ON AN EQUIVALENCE RELATION ON SEMI-ORDERED LINEAR SPACES

By

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§ 1. Let (E, Ω, μ) be a finite measure space with a countably additive non-negative measure μ defined on a σ-field Ω. Two real-valued μ-measurable functions $f(t)$ and $g(t)$ on E are called *mutually equi-measurable* [14], if $\mu\{t; f(t)>r\} = \mu\{t; g(t)>r\}$ holds for each real number r. If we write $f\sim g$, when f and g are mutually equi-measurable, it is observed easily that the relation $\sim$ is an equivalence relation on the space $\mathfrak{M}$ of all measurable functions on E. As is shown in [14], the concept of equi-measurability plays an important rôle in the theory of functions of real variables. Now let $\boldsymbol{X}$ be a linear space consisting of real-valued measurable functions, which is *semi-normal* in the sense of Nakano [11], i.e.

$$(1.1) \qquad 0 \leqq f \in \boldsymbol{X}, \quad |g| \leqq f, \quad g \in \mathfrak{M} \text{ implies } g \in \boldsymbol{X},$$

where $0 \leqq f$ means that $0 \leqq f(t)$ holds almost everywhere. Evidently the function space $\boldsymbol{X}$ is considered as a universally continuous semi-ordered linear space[1)] by this order.

We say that a function space $\boldsymbol{X}$ has the *weak rearrangement invariant property* (*w-RIP*), if $f\in\boldsymbol{X}$, $f\sim g$ always implies $g\in\boldsymbol{X}$, i.e. $\boldsymbol{X}$ is closed under the relation defined by equi-measurability. In the sequel, a function space $\boldsymbol{X}$ on E is termed to be a *Banach function space*[2)] on E, if it is semi-normal and has a complete norm satisfying

$$(1.2) \qquad \|f\| = \sup_{\lambda\in\Lambda} \|f_\lambda\|, \qquad \text{whenever } 0 \leqq f_\lambda \uparrow_{\lambda\in\Lambda} f.$$

A Banach function space $\boldsymbol{X}$ is said to have the *strong rearrangement invariant property* (*s-RIP*), if $f\in X$, $f\sim g$ implies $g\in\boldsymbol{X}$ and $\|g\| \leqq A\|f\|$, where A is a fixed constant independent on f and g. $\boldsymbol{L}^p(E)$ spaces with $1\leqq p$, Orlicz spaces $\boldsymbol{L}_\Phi(E)$ and $\Lambda(\phi)$-spaces established by G. G. Lorentz [5, 6] and I. Halperin

1) A semi-ordered linear space R is called *universally continuous*, if $0 \leqq a_\lambda$ $(\lambda\in\Lambda)$ implies $\bigcap_{\lambda\in\Lambda} a_\lambda \in R$, i.e. a conditionally complete vector lattice in Birkhoff's sense or a K-space in the sense of Vulich [12].

2) For the detailed properties of Banach function spaces see [7] or [13].

[1] independently with much regard to this property, have all *s-RIP* with the majorant 1 obviously. The subject of this note concerns with *RIP* of function spaces, but we deal with abstract semi-ordered linear spaces in the first place, since the theory of semi-ordered linear spaces can throw light on this subject by formalization and by use of representation theory of the spaces.

In §2 we generalize axiomatically the relation of equi-measurability on function spaces, to an equivalence relation (called an $\mathcal{E}$-relation) on abstract semi-ordered linear spaces R. Theorem 1 shows, however, that in case the space R is discrete, the equivalence relation generalized is essentially the same one as is given by equi-measurability on R considered as a discrete measure space. In the next section 3, we treat about a semi-ordered linear space R which has a certain functional ρ together with an $\mathcal{E}$-relation. Utilizing some topological properties of the proper space $\mathfrak{E}$ of R, we derive a result showing that the functional ρ is uniformly bounded with respect to the $\mathcal{E}$-relation in a sense (Theorem 1). In § 4 we return to function spaces and applying this result, we show that if a Banach function space has *w-RIP*, then it must have *s-RIP*, in case E is a non-atomic finite measure space (Theorem 3). Furthermore, as another application of this, we state a theorem characterizing Orlicz spaces among modulared function spaces $\boldsymbol{L}_{M(\xi,t)}(E)$ in terms of *RIP*, i. e. we prove that if a modulared function space $\boldsymbol{L}_{(M\xi,t)}(E)$ has *w-RIP* it reduces to an Orlicz space $\boldsymbol{L}_{\Phi}(E)$ (Theorem 4).

At the end of this paper we extend the equi-measurablity relation on finite measure spaces to the relation between two integrable functions on ρ-finite measure spaces. It is then noted that for function spaces on ρ-finite measure spaces, the above results concerning *w-RIP* and *s-RIP* hold all to be valid.

§ 2. It will be assumed, in the sequel, that R is a *universally continuous semi-ordered linear space* and S^+ $(S\subset R)$ denotes the set of all positive parts of S, i. e. $S^+=\{x\cup 0\,;\ x\in S\}$. A linear lattice manifold M of R is called a *P-manifold*, if $[p]M\subset M$ for any projector $[p]$[3] $(p\in R)$. A P-manifold M is called *full*, if $M\perp x$[4] implies $x=0$. It is obvious that if M is a full P-manifold, $0\leqq x$ is represented as $x=\bigcup_{\lambda\in\Lambda} x_\lambda$, where $x_\lambda\in M$ $(\lambda\in\Lambda)$. A system $\{x_\lambda\}_{\lambda\in\Lambda}$ of elements of M is said to be *M-fundamental with respect to* $x\in R$, if $x=\bigcup_{\lambda\in\Lambda} x_\lambda$ and $[p]x=[p]x_\lambda$ holds for each $\lambda\in\Lambda$, whenever $[p]x\in M$. Now we introduce an equivalence relation on R which can be considered as a generalization of that of equi-measurability in function spaces.

An equivalence relation $\sim$ on R^+ is called an *$\mathcal{E}$-relation*, if it satisfies the

3) A projector $[p]$ is a projection operator on R onto the normal manifold $\{p^\perp\}^\perp$.

4) $M\perp x$ means $|x|\cap|y|=0$ for all $y\in M$. We write $x=x_1\oplus x_2$, if $x=x_1+x_2$ and $x_1\perp x_2$.

following conditions (R. 1) – (R. 4):

(R. 1) $x \sim y$, $x, y \in R^+$ *implies* $\alpha x \sim \alpha y$ *for each* $\alpha > 0$;

(R. 2) *if* $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda}$ *and* $0 \leqq y_\lambda \uparrow_{\lambda \in \Lambda}$ *and* $x_\lambda \sim y_\lambda$ *for each* $\lambda \in \Lambda$, *then* $\bigcup_{\lambda \in \Lambda} x_\lambda \in R^+$ *implies* $\bigcup_{\lambda \in \Lambda} y_\lambda \in R^+$ *with* $\bigcup_{\lambda \in \Lambda} x_\lambda \sim \bigcup_{\lambda \in \Lambda} y_\lambda$;

(R. 3) *if* $x = x_1 \oplus x_2$, $y = y_1 \oplus y_2$ *and* $x_i \sim y_i$ $(\mathrm{i} = 1, 2)$, *then* $x \sim y$;

(R. 4) *there exists a full P-manifold* $M \subset R$ *satisfying the following properties*:

(i) *if* $x \sim y$, $x, y \in R^+$, *then there exists a pair of M-fundamental systems* $\{x_\lambda\}_{\lambda \in \Lambda}$ *and* $\{y_\lambda\}_{\lambda \in \Lambda}$ *with respect to* x *and* y *respectively such that* $x_\lambda \sim y_\lambda$ *holds for each* $\lambda \in \Lambda$;

(ii) *if* $x \sim y$, $x, y \in M^+$ *and* $\{[p_\lambda]\}_{\lambda \in \Lambda}$ *is a mutually orthogonal system of projectors with* $\sum_{\lambda \in \Lambda} [p_\lambda] = [x]$ *there exists also a mutually orthogonal system of projectors* $\{[q_\lambda]\}_{\lambda \in \Lambda}$ *such that*

$$\sum_{\lambda \in \Lambda} [q_\lambda] = [y] \quad \textit{and} \quad [p_\lambda] x \sim [q_\lambda] y \qquad (\lambda \in \Lambda) \ \textit{hold.}$$

The P-manifold M satisfying the conditions (i) and (ii) in (R. 4) is called the *D-manifold* of the $\mathcal{E}$-relation $\sim$.

It is clear that the $\mathcal{E}$-relation on R^+ can be extended to R canonically, i. e. we now induce the relation $\sim$ to be defined on R in such a way that for any $x, y \in R$

$$(2.1) \qquad x \sim y \ \textit{if and only if} \ x^+ \sim y^+ \ \textit{and} \ x^- \sim y^-.$$

This extended $\mathcal{E}$-relation $\sim$ is called an *$\mathcal{E}$-relation* on R, and R associated with an $\mathcal{E}$-relation is called shortly a *space with an $\mathcal{E}$-relation*. It follows from the condition (R. 4) that if $x \sim y$, $x, y \in M$ and $[p]$ is an arbitrary projector, then we can find a projector $[q]$ for which $[p]x \sim [q]y$ and $(1 - [p])x \sim (1 - [q])y$ hold simultaneously, that is, we may say that *the $\mathcal{E}$-relation $\sim$ is decomposable on M.*

In what follows, $\sim$ stands for an $\mathcal{E}$-relation on R always.

Lemma 1. $x \sim 0$ *implies* $x = 0$.

Proof. If $x \sim 0$ and $x \in R^+$, then $nx \sim 0$ for each natural number n by (R. 1). Putting $x_n = 0$ and $y_n = nx$ $(n = 1, 2, \cdots)$, we obtain increasing sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \sim y_n$ $(n \geqq 1)$. Since $0 = \bigcup_{n=1}^{\infty} x_n \in R^+$, we have $\bigcup_{n=1}^{\infty} y_n \in R^+$ on account of (R. 2), which implies $x = 0$, because R is Archimedean[5]. From the

5) Since R is universally continuous, $\bigcap_{\nu=1}^{\infty} \frac{1}{\nu} a = 0$ must hold for any $a \in R^+$.

formula (2.1) it is now evident that Lemma 1 holds.

Lemma 2. *If $x \sim \alpha x$ for some $0<\alpha \neq 1$, then $x \sim 0$.*

Proof. If $0 \leqq x \sim \alpha x$ for some α with $1<\alpha$, then $x \sim \alpha x \sim \alpha^2 x \sim \cdots \sim \alpha^n x \sim \cdots$. Thus, from (R.2) again we have $\bigcup_{n=1}^{\infty} \alpha^n x \in R$. Hence $x=0$ must hold. On account of (R.1) and (2.1) it is obvious that the lemma holds. Q.E.D.

Lemma 3. *If $x \sim y$ and x is an atomic element,*[6] *then y is also such a one.*

Proof. Assume that $x \sim y$, $x, y \in R^+$ and x is an atomic element. If y is decomposed into $y=z_1 \oplus z_2$ with $z_i \neq 0$ $(i=1,2)$, then for each M-fundamental system $\{y_\lambda\}_{\lambda \in \Lambda}$ with respect to y, we can find an index $\lambda \in \Lambda$ with $[z_i]y_\lambda \neq 0$ $(i=1,2)$. On the other hand, as M is full, $x \in M$ and also $x \sim y_\lambda = [z_1]y_\lambda \oplus [z_2]y_\lambda$. This implies that one of the elements $[z_i]y_{\lambda (i=1,2)}$, say $[z_1]y_\lambda$, must be equivalent to 0 by virtue of (R.4, (ii)) and the assumption that x is an atomic element. It follows from Lemma 1 that $[z_1]y_\lambda=0$ and it is a contradiction. Q.E.D.

Lemma 4. *If $x \sim y$ and x is of finite dimension,*[7] *then y is also of the same dimension.*

Proof. Since P-manifold M is full, each atomic element, hence each element of finite dimension belongs to M. Now the proof is easily derived similarly from Lemma 3 and (R.4).

Lemma 5. *If $x \sim y$, $x, y \in R$ and $[p]x$ is of finite dimension, then there exists a projector $[q]$ such that $[p]x \sim [q]y$ and $(1-[p])x \sim (1-[q])y$ hold simultaneously.*

Proof. First suppose that $x, y \in R$ and $[p]x$ is an atomic element. Then in view of (R.4), we can find a pair of M-fundamental systems $\{x_\lambda\}_{\lambda \in \Lambda}$, $\{y_\lambda\}_{\lambda \in \Lambda}$ with respect to x and y and a system of projectors $\{[q_\lambda]\}_{\lambda \in \Lambda}$ such that $[q_\lambda] \leqq [y_\lambda]$, $[p]x=[p]x_\lambda \sim [q_\lambda]y_\lambda$ and $(1-[p])x_\lambda \sim (1-[q_\lambda])y_\lambda$ hold for all $\lambda \in \Lambda$. By Lemma 3 $[q_\lambda]y_\lambda$ is an atomic element, hence $[q_\lambda]y$ is also such a one, and á fortiori $[q_\lambda]y \in M$ and $[q_\lambda]y=[q_\lambda]y_\lambda$ for all $\lambda \in \Lambda$. If $[q_\lambda] \neq [q_{\lambda_1}]$ holds for a fixed $\lambda_1 \in \Lambda$, we have by (R.3)

$$(1-[p])x_\lambda \sim (1-[q_\lambda])y_\lambda = (1-[q_{\lambda_1}]-[q_\lambda])y_\lambda + [q_{\lambda_1}]y_\lambda \sim$$
$$(1-[q_{\lambda_1}]-[q_\lambda])y_\lambda + [q_\lambda]y_\lambda = (1-[q_{\lambda_1}])y_\lambda$$

and $[p]x_\lambda \sim [q_{\lambda_1}]y_\lambda$. Consequently both $[p]x \sim [q_{\lambda_1}]y_\lambda$ and $(1-[p])x_\lambda \sim (1-$

6) An element $x \in R$ is called an *atomic element*, if $x=y \oplus z$, implies always $y=0$ or $z=0$.

7) An element $x \in R$ is called to be *of finite dimension*, if it is represented as $x=\sum_{\nu=1}^{n} \xi_\nu e_\nu$, where e_ν is an atomic element for each $1 \leqq \nu \leqq n$.

$[q_{\lambda_1}])y_\lambda$ hold for all $\lambda\in\Lambda$. Then from (R.2) it follows that $(1-[p])x=\bigcup_{\lambda\in\Lambda}(1-[p])x_\lambda\sim\bigcup_{\lambda\in\Lambda}(1-[q_{\lambda_1}])y_\lambda=(1-[q_{\lambda_1}])y$. In case x is n-dimensional, the proof is similarly obtained by use of induction and the condition (R.3). Q.E.D.

Now we establish a theorem which reveals the structure of an $\mathcal{E}$-relation in the case of discrete spaces. Suppose that R is discrete. Then there exists a mutually orthogonal system of positive atomic elements $\{e_\gamma\}_{\gamma\in\Gamma}$ such that each element $x\in R^+$ is uniquely represented as $x=\sum_{\gamma\in\Gamma}\xi_\gamma e_\gamma$, where $\xi_\gamma\geqq 0$. We call the system $\{e_\gamma\}_{\gamma\in\Gamma}$ above the *natural basis* of R. We put further $I(x,\xi)=\{\gamma\,;\gamma\in\Gamma,\ \xi_\gamma=\xi\}$. For any subset $J\subset\Gamma$ we denote by $n(J)$ the number of elements belonging to J, that is, $n(J)=k$ $(k=0,1,2,\cdots)$ or $+\infty$ if J contains an infinite number of elements.

Theorem 1. *Let R be a discrete space with an $\mathcal{E}$-relation $\sim$. Then there exists a natural basis $\{d_\gamma\}_{\gamma\in\Gamma}$ and a partition of the index set $\Gamma=\sum_{\alpha\in\mathfrak{A}}\Gamma_\alpha$, $\Gamma_\alpha\cap\Gamma_{\alpha'}=\phi$ for $\alpha\neq\alpha'$, such that $x=\sum_{\gamma\in\Gamma}\xi_\gamma d_\gamma$ and $y=\sum_{\gamma\in\Gamma}\eta_\gamma d_\gamma$ stand in the relation if and only if $n(\Gamma_\alpha\frown I(x,\xi))=n(\Gamma_\alpha\frown I(y,\xi))$ for all real number ξ and $\alpha\in\mathfrak{A}$.*

Proof. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary natural basis. Since $\sim$ is an equivalence relation, we can classfy Γ as $\Gamma=\sum_{\alpha\in\mathfrak{A}}\Gamma_\alpha$, $\Gamma_\alpha\frown\Gamma_{\alpha'}=\phi$ for $\alpha\neq\alpha'$ in such a way that for any γ, $\gamma'\in\Gamma_\alpha$, $e_\gamma\sim\xi e_{\gamma'}$ holds with some $\xi>0$, and there exists no real number $\xi>0$ for which $e_\gamma\sim\xi e_{\gamma'}$ holds, whenever γ and γ' do not belong to the same class. Then, by Choice Axiom, we can find a subsystem $\{e_{\gamma_\alpha}\}_{\alpha\in\mathfrak{A}}$ of $\{e_\gamma\}_{\gamma\in\Gamma}$ with $e_{\gamma_\alpha}\in\Gamma_\alpha$ for every $\alpha\in\mathfrak{A}$. Here for any fixed α, we define d_γ $(\gamma\in\Gamma_\alpha)$ by

$$d_\gamma=\xi e_\gamma\,, \tag{2.2}$$

where ξ is a positive number satisfying $e_{\gamma_\alpha}\sim\xi e_\gamma$. d_γ is atomic and uniquely determined for each $\gamma\in\Gamma_\alpha$ on account of Lemma 2 and the construction of Γ_α. Repeating this process to whole $\alpha\in\Gamma$, we obtain a natural basis $\{d_\gamma\}_{\gamma\in\Gamma}$ for which $d_\gamma\sim d_{\gamma'}$ stands if and only if γ and γ' belong to the same class Γ_α. Assume now $x=\sum_{\gamma\in\Gamma}\xi_\gamma d_\gamma$, $y=\sum_{\gamma\in\Gamma}\eta_\gamma d_\gamma$ and $x\sim y$. For any fixed $\alpha_0\in\mathfrak{A}$, $[\{d_{\gamma_1},d_{\gamma_2},\cdots,d_{\gamma_k}\}]x\in M$ $(\gamma_i\in\Gamma_{\alpha_0},\ 1\leqq i\leqq\kappa\,;\ \kappa=1,2,\cdots)$, and furthermore there exists a collection of elements of Γ_{α_0}: $\gamma_1',\gamma_2',\cdots,\gamma_k'$ such that both $\sum_{i=1}^k\xi_{\gamma_i}d_{\gamma_i}\sim\sum_{i=1}^k\eta_{\gamma_i'}d_{\gamma_i'}$ and $(1-[\{d_{\gamma_1},\cdots,d_{\gamma_k}\}])x\sim(1-[\{d_{\gamma_1'},\cdots,d_{\gamma_k'}\}])y$ hold at the same time in accordance with Lemmas 4 and 5. Then, on account of the construction of the basis $\{d_\gamma\}_{\gamma\in\Gamma}$, we can infer from Lemma 4 that $n(\Gamma_{\alpha_0}\frown I(x,\xi))=n(\Gamma_{\alpha_0}\frown I(y,\xi))$ holds for each real number ξ and $\alpha\in\mathfrak{A}$. Q.E.D.

Conversely suppose that a partition of Γ exists. We obtain an equivalence relation $\sim$ on R in such a way that $x=\sum_{\gamma\in\Gamma}\xi_\gamma d_\gamma$ and $y=\sum_{\gamma\in\Gamma}\eta_\gamma d_\gamma$ stand in the relation $\sim$ if and only if $n(\Gamma_\alpha \frown I(x, \xi)) = n(\Gamma_\alpha \frown I(y, \xi))$ for each ξ and $\alpha \in \mathfrak{A}$. For the D-manifold we take a linear subset S of all elements of finite dimension. Now it is evident that the equivalence relation thus defined is an $\mathcal{E}$-relation with the D-manifold S, i.e. it satisfies the conditions (R. 1)–(R. 4).

§ 3. Here we deal with R which has a certain functional ρ together with an $\mathcal{E}$-relation $\sim$. The end of this section is to show, in a sense, uniform boundedness of a ρ-functional with respect to an $\mathcal{E}$-relation (Theorem 2).

A functional ρ defined on R is called a *ρ-functional*, if it satisfies

(ρ. 1) $0 \leqq \rho(x) = \rho(|x|) \leqq +\infty$ *for all* $x \in R$;

(ρ. 2) $\rho(x+y) \leqq \rho(x) + \rho(y)$, *if* $x \perp y$;

(ρ. 3) $\inf_{\alpha>0} \rho(\alpha x) < +\infty$ *for every* $x \in R$;

(ρ. 4) $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ *implies* $\rho(x) = \sup_{\lambda \in \Lambda} \rho(x_\lambda)$;

(ρ. 5) *if* $\{x_\nu\}_{\nu=1}^\infty$ *is a mutually orthogonal sequence with* $\sum_{\nu=1}^\infty \rho(x_\nu) < +\infty$, *then* $x_0 = \sum_{\nu=1}^\infty x_\nu$ *belongs to* R.

From the definition it follows immediately

$$\rho([p]x) \leqq \rho(x) \qquad \text{for every } x, p \in R; \tag{3.1}$$

and

$$[p_\lambda] \uparrow_{\lambda \in \Lambda} [p] \quad \text{implies} \quad \rho([p]x) = \sup_{\lambda \in \Lambda} \rho([p_\lambda]x). \tag{3.2}$$

ρ-functionals thus defined are sufficiently general to include known functionals on semi-ordered linear spaces For instance, the following functionals are all ρ-functionals respectively.

(i) a semi-continuous and complete norm or quasi-norm on R;

(ii) a monotone complete modular in the sense of Nakano [11] or of Orlicz and Musielak [9], a concave modular of Nakano [10] and a quasi-modular in [2, 3]. We shall establish the following basic result on R with a ρ-functional and an $\mathcal{E}$-relation:

Theorem 2. *Let ρ-functional be defined on R with an $\mathcal{E}$-relation. Then there exist positive numbers α, γ, ε and a finite co-dimensional*[8] *normal manifold N of R such that*

$$\rho(x) \leqq \varepsilon \quad \textit{implies} \quad \rho(\alpha y) \leqq \gamma \tag{3.3}$$

8) A linear manifold $N \subset R$ is called a *normal manifold*, if each $x \in R$ is uniquely represented as $x = x_1 + x_2$, $x_1 \in N$ and $x_2 \in N^\perp$. A normal manifold N is called to be finite co-dimensional if $N^\perp$ is of finite dimension.

for any $x, y \in N$ with $x \sim y$.

For the proof of this theorem, we need to prove a number of auxiliary lemmas whose proofs are based on the topological properties of the proper space of semi-ordered linear spaces. In the sequel, $\mathfrak{E}$ denotes the *proper space* of R, i.e. the Boolean lattice of all maximal ideals $\mathfrak{p}$ consisting of normal manifolds $N \subset R$, equipped with the topology generated by the neighbourhood system $\{U_{[N]}\}_{N \subset R}$, where $U_{[N]}$ is the set of all $\mathfrak{p} \in \mathfrak{E}$ such that $N \in \mathfrak{p}$. $U_{[N]}$ is both open and compact in $\mathfrak{E}$ for any normal manifold N, hence $\mathfrak{E}$ is itself compact, because $\mathfrak{E} = U_{[R]}$. An element $\mathfrak{p} \in \mathfrak{E}$ is called *non-atomic*, if for any $N \in \mathfrak{p}$, there exists $M \subsetneq N$ such that $M \in \mathfrak{p}$.

Lemma 6. *Let $x, y \in R$ satisfy $\rho(x) < +\infty$, $\rho(y) < +\infty$ and $x \sim y$. Then for any non-atomic $\mathfrak{p}_0, \mathfrak{p}_0' \in \mathfrak{E}$ and $\varepsilon > 0$, there exist two elements $x_0, y_0 \in R$ such that* (i) $\rho(x_0) > \rho(x) - \varepsilon$, $\rho(y_0) > \rho(y) - \varepsilon$; (ii) $[x_0]R \notin \mathfrak{p}_0$, $[y_0]R \notin \mathfrak{p}_0'$; *and* (iii) $x_0 \sim y_0$ *hold.*

Proof. It is sufficient to prove the lemma, when $[x]R \in \mathfrak{p}_0$ and $[y]R \in \mathfrak{p}_0'$. Assume $x \sim y$, $x, y \in R^+$ and $\varepsilon > 0$. In view of the conditions (R.4) and (ρ.4), there exist elements x', y' of the D-manifold M such that $0 \leqq x' \leqq x$, $0 \leqq y' \leqq y$ and $\rho(x') > \rho(x) - \varepsilon$, $\rho(y') > \rho(y) - \varepsilon$ with $x' \sim y'$. Since $\mathfrak{p}_0$ is non-atomic, we can find a system of mutually orthogonal projectors $\{[p_\gamma]\}_{\gamma \in \Gamma}$ with $\bigcup_{\gamma \in \Gamma} [p_\gamma] = [x']$ and $[p_\gamma]R \notin \mathfrak{p}_0$ $(\gamma \in \Gamma)$. On account of (R.4 (ii)), there exists an orthogonal system $\{[q_\gamma]\}_{\gamma \in \Gamma}$ such that $\bigcup_{\gamma \in \Gamma} [q_\gamma] = [y']$ and $[p_\gamma]x' \sim [q_\gamma]y'$ $(\gamma \in \Gamma)$ hold. By virtue of (ρ.4) we have for suitable chosen $\gamma_1, \gamma_2, \cdots, \gamma_k$ $(\gamma_i \in \Gamma)$

$$\rho\Big(\sum_{i=1}^{k} [p_{\gamma_i}]x'\Big) > \rho(x) - \varepsilon \text{ and } \rho\Big(\sum_{i=1}^{k} [q_{\gamma_i}]y'\Big) > \rho(y) - \varepsilon .$$

Putting $x'' = \sum_{i=1}^{k} [p_{\gamma_i}]x'$ and $y'' = \sum_{i=1}^{k} [q_{\gamma_i}]y'$, we now have by (R.3)

$$x'' \sim y'' , \quad \rho(x'') > \rho(x) - \varepsilon , \quad \rho(y'') > \rho(y) - \varepsilon$$

and $[x'']R \notin \mathfrak{p}_0$. If $[y'']R \in \mathfrak{p}_0'$, then applying the quite same argument (only changing the rôle of x' and y' into y'' and x'' respectively), we can show that we get two elements x_0, y_0 which fulfil the requirement of Lemma 6. Q.E.D.

Lemma 7. *For any non-atomic $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{E}$ there exist normal manifolds $N_{\mathfrak{p}} \in \mathfrak{p}$ and $N_{\mathfrak{p}'} \in \mathfrak{p}'$ and positive numbers $\alpha, \gamma, \varepsilon > 0$ such that $x \in N_{\mathfrak{p}}$, $\rho(x) \leqq \varepsilon$ implies*

$$\rho(\alpha y) \leqq \gamma \quad \textit{for each } y \in N_{\mathfrak{p}'} \textit{ with } x \sim y. \tag{3.3}$$

Proof. Assume that the lemma is not valid. Then for a pair of non-atomic maximal ideals $\mathfrak{p}$, $\mathfrak{p}' \in \mathfrak{E}$, there exists no pair of normal manifolds ($N_{\mathfrak{p}}$, $N_{\mathfrak{p}'}$) and positive numbers α, γ and ε which satisfies (3.3) above. Now we can start with a pair of elements (x', y') with $x' \sim y'$, $\rho(x') < \frac{1}{2}$, $\rho(y') > 1$, $[x']R \in \mathfrak{p}$ and $[y']R \in \mathfrak{p}'$. From Lemma 6 it follows that there exist elements x_1, y_1 ($x_1, y_1 \in M$) such that $\rho(x_1) < \frac{1}{2}$, $\rho(y_1) > 1$, $[x_1]R \notin \mathfrak{p}$, $[y_1]R \notin \mathfrak{p}'$ and $x_1 \sim y_1$. Since $\mathfrak{p}$ and $\mathfrak{p}'$ are maximal ideals of normal manifolds of R, $(1-[x_1])R \in \mathfrak{p}$ and $(1-[y_1])R \in \mathfrak{p}'$ stand. Again we can find also $x'', y'' \in R$ with $x'' \sim y''$, $x'' \in (1-[x_1])R$ and $y'' \in (1-[y_1])R$ satisfying $\rho(x'') < \frac{1}{2^2}$ together with $\rho\left(\frac{1}{2} y''\right) > 2$ by the assmuption. In view of Lemma 6 again, there exists a pair of elements (x_2, y_2) such that $\rho(x_2) < \frac{1}{2^2}$, $\rho\left(\frac{1}{2} y_2\right) > 2$, $[x_2]R \notin \mathfrak{p}$, $[y_2]R \notin \mathfrak{p}$ and $x_2 \sim y_2$. Proceeding this argument, we obtain two sequences of mutually orthogonal positive elements $\{x_\nu\}_{\nu=1}^{\infty}$ and $\{y_\nu\}_{\nu=1}^{\infty}$, for which $x_\nu \sim y_\nu$, $[x_\nu]R \notin \mathfrak{p}$, $[y_\nu]R \notin \mathfrak{p}'$, $\rho(x_\nu) \leqq \frac{1}{2^\nu}$ and $\rho\left(\frac{1}{\nu} y\right) \geqq \nu$ hold for each $\nu \geqq 1$. From (ρ. 4) it follows

$$\bigcup_{\nu=1}^{\infty} x_\nu \in R,$$

which implies $\bigcup_{\nu=1}^{\infty} y_\nu \in R$ on account of (R.2). This is, however, a contradiction, since $\rho\left(\frac{1}{n} \bigcup_{\nu=1}^{\infty} y_\nu\right) \geqq \rho\left(\frac{1}{n} y_n\right) \geqq n$ holds and it is inconsistent with (ρ.3).

Q.E.D.

Lemma 8. *For any non-atomic $\mathfrak{p}_0 \in \mathfrak{E}$, there exists a finite number of normal manifolds $N_0, N_1, \cdots, N_k, N'$ such that $N_0 \in \mathfrak{p}_0$, $R = N_1 \oplus N_2 \oplus \cdots N_k \oplus N'$, N' is of finite dimension and for any $x \in N_0$ with $\rho(x) \leqq \varepsilon$*

$$\text{(3. 4)} \qquad \operatorname*{Max}_{1 \leqq i \leqq k} \left\{ \sup_{x \sim y,\, y \in N_i} \rho(\alpha y) \right\} \leqq \gamma$$

holds, where α, γ and ε are all fixed positive constants.

Proof. Let $N_{\mathfrak{p}_0, \mathfrak{p}}$ and $N_{\mathfrak{p}}$ be two normal manifolds and $\alpha_{\mathfrak{p}}$, $\gamma_{\mathfrak{p}}$ and $\varepsilon_{\mathfrak{p}} > 0$ be positive numbers which satisfy the formula (3.3) corresponding to non-atomic maximal ideals $\mathfrak{p}_0$ and $\mathfrak{p}$. Let $\mathfrak{C}$ denote the set of all non-atomic elements of $\mathfrak{E}$. As the set $(\sum_{\mathfrak{p} \in \mathfrak{C}} U_{[N_{\mathfrak{p}}]})^-$ is both open and compact in $\mathfrak{E}$, $(\sum_{\mathfrak{p} \in \mathfrak{C}} U_{[N_{\mathfrak{p}}]})^- = U_{[N]}$ holds for a normal manifold $N \subset R$ and clearly $(1-[N])R$ is of finite dimension.

On the other hand, if $\mathfrak{p}$ belongs to the set $U_{[N]} - \sum_{\mathfrak{p} \in \mathfrak{C}} U_{[N_{\mathfrak{p}}]}$ it must be non-atomic as easily seen, whence it follows $U_{[N]}=\sum_{\mathfrak{p}\in\mathfrak{C}} U_{[N\mathfrak{p}]}$. Thus we can find a finite number of $\mathfrak{p}\in\mathfrak{C}$, say $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k$, such that $U_{[N]}=\sum_{\nu=1}^{k} U_{[N_{\mathfrak{p}_\nu}]}$ holds. Now we put $N_0=\bigcap_{\nu=1}^{k} N_{\mathfrak{p}_0,\mathfrak{p}_\nu}$, $N'=(1-[N])R$, $\varepsilon=\underset{1\leqq\nu\leqq k}{\mathrm{Min}}\{\varepsilon_{\mathfrak{p}_\nu}\}$, $\alpha=\underset{1\leqq\nu\leqq k}{\mathrm{Min}}\{\alpha_{\mathfrak{p}_\nu}\}$ and $\gamma=\underset{1\leqq\nu\leqq k}{\mathrm{Max}}\{\gamma_{\mathfrak{p}_\nu}\}$, and also we choose a set of mutually orthogonal normal manifolds: $\{N_\nu\}_{\nu=1}^{k'}$ such that $N=N_1\oplus N_2\oplus\cdots\oplus N_{k'}$ and $N_\nu\subset N_{\mathfrak{p}_\nu}$ for each ν with $1\leqq\nu\leqq k'\leqq k$, by use of the usual orthogonalization method. It is now clear that (3.4) is valid for normal manifolds and the positive numbers thus constructed.

Q. E. D.

Lemma 9. *For any non-atomic $\mathfrak{p}_0\in\mathfrak{C}$ there exists a normal manifold $N_{\mathfrak{p}}$, a finite co-dimensional normal manifold $N'_{\mathfrak{p}}$ and positive numbers $\alpha, \gamma, \varepsilon$ such that $x\in N_{\mathfrak{p}}$, $\rho(x)\leqq\varepsilon$ implies*

$$\sup_{x\sim y,\, y\in N'_{\mathfrak{p}}} \rho(\alpha y)\leqq\gamma' . \tag{3.5}$$

Proof. Let $N_0, N_1, \cdots, N_k, N'$ be normal manifolds and α, γ and ε be positive numbers which statisfy (3.4) in the preceding lemma. Suppose $x\sim y$, $x\in[N_0]M$, $y\in[N'^{\perp}]M$ and $\rho(x)\leqq\varepsilon$. Then, on account of (R.4), we can find a mutually orthogonal set of projectors $\{[p_\nu]\}_{\nu=1}^{k}$ for which $[p_\nu]x\sim[N_\nu]y$ holds $(1\leqq\nu\leqq k)$. Since $\rho([p_\nu]x)\leqq\varepsilon$ for all $1\leqq\nu\leqq k$, we have $\rho(\alpha[N_\nu]y)\leqq\gamma$ and

$$\rho(\alpha y)\leqq\sum_{\nu=1}^{k}\rho([N_\nu]y)\leqq k\cdot\gamma .$$

Since ρ is semi-continuous and M is full, putting $\gamma'=k\gamma$ and $N'_{\mathfrak{p}}=(N')^{\perp}$, we obtain the proof.

Q. E. D.

Proof of Theorem 2. For any $\mathfrak{p}\in\mathfrak{C}$, we denote by $N_{\mathfrak{p}}$, $N'_{\mathfrak{p}}$, $\alpha_{\mathfrak{p}}$, $\gamma_{\mathfrak{p}}$ and $\varepsilon_{\mathfrak{p}}$ be the same as in Lemma 9, corresponding with $\mathfrak{p}$. Then, as above, there exists a finite co-dimensional normal manifold N_0 satisfying $U_{[N_0]}=\sum_{\mathfrak{p}\in\mathfrak{C}} U_{[N_{\mathfrak{p}}]}$. Hence we can find a finite number of $N_{\mathfrak{p}_\nu}$, $\mathfrak{p}_\nu\in\mathfrak{C}$ $(\nu=1, 2, \cdots, k)$ for which $U_{[N_0]}=\sum_{\nu=1}^{k} U_{[N_{\mathfrak{p}_\nu}]}$ hold. Now we put $N=N_0\frown\bigcap_{\nu=1}^{k} N'_{\mathfrak{p}_\nu}$, $\alpha=\underset{1\leqq\nu\leqq k}{\mathrm{Min}}\{\alpha_{\mathfrak{p}_\nu}\}$, $\gamma=\sum_{\nu=1}^{k}\gamma_{\mathfrak{p}_\nu}$ and $\varepsilon=\underset{1\leqq\nu\leqq k}{\mathrm{Min}}\{\varepsilon_{\mathfrak{p}_\nu}\}$ respectively. It is evident that N is a finite co-dimensional normal manifold, and we can find again normal manifolds $M_1, M_2, \cdots, M_{k'}$ such that $N=M_1\oplus M_2\oplus\cdots\oplus M_{k'}$, $M_\nu\subset N_{\mathfrak{p}_\nu}$ and $M_\nu\frown M_\mu=\{0\}$ for $\nu\neq\mu$. Now suppose that $x\sim y$, $x, y\in[N]M$ and $\rho(x)\leqq\varepsilon$. Since $x=\sum_{\nu=1}^{k'}[M_\nu]x$ and $[M_\nu]x\in N_{\mathfrak{p}_\nu}$, there exists mutually orthogonal projectors $\{[p_\nu]\}_{\nu=1}^{k'}$ such that $[M_\nu]x\sim[p_\nu]y$, $\sum_{\nu=1}^{k'}[p_\nu]=[y]$. As $[M_\nu]x\in N_{\mathfrak{p}_\nu}$ and $[p_\nu]y\in N'_{\mathfrak{p}_\nu}$, we have by the preceding lemma

$$\rho\left(\alpha[p_\nu]y\right) \leqq \rho\left(\alpha_{\mathfrak{p}_\nu}[p_\nu]y\right) \leqq \gamma_{\mathfrak{p}_\nu} \qquad (1 \leqq \nu \leqq k').$$

Therefore we get

$$\rho(\alpha y) \leqq \sum_{\nu=1}^{k'} \rho(\alpha_{\mathfrak{p}_\nu}[p_\nu]y) \leqq \sum_{\nu=1}^{k'} \gamma_{\mathfrak{p}_\nu} \leqq \gamma,$$

which implies (3.3), because of the semi-continuity of ρ. Q.E.D.

Remark 1. If R is non-atumic,[9] there is no finite dimensional normal manifold. Hence the formula (3.3) in Theorem 2 holds valid in the whole space R in this case.

Corollary 1. *Let R have a complete semi-continuous*[10] *norm $\|\cdot\|$ together with an $\mathcal{E}$-relation. Then there exists a positive number γ such that*

$$\frac{1}{\gamma}\|y\| \leqq \|x\| \leqq \gamma\|y\| \tag{3.6}$$

halds for each pair of elements $x, y \in R$ with $x \sim y$.

Proof. A complete semi-continuous norm $\|\cdot\|$ is a ρ-functional, hence by virtue of Theorem 2 there exist a finite co-dimensional normal manifold N and a positive number γ_1, for which $\frac{1}{\gamma_1}\|y\| \leqq \|x\| \leqq \gamma_1\|y\|$ holds for $x, y \in N$ with $x \sim y$. On the other hand, $N^\perp$ being of finite dimension, there exists also $\gamma_2 > 0$ such that $\frac{1}{\gamma_2}\|y\| \leqq \|x\| \leqq \gamma_2\|y\|$ holds for $x, y \in N^\perp$ with $x \sim y$. Let $\{e_1, \cdots, e_n\}$ be a natural basis of $N^\perp$ with $\|e_\nu\| = 1$ $(1 \leqq \nu \leqq n)$. We put now $\alpha_\nu = \inf_{x \in N, x \sim e_\nu} \|x\|$, and $\beta_\nu = \sup_{x \in N, x \sim e_\nu} \|x\|$. It is evident that both $\alpha_\nu > 0$ and $\beta_\nu < +\infty$ hold for all ν $(1 \leqq \nu \leqq n)$ from above. If $x \sim y$, $x \in N^\perp$ and $y \in N$, we can verify easily that $\frac{1}{\gamma_3}\|y\| \leqq \|x\| \leqq \gamma_3\|y\|$ holds, where $\gamma_3 = n \cdot \mathrm{Max}_{1 \leqq \nu \leqq n}\left\{\frac{1}{\alpha_\nu}, \beta_\nu\right\}$. From these facts it follows immediately that there exists $\gamma > 0$ which satisfies (3.6) in the whole space. Q.E.D.

Corollary 2. *Let R be a modulared (quasi-modulared) semi-ordered linear space with a monotone complete modular*[11] *(quasi-modular) m. If R is non-atomic and has an $\mathcal{E}$-relation, then we can find positive numbers α, γ' and ε such that*

9) R is called to be non-atomic, if R has no atomic element.

10) A norm $\|\cdot\|$ on R is called semi-continuous, if $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $\|x\| = \sup_{\lambda \in \Lambda} \|x_\lambda\|$.

11) For the definition of a modular see [11]. Here we use the term of modular in the sense of Nakano.

(3.7) $\qquad m(x) > \varepsilon$ *implies* $m(\alpha y) \leqq \gamma' m(x)$

for any pair of elements $x, y \in R$ *with* $x \sim y$.

Proof. Since a monotone complete modular (or quasi-modular) m satisfies the conditions (ρ.1)–(ρ.5) [11, 3], it is a ρ-functional. Thus there exist positive numbers α, γ and ε such that $m(x) \leqq \varepsilon$, $x \sim y$ yields $m(\alpha x) \leqq \gamma$. If $x \sim y$, $x, y \in M$ and $m(x) > \varepsilon$, then we can find two sets of mutually orthogonal projectors $\{[p_\nu]\}_{\nu=1}^{k+1}$, $\{[q_\nu]\}_{\nu=1}^{k+1}$ such that $\sum_{\nu=1}^{k+1} [p_\nu] = [x]$, $\sum_{\nu=1}^{k+1} [q_\nu] = [y]$, $[p_\nu] x \sim [q_\nu] y$ $(1 \leqq \nu \leqq k+1)$, $m([p_\nu] x) = \varepsilon$ $(1 \leqq \nu \leqq k)$ and $m([p_{k+1}] x) < \varepsilon$ on account of the non-atomicity of R and the condition (R.4, (ii)). Hence we get

$$m(\alpha y) = \sum_{\nu=1}^{k+1} m(\alpha [q_\nu] y) \leqq \gamma (k+1) \leqq 2 \frac{\gamma}{\varepsilon} m(x),$$

which yields (3.7), since a modular (or quasi-modular) is semi-continuous.

Q. E. D.

§ 4. Throughout this section let E be a non-atomic finite measure space and $\boldsymbol{X}(E)$ be a Banach function space with a semi-continuous norm $\|\cdot\|$. It is well known that $\boldsymbol{X}$ constitutes a superuniversally continuous semi-ordered linear space[12] by the usual order and addition of measurable functions. When $\boldsymbol{X}$ has *w-RIP*, the relation of equi-measurability between two functions belonging to $\boldsymbol{X}$ can be regarded as an $\mathcal{E}$-relation on the space $\boldsymbol{X}$. Indeed, the conditions (R.1) and (R.3) are evidently satisfied. Since μ is assumed to be countably additve and $\boldsymbol{X}$ has *w-RIP*, the condition (R.2) is fulfilled. As a D-manifold M, we can take the set of all simple functions[13] on E and it is now clear that the relation of equi-measurability satisfies also (R.4), hence an $\mathcal{E}$-relation on $\boldsymbol{X}$.

Consequently, in view of Corollary 1 we have

Theorem 3. *In order that a Banach function space* $\boldsymbol{X}(E)$ *on a finite non-atomic measure space* E *has w-RIP, it is necessary and sufficient that* $\boldsymbol{X}(E)$ *has s-RIP, that is,*

(4.1) $$\frac{1}{\gamma} \|g\| \leqq \|f\| \leqq \gamma \|g\|$$

for any two mutually equi-measurable functions f *and* $g \in \boldsymbol{X}$, *where* γ *is a*

12) R is called *superuniversally continuous*, if for any system of positive elements $\{a_\lambda\}_{\lambda \in \Lambda}$ there exists a sequence of elements: $\{a_{\lambda_\nu}\}_{\nu=1}^{\infty} \subset \{a_\lambda\}_{\lambda \in \Lambda}$ such that $\bigcap_{\nu=1}^{\infty} a_{\lambda_\nu} = \bigcap_{\lambda \in \Lambda} a_\lambda$ holds.

13) A function on E is called a *simple function* if it is represented as $\sum_{\nu=1}^{n} \xi_\nu x_{e_\nu}$. where x_{e_ν} is the characteristic function of a measurable set $e_\nu \subset E$ for each ν with $1 \leqq \nu \leqq n$.

positive number.

Corollary 3. *If a Banach function space $\boldsymbol{X}(E)$ has w-RIP, there exists an equivalent norm $\|\cdot\|_1$ on $\boldsymbol{X}(E)$ having the rearrangement majorant* 1, *i.e.* $\|f\|_1=\|g\|_1$ *for* $f, g\in\boldsymbol{X}$ *with* $f\sim g$.

Proof. On account of Theorem above, we can define a finite valued functional $\|\cdot\|_1$ as

$$(4.2) \qquad \|f\|_1=\sup_{f\sim g}\|g\| \qquad (f\in\boldsymbol{X}).$$

It is now evident from the definition that the functional $\|\cdot\|_1$ satisfies all the conditions of semi-continuous norm except for the subadditivity. For any simple functions f, g: $f=\sum_{\nu=1}^{k}\xi_\nu\chi_{e_\nu}$, $g=\sum_{\nu=1}^{k}\eta_\nu\chi_{e_\nu}$ with $e_\nu\frown e_\mu=\phi$ for $\nu\neq\mu$, we have

$$\|f+g\|_1=\sup_{\substack{e_1'\oplus\cdots\oplus e_k'=E,\ \mu(e_i)=\mu(e_i') \\ (1\leqq\nu\leqq k)}}\Big\|\sum_{\nu=1}^{k}(\xi_\nu+\eta_\nu)\chi_{e_\nu'}\Big\|.$$

Let h be a simple function such that $h=\sum_{\nu=1}^{k}(\xi_\nu+\eta_\nu)\chi_{e_\nu'}$ and $\mu(e_\nu')=\mu(e_\nu)$, $e_\nu'\frown e_\mu'=\phi$ for $\nu\neq\mu$. Then $|h|\leqq\sum_{\nu=1}^{k}|\xi_\nu|\chi_{e_\nu'}+\sum_{\nu=1}^{k}|\eta_\nu|\chi_{e_\nu'}$ and $|f|\sim\sum_{\nu=1}^{k}|\xi_\nu|\chi_{e_\nu'}$, $|g|\sim\sum_{\nu=1}^{k}|\eta_\nu|\chi_{e_\nu'}$, which imples

$$\|h\|_1\leqq\|f\|_1+\|g\|_1.$$

Consequently we have $\|f+g\|_1\leqq\|f\|_1+\|g\|_1$ for arbitrary $f, g\in\boldsymbol{X}$ on account of the semi-continuity of $\|\cdot\|_1$. Q.E.D.

Next we turn to prove a theorem concerning characterization of Orlicz spaces among the classes of modulared function spaces.

Now let $M(\xi, t)$ be a *modular function*, i.e. $M(\xi, t)$ be a real-valued function on $[0, +\infty)\times E$ satisfying (i) it is a non-decreasing convex function of $\xi\geqq0$ which is left hand continuous for each $t\in E$; (ii) it is measurable on E for each $\xi\geqq0$; (iii) $\lim_{\xi\to0}M(\xi, t)=0$, $\sup_{0\leqq\xi}M(\xi, t)=+\infty$ and $M(0, t)=0$ for all $t\in E$. Then a *modulared function space* $\boldsymbol{L}_{M(\xi,t)}$ is the set of all measurable functions f on E such that $\int_E M(\xi|f(t)|, t)d\mu(t)<+\infty$ for some $\xi>0$. $\boldsymbol{L}_{M(\xi,t)}$ is a modulared space with the modular m:

$$(4.3) \qquad m(f)=\int_E M(|f(t)|, t)d\mu(t) \qquad (f\in\boldsymbol{L}_{M(\xi,t)}),$$

hence, as is well known, it is a Banach function space with the norm:

$\|f\| = \inf_{m(\xi f) \leq 1} \frac{1}{|\xi|}$ $(f \in \boldsymbol{L}_{M(\xi,t)})$. Evidently Orlicz spaces[14] constitute a special class in the modulared function spaces.

Now we have

Theorem 4. *If a modulared function space $\boldsymbol{L}_{M(\xi,t)}(E)$ on a non-atomic finite measure space E has w-RIP, then it reduces to an Orlicz space $\boldsymbol{L}_\Phi(E)$.*

Proof. It is obvious that we may assume $\mu(E)=1$, without loss of generality. Putting $\Phi(\xi)=m(\xi\chi_E)$ for $\xi \geqq 0$, we obtain a non-decreasing left hand continuous convex function $\Phi(\xi)$ on $[0, +\infty)$ satisfying $\lim_{\xi \to +\infty} \Phi(\xi) = +\infty$, $\inf_{0<\xi} \Phi(\xi)=0$ and $\Phi(0)=0$. Here we shall show that $\boldsymbol{L}_{M(\xi,t)}$ coincides with the Orlicz space[15] $\boldsymbol{L}_\Phi$ defined by the function Φ as a Banach function space.

By virtue of Theorem 2 and Corollary 2 (3.7), we can find positive numbers γ_1, γ_2 and α satisfying both the conditions:

(i) $m(f) \leqq \varepsilon$, $f \sim g$ implies $m(\alpha g) \leqq \gamma_1$;

(ii) $m(f) > \varepsilon$, $f \sim g$ implies $m(\alpha g) \leqq \gamma_2 m(f)$.

We put further for any $f \in \boldsymbol{L}_{M(\xi,t)}$

$$m^*(f) = \sup_{f \sim g} m(g) \quad \text{and} \quad m_*(f) = \inf_{f \sim g} m(g) .$$

It follows from above that for each $f \in \boldsymbol{L}_{M(\xi,t)}$

$$(4.4) \qquad m_*(\alpha f) \leqq m(\alpha f) \leqq m^*(\alpha f) \leqq \gamma_2 m_*(f) + \gamma_1$$

holds. Let $\mathfrak{M}_0$ be the set of all simple functions $h = \sum_{\nu=1}^{k} \xi_\nu \chi_{e_\nu}$ such that

$$(4.5) \qquad e_\nu \frown e_\mu = \phi \text{ for } \nu \neq \mu, \quad E = \sum_{\nu=1}^{k} e_\nu \text{ and } \mu(e_\nu) = \frac{1}{k} \text{ for all } \nu \geqq 1.$$

Here we denote by $P_n(\nu)$ a permutation of the set: $\{1, 2, \cdots, k\}$ defined by $P_n(\nu) = \nu + n$ (mod. k) for each n. Then, for any $h \in \mathfrak{M}_0$ we put $h^{(n)} = \sum_{\nu=1}^{k} \xi_\nu \chi_{e_\nu^n}$ $(0 \leqq n \leqq k-1)$, where $e_\nu^n = e_{P_n(\nu)}$. Evidently we have $h = h^{(0)} \sim h^{(1)} \sim \cdots \sim h^{(k-1)}$ and $\sum_{n=0}^{k-1} m(h^{(n)}) = \sum_{n=0}^{k-1} \sum_{\nu=1}^{k} m(\xi_\nu \chi_{e_\nu^n}) = \sum_{\nu=1}^{k} \sum_{n=0}^{k-1} m(\xi_\nu \chi_{e_\nu^n}) = \sum_{\nu=1}^{k} m(\xi_\nu \chi_E) = \sum_{\nu=1}^{k} \Phi(\xi_\nu) = k \cdot m_\Phi(h)$. Therefore there exists at least a pair of integers (m_0, n_0) $(0 \leqq m_0, n_0 \leqq k-1)$ such that

$$m(h^{(m_0)}) \leqq m_\Phi(h) \leqq m(h^{(n_0)}) ,$$

14) For the details of Orlicz spaces see [4], [7] or [13].

15) $m_\Phi(f)$ denotes the modular of the space $\boldsymbol{L}_\Phi$, *i.e.* for $f \in \boldsymbol{L}_\Phi$ $m_\Phi(f) = \int_E \Phi(|f(t)|)\, d\mu(t)$. Since $\boldsymbol{L}_{M(\xi,t)}$ has *w-RIP*, $\mathbf{1} \in \boldsymbol{L}_{M(\xi,t)}$.

which implies

$$m_*(h) \leqq m_\Phi(h) \leqq m^*(h).$$

From this and (4.4) it follows that

$$(4.6) \qquad m(\alpha h) \leqq \gamma_2 m_\Phi(h) + \gamma_1 \quad \text{and} \quad m_\Phi(\alpha h) \leqq \gamma_2 m(h) + \gamma_1.$$

Since E is non-atomic, for any $f \in \boldsymbol{L}_{M(\xi,t)}$ there exists a sequence $\{h_n\}_{n=1}^\infty$ of elements of $\mathfrak{M}_0$ such that $h_n \uparrow_{n=1}^\infty |f|$ holds. Consequently, by the semi-continuity of m and m_Φ, (4.6) implies

$$(4.7) \qquad m(\alpha f) \leqq \gamma_2 m_\Phi(f) + \gamma_1 \quad \text{and} \quad m_\Phi(\alpha f) \leqq \gamma_2 m(f) + \gamma_1$$

for any $f \in \boldsymbol{L}_{M(\xi,t)}$. It is now evident that the Banach function spaces $\boldsymbol{L}_{M(\xi,t)}$ and $\boldsymbol{L}_\Phi$ coincide. Q. E. D.

Remark 2. As this proof shows, the convexity of modular m and m_Φ is not used. Therefore, it is verified in the quite same way, that if a (non-convex) quasi-modular function space $\boldsymbol{L}_{N(\xi,t)}$ [2] has *w-RIP*, then it reduces to a generalized Orlicz space $\boldsymbol{L}_N$ considered by S. Mazur and W. Orlicz in [8].

Lastly let E be a σ-finite (or locally finite) measure space with a countably additive measure μ. The relation defined by equi-measurability has essentially the sense on the set of finite measure only, in fact, it can not be extended naturally to the whole space of all measurable functions on E without loss of the original significance. Only we can define an equivalence relation $\sim$ on the set $\mathfrak{J}$ of all integrable functions on E in the following way. Two positive functions f, g belonging to $\mathfrak{J}$ are called equi-measurable if $\mu\{t; f(t) > r\} = \mu\{t, g(t) > r\}$ holds for every positive number r. Next two functions f, g of $\mathfrak{J}$ is called equi-measurable (in the extended sense) and written as $f \sim g$, if both f^+ and f^- [16] are equi-measurable to g^+ and g^- respectively. Then the relation $\sim$ comes to be an equivalence relation on the space $\mathfrak{J}$. Thus, if a Banach function space $\boldsymbol{X}$ consisting of integrable functions on E has *w-RIP* with respect to the relation $\sim$ of equi-measurability in the extended sense, the relation $\sim$ is an $\mathcal{E}$-relation on $\boldsymbol{X}$ as is easily seen. Hence, on account of Theorem 2, we have as similarly as Theorem 3

Theorem 3′. *If a Banach function space $\boldsymbol{X}$ consisting of integrable functions on a σ-finite (or locally finite) measure space E has w-RIP, then it has s-RIP.*

We obtain also

Theorem 4′. *Let $\boldsymbol{L}_{M(\xi,t)}(E)$ be a modulared function space consisting*

16) $f^+(t) = \mathrm{Max}(f(t), 0)$ and $f^-(t) = \mathrm{Max}(-f(t), 0)$ for all $t \in E$.

of integrable functions on a non-atomic σ-finite measure space E. If $\boldsymbol{L}_{M(\xi,t)}$ has w-RIP, then it reduces to an Orlicz space $\boldsymbol{L}_{\Phi}$.

Proof. Let $\{E_\nu\}_{\nu=1}^{\infty}$ be a sequence of measurable sets of finite measure such that $E_\nu \uparrow_{\nu=1}^{\infty} E$ holds. Now we put

$$\Phi^*(\xi) = \sup_{0 \leqq \eta < \xi} \varlimsup_{\nu \to \infty} \frac{m(\eta \chi_{E_\nu})}{\mu(E_\nu)} \quad \text{and} \quad \Phi_*(\xi) = \varliminf_{\nu \to \infty} \frac{m(\xi \chi_{E_\nu})}{\mu(E_\nu)}.$$

Then, by virtue of Corollary 2 in §3 and the non-atomicity of E, we can find positive numbers α and γ for which $\Phi_*(\alpha\xi) \leqq \Phi^*(\alpha\xi) \leqq \gamma\Phi_*(\xi)$ holds for each $\xi \geqq 0$. From this we can verify as similarly as in Theorem 4 that $\boldsymbol{L}_{M(\xi,t)}$ coincides with the Orlicz space $\boldsymbol{L}_{\Phi^*}$. Q.E.D.

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