ON NAGAHARA'S THEOREM

By

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Throughout the present note, $A=\sum De_{ij}$ will represent a simple ring ($\{e_{ij}$'s$\}$ is a system of matrix units and $D=V_A(\{e_{ij}\text{'s}\})$ a division ring), B a simple subring of A containing the identity 1 of A, and $\mathfrak{G}$ the group of all the B-(ring) automorphisms of A. And we set $V=V_A(B)$ and $H=V_A^2(B)=V_A(V_A(B))$. As to notations and terminologies used in this note, we follow [3][1] and [4].

In [3], A/B was called h-Galois if (1) B is regular, (2) A is Galois over B' and $V_A^2(B')$ is simple for every regular subring B' of A left finite over B, and if (3) $A'=V_A^2(A')$ and $[A':H]_l=[V: V_A(A')]_r$ for every regular subring A' of A left finite over H. Recently, in his paper [1], T. Nagahara has obtained the following theorem.

Theorem 1. (i) *A/B is h-Galois and left locally finite if and only if any of the following conditions $(A_l)-(B_r)$ is satisfied*:

(A_l) (1) *B is a regular subring of A and $\mathfrak{G}A_r$ is dense in* $\mathrm{Hom}_{B_l}(A, A)$.
(2) *A/B is left locally finite.*

(A_r) (1) *B is a regular subring of A and $\mathfrak{G}A_l$ is dense in* $\mathrm{Hom}_{B_r}(A, A)$.
(2) *A/B is right locally finite.*

(B_l) (1) *A/B is Galois and A is BV-A-irreducible.*
(2) *A/B is left locally finite.*

(B_r) (1) *A/B is Galois and A is A-BV-irreducible.*
(2) *A/B is right locally finite.*

(ii) *If A/B is h-Galois and left locally finite, then $[B':B]$*[2] *$\geqslant [V: V_A(B')]=[V_A^2(B'): H]=[B': H\cap B']$ for every regular subring B' of A left finite over B.*

And by the aid of Theorem 1, he has obtained also the next important theorem.

Theorem 2. *Let A be h-Galois and left locally finite over B. If $\mathfrak{H}$ is a $(*_f)$-regular subgroup of $\mathfrak{G}$ then $\mathfrak{H}$ is f-regular.*

One of the purposes of this note is to give a rather direct proof to Theorem 2. To this end, we shall prove first the following brief lemma.

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1) Numbers in brackets refer to the references cited at the end of this note.

2) If $[B':B]_l=[B':B]_r$, they are represented as $[B':B]$.

Lemma 1. *Let A be left locally finite over B, and T an intermediate ring of A/B with $[T: B]_l<\infty$ such that $V_A(T)$ coincides with the center of A. If there exists an automorphism group $\mathfrak{H}$ of A such that $J(\mathfrak{H}, A)=T$ then T is a simple ring.*

Proof. If $B'=T[\{e_{ij}\text{'s}\}]$ then $(\mathfrak{H}|B')A_r=\oplus_1^m(\sigma_i|B')A_r$ with some $\sigma_i\in\mathfrak{H}$. By [3, Corollary 1.1],[3] we have $\mathfrak{H}|B'=\{\sigma_1|B',\cdots,\sigma_m|B'\}$. Accordingly, $B_0=B'[\cup B'\sigma_i]$ is an $\mathfrak{H}$-invariant simple subring of A left finite over B and $\sharp(\mathfrak{H}|B_0)<\infty$. Since $J(\mathfrak{H}|B_0, B_0)=T$ and $V_{B_0}(T)=V_{B_0}(B_0)$, it is well-known that T is a simple ring.

Lemma 2. *Let A be h-Galois and left locally finite over B, and $\mathfrak{H}$ a $(*_f)$-regular subgroup of $\mathfrak{G}$. If $T=J(\mathfrak{H}, A)$ then $[T: H\cap T]_l<\infty$.*

Proof. Let N be a $\mathfrak{G}(H/B)$-invariant shade of $\{d_{hk}\text{'s}\}$ (a system of matrix units of H such that the centralizer of $\{d_{hk}\text{'s}\}$ in H is a division ring), $\mathfrak{H}^*=\mathfrak{G}(N)\cap\mathfrak{H}=\mathfrak{H}(N)$, $T^*=J(\mathfrak{H}^*, A)$ $(\supseteq T[N])$, and $H^*=H\cap T^*$. Then, $\mathfrak{H}^*$ is an invariant subgroup of $\mathfrak{H}$ and $(\mathfrak{H}:\mathfrak{H}^*)=\sharp(\mathfrak{H}|N)\leqslant\sharp\mathfrak{G}(N/B)=[N: B]<\infty$. As $H^*=H\cap T^*$ is an $\mathfrak{H}$-invariant simple subring of H by [2, Theorem 1.1], $J(\mathfrak{H}|H^*, H^*)=H^*\cap T=H\cap T$ and $\infty>(\mathfrak{H}:\mathfrak{H}^*)\geqslant\sharp(\mathfrak{H}|H^*)$, we see that $H^*/H\cap T$ is outer Galois and $[H^*: H\cap T]<\infty$. On the other hand, in virtue of [1, Lemma 2], there holds $[T^*: H^*]_l=[H\cdot T^*: H]_l$. And further, we can see that $V_{\mathfrak{H}^*}=V_{\mathfrak{H}}=V_A(B[E])$ for some finite subset E of A. Now, let B' be a regular subring of A containing $B[E]$ such that $[B': B]_l<\infty$. If $A'=V_A^2(B')$ then $V_{\mathfrak{H}^*}=V_A(B[E])\supseteq V_A(B')$ yields $H[T^*]\subseteq V_A(V_{\mathfrak{H}^*})\subseteq A'$, so that $[H[T^*]: H]_l\leqslant[A': H]_l<\infty$. Combining this with $[T^*: H^*]_l=[H\cdot T^*: H]_l$, we obtain eventually $[T: H\cap T]_l\leqslant[T^*: H^*]_l\cdot[H^*: H\cap T]\leqslant[H[T^*]: H]_l\cdot[H^*: H\cap T]<\infty$.

The next lemma is proved essentially in [1, Lemma 3 (ii)]. However, for the sake of completeness, we shall give here a slight simplified proof.

Lemma 3. *Let A be h-Galois and left locally finite over B, and V' a simple subring of V with $[V: V']_r<\infty$. If $V_A(V_A(V')[E])\subseteq V'$ for some finite subset E of A, then $B'=V_A(V')$ is a simple ring.*

Proof. By the light of Theorem 1, we may assume that $H=B$. If $B=\sum Kd_{hk}$ with a system of matrix units $\{d_{hk}\text{'s}\}$ such that $K=V_B(\{d_{hk}\text{'s}\})$ is a division ring, then we have $BV=K\cdot\sum Vd_{hk}$, $V_A(K)=\sum Vd_{hk}$(simple) and $V_A(\sum Vd_{hk})=K$. Hence, again by Theorem 1, we may assume further that B is a division ring. Since A is A-BV-irreducible, $[B': B]_l\leqslant[V: V']_r<\infty$ by [1, Lemma 1]. If $B''=B'[E, \{e_{ij}\text{'s}\}]$ and $V''=V_A(B'')(\subseteq V')$, then $V_A(V'')=B''$ and $\infty>[B'': B]=[V: V'']=[V: V_A^2(V'')\cap V]$ by Theorem 1. Noting here that

3) [3, Corollary 1.1] is valid without the assumption that B is regular.

A is V-A-irreducible by [1, Lemma 1], we obtain $\mathrm{Hom}_{V_l'}(V, A)=(\widetilde{B''}|V)A_r = \oplus_1^t(\sigma_i|V)A_r(\sigma_i\in\widetilde{B''})$ by [3, Lemma 3.1], where each $(\sigma_i|V)A_r$ is V_r-A_r-irreducible and $\sigma_i|V$ is linearly independent over A_r. And then, $\mathrm{Hom}_{V_l'}(V, A)$ being a V_r-A_r-submodule of the completely reducible $\mathrm{Hom}_{V_l''}(V, A)$, there holds $\mathrm{Hom}_{V_l'}(V, A)=\oplus_1^s\mathfrak{M}_j$ with V_r-A_r-irreducible $\mathfrak{M}_j$. To be easily verified, each $\mathfrak{M}_j$ is then of the form $(\sigma u_l|V)A_r$ with som σ in $\{\sigma_i\text{'s}\}\,(\subseteq\widetilde{B''})$ and non-zero u, and so we may set $\mathfrak{M}_j=(b_{jl}|V)A_r$ with some non-zero b_j. Noting here that $b_{jl}|V$ is contained in $\mathrm{Hom}_{V_l'}(V, A)$, it will be easy to see that b_j is contained in B'. Now, let $M=V'vA$ $(v\in V)$ be a V'-A-submodule of A such that the length $[M|A_r]$ of the composition series of M as right A-module is minial among non-zero V'-A-submodules of A of the form $V'xA$ $(x\in V)$. Evidently, M is BV'-A-admissible. If M' is a minimal BV'-A-submodule of M then $M'=uA$ with some $u\in V$ as a direct summand of the completely reducible B-A-module A (cf. [1, Lemma 1]), and so $M\supseteq M'=V'uA$. Hence, by the minimality of $[M|A_r]$, it follows that $M=M'$, that is, M is BV'-A-irreducible. Consequently, for an arbitrary V'-A-minimal submodule VxA of M, there holds $M=BV'xA=\sum_{b\in B}V'bxA=\oplus_1^q V'x_iA$, where each $V'x_iA$ is V'-A-isomorphic to the V'-A-irreducible $V'xA$. Since $V'v\subseteq V$ and A is $V_l'\cdot\mathrm{Hom}_{V_l'}(A, A)$-irreducible, it follows that $A=v(V_l'\cdot\mathrm{Hom}_{V_l'}(A, A))=(V'v)\mathrm{Hom}_{V_l}(V, A)=\sum_j V'v\mathfrak{M}_j=\sum_{i,j}b_j(V'x_iA)$. Now, b_j being contained in B', each $b_j(V'x_iA)$ is V'-A-homomorphic to $V'x_iA\cong V'xA$. Hence, A is homogeneously V'-A-completely reducible, and consequently B' is a simple ring.

Now, combining Lemmas 1, 2 and 3, the proof of Theorem 2 will be completed at once.

Proof of Theorem 2. We set $T=J(\mathfrak{H}, A)$. As $[V:V_{\mathfrak{H}}]_r<\infty$ and $V_{\mathfrak{H}}=V_A^2(V_{\mathfrak{H}})$, $V_A^2(T)=V_A(V_{\mathfrak{H}})$ is simple by Lemma 3. Further, by Lemma 2, there holds $[T:H\cap T]_l<\infty$. Since $A/H\cap T$ is locally finite by [1, Corollary 4], $V_{V_A^2(T)}(T)$ coincides with the center of $V_A^2(T)$ and $J(\mathfrak{H}|V_A^2(T), V_A^2(T))=T$, Lemma 1 proves that T is a simple ring.

Next, concerning [1, Lemma 3 (i)], the method used in the proof of Lemma 3 enables us to see the following improvement.

Theorem 3. *Let A be h-Galois and left locally finite over B. If A' is a simple intermediate ring of A/H with $[A':H]_l<\infty$, then $V'=V_A(A')$ is simple and $[A':H]=[V:V']$.*

Proof. By Theorem 1, without loss of generality, we may assume that $B=H$. And so, by [3, Lemma 3.1], $\widetilde{V}A_r$ is dense in $\mathrm{Hom}_{B_l}(A, A)$. Now, let M be an arbitrary minimal A'-A-submodule of A. Then, $M=eA\,(=A'eA)$

with some non-zero idempotent e. We set here $A''=A'[e, \{e_{ij}\text{'s}\}]$, which is a regular subring of A left finite over B. And, as A is A''-A-irreducible, $\mathrm{Hom}_{B_l}(A'', A)=(\tilde{V}|A'')A_r$ is A''_r-A_r-completely reducible. Accordingly, the A''_r-A_r-submodule $\mathrm{Hom}_{A'_l}(A'', A)$ is completely reducible: $\mathrm{Hom}_{A'_l}(A'', A)=\oplus_1^s\mathfrak{M}_j$ with A''_r-A_r-irreducible $\mathfrak{M}_j$. To be easily seen, each $\mathfrak{M}_j=(\tilde{v}_j u_{jl}|A'')A_r$ with some $\tilde{v}_j\in\tilde{V}$ and non-zero u_j, so that we may set $\mathfrak{M}_j=(a_{jl}|A'')A_r$. Recalling here that $a_{jl}|A''$ is contained in $\mathrm{Hom}_{A'_l}(A'', A)$, it will be easy to see that a_j has to be contained in V'. Since A is $A'_l\cdot\mathrm{Hom}_{A'_l}(A, A)$-irreducible and $A'e\subseteq A''$, it follows $A=e(A'_l\cdot\mathrm{Hom}_{A'_l}(A, A))=(A'e)\,\mathrm{Hom}_{A'_l}(A'', A)=\sum_j(A'e)\mathfrak{M}_j$ $=\sum_j a_j M$. Evidently, a_j being contained in V', each $a_j M$ is A'-A-homomorphic to M. We have proved therefore that A is homogeneously A'-A-completely reducible, and consequently V' is simple. The final assertion is then a consequence of Theorem 1.

Now, combining Theorems 2, 3 with [3, Corollary 3.3], one will readily see that if A is h-Galois and left locally finite over B with $B=V_A^2(B)$, then there exists a 1–1 dual correspondence between simple subrings of A left finite over B and closed $(*_f)$-regular subgroups of $\mathfrak{G}$. Further, as another corollary to Theorem 3, we can prove the next:

Corollary. *Let A be inner Galois and locally finite over B, and V finite over the center of V. And let B' be a simple intermediate ring of A/B with $[B':B]_l<\infty$. If B'/B is inner Galois then the center Z' of B' is contained in the center Z of B, and conversely.*

Proof. If B'/B is inner Galois then $V_{B'}^2(B)=B$ yields at once $Z'\subseteq B\cap V$ $=Z$. Now, assume conversely $Z'\subseteq Z$. Then, V is evidently an algebra over Z'. Since A/B is h-Galois by [3, Theorems 2.2, 2.4], Theorem 3 yields $V_A(B')\cap V_A^2(B')=V_A(B')\cap B'=Z'$, so that $V_A(B')$ is a central simple algebra of finite rank over Z' by [5, Lemma]. Hence, by Wedderburn's theorem, we obtain $V=V_A(B')\otimes_{Z'}V_V(V_A(B'))=V_A(B')\otimes_{Z'}V_{B'}(B)$. From the last relation, we see that $V_{B'}(B)$ is a simple ring. And finally, $J(\widetilde{V_{B'}(B)}|B', B')=V_A(V_{B'}(B))\cap V_A^2(B')=V_A(V)=B$.

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