. Under the assumption $\pi_-(a)<\infty$, by (6) we can see without difficulty that

(20) $$\pi_-([p]a)+\pi_-([q]a)=\pi_-([p]a+[q]a) \quad \text{for } p,q \quad p\frown q=0.$$

Put $$\overline{a}([p]a)\equiv\pi_-([p]a) \quad \text{for all } p\in R.$$

Then by (4) we have $m([p]a)-m(\xi[p]a)\leqq(1-\xi)\overline{a}([p]a)$ for all ξ,

consequently $\xi\bar{a}([p]a)-m(\xi[p]a)\leqq\bar{a}([p]a)-m([p]a)\leqq\bar{a}(a)-m(a)$.
Hence extending $\bar{a}$ for so-called *step-elements*, we define

$$\bar{a}\Big(\sum_{\nu=1}^{\kappa}\xi_\nu[p_\nu]a\Big)\equiv\sum_{\nu=1}^{\kappa}\xi_\nu\bar{a}([p_\nu]a)\ \text{ for } \xi_\nu\geqq 0\ \ p_\nu\frown p_\mu=0\ \ (\nu,\ \mu=1,2,\cdots,\kappa).$$

Then by (6) and (20) we have

$$\bar{a}(x)-m(x)\leqq\bar{a}(a)-m(a) \qquad \text{for all step-element } x.$$

For any $0\leqq b\in[a]R$ there exist step-elements $\{x_\lambda\}_{\lambda\in\Lambda}$ such that $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda}b$.
Extend $\bar{a}$ for b by

$$\bar{a}(b)\equiv\sup_\lambda\bar{a}(x_\lambda)$$

(this is possible, because by (3) $m(\alpha x_\lambda)\leqq m(\alpha b)<\infty$ and

$$\alpha\bar{a}(x_\lambda)\leqq m(\alpha x_\lambda)+\bar{a}(a)-m(a) \quad \text{for some } \alpha>0).$$

Now extend $\bar{a}$ over all R by the formula:

$$\bar{a}(x)\equiv\bar{a}([a]x^+)-\bar{a}([a]x^-) \qquad \text{for all} \quad x\in R.$$

Then by (7) and (20) we have $\bar{a}\in\bar{R}$ and

$$\bar{a}(x)-m(x)\leqq\bar{a}(a)-m(a) \qquad \text{for all} \quad x\in R,$$

consequently from the definition (10)

$$\bar{m}(\bar{a})=\bar{a}(a)-m(a). \qquad \text{Q.E.D.}$$

Lemma 1.2. *In order that m be even, it is necessary and sufficient that for any $0\leqq a\in R$ $\pi_-(a)<\infty$ there exists uniquely $\bar{a}\in\bar{R}$ such that*

$$(\Delta) \qquad m(a)+\bar{m}(\bar{a})=\bar{a}(a).$$

Proof. Necessity. For a and $\bar{a}$ satisfying together (Δ) and for any $\varepsilon>0$ and $x\in R$, from (11) we obtain

$$\bar{a}(a\pm\varepsilon x)\leqq m(a\pm\varepsilon x)+\bar{m}(\bar{a})=\bar{a}(a)-m(a)+m(a\pm\varepsilon x),$$

hence

$$\frac{m(a)-m(a-\varepsilon x)}{\varepsilon}\leqq\bar{a}(x)\leqq\frac{m(a+\varepsilon x)-m(a)}{\varepsilon}.$$

Finally we obtain from the definition (17)

$$(21) \qquad \pi_-([p]a)\leqq\bar{a}([p]a)\leqq\pi_+([p]a) \qquad \text{for all} \quad p\in R.$$

Now if a is domestic, by (e)

$$\pi_-([p]a)=\bar{a}([p]a)=\pi_+([p]a) \qquad \text{for all} \quad p\in R.$$

Monotony of m implies $\qquad \bar{a}(x)=0 \qquad$ for all $x\in(I-[a])R$.
Since $\bar{a}$ is universally continuous, it is uniquely determined, when a is domestic. If a is not domestic, from the definition (e) and $\pi_-(a)<\infty$ there exist $\{a_\lambda\}_{\lambda\in\Lambda}\subseteqq R$ such that $[a_\lambda]\uparrow_{\lambda\in\Lambda}[a]$ $\pi_+([a_\lambda]a)<\infty$ $(\lambda\in\Lambda)$.

Then by (18) a_λ and $\bar{a}[a_\lambda]$ together satisfy (Δ) $(\kappa \in \Lambda)$, hence by the above method, $\bar{a}([a_\lambda]+(I-[a_\lambda]))$ $(\lambda \in \Lambda)$ are determined uniquely, consequently so $\bar{a}$ is.

Sufficiency. By Lemma 1.1 and the result of H. Nakano stated above, we have $\pi_+(a) = \pi_-(a)$ for all domestic $a \in R$.

If $\pi_-(a) < \infty$, from the assumption there exists *uniquely* $\bar{a} \in \bar{R}$ such that

$$m(a)+\bar{m}(\bar{a})=\bar{a}(a).$$

If $\pi_+([p]a)=\infty$ for all $0 \neq [p] \leqq [a]$,

$$\begin{aligned}\bar{m}(2\bar{a}) &= \sup_{x \in R}\{2\bar{a}(x)-m(x)\}\\ &=\sup_{x \in R}\{\bar{a}([a]x)+(\bar{a}([a]x)-m(x))\}=2\bar{a}(a)-m(a)\end{aligned}$$

because $m(x)<\infty$ implies $[a]x \leqq |a|$. This contradicts uniqueness of $\bar{a}$. The proof of monotony is easy. Q.E.D.

Now we shall state a relation between strict convexity and evenness.

Lemma 1.3. *Strict convexity and evenness are conjugate.*

Proof. Let m be strictly convex. If for $0 \leqq \bar{a} \in \bar{R}$ $\pi_-(\bar{a}) < \infty$ there exist $\bar{\bar{a}}_1, \bar{\bar{a}}_2 \in \bar{\bar{R}}$ such that $\bar{m}(\bar{a})+\bar{\bar{m}}(\bar{\bar{a}}_1)=\bar{\bar{a}}_1(\bar{a})$, and $\bar{m}(\bar{a})+\bar{\bar{m}}(\bar{\bar{a}}_2)=\bar{\bar{a}}_2(\bar{a})$, then there exist $\{p_\lambda\}_{\lambda \in \Lambda} \subseteqq R$ such that

$$[p_\lambda]_{\lambda \in \Lambda} \uparrow I \quad \text{and} \quad [p_\lambda]\bar{\bar{a}}_\nu \in R \qquad (\nu=1, 2;\ \lambda \in \Lambda),$$

because R is a complete semi-normal manifold of $\bar{\bar{R}}$ (cf. Introduction). Then by (11), (18) and (19) we have

$$m\left(\frac{[p_\lambda]\bar{\bar{a}}_1+[p_\lambda]\bar{\bar{a}}_2}{2}\right)=\frac{m([p_\lambda]\bar{\bar{a}}_1)+m([p_\lambda]\bar{\bar{a}}_2)}{2} \qquad (\lambda \in \Lambda),$$

so by (s′) $[p_\lambda]\bar{\bar{a}}_1=[p_\lambda]\bar{\bar{a}}_2$ $(\lambda \in \Lambda)$, consequently $\bar{\bar{a}}_1=\bar{\bar{a}}_2$. Thus $\bar{m}$ is even by Lemma 1.2.

Now suppose that m is even. If for some $\alpha \geqq \beta \geqq 0$ $0 \leqq \bar{a} \in \bar{R}$,

$$\bar{m}(\alpha\bar{a})<\infty \quad \text{and} \quad \bar{m}\left(\frac{\alpha+\beta}{2}\bar{a}\right)=\frac{\bar{m}(\alpha\bar{a})+\bar{m}(\beta\bar{a})}{2},$$

then for $\bar{\bar{a}} \in \bar{\bar{R}}$ satisfying (Δ) together with $\frac{\alpha+\beta}{2}\bar{a}$ we have by (11) and

$$(4) \qquad \bar{m}(\alpha\bar{a})+\bar{\bar{m}}(\bar{\bar{a}})=\bar{\bar{a}}(\alpha\bar{a}) \quad \text{and} \quad \bar{m}(\beta\bar{a})+\bar{\bar{m}}(\bar{\bar{a}})=\bar{\bar{a}}(\beta\bar{a}).$$

As above, there exist $\{p_\lambda\}_\lambda \subseteqq R$ such that

$$[p_\lambda]\uparrow_{\lambda \in \Lambda} I \quad \text{and} \quad [p_\lambda]\bar{\bar{a}} \in R \qquad (\lambda \in \Lambda),$$

hence by (18) and Lemma 1.2 we have $\alpha[p_\lambda]\bar{a}=\beta[p_\lambda]\bar{a}$ $(\lambda \in \Lambda)$ consequently $\alpha\bar{a}=\beta\bar{a}$. Thus $\bar{m}$ is strictly convex by definition. Q.E.D.

Corresponding to strict convexity and evenness of a modular, those of norms are defined (cf. [9; Chap VII]). Let $(S, \|.\|)$ be a normed linear space[4]. A norm $\|.\|$ on S is said to be *strictly convex*, if

(S) $\quad \|a\|=\|b\|=1,\ \|a+b\|=2 \quad$ implies $\quad a=b.$

It is easy to see that (S) is equivalent to

(S′) $\quad \|a+b\|=\|a\|+\|b\|,\ a\neq 0 \quad$ implies $\quad b=\xi a$ for some ξ.

A norm is said to be *even*, if for any $a, b\in S$ $\|a\|=\|b\|=1$

(E) $$\lim_{\xi\to 0}\frac{\|a+\xi b\|-\|a\|}{\xi} \quad \text{exists.}$$

S. Mazur (see [9; p. 112]) proved that (E) is equivalent to

(E′) $\begin{cases}\text{for any } a\in S\ \ \|a\|=1 & \text{there exists } \textit{uniquely}\ \ \tilde{a}\in\tilde{S} \\ \text{such that}\ \ \tilde{a}(a)=\|\tilde{a}\|=1. & \end{cases}$

The following is known (see [9]):

Lemma 1.4. *Let $(S, \|.\|)$ be a normed linear space. If the associated norm $\|\tilde{.}\|$ on $\tilde{S}$ is strictly convex (resp. even), then the original norm on S is even (resp. strictly convex).*

We conclude this section with some results on *non-atomic* spaces.

Lemma 1.5. *Let S be a non-atomic semi-ordered linear space and φ be a functional defined on S satisfying the following conditions:*

(i) $\quad 0\leqq\varphi(a)\leqq\infty \quad$ *for all* $a\in S$;

(ii) $\quad |a|\leqq|b| \quad$ implies $\quad \varphi(a)\leqq\varphi(b)$;

(iii) $\quad [p_\lambda]\uparrow_{\lambda\in\Lambda}$ *or* $[p_\lambda]\downarrow_{\lambda\in\Lambda}[p]$ $\ \varphi(a)<\infty$ *implies* $\lim_\lambda\varphi([p_\lambda]a)=\varphi([p]a)$.

Then for any $a_1, a_2\in S$ satisfying $|a_1|\leqq|a_2|$ $\varphi(a_2)<\infty$ there exist $b, c\in S$ such that $b+c=a_1+a_2$, $|b-c|=|a_2-a_1|$ and $\varphi(b)=\varphi(c)$.

If further φ satisfies the additional condition:

(iv) $\quad x\frown y=0 \quad$ *implies* $\quad \varphi(x+y)=\varphi(x)+\varphi(y)$,

then we can add to the conclusion

$$\varphi(b)=\varphi(c)=\frac{\varphi(a_1)+\varphi(a_2)}{2}.$$

Proof. Putting

$$\psi([p])=\varphi([p]a_2+([a_2]-[p])a_1) \qquad \text{for all } p\in R$$

and using (iii), by so-called *exhaustion method* we can find $p_0\in R$ such that $\quad \psi([p_0])=\psi([a_2]-[p_0])$

4) Not necessarily semi-ordered.

i.e. $\varphi([p_0]a_2+([a_2]-[p_0])a_1)=\varphi(([a_2]-[p_0])a_2+[p_0]a_1)$.

If we put $b=[p_0]a_2+([a_2]-[p_0])a_1$ and $c=([a_2]-[p_0])a_2+[p_0]a_1$, b, c satisfy the required conditions. Q.E.D.

Lemma 1.6. *Let (R, m) be non-atomic. Then*

(i) *The modular norm is continuous, if and only if m is finite.*

(ii) *Under the additional condition that (R, m) is monotone complete, the modular norm is continuous, if and only if m is uniformly finite.*

For the proof, see [1].

§2. Strict convexity and evenness of the norms[5)]

In this section we shall give necessary and sufficient conditions for that the norms by a modular are strictly convex or even. We begin with a comment on the first norm.

Lemma 2.1. *If a modular m is infinitely increasing, for any $0\neq a\in R$ there exists $\xi_0>0$ such that $\|\xi_0 a\|=1+m(\xi_0 a)$.*

Proof. If $\sup\limits_{\xi>0}\frac{m(\xi a)}{\xi}=\|a\|$, we have by (8) $\sup\limits_{\xi>0}\{\xi\|a\|-m(\xi a)\}\leqq 1$, contradicting infinite increasingness. Hence by [2; Lemma 3.2] ξ_0, satisfying the required condition, exists. Q.E.D.

We shall use frequently the following version of (10), proved in [2],

(22) $a\in R$ $\bar{a}\in\bar{R}$ $m(a)=1$ $1+\bar{m}(\bar{a})=\bar{a}(a)$ implies $\|\bar{a}\|=\bar{a}(a)$.

Theorem 2.1. *If m is strictly convex and infinitely increasing, the first norm is strictly convex. If R is non-atomic, the converse is also true.*

Proof. Let m be strictly convex and infinitely increasing. For any $a, b\in R$ such that $\|a\|=\|b\|=1$ $\|a+b\|=2$ by Lemma 2.1 there exist $\xi, \eta>0$ such that $\xi=1+m(\xi a)$ and $\eta=1+m(\eta b)$, hence we have by (4) and (8)

$$\frac{\xi+\eta}{2}=\frac{m(\xi a)+m(\eta b)}{2}+1\geqq m\left(\frac{\xi a+\eta b}{2}\right)+1\geqq\left\|\frac{\xi a+\eta b}{2}\right\|.$$

Since $\|a\|=\|b\|=\left\|\frac{a+b}{2}\right\|=1$ implies $\|\xi a+\eta b\|=\xi+\eta$, we obtain

5) In this section, non-atomicity is not essential. We can obtain necessary and sufficient conditions for strict convexity and eveness, without assumption of non-atomicity, in somewhat complicated forms cf. [2].

$$m\left(\frac{\xi a+\eta b}{2}\right)=\frac{m(\xi a)+m(\eta b)}{2},$$

consequently by (s′) $\xi a = \eta b$ i.e. $a=b$. Thus the first norm is strictly convex by definition.

Now conversely, let R be non-atomic and the first norm be strictly convex. We shall prove first that m is infinitely increasing. Otherwise, there exist $0 \neq a \in R$ such that

$$\sup_{\xi>0} \frac{m(\xi a)}{\xi}=\gamma_1<\infty \quad \text{and} \quad \sup_{\xi>0}\{\xi\gamma_1-m(\xi a)\}=\gamma_2<\infty .$$

Using non-atomicity of R, we may assume that $\gamma_2 \leqq 1$. Then by the definition (8) we have

$$\|[p]a\|=\sup_{\xi>0} \frac{m(\xi[p]a)}{\xi} \qquad \text{for all } p\in R.$$

consequently by [2; Lemma 3.3] the first norm is of L^1-type on $[a]R$, *a fortiori* is not strictly convex. Next we shall prove strict convexity. Suppose that for $a\in R$ $\alpha \geqq \beta \geq 0$, $\frac{\alpha+\beta}{2}a$ is domestic and

$$m\left(\frac{\alpha+\beta}{2}a\right)=\frac{m(\alpha a)+m(\beta a)}{2}<\infty .$$

Then there exists $\bar{a}\in\bar{R}$ satisfying (Δ) together with $\frac{\alpha+\beta}{2}$. It follows that $\qquad m(\alpha a)+\bar{m}(\bar{a})=\bar{a}(\alpha a) \quad$ and $\quad m(\beta a)+\bar{m}(\bar{a})=\bar{a}(\beta a)$.

We may assume that $[\bar{a}]^R \leqq [a]$ and $\bar{m}(\bar{a})<1$, because of non-atomicity of R and (18). Since the conjugate modular is continuous by Theorem A and R is a complete semi-normal manifold of $\bar{\bar{R}}$, we can find $b\in(I-[a])R$ $\bar{b}\in(1-[a])\bar{R}$ satisfying together (Δ) and $\bar{m}(\bar{a}+\bar{b})=1$. Then we have

$$m(\alpha a+b)+\bar{m}(\bar{a}+\bar{b})=(\bar{a}+\bar{b})(\alpha a+b)$$

and

$$m(\beta a+b)+\bar{m}(\bar{a}+\bar{b})=(\bar{a}+\bar{b})(\beta a+b)$$

from this by (22) we obtain

$$\begin{aligned}&\|\alpha a+b\| + \|\beta a+b\| \\ &= (\bar{a}+\bar{b})(\alpha a+b+\beta b+b) \\ &\leqq \|(\alpha a+b)+(\beta a+b)\| .\end{aligned}$$

Finally (S′) tells us $\alpha a=\beta a$. Thus m is strictly convex. Q.E.D.

Theorem 2.2. *If m is strictly convex and*

$$(*) \qquad \inf_{x\neq 0} \omega(1|x) = 1,$$

the second norm is strictly convex. If R is non-atomic, the converse is also true.

Poof. Let m be strictly convex and $\inf_{x \neq 0} \omega(1|x)=1$.
The latter assumption shows that "$|||x|||=1$" is equivalent to "$m(x)=1$".
If $a, b \in R$ $|||a|||=|||b|||=1$ $|||a+b|||=2$, then $m(a)=m(b)=m\left(\frac{a+b}{2}\right)=1$, hence by (s') $a=b$. Thus the second norm is strictly convex. Now conversely, let R be non-atomic and the second norm be strictly convex. Since the norm is monotone[6], by [2; Theorem 3.3] we obtain $\inf_{x \neq 0} \omega(1|x)=1$. Next suppose that for $0 \neq a \in R$ $\alpha \geqq \beta \geqq 0$

$$m\left(\frac{\alpha+\beta}{2} a\right)=\frac{m(\alpha a)+(\beta a)}{2}<\infty .$$

From (19) we may assume that $[a]<I$ $m(\alpha a) \leqq 1$, because R is non-atomic. By Lemma 1.5 there exist $b, c \in R$ such that

$$b+c=(\alpha+\beta)a, \qquad |b-c|=(\alpha-\beta)|a|$$

and

$$m(b)=m(c)=\frac{m(\alpha a)+m(\beta a)}{2}$$

Since there exists $d \in (I-[a])R$ such that $m(b)+m(d)=1$, we have

$$m\left(\frac{b+d+c+d}{2}\right)=m\left(\frac{\alpha+\beta}{2} a\right)+m(d)$$

$$=\frac{m(\alpha a)+m(\beta a)}{2}+m(d)=m(b)+m(d)=1,$$

or equivalently by (*) $\quad |||b+d|||=|||c+d|||=\left|\left|\left|\frac{b+c}{2}+d\right|\right|\right|=1$,

then by (S) we can conclude $b=c$, i.e. $\alpha a=\beta a$. Thus m is strictly convex. Q.E.D.

Remark 2.1. *Since we proved in [5] that (*) implies continuity of the norm, we can state "when R is non-atomic, the second norm is strictly convex, if and only if m is normal in the sense of H. Nakano".*

Next we shall consider evenness.

Remark 2.2. (see [3]) *Evenness of a norm implies its continuity.*

Theorem 2.3. *If m is even, finite and infinitely increasing, then the first norm is even. If R is non-atomic, the converse is also true.*

Proof. Let m be even, finite and infinitely increasing. For any

6) i.e. $0 \leqq a<b$ implies $|||a|||<|||b|||$.

$a \in R$ $\bar{a} \in \bar{R}$ satisfying $\|a\| = |||\bar{a}||| = \bar{a}(a) = 1$ by Lemma 2.1 there exists $\xi > 0$ such that $m(\xi a) + 1 = \xi$, i.e. $m(\xi a) + \bar{m}(\bar{a}) = \xi\bar{a}(a)$. Then by Lemma 1.2 $\bar{a}$ is uniquely determined for a. Thus (E′) is satisfied, because finiteness implies $\bar{R} = \tilde{R}$ by Theorem C. Now conversely let R be non-atomic and the first norm be even. m is infinitely increasing, as shown in the proof of Theorem 2.1. m is finite by Lemma 1.6 and Remark 2.2. Finally suppose that for $a \in R$ $0 < [a] < I$ there exist $\bar{a}_1, \bar{a}_2 \in \bar{R}$ such that

$$m(a) + \bar{m}(\bar{a}_\nu) = \bar{a}_\nu(a) \qquad (\nu = 1, 2).$$

Using (18), we may assume $|\bar{a}_1| \leqq |\bar{a}_2|$. As in the proof of Theorem 2.1 we can find $d \in (I - [a])R$ $\bar{d} \in (I - [a])\bar{R}$ such that $\bar{d} \frown |\bar{a}_2| = 0$,

$$m(d) + \bar{m}(\bar{d}) = \bar{d}(d) \quad \text{and} \quad \frac{\bar{m}(\bar{a}_1) + \bar{m}(\bar{a}_2)}{2} + \bar{m}(\bar{d}) = 1.$$

By Lemma 1.5 there exist $\bar{b}, \bar{c} \in R$ such that $\bar{b} + \bar{c} = \bar{a}_1 + \bar{a}_2$,

$$|\bar{b} - \bar{c}| = |\bar{a}_2 - \bar{a}_1| \quad \text{and} \quad \bar{m}(\bar{b}) = \bar{m}(\bar{c}) = \frac{\bar{m}(\bar{a}_1) + \bar{m}(\bar{a}_2)}{2},$$

hence we have $\quad m(a+d) + \bar{m}(\bar{b} + \bar{d}) = (\bar{b} + \bar{d})(a+d)$

and $\quad m(a+d) + \bar{m}(\bar{c} + \bar{d}) = (\bar{c} + \bar{d})(a+d)$.

From this by (22) we obtain

$$\|a+d\| = (\bar{b} + \bar{d})(a+d) = (\bar{c} + \bar{d})(a+d).$$

Finally by (E′) we can conclude $\bar{b} = \bar{c}$, i.e. $\bar{a}_1 = \bar{a}_2$. Thus m is even by Lemma 1.2. Q.E.D.

Theorem 2.4. *If m is finite and even, the second norm is even. If R is non-atomic, the converse is also true.*

Proof. Let m be finite and even. Then $\bar{R} = \tilde{R}$ by Theorem C, and the conjugate modular $\bar{m}$ is strictly convex and increasing by Lemma 1.3 and Theorem A. Thus by Theorem 2.1 the first norm by $\bar{m}$ is strictly convex, consequently by (15) and Lemma 1.4 the second norm by m is even. Now conversely let R be non-atomic and the second norm be even. Finiteness of m follows from Remark 2.2. For $a \in R$ $\bar{a} \in \bar{R}$ satisfying $m(a) = 1$ $1 + \bar{m}(\bar{a}) = \bar{a}(a)$, it follow from (22) $\|\bar{a}\| = \bar{a}(a)$. (E′) show that $\dfrac{\bar{a}}{\|\bar{a}\|}$ is uniquely determined for a. We can prove further that $\bar{a}$ itself is uniquely determined. From this, through easy arguments we can conclude that m is even. Q.E.D.

§ 3. Uniform convexity and uniform evenness of the norms

In this section we shall treat uniform convexity and uniform evenness of the norms by a modular. As uniformization of convexity H. Nakano [12] defined that a modular m is said to be *uniformly convex*, if for any $\gamma, \varepsilon > 0$ there exists $\delta = \delta(\gamma, \varepsilon) > 0$ such that

$$\text{(uc)} \qquad \begin{cases} \gamma \geqq \alpha > \beta \geqq 0 \qquad \alpha - \beta \geqq \varepsilon & \text{implies} \\ \dfrac{\omega(\alpha \mid a) + \omega(\beta \mid a)}{2} \geqq \omega\left(\dfrac{\alpha+\beta}{2}\middle| a\right) + \delta & \text{for all} \quad a \neq 0. \end{cases}$$

It is easy to see that uniform convexity implies uniform simplicity. A modular m is said to be *uniformly even*, if for any $\gamma, \varepsilon > 0$ there exists $\delta = \delta(\gamma, \varepsilon) > 0$ such that

$$\text{(ue)} \qquad \begin{cases} \gamma \geqq \alpha \geqq |\beta|, \qquad \alpha - \beta \leqq \delta & \text{implies} \\ \dfrac{\omega(\alpha \mid a) + \omega(\beta \mid a)}{2} \leqq \omega\left(\dfrac{\alpha+\beta}{2}\middle| a\right) + (\alpha - \beta)\varepsilon & \text{for all} \quad a \neq 0. \end{cases}$$

Uniform evenness implies uniform finiteness and uniform monotony.

As a relation between uniform convexity and uniform evenness, H. Nakano [12; § 51] proved:

Lemma 3.1. *If m is uniformly convex and uniformly increasing, the associated modular is uniformly even. If m is uniformly even and uniformly increasing, then the associated modular is uniformly convex.*

Corresponding to these definitions, a norm $\|.\|$ on a normed linear space S is said to be *uniformly convex*, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\text{(UC)} \qquad \|a\| = \|b\| = 1 \qquad \|a - b\| \geqq \varepsilon \qquad \text{implies} \qquad \|a + b\| \leqq 2 - \delta.$$

A norm is said to be *uniformly even*, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\text{(UE)} \qquad \begin{cases} \|a\| = \|b\| = 1 \qquad 0 \leqq \xi \leqq \delta & \text{implies} \\ \qquad \|a + \xi b\| + \|a - \xi b\| \leqq 2 + \xi\varepsilon. \end{cases}$$

Lemma 3.2. ([13; § 77]) *Uniform convexity and uniform evenness of norms are associated.*

H. Nakano [13; §§ 87–88] derived uniform convexity of the norms from that of m.

Theorem 3.1. (1) *If m is uniformly convex and uniformly increasing, the first norm is uniformly convex.*

(2) *If m is uniformly convex and uniformly finite, the second norm is uniformly convex.*

Now we shall prove the converse in the case R is non-atomic. The following remark is useful.

Remark 3.1. *If R is non-atomic and the norm is uniformly convex, the associated norm together with it is continuous, hence by Lemma* 1.6, *Theorem* B *and Theorem* C *m is uniformly finite and uniformly increasing.*

Theorem 3.2. *Let R be non-atomic. If the first norm is uniformly convex, then m is uniformly convex.*

Proof. First fix γ $\varepsilon>0$. For any $0\leqq a\in R$ $m(a)=1$ and for any $\alpha, \beta\geqq 0$ $\gamma\geqq\alpha\geqq\beta$ $\alpha-\beta\geqq\varepsilon$ by Lemma 2.1 there exists $\bar{a}\in\overline{R}$ such that

$$[\bar{a}]^R\leqq[a]<I \quad \text{and} \quad m\Big(\frac{\alpha+\beta}{2}a\Big)+\overline{m}(\bar{a})=\bar{a}\Big(\frac{\alpha+\beta}{2}a\Big).$$

If $\overline{m}(\bar{a})\geqq 1$, by non-atomicity of R we can find $p\in R$ such that $\overline{m}([p]\bar{a})=1$. If $\overline{m}(\bar{a})<1$ there exist $d\in(1-[a])R$ $\bar{d}\in(I-[a])\overline{R}$ such that

$$\overline{m}(\bar{a}+\bar{d})=1 \quad \text{and} \quad m(d)+\overline{m}(\bar{d})=\bar{d}(d).$$

Put $\bar{x}=[p]\bar{a}$, $x_1=\alpha[p]a$ $x_2=\beta[p]a$ in the former case,

or $\bar{x}=\bar{a}+\bar{d}$, $x_1=\alpha a+d$ $x_2=\beta a+d$, in the latter case.

By Lemma 1.5 there exists $b, c\in R$ such that

$$b+c=x_1+x_2, \quad |b-c|=|x_2-x_1|\geqq(\alpha-\beta)[p]\,|a|$$

and

$$\|b\|=\|c\|,$$

because the first norm is continuous. From the definition of p and by (22) we obtain $\frac{\alpha+\beta}{2}\|[p]a\|=\bar{a}\Big(\frac{\alpha+\beta}{2}[p]a\Big)\geqq\pi_-([p]\bar{a})\geqq 1$, consequenly

$$\|[p]a\|\geqq 1/\gamma \quad \text{and} \quad \|b\|=\|c\|\geqq\frac{1}{2\gamma}.$$

On the other hand, by Remark 3.1 and Theorem B $\overline{m}$ is uniformly finite, hence there exists a constant $\rho>0$ such that

$$\bar{\pi}_+(\bar{y})\leqq\rho \qquad \text{for all } \bar{y}\in R \quad \overline{m}(\bar{y})\leqq 1.$$

Since the first norm is uniformly convex by assumption, there exists $\delta>0$ such that $\quad 2\rho\geqq\|x\|=\|y\|\geqq 1/2\gamma \quad \|x-y\|\geqq\frac{\varepsilon}{\gamma}$

implies $\quad \|x+y\|\leqq 2(\|x\|-\delta)$ (cf. [13: §76]).

If $\bar{x}(b)+\delta\geqq m(b)+\overline{m}(\bar{x})\equiv m(b)+1$ and $\bar{x}(c)+\delta\geqq m(c)+\overline{m}(\bar{x})\equiv m(c)+1$, it follows from (8) and (15)

$$||b+c||+2\delta \geqq ||b||+||c||=2||b||$$

contradicting the above, because $\quad ||b-c|| \geqq (\alpha-\beta)||[p]a|| \geqq \frac{\varepsilon}{\gamma}$

and $$\left\|\frac{b+c}{2}\right\| \leqq \bar{\pi}_{+}(\bar{x}) \leqq \rho .$$

Thus we have, say, $\bar{x}(b)+\delta \leqq m(b)+1$. On the other hand, by (11) we know

$$\bar{x}(c) \leqq m(c)+\bar{m}(\bar{x})=m(c)+1,$$

hence

$$\bar{x}\left(\frac{b+c}{2}\right)+\frac{\delta}{2} \leqq \frac{m(b)+m(c)}{2}+1 .$$

Since from the definition we have

$$m\left(\frac{b+c}{2}\right)+\bar{m}(\bar{x})=\bar{x}\left(\frac{b+c}{2}\right)$$

we can conclude

$$m\left(\frac{b+c}{2}\right)+\frac{\delta}{2} \leqq \frac{m(b)+m(c)}{2} .$$

From this by (4) and (6) we can deduce

$$m\left(\frac{\alpha+\beta}{2} a\right)+\frac{\delta}{2} \leqq \frac{m(\alpha a)+m(\beta a)}{2}.$$

Thus m is uniformly convex, because for any $0<x \in R$ $|||x|||=1$ there exist $0<x_\nu \in R$ such that $|||x_\nu|||=1$ $[x_\nu]<I$ $(\nu=1,2,\cdots)$ and $\lim_{\nu\to\infty} m(\xi x_\nu)=m(\xi x)$ for all $\xi \geqq 0$. Q.E.D.

Theorem 3.3. *Let R be non-atomic. If the second norm is uniformly convex, then m is uniformly convex.*

Proof. For any $\gamma>1>\varepsilon>0$ and for any $0 \leqq a \in R$ $m(a)=1$ there exists $p \in R$ such that $[p]<I$ and $\frac{1}{2\gamma} \leqq |||[p]a||| \leqq \frac{1}{\gamma}$. Since the second norm is uniformly convex by assumption, there exists $\delta>0$ such that

$$|||x|||=|||y|||=1, \quad |||x-y||| \geqq \varepsilon/2\gamma$$

implies $$|||x+y||| \leqq 2(1-\delta) .$$

For any $\alpha, \beta \geqq 0$ $\gamma \geqq \alpha \geqq \beta$, $\alpha-\beta \geqq \varepsilon$, by Lemma 1.5 there exist $b, c \in R$ such that $\quad b+c=(\alpha+\beta)[p]a \quad |b-c|=(\alpha-\beta)[p]a$

and $$m(b)=m(c)=\frac{m(\alpha[p]a)+m(\beta[p]a)}{2}$$

Choose $d \in (1-[p])R$ such that $m(b)+m(d)=1$, then we have

$$|||b+d||| = |||c+d||| = 1$$

and $$|||(b+d)-(c+d)|||=|||b-c|||=(\alpha-\beta)|||[p]a|||\geqq \frac{\varepsilon}{2\gamma},$$

hence we have

$$m\Big(\frac{\alpha+\beta}{2}[p]a\Big)+m(d)\leq\Big|\Big|\Big|\frac{(b+d)+(c+d)}{2}\Big|\Big|\Big|$$

$$\leqq 1-\delta=\frac{m(\alpha[p]a)+m(\beta[p]a)}{2}+m(d)-\delta,$$

consequently $$m\Big(\frac{\alpha+\beta}{2}[p]a\Big)+\delta\leqq\frac{m(\alpha[p]a)+m(\beta[p]a)}{2}.$$

Finally by (6) and (11) we can conclude

$$m\Big(\frac{\alpha+\beta}{2}a\Big)+\delta\leqq\frac{m(\alpha a)+m(\beta a)}{2},$$

thus m is uniformly convex. Q.E.D.

Combining the above two theorems, we obtain a quite simple relation.

Theorem 3.4. *Let R be non-atomic. Then the following conditions are mutually equivalent:*

(1) *m is uniformly convex.*

(2) *The first norm is uniformly convex.*

(3) *The second norm is uniformly convex.*

Proof. If m is uniformly convex, it is uniformly simple, hence by Lemma 1.6 uniformly finite. Then Theorem 3.1 is applicable, and the second norm is uniformly convex, i.e. (1)$\rightarrow$(3). If the second norm is uniformly convex, by Remark 3.1 and Theorem 3.3 m is uniformly convex and uniformly increasing, consequently by Theorem 3.1 the first norm is uniformly convex, i.e. (3)$\rightarrow$(1)$\rightarrow$(2). (2)$\rightarrow$(1) follows from Theorem 3.2. Q.E.D.

Theorem 3.5. *Let R be non-atomic. Then the following conditions are mutually equivalent:*

(1) *m is uniformly even.*

(2) *The first norm is uniformly even.*

(3) *The second norm is uniformly even.*

The assertion follows from (15), Lemma 3.1, Lemma 3.2 and Theorem 3.4.

Chapter II. Equivalent Modulars

§4. Some topological properties

When Banach spaces are considered, usually reflexivity and weak completeness are discussed. But when we consider semi-ordered Banach space, further *continuity of norms* comes into question. In this section we shall investigate these properties in relation to concrete spaces and order-structures of the space. Throughout this section $(S, \|.\|)$ denotes a semi-ordered Banach space.

$l_\infty(\Lambda)$ denotes the Banach space, consisting of all bounded real function on Λ with the norm: $\|a\|=\sup_{\lambda\in\Lambda}|\xi_\lambda|$ for $a=(\xi_\lambda)$, and $l_1(\Lambda)$ denotes the Banach space, consisting of all real functions on Λ such that $\sum_\lambda|\xi_\lambda|<\infty$, with the norm: $\|a\|=\sum_{\lambda\in\Lambda}|\xi_\lambda|$ for $a=(\xi_\lambda)$. When Λ is a set of countable points, we write simply l_∞ and l_1. c_0 denotes the closed linear subspace of l_∞, consisting of all null sequences.

S, considered as a semi-ordered linear space, is said to be *uniformly continuous,* if for any $a_\nu\downarrow_{\nu=1}^\infty 0$ there exists $a_0\in S$ and $\{\varepsilon_\nu\}_{\nu=1}^\infty$ such that $\varepsilon_\nu\downarrow_{\nu=1}^\infty 0$ $a_\nu\leqq\varepsilon_\nu a_0$ $(\nu=1,2,\cdots)$. A subset A of S is said to be *C-bounded,* if $\{a_\nu\}_{\nu=1}^\infty\subseteq A$ $\sum_{\nu=1}^\infty\alpha_\nu<\infty$[7] implies $\sum_{\nu=1}^\infty\alpha_\nu a_\nu<+\infty$[8]. S is said to be *K-bounded,* if every C-bounded subset $\{a_\lambda\}_{\lambda\in\Lambda}$ such that $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda}$ is *bounded,* i.e. $a_\lambda\leqq a_0$ for some a_0 $(\lambda\in\Lambda)$. It is known (see [12; §30] and [16]) that when $(S, \|.\|)$ is complete, C-boundness is equivalent to boundedness under the norm, and K-boundedness is equivalent to monotone completeness. S is said to be *contractile,* if for any orthogonal system of positive elements $\{a_\lambda\}_{\lambda\in\Lambda}$, where Λ is a set of uncountable indices, there exist $\{\alpha_\nu\}_{\nu=1}^\infty$ and $\{a_{\lambda_\nu}\}_{\nu=1}^\infty\subseteq\{a_\lambda\}_{\lambda\in\Lambda}$ such that $\sum_{\nu=1}^\infty\alpha_\nu=\infty$ and $\sum_{\nu=1}^\infty\alpha_\nu a_{\lambda_\nu}<+\infty$. S is said to be *strictly contractile,* if for any C-bounded orthogonal sequence of positive elements $\{a_\nu\}_{\nu=1}^\infty$ there exist $\{\alpha_\nu\}_{\nu=1}^\infty$ such that $\sum_{\nu=1}^\infty\alpha_\nu=\infty$ and $\sum_{\mu=1}^\infty\alpha_\nu a_\nu<+\infty$. We call a subspace V of a Banach space S a *direct factor,* if there exists a bounded linear projection from S onto V.

Theorem 4.1. *The following conditions are mutually equivalent:*

7) α,β denote positive real numbers, and a, b positive elements of S.

8) $\sum_{\nu=1}^\infty a_\nu<+\infty$ means that $\{\sum_{\nu=1}^\kappa a_\nu\}_{\kappa=1}^\infty$ is (order)-bounded.

(1.1) *The norm on S is continuous.*

(1.2) *S is uniformly continuous.*

(1.3) *$(S, \|.\|)$ does not contain any subspace isomorphic to l_∞.*

Proof. (1.1)$\rightleftarrows$(1.2) is proved in [12; §30]. For (1.3)$\to$(1.1), see [3]. Finally if $(S, \|.\|)$ contains a subspace M isomorphic to l_∞, denote by V the closed subspace of M corresponding to c_0 under the isomorphic mapping. A. Sobszyk [15] showed that for any closed *separable* subspace U such that $V \subseteqq U \subseteqq M$, V is a direct factor of U but U is never that of M. Since M is a direct factor of S (see [9; p. 94]), any *separable* closed subspace of S containing V can not be a direct factor of S. Hence (1.1)$\to$(1.3) is a consequence of the following lemma.

Lemma 4.1. *If the norm on S is continuous, for any $\{a_\nu\}_{\nu=1}^\infty \subseteq S$ there exists a separable closed subspace $V \supseteqq \{a_\nu\}_{\nu=1}^\infty$, which is a direct factor of S.*

Proof. We may assume that there exists $0 \leqq \bar{a}_0 \in \bar{S}$ with $[\bar{a}_0]^R = I$. Let V be the least order-closed (*a fortiori* closed) linear lattice manifold containing $\{a_\nu\}_{\nu=1}^\infty$, then by continuity of the norm V is *separable.* We may assume that $[V] = I$. Let $\hat{U}$ be the least order-closed linear manifold of $\bar{S}$ containing all $[p]\bar{a}_0$. $(p \in V)$. Then for any $\bar{a} \in \hat{U}$ there exists $a \in V$ with $[\bar{a}]^R = [a]$, hence the mapping $\hat{U} \ni \bar{a} \to \bar{a}' \in \bar{V}$ defined by

$$\bar{a}'(x) = \bar{a}(x) \qquad \text{for all} \quad x \in V$$

is one-to-one. On the other hand, by Hahn-Banach's theorem (see [9; §3]) and Theorem C for any $0 \leqq \bar{b}' \in \bar{V}$ there exists $0 \leqq \hat{b} \in \bar{S}$ such that

$$\hat{b}(x) = \bar{b}'(x) \qquad \text{for all} \quad x \in V.$$

Put $\bar{b}_0 = \bigcup_{\substack{0 \leqq \bar{x} \leqq \hat{b} \\ \bar{x} \in \hat{U}}} \bar{x}$ then we can prove $\bar{b}_0' = \bar{b}'$. Thus we have proved that $\hat{U}$ is order-isomorphic to the conjugate space $\bar{V}$ of V under the mapping indicated. Since $(\hat{U}, \|.\|)$ and $(\bar{V}, \|.\|_*)$ are both Bach spaces, where $\|\bar{x}'\|_* = \sup_{x \in V, \|x\| \leqq 1} |\bar{x}'(x)|$ for $\bar{x}' \in \bar{V}$, by Theorem C there exists $\alpha > 0$ such that

$$\|\bar{a}\| \leqq \alpha \|\bar{a}'\|_* \qquad \text{for all} \quad \bar{a} \in \hat{U}.$$

We can define a mapping T from S to $\bar{\bar{V}}$ by

$$\mathrm{T}a(\bar{x}') = \bar{x}(a) \qquad \text{for} \quad a \in S,\ \bar{x} \in \hat{U},$$

then we have $\|\mathrm{T}a\|_* \leqq \alpha \|a\|$ for all $a \in S$, where $\|\cdot\|_*$ is the norm on $\bar{\bar{V}}$. Since V is a complete semi-normal manifold of $\bar{\bar{V}}$, we have from the definition of T, $\mathrm{T}a \in V$ for all $a \in S$ satisfying $|a| \leqq x$ for some $x \in V$. Since for any $0 \leqq a \in S$ there exist $0 \leqq a_\nu \uparrow_{\nu=1}^\infty a$ such that $a_\nu \leqq x_\nu$ for some $x_\nu \in V$ $(\nu=1, 2, \cdots)$, we have

$$\| \mathrm{T}a_\nu - \mathrm{T}a \|_* \leqq \alpha \| a - a_\nu \| \qquad (\nu=1, 2, \cdots).$$

Since the norm on S is continuous, $\lim_{\nu\to\infty} \| a_\nu - a \| = 0$. Since V is closed, we can conclude $\mathrm{T}a \in V$. Thus T is a bounded linear projection from S onto the separable subspace V. Q.E.D.

Theorem 4.2. *The following conditions are mutually equivalent:*

(2.1) $(S, \|.\|)$ *is weakly complete.*[9]

(2.2) S *is uniformly continuous and K-bounded.*

(2.3) $(S, \|.\|)$ *does not contain any subspace isomorphic to* c_0.

Proof. (2.1)$\to$(2.3), for c_0 is not weakly complete. (2.3)$\to$(2.2) If $(S, \|.\|)$ contains no subspace isomorphic to c_0, the norm is continuous and S is monotone complete, hence S is uniformly continuous and K-bounded. (2.2)$\to$(2.1) If S is uniformly continuous and K-bounded by [12; Theorem 32.7] it is order-reflexive. Then by [12; Theorem 27.5] $(S; \|.\|)$ is weakly complete, because $\tilde{S} = \bar{S}$ by Theorem C. Q.E.D.

Before considering reflexivity, we shall treat continuity of the associated norm for the later use.

Lemma 4.2. *If the associated space $\tilde{S}$ is uniformly continuous, S is strictly contractile. Conversely if S is strictly contractile, the conjugate space $\bar{S}$ is uniformly continuous.*

Proof. Let $\tilde{S}$ be uniformly continuous and $\{a_\nu\}_{\nu=1}^\infty$ be a C-bounded orthogonal sequence of positive elements of S. For any $0 \leqq \tilde{a} \in \tilde{S}$ we have $\lim_{\nu\to\infty} \tilde{a}(a_\nu) = 0$, because otherwise $\lim_{\nu\to\infty} \| [a_\nu]\tilde{a} \| \neq 0$ contradicting continuity of the associated norm. Devide the set of all natural number into a countable subsets $\{I_\nu\}_{\nu=1}^\infty$, each consisting of countable elements. From the above and by a theorem on separation of convex sets (see [9; §6]) there exist $\kappa_\nu \in I_\kappa$ $(\nu=1, 2, \cdots, \rho_\kappa,\ \kappa=1, 2, \cdots)$ and $0 \leqq \alpha_{\nu,\kappa}$ $(\nu=1, 2, \cdots, \rho_\kappa;\ \kappa=1, 2, \cdots)$ such that

$$\sum_{\nu=1}^{\rho_\kappa} \alpha_{\nu,\kappa} = 1 \quad \text{and} \quad \left\| \sum_{\nu=1}^{\rho_\kappa} \alpha_{\nu,\kappa} a_{\kappa_\nu} \right\| \leqq \frac{1}{2^\kappa} \qquad (\kappa=1, 2, \cdots).$$

Arranging $\{\kappa_\nu\}_{\nu,\kappa}$ in a line, we have $\sum_{\nu,\kappa} \alpha_{\nu,\kappa} = \infty$ $\sum_{\kappa=1}^\infty \sum_{\nu=1}^{\rho_\kappa} \alpha_{\nu,\kappa} a_{\kappa_\nu} < +\infty$,

9) Sequentially complete under the weak topology.

because $\sum_{\kappa=1}^{\infty}\left\|\sum_{\nu=1}^{\rho_\kappa}\alpha_{\nu,\kappa}a_{\kappa_\nu}\right\|\leqq 1$. Thus S is strictly contractile. Conversely let S be strictly contractile. If the conjugate space $\bar{S}$ is not uniformly continuous, by Theorem 4.1 there exist $0<\bar{a}\in\bar{S}$ and orthogonal $\{p_\nu\}_{\nu=1}^{\infty}$ such that

$$\inf_{\nu=1,2,\cdots}\|[p_\nu]\bar{a}\|>\varepsilon>0 \qquad \text{for some } \varepsilon.$$

Further there exist $\{a_\nu\}_{\nu=1}^{\infty}$ such that

$$[p_\nu]a_\nu=a_\nu,\quad \|a_\nu\|=1,\quad \bar{a}(a_\nu)>\varepsilon \qquad (\nu=1,2,\cdots).$$

Since $\{a_\nu\}_{\nu=1}^{\infty}$ is C-bounded, by assumption there exist $\{\alpha_\nu\}_{\nu=1}^{\infty}$ such that

$\sum_{\nu=1}^{\infty}\alpha_\nu=\infty$ and $\sum_{\nu=1}^{\infty}\alpha_\nu a_\nu<+\infty$. Then we have

$$\frac{\kappa\varepsilon}{2}\leqq\sum_{\nu=1}^{\infty}\alpha_\nu\bar{a}(a_\nu)\leqq\bar{a}\Big(\sum_{\nu=1}^{\infty}\alpha_\nu a_\nu\Big)<\infty \qquad (\kappa=1,2,\cdots).$$

Clearly this is a contradiction. Q.E.D.

A semi-ordered linear space S is said to be *superuniversally continuous*, if $a_\lambda\frown a_\mu=0$ $(\lambda\neq\mu)$, $a_\lambda\leqq a$ implies $a_\lambda=0$ except for countable indices. Then quite similarly as in Lemma 4.2 we can prove:

Lemma 4.3. *If the associated space $\tilde{S}$ is superuniversally continuous, S is contractile. Conversely if S is contractile, the conjugate space $\bar{S}$ is superuniversally continuous.*

Remark 4.1. *If $\bar{S}=\tilde{S}$, namely the norm on S is continuous, strict contractility (resp. contractility) is a necessary and sufficient condition for the uniform continuity (resp. superuniversal continuity) of the associated space.*

Remark 4.2. *As is shown in [4], if S is a modulared semi-ordered linear space, uniform continuity (resp. superuniversal continuity) of the conjugate space $\bar{S}$ implies that of the associated space $\tilde{S}$.*

Theorem 4.3. *The following conditions are mutually equivalent.*

(3.1) *$(S,\|.\|)$ is reflexive (as a Banach space).*

(3.2) *S is K-bounded, uniformly continuous and strictly contractile.*

(3.3) *$(S,\|.\|)$ does not contain any subspace isomorphic to c_0 or to l_1.*

Proof. (3.1)→(3.3), for neither c_0 nor l_1 is reflexive. (3.3)→(3.2) If $(S;\|.\|)$ does not contain any subspace isomorphic to c_0 nor to l_1, then by Theorem 4.2 S is K-bounded and uniformly continuous. If S is not strictly contractile, there exists a orthogonal sequence of positive elements $\{a_\nu\}_{\nu=1}^{\infty}$ of S such that $\alpha_\nu>0$ $\sum_{\nu=1}^{\infty}\alpha_\nu a_\nu<+\infty$ implies $\sum_{\nu=1}^{\infty}\alpha_\nu<\infty$. This means that the subspace generated by $\{a_\nu\}_{\nu=1}^\infty$ is isomorphic to l_1, this contradict the assumption. (3.2)→(3.1) If S is K-bounded, uniformly continuous and strictly contractile, by Theorem 4.2 and Lemma 4,2 $(S;\ \|.\|)$ is weakly complete and the associated space is uniformly continuous, because $\bar{S}=\tilde{S}$. Then as is well-known, $(S;\ \|.\|)$ is reflexive. Q.E.D.

§ 5. Methods of construction of new modulars

Two modulars m_1, m_2 on a semi-ordered linear space is said to be *equivalent*, if their modular norms are equivalent. In this section R denotes a modulared semi-ordered linear space with a modular m. The most simple principles of construction of equivalent modulars are the following:

(i) If m_1 is another modular on R whose modular norm is weaker than that by m, then the functional $m+m_1$ is an equivalent modular.

(ii) If $\overline{m}^1$ is a modular on $\overline{R}$, equivalent to the conjugate modular $\overline{m}$ of m, the conjugate modular m_1 of $\overline{m}^1$, considered on R, is a modular on R equivalent to m.

H. Nakano [12; Chap. IX] discovered an intrinsic method of construction of a modular. In the remainder of this section, we shall reformulate some of his results in a convenient form.

Let $\overline{m}^1$ be a modular on $\overline{R}$ equivalent to the conjugate modular $\overline{m}$. Then there exists an operator T from the domain

$F=\{x;\ x\geqq 0\ m(x)<\infty\}$ to $\overline{F}=\{\bar{x};\ \overline{R}\ni\bar{x}\geqq 0\ \overline{m}^1(\bar{x})<\infty\}$ such that

$$\overline{m}^1([p]\mathrm{T}a)^{10)}\leqq m([p]a) \qquad \text{for all } p\in R \quad a\in F$$

and "$\overline{m}^1([p]\bar{x})\leqq m([p]a)$ for all $p\in R$" implies $\bar{x}\leqq \mathrm{T}a$. In fact, we can define T by the formula:

$$\mathrm{T}a=\bigcup\{\bar{x};\ \overline{m}^1([p]\bar{x})\leqq m([p]a) \ \text{ for all } p\in R\}. \tag{23}$$

If $\overline{m}^1$ is *continuous*, we have further

$$\overline{m}^1([p]\mathrm{T}a)=m([p]a) \qquad \text{for all } p\in R \quad a\in F. \tag{24}$$

For, if $\overline{m}^1([p_0]\mathrm{T}a)<m([p_0]a)$ for some $0\neq[p_0]\leqq[a]$, there exists $0<\bar{a}\in\overline{R}$ such that $[p_0]\bar{a}=\bar{a}$ and $\overline{m}^1([p_0]\mathrm{T}a+\bar{a})<\infty$ because of continuity of $\overline{m}^1$. Suitably modifying $\bar{a}$, we may assume that

$$\overline{m}^1([p]\mathrm{T}a+[p]\bar{a})\leqq m([p]a) \qquad \text{for all } [p]\leqq[p_0],$$

Then from the definition of T we have $[p_0](\mathrm{T}a+\bar{a})\leqq \mathrm{T}a$, contradicting $\bar{a}>0$.

10) We use $[p]\bar{x}$ insteated of $\bar{x}[p]$.

Lemma 5.1. *If m is continuous and $\overline{m}^1$ is a modular on $\overline{R}$ equivalent to the conjugate modular $\overline{m}$, then there exists a modular m_* on R which is equivalent to m and satisfies*

$$m_*(a)=\int_0^1 \mathrm{T}\xi a(a)d\xi \qquad \text{for all} \quad a\in F,$$

where T *is defined by* (23).

Proof. Without loss of generality we may assume that (R, m) is monotone complete. We define first a functional m_1 on F by

$$m_1(a)=\int_0^1 \mathrm{T}\xi a(a)d\xi \qquad \text{for all} \quad a\in F.$$

Since $\mathrm{T}\xi a(a)$ is an increasing function of $1\geqq\xi\geqq 0$, $m_1(\xi a)$ is a convex function of $1\geqq\xi\geqq 0$ for every $a\in F$, hence for any $a, b\in F$ and $0\leqq\alpha\leqq 1$

$$m_1(\alpha a+(1-\alpha)b) \leqq \alpha m_1(a)+(1-\alpha)m_1(b).$$

From the integration theory, $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda} a\in F$ implies $\sup_{\lambda\in\Lambda} m_1(a_\lambda)=m_1(a)$. If $a, b\in F$ $a\frown b=0$, then

$$\mathrm{T}\xi(a+b)(a+b)=\mathrm{T}\xi a(a)+\mathrm{T}\xi b(b) \qquad \text{for} \quad 0\leqq\xi\leqq 1,$$

consequently $\qquad m_1(a+b)=m_1(a)+m_1(b).$

Define a functional $\overline{m}^*$ on $\overline{R}$ by the formula:

$$\overline{m}^*(\overline{a}) = \sup_{x\in F}\{|\overline{a}|(x)-m_1(x)\} \qquad \text{for} \quad \overline{a}\in\overline{R}, \tag{25}$$

then we shall show that $\overline{m}^*$ satisfies all the modular conditions. The conditions except (3) are easily tested (see [12; §38]). Generally we can prove (see. [12; §58]): $\quad \mathrm{T}\tfrac{1}{2}a(\tfrac{1}{2}a)\leqq m_1(a)$

and $$\mathrm{T}a(b)+\mathrm{T}b(a) \leqq \mathrm{T}a(a)+\mathrm{T}b(b) \qquad \text{for} \quad a, b\in F.$$

For any $0\leqq\overline{a}\in\overline{R}$ there exists $\alpha>0$ such that $\overline{m}^1(\alpha\overline{a})<\infty$. By continuity of m, using (24) we can find $a\in F$ such that

$$\overline{m}^1(\alpha[p]\overline{a}) = m([p]a) \qquad \text{for all} \quad p\in R,$$

hence by the definition (23) we have $\alpha\overline{a}\leqq\mathrm{T}a$, hence it follows that

$$\begin{aligned}\alpha\overline{a}(\tfrac{1}{2}x) &\leqq \mathrm{T}a(\tfrac{1}{2}x) \leqq \{\mathrm{T}a(a)+\mathrm{T}\tfrac{1}{2}x(\tfrac{1}{2}x)\}\\ &\leqq \mathrm{T}a(a)+m_1(x) \qquad \text{for all} \quad x\in F,\end{aligned}$$

consequently $\quad \overline{m}^*\left(\dfrac{\alpha}{2}\overline{a}\right) = \sup_{x\in F}\left\{\dfrac{\alpha}{2}\overline{a}(x)-m_1(x)\right\}\leqq\mathrm{T}a(a)<\infty.$

Since for any $0\leqq x\in R$ there exists $\beta>0$ such that $m(\beta x)<\infty$, from the above consideration we can conclude that $(\overline{R}, \overline{m}^*)$ is monotone complete, hence by Theorem C $\overline{m}^*$ is equivalent to $\overline{m}$. Define m_* on R as the conjugate modular of $\overline{m}^*$. We can prove as in [12; §39] that

$$m_*(a)=m_1(a) \qquad \text{for all} \quad a\in F. \qquad \text{Q.E.D.}$$

We conclude this section with an example of change of a modular.

Therem 5.1. *Every modular is equivalent to one which is continuous and infinitely increasing at the same time. If the original modular is strictly convex (or even), the obtained one can be also.*

The proof is not difficult and is left to the readers.

§ 6. Strict convexity

In §§ 6–7 let (R, m) be *complete* under the modular norm. Our aim in this section is to investigate conditions under which we can introduce a new modular on R that is equivalent to the original and possesses some prescribed properties (for example, strict convexity, uniform convexity, etc.). We wish to express these conditions in connection with topological and order structures of R. Our investigation was motivated by the question: "what kinds of properties are invariant under an isomorphism[11] between two modulared semi-ordered linear spaces?" Since, as is shown in Chapter I, convexity and evenness of a modular are in close connection with those of their norms, our results give answers for the question: under what conditions the modular norm is equivalent to a strictly convex (uniformly convex, etc.) norm. Problems of this type were treated in the theory of Banach spaces by M. M. Day [8]. Supported by some of his results, we shall give almost complete answers, so far as modulared semi-ordered linear spaces are considered.

In the sequel, (M.), (O.), (T.) and (N.) denote respectively a condition related to the modular structure of R, to the order one, to the topological one and to the type of norms.

We begin with *almost finiteness,* though it has no direct connection with strict convexity and evenness. Consider the atomic part R_d of R. In R_d there exists a system of atoms $\{d_\lambda\}_{\lambda\in\Lambda}$ such that

$$\bigcup_{\lambda\in\Lambda}[d_\lambda]=[R_d], \qquad d_\lambda\frown d_\mu=0 \quad (\lambda\neq\mu) \qquad |||d_\lambda|||=1 \qquad (\lambda,\mu\in\Lambda).$$

We call $\{d_\lambda\}_{\lambda\in\Lambda}$ the *complete system of atoms* (with respect to m). Let $\{\overline{d}_\lambda\}_{\lambda\in\Lambda}$ be the complete system of atoms of $\overline{R}$ with respect to the conjugate modular $\overline{m}$. If we put

11) Isomorphic *as Banach spaces.*

$$\varphi_\lambda(\xi)=\begin{cases}\overline{m}(\xi\overline{d}_\lambda) & \text{for } 0\leqq\xi\leqq1,\\ (\xi-\overline{m}(\overline{d}_\lambda))^2+\overline{m}(\xi\overline{d}_\lambda) & \text{for } 1<\xi,\end{cases}$$

each $\varphi_\lambda(\xi)$ is a convex function of $\xi\geqq0$. Defining a new functional $\overline{m}_d$ on $\overline{R}_d$ by

$$\overline{m}_d(\overline{a})=\sum_{\lambda\in\Lambda}\varphi_\lambda(\xi_\lambda) \qquad \text{where } \xi_\lambda\overline{d}_\lambda=[\overline{d}_\lambda]|\overline{a}| \quad (\lambda\in\Lambda),$$

we obtain a modular satisfying

$$\overline{m}_d(\overline{a})\geqq\overline{m}(\overline{a}) \text{ for all } \overline{a}\in\overline{R}_d \text{ and } \overline{m}_d(\overline{a})=\overline{m}(\overline{a}) \text{ for } \overline{a}\in\overline{R}_d\ \overline{m}(\overline{a})\leqq1.$$

Thus $\overline{m}_d$ is a modular on $\overline{R}_d$, equivalent to $\overline{m}$. From the definition (26)–(27) we can see without difficulty that $\overline{m}$ is increasing. Hence by Theorem A the conjugate modular of $\overline{m}_d$ (considered on R_d) is almost finite and equivalent to m. Thus any modular is always equivalent to an almost finite one on the atomic part.

On the other hand, there exists the decomposition of the non-atomic part $R_c: R_c=R_{c,1}\oplus R_{c,2}$ such that m is almost finite on $R_{c,1}$ and *infinite* on $R_{c,2}$ (see [12; §44]). Since we may assume by Theorem 5.1 continuity of m, by [12, Theorem 44.16] there exists $0<a_0\in R_{c,2}$ such that for any $x\in R$ there exists $\alpha>0$ satisfying $[a_0]|x|\leqq\alpha a_0$.

Theorem 6.1. *The following conditions are mutually equivalent:*

(M_{af}) *m is equivalent to an almost finite modular.*

(O_{af}) *For any non-atomic $0<a\in R$ there exists $b\in[a]R$ such that*

$$(b-\nu a)^+\neq 0 \qquad (\nu=1,2,\cdots).$$

Proof. $(O_{af})\to(M_{af})$ is proved above. If for some non-atomic $0<a\in R$ and for any $x\in[a]R$ there exists $\alpha>0$ such that $|x|\leqq\alpha a$, any non-trivial semi-normal manifold of $[a]R$ is not uniformly continuous as itself, hence by Theorem 4.1 on any semi-normal manifold of $[a]R$ the modular norm is not continuous. This means that R does not admit any equivalent modular which is almost finite, i.e. $(M_{af})\to(O_{af})$. Q.E.D.

Theorem 6.2. *The following conditions are mutually equivalent:*

(M_i) *m is equivalent to an increasing modular.*

(O_i) *For any non-atomic $0<a\in R$ there exist $\{\alpha_\nu\}_{\nu=1}^\infty$ and $\{a_\nu\}_{\nu=1}^\infty\subseteqq[a]R$ such that $\sum_{\nu=1}\alpha_\nu=\infty$ and $\sum_{\nu=1}\alpha_\nu\beta_\nu a_\nu<+\infty$ for all C-bounded set of the form $\{\beta_\nu a_\nu\}_{\nu=1}^\infty$.*

Proof. If m is not equivalent to any increasing modular, by Theorem A the conjugate modular $\overline{m}$ is not equivalent to any almost finite modular.

Then by Theorem 6.1 there exists a non-atomic $0<\bar{a}\in\bar{R}$ such that for any $\bar{x}\in[\bar{a}]\bar{R}$ there exists $\alpha>0$ with $|\bar{x}|\leqq\alpha\bar{a}$. Then clearly the norm defined on $[\bar{a}]^R R$ by

(29) $$\|x\|_* = \bar{a}(|x|) \qquad \text{for } x\in[\bar{a}]^R R$$

is equivalent to the modular norm by m. Let $\{a_\nu\}_{\nu=1}^{\infty}\subseteq[\bar{a}]R$,

$$\sum_{\nu=1}^{\infty}\alpha_\nu=+\infty, \quad |||a_\nu|||=1 \quad \text{and} \quad \alpha_\nu\geqq 0 \qquad (\nu=1, 2, \cdots),$$

then we have $$\left\|\sum_{\nu=1}^{\kappa}\alpha_\nu a_\nu\right\|_* = \sum_{\nu=1}^{\kappa}\alpha_\nu\|a_\nu\|_* \xrightarrow{\kappa\to\infty} \infty$$

hence $\left\{\sum_{\nu=1}^{\kappa}\alpha_\mu a_\nu\right\}_{\kappa=1}^{\infty}$ is not order-bounded. This shows $(\mathrm{O}_i)\to(\mathrm{M}_i)$.

If m is equivalent to an increasing modular, we can find $\{a_\nu\}_{\nu=1}^{\infty}\subseteq R$ such that $\sum_{\nu=1}^{\infty}a_\nu<+\infty$ and $\sum_{\nu=1}^{\infty}\|a_\nu\|=+\infty$.

because, otherwise, putting

$$\hat{a}(a) = \sup\sum_{\nu=1}^{\infty}\|x_\nu\| \qquad \text{for } 0\leqq a\in R$$

where supremum is formed over all the family $\{x_\nu\}_{\nu=1}^{\kappa}$ such that

$$x_\nu\frown x_\mu=0 \quad (\nu\neq\mu), \qquad \sum_{\nu=1}^{\kappa}x_\nu\leqq a.$$

$\hat{a}(.)$ is finite valued and can be extended as a universally continuous linear functional over R. It is not difficult to see that for any $\bar{x}\in\bar{R}$ there exists $\alpha>0$ with $|\bar{x}|\leqq\alpha\hat{a}$, contradicting the assumption by Theorem 6.1. Putting $\alpha_\nu=\|a_\nu\|$ $(\nu=1, 2, \cdots)$ we have $\sum_{\nu=1}^{\infty}\alpha_\nu=\infty$ and $\sum_{\nu=1}^{\infty}\alpha_\nu\beta_\nu a_\nu<+\infty$ for all C-bounded set of the form $\{\beta_\nu a_\nu\}_{\nu=1}^{\infty}$ i.e. $(\mathrm{M}_i)\to(\mathrm{O}_i)$. Q.E.D.

We could not find any *topological invariance* for almost finiteness.

Theorem 6.3. *The following conditions are mutually equivalent:*

(M_f) *m is equivalent to a finite modular.*

(O_f) *R is uniformly continuous.*

(T_f) *$(R, |||.|||)$ does not contain any subspace isomorphic to l_∞.*

Proof. $(\mathrm{O}_f)\rightleftarrows(\mathrm{T}_f)$ follows from Theorem 4.1. $(\mathrm{M}_f)\to(\mathrm{O}_f)$ is clear. If the modular norm is continuous, by Lemma 1.6 m is finite on the non-atomic part R_c and is equivalent to an almost finite modular on the atomic part by Theorem 6.1. Since the *finite manifold* i.e. the totality of all finite elements, is closed under the norm, the almost finite modular obtained above is finite, because under continuity of the norm a closed complete semi-normal manifold coincides with the whole space, i.e. $(\mathrm{O}_f)\to(\mathrm{M}_f)$. Q.E.D.

Now let us take up strict convexity and evenness. Let m be *simple*. By Theorem 5.1 we may assume that m is further *continuous* and *infinitely increasing*. By Theorem A the conjugate modular $\overline{m}$ on $\overline{R}$ is continuous. The operator T, defined by (23) for $\overline{m}^1=\overline{m}$, satisfies

$$\overline{m}([p]\mathrm{T}a) = m([p]a) \qquad \text{for all} \quad p \in R,\ a \in F,$$

and is one-to-one from $F=\{x;\ 0 \leqq x \in R\ \ m(x)<\infty\}$ to $\overline{F}=\{\overline{x};\ 0 \leqq \overline{x} \in \overline{R}\ \ \overline{m}(\overline{x})<\infty\}$, because $x, y \in F\ \ x \neq y$ implies $m([p]x) \neq ([p]y)$ for some p by virtue of simplicity of m, hence $\mathrm{T}\xi a(a)$ is a strictly increasing function of $1 \geqq \xi \geqq 0$, consequently m_* defined by

$$m_*(a) = \int_0^1 \mathrm{T}\xi a(a)\xi \qquad \text{for} \quad a \in F$$

is *strictly convex* on F. By Lemma 5.1 we can extend m_* over R as an equivalent modular. We shall show strict convexity of m_* on R. If αa is a domestic element (with respect to m_*), we can prove that there exists $0 \neq [p_0] \leqq [a]$ such that $\alpha[p_0]a \in F$ in fact, for $\varepsilon>0$ satisfying $m_*((1+\varepsilon)\alpha a)<\infty$, put $e = \bigcup_{x \in F, x \leqq (1+\varepsilon)\alpha a} x$. If $[((1+\varepsilon)\alpha a-e)^+]e \equiv p \neq 0$, then for $0 \leqq \overline{a} \in \overline{R}$ with $\overline{a}(p)>0$ by (25) we have

$$\overline{m}_*(\xi[p]\overline{a}) \leqq \xi\overline{a}(p) \qquad \text{for all} \quad \xi \geqq 0$$

(because, from the definition of p, $x \in F$ implies $[p]x \leqq p$), hence

$$\begin{aligned} m_*((1+\varepsilon)\alpha[p]a) &= \sup_{\overline{x} \in \overline{R}} \{(1+\varepsilon)\alpha\overline{x}([p]a) - \overline{m}_*(\overline{x})\} \\ &\geqq (1+\varepsilon)\alpha\xi\overline{a}([p]a) - \overline{m}_*(\xi[p]\overline{a}) \\ &\geqq \xi\overline{a}([p]((1+\varepsilon)\alpha a-e))>0 \qquad \text{for all} \quad \xi>0, \end{aligned}$$

clearly this is a contradiction, thus we have proved $e=(1+\varepsilon)\alpha a$, consequently

$$0<\alpha[p_0]a \in F \qquad \text{for some} \quad p_0 \in R.$$

If

$$\alpha \geqq \beta \geqq 0 \quad \text{and} \quad m_*\left(\frac{\alpha+\beta}{2}\right) = \frac{m_*(\alpha a)+m_*(\beta a)}{2},$$

then by (19) we have

$$m_*\left(\frac{\alpha+\beta}{2}[p_0]a\right) = \frac{m_*(\alpha[p_0]a)+m_*(\beta[p_0]a)}{2}.$$

Strict convexity of m_* on F implies $\alpha[p_0]a=\beta[p_0]a$ i.e. $\alpha=\beta$. Thus we have proved that every simple modular is equivalent to a strictly convex one.

Theorem 6.4. *Let (R, m) be monotone complete. The following conditions are mutually equivalent:*

(M_s) *m is equivalent to a simple modular.*

(M_c) *m is equivalent to a strictly convex modular.*

(O_c) *R is superuniversally continuous.*

(T_c) *(R, $|||.|||$) does not contain any subpace isomorphic to $l_\infty(\Lambda)$, where Λ is a set of uncountable indices.*

(N_c) *The modular norm is equivalent to a strictly convex norm*[12].

Proof. (M_s)$\rightleftarrows$(M_c) is proved above. (M_c)$\rightarrow$(N_c) follows from Theorem 2.2, because we may assume that m_* defined above is infinitely increasing. M. M. Day [8] proved that $l_\infty(\Lambda)$ does not admit any equivalent norm which is strictly convex, hence we obtain (N_c)$\rightarrow$(T_c). (T_c)$\rightarrow$(O_c) is clear. It remains only to prove (O_c)$\rightarrow$(M_s). Putting $e = \bigcup_{m(x)=0} x$ (e exists by virtue of monotone completeness), we are in the situation that m is simple on $(1-[e])R$ and there exist $\{\bar{a}_\nu\}_{\nu=1}^\infty \subseteqq \bar{R}$ such that $\bigcup_{\nu=1}^\infty [\bar{a}_\nu]^R \geqq [e]$, because R is superuniversally continuous and semi-regular by assumption. Putting

$$\bar{a} = \sum_{\nu=1}^\infty \frac{|\bar{a}_\nu|}{2^\nu \|\bar{a}_\nu\|} \quad \text{and} \quad m_1(a) = m(a) + \bar{a}(|a|) \quad \text{for} \quad a \in R,$$

we obtain a simple modular equivalent to m. Q.E.D.

Remark 6.1. *In Theorem 6.4 we can not replace monotone completeness by completeness.*

In fact, generally we can prove that a semi-ordered Banach space with a continuous norm admits an equivalent, strictly convex norm. If Theorem 6.4 is true under *completeness*, a modulared semi-ordered linear space with a continuous norm has the second conjugate space $\bar{\bar{R}}$ which is superuniversally continuous. But this is not the case.

Theorem 6.5. *The following conditions are mutually equivalent.*

(M_m) *m is equivalent to a monotone modular.*

(M_e) *m is equivalent to an even modular.*

(O_m) *R is contractile.*

(T_m) *(R, $|||.|||$) does not admit any direct factor isomorphic to $l_1(\Lambda)$, where Λ is any set of uncountable indices.*

(N_m) *The associated norm on $\tilde{R}$ is equivalent to a strictly convex one.*

Proof. (M_m)$\rightleftarrows$(M_e) follows from Theorem 6.4 and Theorem A. If m is monotone, the associated modular $\tilde{m}$ on $\tilde{R}$ is simple by Theorem A,

12) In this section, equivalent norms need not satisfy the condition (★).

hence $(\mathrm{M}_m) \rightarrow (\mathrm{N}_m)$ follows from Theorem 6.4. $(\mathrm{N}_m) \rightarrow (\mathrm{O}_m) \rightarrow (\mathrm{M}_m)$ follows from Lemma 4.3, Theorem 6.4 and Theorem A. If $\overline{R}$ is not super-universally continuous, there exist $0 < \overline{a} \in \overline{R}$ and $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq R$ (Λ being a set of uncountable indices) such that

$$a_\lambda \frown a_\mu = 0 \quad (\lambda \neq \mu), \qquad |||a_\lambda||| = 1 \qquad (\lambda \in \Lambda)$$

and

$$\inf_{\lambda \in \Lambda} \cdot \overline{a}(a_\lambda) > 0 .$$

Putting

$$\alpha_\lambda = \overline{a}(a_\lambda), \qquad \overline{b}_\lambda = [a_\lambda]\overline{a} \qquad (\lambda \in \Lambda),$$

we define a linear operator P by

$$\mathrm{P}a = \sum_{\lambda \in \Lambda} \frac{\overline{b}_\lambda(a)}{\alpha_\lambda} a_\lambda \qquad \text{for} \quad a \in R.$$

Then clearly P is a bounded linear projection from R onto the subspace V generated by $\{a_\lambda\}_{\lambda \in \Lambda}$. On the other hand, V is isomorphic to $l_1(\Lambda)$. Thus we have proved $(\mathrm{T}_m) \rightarrow (\mathrm{O}_m)$ by virtue of Lemme 4.3 and Remark 4.2. Finally if $(R, |||.|||)$ contains a direct factor isomorphic to $l_1(\Lambda)$, then $(\widetilde{R}, \|\sim\|)$ contains a subspace isomorphic to $l_\infty(\Lambda)$, thus $(\mathrm{M}_m) \rightarrow (\mathrm{T}_m)$ follow from Theorem 6.4 and Theorem A. Q.E.D.

Remark 6.2. *In Theorem* 6.5 *we can not replace* (T_m) *by*

(T_e) $(R, |||.|||)$ *does not contain any subspace isomorphic to* $l_1(\Lambda)$.

In fact, $l_1(\Lambda)$ is isometrically imbedded into $l_\infty(\Gamma)$ for some Γ and $l_\infty(\Gamma)$ admits a monotone modular as a modulared semi-ordered linear space.

Remark 6.3. *In Theorem* 6.5 *we can not replace* (N_m) *by*

(N_e) *The modular norm is equivalent to an even norm.*

For evenness of the norm implies its continuity.

Finally we shall give conditions for evenness of the modular norm.

Theorem 6.6. *The following conditions are mutually equivalent.*

$(\mathrm{M}_{f,m})$ *m is equivalent to a finite, monotone modular.*

$(\mathrm{M}_{f,e})$ *m is equivalent to a finite, even modular.*

$(\mathrm{O}_{f,m})$ *R is uniformly continuous and contractile.*

$(\mathrm{T}_{f,e})$ $(R, |||\ |||)$ *does not contain any subspace isomorphic to* l_∞ *or to* $l_1(\Lambda)$, *where* Λ *is any set of uncountable indices.*

(N_e) *The modular norm is equivalent to an even norm.*

Proof. Using Theorem 6.3–6.5, we can prove: $(\mathrm{M}_{f,e}) \rightleftarrows (\mathrm{M}_{f,e}) \rightleftarrows (\mathrm{O}_{f,m})$. $(\mathrm{T}_{f,e}) \rightarrow (\mathrm{O}_{f,e})$ is clear from Theorems 6.4–5. M. M. Day [8] proved that $l_1(\Lambda)$ does not admit any even norm equivalent to the original one, hence we obtain $(\mathrm{N}_e) \rightarrow (\mathrm{T}_{f,e})$. $(\mathrm{M}_{f,e}) \rightarrow (\mathrm{N}_e)$ follows from Theorem 2.3. Q.E.D.

§ 7. Uniform convexity

For normed semi-ordered linear spaces, *completeness* and *monotone completeness* are different properties. Some special aspect of a modulared semi-ordered linear space which is monotone complete and has a continuous modular norm was studied in [1], [14] and [16]. For example, S. Yamamuro [16] proved

Lemma 7.1. *If (R, m) is monotone complete, m is simple and the modular norm is continuous, then m is uniformly simple.*

Now let (R, m) be *monotone complete* and the modular norm be *continuous*. Since continuity of the modular norm implies superuniversal continuity of R, by Theorems 6.3–6.4 we may assume that m is simple and finite. Then by Lemma 7.1 m is uniformly simple. There exist $\gamma>0$ such that $\quad m(x)\leqq\frac{1}{\gamma} \quad$ implies $\quad |||x|||\leqq\frac{1}{2}$.

Define a new modular m_1 by $m_1=\gamma m$, then we have:

$$m_1(x)\leqq 1 \qquad \text{implies} \qquad m_1(2x)\leqq\gamma .$$

Let $\{d_\lambda\}_{\lambda\in\Lambda}$ be a complete system of atoms of R (with respect to m_1). Define real convex functions φ_λ by

$$\varphi_\lambda(\xi)=\begin{cases} m_1(\xi d_\lambda) & \text{for } 0\leqq\xi\leqq 1,\\ 1+(\xi-1)\pi_{1,+}(d_\lambda) & \text{for } 1\leqq\xi,\end{cases}$$

and a functional m_2 on R by

$$m_2(a)=m_1([R_c]a)+\sum_{\lambda\in\Lambda}\varphi_\lambda(\xi_\lambda) \quad \text{where} \quad \xi_\lambda d_\lambda=[d_\lambda]\,|a| \qquad (\lambda\in\Lambda),$$

then m_2 is a modular equivalent to m. Further we have

$$m_2(2\xi d_\lambda)\leqq\gamma_1 m_2(\xi d_\lambda) \qquad \text{for some } \gamma_1>0 \text{ and for all } \xi\geqq 1.$$

On the other hand, by Lemma 1.6 m_2 is uniformly finite on R_c. Combining these, we can conclude uniform finiteness of m_2, (see[1]) in fact,

$$m_2(a)\geqq 1 \quad \text{implies} \quad m_2(2a)\leqq\gamma_2 m_2(a) \quad \text{for some constant } \gamma_2.$$

The subset

$$U=\{x;\ m_2(2[p]x)\geqq 2\gamma_2 m_2([p]x) \quad \text{for all } p\in R\}$$

is *directed*, i.e. $x, y\in U$ implies $x\smile y\in U$ and by the definition of γ_2

$$x\in U \qquad \text{implies} \qquad m_2(x)\leqq 1 .$$

By monotone completeness of (R, m_2), there exists $e=\bigcup_{x\in U} x$ and for $e_0=2e$

$$\omega_2(2\xi, e_0, \mathfrak{p})\leqq 3\gamma_2\omega_2(\xi, e_0, \mathfrak{p}) \qquad \text{for all } \xi\geqq 1,\ \mathfrak{p}\in U_{[e]},$$

where $\mathfrak{p}$ is a point of the *proper space* of R and

$$\omega_2(\xi, e_0, \mathfrak{p}) = \lim_{[p]\to\mathfrak{p}} \frac{m_2(\xi[p]e_0)}{m_2([p]e_0)}$$

(for the detailed discussion, see [12]). Put, for $\mathfrak{p}\in U_{[e]}$,

$$\omega_*(\xi, \mathfrak{p}) = \begin{cases} \xi & \text{for } 0 \leqq \xi \leqq 1, \\ \omega_2(\xi, e_0, \mathfrak{p}) & \text{for } \xi \geqq 1, \end{cases}$$

then $\omega_*(\xi, \mathfrak{p})$ is a convex function of $\xi \geqq 0$ and a continuous one of $\mathfrak{p}$ and satisfies the condition: $\omega(2\xi, \mathfrak{p}) \leqq (2+3\gamma_2)\omega_*(\xi, \mathfrak{p})$ for $\xi \geqq 0$, $\mathfrak{p}\in U_{[e]}$. Now we define a new modular m_* by

$$m_*(a) = \int_{[e]} \omega_*\Big(\Big(\frac{|a|}{e_0}, \mathfrak{p}\Big), \mathfrak{p}\Big) m(d\mathfrak{p}e_0) + m_2((1-[e_0])a).$$

Then we have $\quad m_2(a) \leqq m_*(a) \leqq m_2(a) + m_2(e) \quad$ for all $a\in R$.

Since R is monotone complete under both m and m_*, they are equivalent by Theorem C. From the definition of e_0 we have

$$m_2(2x) < 2\gamma_2 m_2(x) \qquad \text{for all } x\in(1-[e_0]))R,$$

hence

$$m_*(2a) \leqq (2+3\gamma_2) m_*(a) \qquad \text{for all } a\in R,$$

that is, m_* is upper bounded.

Combining this result with Theorem 4.2, we obtain

Theorem 7.1. *The following conditions are mutually equivalent:*

(M_{us}) *m is equivalent to a uniformly simple (or uniformly finite) modular.*

(M_{ub}) *m is equivalent to an upper bounded modular.*

(O_{us}) *R is K-bounded and uniformly continuous.*

(T_w) *$(R, |||.|||)$ is weakly complete.*

(T_{us}) *$(R, |||.|||)$ does not contain any subspace isomorphic to c_0.*

We could not succeed in characterizing (M_{ub}) by the type of norms.

Theorem 7.2. *The following conditions are mutually equivalent.*

(M_{um}) *m is equivalent to a uniformly monotone (or uniformly increasing) modular.*

(M_{lb}) *m is equivalent to a lower bounded modular.*

(O_{um}) *R is strictly contractile.*

(T_{um}) *$(R, |||.|||)$ does not admit any direct factor isomorphic to l_1.*

The proof is similar to that of Theorem 6.5.

Remark 7.1. *In Theorem 7.1 we can not replace* (T_{um}) *by*

(T'_{um}) *$(R, |||.|||)$ does not contain any subspace isomorphic to l_1.*

In this direction, corresponding to Theorem 6.6, we have

Theorem 7.3. *The following conditions are mutually equivalent*

($M_{f,lf}$) *m is equivalent to a finite, lower bounded modular.*

($O_{f,um}$) *R is uniformly continuous and strictly contractile.*

($T_{f,um}$) *$(R, |||.|||)$ does not contain any subspace isomorphic to l_1.*

Finally we shall consider *uniform convexity* and *uniform evenness.* It is well-known that a uniformly convex (or uniformly even) Banach space is reflexive (see [13; §§76–77]). In case of modulared semi-ordered linear space, if m is uniformly convex (or uniformly even), $(R, |||.|||)$ is reflexive.

Now let $(R, |||.|||)$ be *reflexive.* Then by Theorem 4.3 and Theorem 7.1 m (resp. its conjugate modular $\overline{m}$) is equivalent to an upper bounded modular m_1 (resp. to $\overline{m}^1$). Since m_1 and $\overline{m}^1$ are both simple and finite, by Lemma 5.1 there exists a one-to-one mapping T from $R^+=\{x; x\geqq 0\}$ onto $\overline{R}^+=\{\overline{x}; \overline{x}\geqq 0\}$ defined by the condition:

$$\overline{m}^1([p]\mathrm{T}a) = m_1([p]a) \qquad \text{for all} \quad p\in R,$$

and we can define a new modular m_* on R, equivalent to m, by

$$m_*(a) = \int_0^1 \mathrm{T}\xi|a|(|a|)d\xi \qquad \text{for} \quad a\in R.$$

First we shall show uniform convexity of m_*. Let $\gamma, \varepsilon>0$ $\gamma\geqq\alpha\geqq\beta\geqq 0$, $\alpha-\beta\geqq\varepsilon$ and $m_*(a)=1$. By the definition of m_* we have

$$\frac{m_*(\alpha a)+m_*(\beta a)}{2} = m_*\Big(\frac{\alpha+\beta}{2}a\Big)+\frac{\alpha-\beta}{4}\int_0^1(\mathrm{T}a_\xi-\mathrm{T}a^\xi)(a)d\xi,$$

where

$$a_\xi=\Big(\frac{\alpha+\beta}{2}+\xi\frac{\alpha-\beta}{2}\Big)a \quad \text{and} \quad a^\xi=\Big(\frac{\alpha+\beta}{2}-\xi\frac{\alpha-\beta}{2}\Big) \qquad \text{for} \quad 0\leqq\xi\leqq 1,$$

Since

$$a_\xi\geqq a^\xi \quad \text{and} \quad \mathrm{T}a_\xi\geqq\mathrm{T}a^\xi \qquad \text{for} \quad 0\leqq\xi\leqq 1,$$

$$\frac{m_*(\alpha a)+m_*(\beta a)}{2}\geqq$$

$$\geqq m_*\Big(\frac{\alpha+\beta}{2}a\Big)+\frac{\alpha-\beta}{4}\int_{\frac{1}{2}}^1(\mathrm{T}a_\xi-\mathrm{T}a^\xi)(a)d\xi.$$

Since

$$|||a_\xi-a^\xi|||_*=\frac{\xi(\alpha-\beta)}{2}\geqq\frac{\varepsilon}{4} \qquad \text{for} \quad 1\geqq\xi\geqq\frac{1}{2},$$

by uniform simplicity and uniform finiteness of m_1 there exist κ, $\rho>0$ such that

$$m_1(a_\xi - a^\xi) \geqq \rho \qquad \text{for} \quad \tfrac{1}{2} \leqq \xi \leqq 1,$$

$$\overline{m}^1(\mathrm{T}a^\xi) \leqq \overline{m}^1(\mathrm{T}a_\xi) = m_1(a_\xi) \leqq \kappa \qquad \text{for} \quad 0 \leqq \xi \leqq 1,$$

hence we have $\qquad |||\mathrm{T}a^\xi|||^1 \leqq |||\mathrm{T}a_\xi|||^1 \leqq \kappa + 1 \qquad \text{for} \quad 0 \leqq \xi \leqq 1.$

By uniform finiteness of $\overline{m}^1$ there exists $\mu > 0$ such that $|||\overline{x}|||^1 \leqq 2(\kappa+1)$ implies $\overline{m}^1(\overline{x}) \leqq \mu$. We shall prove that

$$|||\mathrm{T}a_\xi - \mathrm{T}a^\xi|||^1 \geqq \frac{\rho(\kappa+1)}{2\mu} \qquad \text{for} \quad \tfrac{1}{2} \leqq \xi \leqq 1.$$

Suppose the contrary, i.e. for some $\frac{1}{2} \leqq \xi_0 \leqq 1$ $|||\mathrm{T}a_{\xi_0} - \mathrm{T}a_{\xi_0}|||^1 < \dfrac{\rho(\kappa+1)}{2\mu}$.

Since

$$\left|\left|\left| \mathrm{T}a^{\xi_0} + \frac{2\mu}{\rho}(\mathrm{T}a_{\xi_0} - \mathrm{T}a^{\xi_0}) \right|\right|\right|^1 \leqq \kappa + 1 + \frac{2\mu}{\rho} \cdot \frac{\rho(\kappa+1)}{2\mu} = 2(\kappa+1),$$

we have

$$\begin{aligned} \rho &\leqq m_1(a_{\xi_0} - a^{\xi_0}) \\ &\leqq m_1(a_{\xi_0}) - m_1(a^{\xi_0}) = \overline{m}^1(\mathrm{T}a_{\xi_0}) - \overline{m}^1(\mathrm{T}a^{\xi_0}) \\ &\leqq \frac{\rho}{2\mu}\overline{m}^1\left(\mathrm{T}a^{\xi_0} + \frac{2\mu}{\rho}\left(\mathrm{T}a_{\xi_0} - \mathrm{T}a^{\xi_0}\right)\right) \leqq \frac{\rho}{2\mu} \cdot \mu = \frac{\rho}{2}, \end{aligned}$$

$\left(\text{because} \quad m(x) \leqq m(y) + \varepsilon \cdot m\left(y + \frac{1}{\varepsilon}(x-y)\right) \qquad \text{for all} \quad x, y \quad 0 < \varepsilon < 1\right)$

clearly this is a contradiction. On the other hand, it is known (see [12; § 60]) that

$$m_*(a) + \overline{m}_*(\mathrm{T}a) = \mathrm{T}a(a) \qquad \text{for all} \quad a \in R$$

and

$$\overline{m}_*(\overline{a}) = \int_0^1 \overline{a}(\mathrm{T}^{-1}\xi\overline{a})d\xi \qquad \text{for all} \quad 0 \leqq \overline{a} \in \overline{R},$$

hence $\overline{m}_*$ is uniformly simple by Lemma 7.1, consequently there exists $\delta > 0$ such that

$$\overline{m}_*(\mathrm{T}a_\xi - \mathrm{T}a^\xi) \geqq \delta \qquad \text{for} \quad \tfrac{1}{2} \leqq \xi \leqq 1.$$

Thus we obtain

$$\begin{aligned} \delta &\leqq \overline{m}_*(\mathrm{T}a_\xi - \mathrm{T}a^\xi) \leqq (\mathrm{T}a_\xi - \mathrm{T}a^\xi)(\mathrm{T}^{-1}(\mathrm{T}a_\xi - \mathrm{T}a^\xi)) \\ &\leqq (\mathrm{T}a_\xi - \mathrm{T}a^\xi)(\mathrm{T}^{-1}\mathrm{T}a^\xi) \leqq \gamma(\mathrm{T}a_\xi - \mathrm{T}a^\xi)(a), \end{aligned}$$

hence

$$\frac{\delta}{\gamma} \leqq (\mathrm{T}a_\xi - \mathrm{T}a^\xi)(a) \qquad \text{for} \quad \tfrac{1}{2} \leqq \xi \leqq 1.$$

Finally we can conclude

$$\frac{m_*(\alpha a) + m_*(\beta a)}{2} \geqq m_*\left(\frac{\alpha+\beta}{2}a\right) + \frac{(\alpha-\beta)\delta}{4\gamma}.$$

Thus m_* is uniformly convex by definition. Similarly $\overline{m}_*$ is uniformly convex. It is not difficult to see that both m_* and $\overline{m}_*$ are upper bounded, hence by Theorem B m is bounded. Finally by Lemma 3.1 m is uniformly even.

Summarizing these results, we obtain.

Theorem 7.4. *The following conditions are mutually equivalent.*

(M_b) *m is equivalent to a bounded modular.*

(M_{uce}) *m is equivalent to a uniformly convex, uniformly even modular.*

(O_r) *R is K-bounded, uniformly continuous and strictly contractile.*

(T_r) *$(R, |||.|||)$ is reflexive.*

(T_{usm}) *$(R, |||.|||)$ does not contain any subspace isomorphic to c_0 or to l_1.*

(N_{uce}) *The modular norm is equivalent to a uniformly convex and uniformly even one.*

Remark 7.2. *The condition* (N_{uce}) *is one of the characteristic properties of a reflexive modulared semi-ordered linear spaces.*

In fact, M. M. Day [6] gave an example of a relexive semi-ordered Banach space which does not admit any uniformly convex norm, equivalent to the original one.

Appendix

A non-modularable semi-ordered Banach space with a uniformly convex and uniformly even norm

From the consideration in Chapter II, it is natural to ask whether a semi-ordered Banach space, with a uniformly convex and uniformly even norm, is *modularable*, i.e. its norm is equivalent to a modular norm by some modular. Recently T. Shimogaki [14] gave a negative answer to the conjecture that a semi-ordered Banach space is modularable. The function space used in his counter example has a continuous norm, but is not reflexive. We shall give an example of a semi-ordered Banach space, with a uniformly convex and uniformly even norm, which is not modularable.

Lemma. *If two finite modulars m_1, m_2 on a non-atomic semi-ordered linear espace R are equivalent, there exist $\alpha, \beta > 0$ such that*

$$R \ni a \quad m_1(a) \geqq 1 \quad \textit{implies} \quad m_1(a) \leqq m_2(\alpha a) \leqq m_1(\alpha\beta a).$$

Proof. (cf. [1]). Since R is non-atomic and m_1 is finite, for any $0 \leqq a \in R$ $m_1(a) \geqq 1$ there exist $\{a_\nu\}_{\nu=1}^{\kappa}$ such that $a_\nu \frown a_\mu = 0$ $(\nu \neq \mu)$,

$$\sum_{\nu=1}^{\kappa} a_\nu = a \qquad \tfrac{1}{2} \leqq m_1(a_\nu) \leqq 1 \qquad (\nu = 1, 2, \cdots, \kappa).$$

On the other hand, there exist $\alpha', \beta' > 0$ such that

$$\alpha' |||x|||_1 \leqq |||x|||_2 \leqq \beta' |||x|||_1 \qquad \text{for all} \quad x \in R,$$

because these two norms are equivalent by assumption. Since

$$|||a_\nu|||_2 \geqq \alpha' |||a_\nu|||_1 \geqq \frac{\alpha'}{2} \qquad (\nu=1, 2, \cdots, \kappa),$$

we have

$$m_2\left(\frac{2a_\nu}{\alpha'}\right) \geqq 1 \qquad (\nu=1, 2, \cdots, \kappa)$$

hence

$$m_1(a) = \sum_{\nu=1}^{\kappa} m_1(a_\nu) \leqq \kappa \leqq \sum_{\nu=1}^{\kappa} m_2\left(\frac{2a_\nu}{\alpha'}\right) = m_2\left(\frac{2a}{\alpha'}\right).$$

Similarly we have $\quad m_2\left(\frac{2a}{\alpha'}\right) \leqq m_1\left(\frac{4\beta'}{\alpha'} a\right).$ Q.E.D.

$\mathbf{L}_p$ $(1<p<\infty)$ is the totality of all p-integrale function f on the unit interval, i.e. $\int_0^1 |f(t)|^p dt < \infty$. It is a modulared semi-ordered linear space with the modular

$$m_p(f) = \int_0^1 |f(t)|^p dt$$

and the modular norm

$$|||f|||_p = \left(\int_0^1 |f(t)|^p dt\right)^{1/p}$$

Let $\mathbf{L}_p^q$ denote the totality of all measurable function F on the unit square such that

$$(\sharp) \qquad \|F\| = \left\{\int_0^1 \left(\int_0^1 |F(t, s)|^p dt\right)^{q/p} ds\right\}^{1/q} < \infty .$$

$\mathbf{L}_p^q$ is a semi-ordered Banach space with the norm ($\sharp$). It is known that the norm on $\mathbf{L}_p^q$ is both uniformly convex and uniformly even (see [7]).

We shall show that $\mathbf{L}_p^q$ is not modularable, if $p \neq q$. Suppose that $\mathbf{L}_p^q$ is modularable by a modular m. Then the linear operator T_1 from $\mathbf{L}_p$ to $\mathbf{L}_p^q$ and T_2 from $\mathbf{L}_q$ to $\mathbf{L}_p^q$ are defined by

$$\mathbf{L}_p \ni f \longrightarrow (\mathrm{T}_1 f)(t, s) \equiv f(t) \qquad (\text{for} \quad 0 \leqq t,\ s \leqq 1),$$

and

$$\mathbf{L}_q \ni g \longrightarrow (\mathrm{T}_2 g)(t, s) \equiv g(s) \qquad (\text{for} \quad 0 \leqq t,\ s \leqq 1).$$

We can prove easily that

$$\|\mathrm{T}_1 f\| = |||f|||_p \quad \text{for all } f \in \mathbf{L}_p \quad \text{and} \quad \|\mathrm{T}_2 g\| = |||g|||_q \quad \text{for all } g \in \mathbf{L}_q.$$

Further $\quad \bigcap_{\nu=1}^{\infty} f_\nu = 0$ in $\mathbf{L}_p$ $\quad$ implies $\quad \bigcap_{\nu=1}^{\infty} \mathrm{T}_1 f_\nu = 0$ in $\mathbf{L}_p^q$, and $\bigcap_{\nu=1}^{\infty} g_\nu = 0$ in $\mathbf{L}_q$ implies $\bigcap_{\nu=1}^{\infty} \mathrm{T}_2 g_\nu = 0$ in $\mathbf{L}_p^q$.

Since $\mathbf{L}_p^q$ is non-atomic and the norm on it is continuous, m is finite by Lemma 1.6. Putting

$$m_*(f) = m(\mathrm{T}_1 f) \quad \text{for } f \in \mathbf{L}_p \quad \text{and} \quad m^*(g) = m(\mathrm{T}_2 g) \quad \text{for } g \in \mathbf{L}_q$$

we obtain a modular m_* on $\mathbf{L}_p$ (and m^* on $\mathbf{L}_q$) equivalent to m_p (and to m_q respectively). By Lemma there exist $\alpha, \beta > 0$ such that

$$m_p(f) \geqq 1 \quad \text{implies} \quad m_p(f) \leqq m_*(\alpha f) \leqq m_p(\alpha\beta f)$$

and $$m_q(g) \geqq 1 \quad \text{implies} \quad m_q(g) \leqq m^*(\alpha g) \leqq m_q(\alpha\beta g).$$

Putting $$f_0(t) = g_0(s) \equiv 1 \qquad (\text{for all } 0 \leqq t, s \leqq 1),$$

we have $$\mathrm{T}_1 f_0 = \mathrm{T}_2 g_0 \quad \text{and} \quad m_p(f_0) = m_q(g_0) = 1,$$

hence

$$m_p(\xi f_0) = \xi^p \leqq m_*(\alpha \xi f_0) \leqq m_p(\alpha\beta\xi f_0) = (\alpha\beta)^p \xi^p$$

and $$m_q(\xi g_0) = \xi^q \leqq m^*(\alpha \xi g_0) \leqq m_q(\alpha\beta\xi g_0) = (\alpha\beta)^q \xi^q$$

consequently $$\left(\frac{1}{\alpha\beta}\right)^p \leqq \xi^{p-q} \leqq (\alpha\beta)^q \qquad \text{for all } \xi \geqq 1.$$

This inequality is possible, only when $p = q$.

It remains a fundamental question "under what conditions a semi-ordered linear space is modularable". We shall treat this problem in another paper.

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Added in proof. After we had prepared this paper, we received S. Yamamuro's paper: *On conjugate spaces of Nakano spaces*, Trans. Amer. Math. Soc. vol. 90 (1959) pp. 291–311. It has contact with ours in some extent, in particular, parts of Theorems 2.1–2 are proved and the assertion (22) is discussed from the another direction.