ON THE CLASS OF FUNCTIONS INTEGRABLE IN A CERTAIN GENERALIZED SENSE

Dedicated to Professor Kinjiro Kunugi on the occasion of his sixtieth birthday

By

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1. Introduction. Let us consider the Lebesgue integral on a bounded interval I of the real line. There have been many attempts of extending the concept of integral for a larger class of measurable functions; some of them depend heavily on special properties of the derivative on the real line, preventing to be brought in general settings, but some are based only on the notions of measure theory, admitting investigation in general cases. We will follow the latter direction.

If a measurable function is not integrable, it has points of singularity, i.e., roughly speaking, it is not integrable on any neighborhood containing the points. If functions with a fixed point of singularity are in question, a generalized integral may be defined as something like Cauchy's principal value. However, if points of singularity are distributed over some region, depending on a function $f(x)$, such an approach must undergo some modification. A natural generalization will be

$$\lim_{n\to\infty}\int_{F_n} f(x)\,dx\,,$$

where $\{F_n\}$ is a sequence of measurable sets such that

$$F_n \subset F_{n+1} \qquad (n=1,2,\cdots) \qquad \text{and} \qquad \lim_{n\to\infty} \operatorname{meas}(F_n^c)=0$$

with F_n^c denoting the complement of F_n. Since the region of singularity varies along the function $f(x)$, we can not take one and the same sequence $\{F_n\}$ for the definition of a generalized integral, and even for a single function the limit on a sequence $\{F_n\}$ may differ from that on another $\{F_n'\}$. In order to avoid these inconvenience and ambiguity, sequences used for the definition must obey some additional requirements. In this respect, Kunugi [3] proposed a definition

*) A part of this work was done while the author was at California Institute of Technology under grant NSF-G 19914.

of a generalized integral which is independent of the special choice of an admissible sequence and proved that the class of functions integrable in the generalized sense is linear and the generalized integral is a linear functional on it. Then there arises a question whether the generalized integral can be calculated more directly from the function $f(x)$ without any reference to a sequence $\{F_n\}$. In this connection we are interested in the generalized integral, proposed by Titchmarsh [6], which is defined by

$$\lim_{n\to\infty}\int_I f^{(n)}(x)\,dx$$

where the function $f^{(\gamma)}(x)$, the *truncation* of $f(x)$ by the positive number γ, is defined by

$$f^{(\gamma)}(x) = \begin{cases} f(x) & \text{if } |f(x)|\leqslant\gamma\,, \\ \gamma\,\operatorname{sign} f(x) & \text{if } |f(x)|>\gamma\,. \end{cases}$$

The class of functions integrable in this sense is not linear, nor the generalized integral is a linear functional. In order to guarantee both linearities, we have to impose additional conditions on functions. The detailed definitions of both generalized integrals are given in the next section.

We will prove that Kunugi's generalized integral is identical with Titchmarsh's (Theorem 1). This gives an intrinsic characterization of Kunugi's generalized integral and, at the same time, an approach to Titchmarsh's from the standpoint of Cauchy's principal value. The class of functions integrable in the generalized sense can be characterized from various points of view. It is shown to be the maximum extension of the Lebesgue integral under the suitable restrictions (Theorem 2). As the Lebesgue class can be obtained from the class of continuous functions by the procedure of completion with respect to the L_1-norm

$$\|f\| = \int_I |f(x)|\,dx\,,$$

the class of functions integrable in the generalized sense is the completion of the Lebesgue class with respect to the quasi-norm

$$\|f\|_* = \sup_{\gamma>0}\left|\int_I f^{(\gamma)}(x)\,dx\right| + \sup_{\gamma>0}\gamma\ \mathrm{meas}(x\colon |f(x)|\geqslant\gamma)$$

(Theorem 3). The quasi-norm introduces the linear metric topology on the class, under which the generalized integral becomes a continuous linear functional, and moreover, in contrast to the Lebesgue case, it is the only continuous one up to scalar (Theorem 4). The last section is a survey of the usefulness of the generalized integral in connection with conjugate functions in the theory of Fourier series. Except in the last section, we will discuss generalized integrals on an abstract measure space.

2. Generalized Integrals. We shall use the terminologies of [2] concerning measure and integration theory. Let $(X, \mathfrak{B}, m)$ be a non-atomic, finite measure space. Every functions $f, g, \cdots$ in this paper will be assumed to be real valued and measurable. Any two functions which coincide with each other almost everywhere are identified, and $\mathfrak{L}$ stands for the class of all integrable functions. Capitals $E, F, \cdots$ denote measurable sets and Greek letters $\alpha, \beta, \cdots$ are reserved for real numbers. The following property of the non-atomic measure will be used without any reference: if $m(E)>\gamma>0$, there is a set F such that $F\subset E$ and $m(F)=\gamma$. We shall use the simplified notations $\int f dm$ and $\{f\geqslant \gamma\}$ for $\int_X f(x)dm(x)$ and $\{x: f(x)\geqslant\gamma\}$ respectively.

Kunugi [3] introduced the notion of a generalized integral based on his theory of ranked spaces and called it the (E. R.) integral, the definition of which in an abstract measre space can be found in [5]. We can formulate it in a slightly different from a function; f is (E. R.) *integrable*, if there are a sequence $\{F_n\}$, a decreasing sequence $\{\varepsilon_n\}$ with the limit 0, and a constant $\rho>1$ which obey jointly the following requirements:

(**k 1**) f is integrable on every F_n and the limit

$$\lim_{n\to\infty}\int_{F_n} f dm$$

exists,

(**k 2**) $$F_n\subset F_{n+1},\quad m(F_n{}^c)\leqslant \rho\cdot m(F_{n+1}{}^c)\quad n=1,2,\cdots,$$

and $$\sum m(F_n{}^c)<\infty,$$

(**k 3**) $\int_E |f|\,dm<\varepsilon_n$ whenever $E\subset F_n$ and $m(E)\leqslant m(F_n{}^c)$.

The (E.R.) *integral* of the function f is defined by

$$(\text{E. R.})\int f dm = \lim_{n\to\infty}\int_{F_n} f dm$$

It is proved in [3], [5] that the limit does not depend on any special choice of the sequence $\{F_n\}$ obeying (**k1**) to (**k3**) and that the class of all (E.R.) integrable functions forms a linear set and the (E.R.) integral is a linear functional on it. If a function is integrable, it is obviously (E.R.) integrable and its (E.R.) integral coincides with its integral.

Titchmarsh [6] introduced the notion of the Q integral, the precise definition of which with the condition (**q 2**) is due to Očan [4] and Ulyanov [7]. A function f is *Q integrable*, if

(**q 1**) the limit of the integrals of truncated functions

$$\lim_{\gamma\to\infty}\int f^{(\gamma)}\,dm$$

exsts, and

(**q 2**)
$$m(|f|\geqq\gamma)=o\left(\frac{1}{\gamma}\right).$$

The *Q integral* of the function f is defined by

$$(Q)\int f\,dm=\lim_{\gamma\to\infty}\int f^{(\gamma)}\,dm\,.$$

It is proved in [6] that the class of all Q integrable functions in a linear set and the Q integral is a linear functional on it. It is obvious that the Q-integral coincides with the integral on the class $\mathfrak{L}$. The parameter γ in (**q 1**) may be confined to the sequence of all positive integers n view of (**q 2**). In this connection, it is interesting to observe that Očan [4] proved that the Q integrability can be characterized by the single condition absorbing (**q 1**) and (**q 2**): f is Q integrable if and only if the limit

$$\lim_{\gamma\to\infty}\int f^{(n_1\gamma,n_2\gamma)}\,dm$$

exists for every pair of positive integers n_1, n_2, and does not depend on the choice of n_1 and n_2, where

$$f^{(n_1\gamma,n_2\gamma)}(x)=\max\left\{-n_2\gamma,\ \min\left\{f(x),\ n_1\gamma\right\}\right\}.$$

Let us state some of immediate consequences of the definition of the Q integral. First of all, if a non-negative function is Q integrable, it is integrable. If a function f is Q integrable, every p-th power of $|f|$ is integrable with $0<p<1$, but $|f|$ itself is not necessarily integrable. If two Q integrable functions f and g are *equi-measurable* in the sense:

$$m(\alpha>f>\beta)=m(\alpha>g>\beta)\qquad\text{for every pair } \alpha,\beta,$$

then they have one and the same Q integral.

Theorem 1. *The* (*E.R.*) *integrability coincides with the Q integrability, and the* (*E.R.*) *integral is equal to the Q integral.*

Proof. Let f be (E.R.) integrable with the defining sequences $\{F_n\}$ $\{\varepsilon_n\}$ and ρ, then each set $F_{n+1}-F_n$ can be written as the sum of mutually disjoint subsets E_j $(j=1,\cdots,k_n)$ such that

$$m(E_j)=m(F_{n+1}{}^c)\quad j=1,\cdots,k_n-1 \quad\text{and}\quad m(E_{k_n})\leqslant m(F_{n+1}{}^c).$$

Since by (**k 2**)

$$m(F_{n+1}-F_n)=m(F_n{}^c)-m(F_{n+1}{}^c)\leqslant(\rho-1)m(F_{n+1}{}^c)$$

and

$$m(F_{n+1}-F_n)=\sum_{j=1}^{k_n}m(E_j)\geqslant(k_n-1)m(F_{n+1}{}^c)$$

we have $k_n\leqslant\rho$, hence from (**k 3**) it follows that

$$\int_{F_{n+1}-F_n}|f|\,dm=\sum_{j=1}^{k_n}\int_{E_j}|f|\,dm\leqslant k_n\varepsilon_n\leqslant\rho\cdot\varepsilon_n\,.$$

If $\sum\varepsilon_n<\infty$, f is integrable, a fortiori, Q integrable, for

$$\int|f|\,dm=\sum\int_{F_{n+1}-F_n}|f|\,dm+\int_{F_1}|f|\,dm\leqslant\rho\sum\varepsilon_n+\int_{F_1}|f|\,dm<\infty\,.$$

If $\sum\varepsilon_n=\infty$, putting

$$\alpha_n=\frac{\rho\varepsilon_n}{m(F_n{}^c)}$$

we have

$$\alpha_{n+1}\leqslant\rho\alpha_n\quad n=1,2,\cdots\quad\text{and}\quad \sup_{j=1,2,\dots}\alpha_j=\infty\,,$$

for the first inequality is an immediate consequence of (**k 2**) and if $\{\alpha_n\}$ is bounded with a positive number γ, by (**k 2**)

$$\sum\varepsilon_n\leqslant\gamma\sum m(F_n{}^c)<\infty$$

contrary to the hypothesis. Now given any positive number $\gamma\geqslant\alpha_1$, taking the minimum index n such that

$$\alpha_n\leqslant\gamma<\rho\alpha_n\,,$$

the existence of which is guaranteed by the above consideration, let us prove that

$$m(G_n)\leqslant m(F_n{}^c)\qquad\text{with}\quad G_n=F_n\frown\{|f|\geqslant\alpha_n\}\,,$$

for, otherwise, there is a set E such that

$$E\subset G_n\quad\text{and}\quad m(E)=m(F_n{}^c)\,,$$

then by (**k 3**) we have

$$\varepsilon_n>\int_E|f|\,dm\geqslant\alpha_n m(E)=\alpha_n m(F_n{}^c)=\rho\varepsilon_n\,,$$

leading to the contradiction that $\varepsilon_n > \rho\varepsilon_n$ with $\rho > 1$. Now it follows that

$$\gamma m(|f| \geqslant \gamma) \leqslant \rho\alpha_n \left\{m(F_n^c) + m(G_n)\right\} \leqslant 2\rho\alpha_n m(F_n^c) = 2\rho^2\varepsilon_n$$

and finally

$$m(|f| \geqslant \gamma) = o\left(\frac{1}{\gamma}\right)$$

proving the condition (**q 2**) for f. With the same γ and n we have

$$\left|\int_{F_n} f dm - \int f^{(\gamma)} dm\right| \leqslant \int_{F_n} |f - f^{(\gamma)}| dm + \gamma m(F_n^c)$$

$$= \int_{G_n} |f - f^{(\gamma)}| dm + \gamma m(F_n^c) \leqslant \int_{G_n} |f| dm + 2\rho\alpha_n m(F_n^c) \leqslant \varepsilon_n + 2\rho^2\varepsilon_n$$

because $m(G_n) \leqslant m(F_n^c)$. This shows that

$$\lim_{n\to\infty} \int_{F_n} f dm = \lim_{\gamma\to\infty} \int f^{(\gamma)} dm\,.$$

Thus we have proved that every (E.R.) integrable function is Q integrable and its Q integral is equal to its (E.R.) integral.

Conversely now let f be Q integrable. Without loss of generality, we may assume that $m(X) \geqslant 1$ and f is not essentially bounded. We can take a sequence $\{\gamma_n\}$ of positive numbers and a sequence $\{F_n\}$ in the following way: $\gamma_n \leqslant \gamma_{n+1}$, $F_n \subset F_{n+1}$, $m(F_n^c) = \dfrac{1}{2^n}$

$$F_n \subset \left\{|f| \leqslant \gamma_n\right\} \quad \text{and} \quad F_n^c \subset \left\{|f| \geqslant \gamma_n\right\} \qquad n = 1, 2, \cdots$$

Since

$$\left|\int_{F_n} f dm - \int f^{(\gamma_n)} dm\right| \leqslant \gamma_n \cdot m(|f| \geqslant \gamma_n - 1)\,,$$

(**k 1**) follows from (**q 1**) and (**q 2**). (**k 2**) is obviously satisfied with $\rho = 2$. Finally with the definition

$$\varepsilon_n = \sup_{j \geqslant n} \gamma_j m(|f| \geqslant \gamma_j)\,,$$

if $E \subset F_n$ and $m(E) \leqslant m(F_n^c)$

$$\int_E |f| dm \leqslant \gamma_n m(F_n^c) \leqslant \gamma_n m(|f| \geqslant \gamma_n) \leqslant \varepsilon_n\,,$$

showing that (**k 3**) is also satisfied. Thus f is (E. R.) integrable. This completes the proof.

3. Characterizations. A pair $(\mathfrak{K}, J)$ of a class $\mathfrak{K}$ of functions and a functional J on it is called an *extension* of the integral, if $\mathfrak{R}$ is a linear class containg $\mathfrak{L}$ and J is a linear functional which coincides with the integral on $\mathfrak{L}$. An extension, $\mathfrak{R}$, J is said to be larger than $(\mathfrak{R}_2, J_2)$ if $\mathfrak{R}_1 \supset \mathfrak{R}_2$ and J_1 coincides with J_2 on $\mathfrak{R}_2$. If we denote by $\mathfrak{Q}$ the class of all Q integrable functions, the pair $(\mathfrak{Q}, (Q)\int)$ is obviously an extension.

Theorem 2. *The Q integral, i.e., the pair $(\mathfrak{Q}, (Q)\int)$, is the largest of all extensions $(\mathfrak{R}, J)$ obeying the following restrictions*:

(**e 1**) $$J(f) = \lim_{\gamma\to\infty} \int f^{(\gamma)}\, dm \qquad f\in\mathfrak{R}$$

(**e 2**) *a function g is contained in $\mathfrak{R}$, whenever $|g| \leqslant |f|$ for some $f\in\mathfrak{R}$ and $\lim_{\gamma\to\infty}\int g^{(\gamma)}\, dm$ exists.*

Before going into the proof, let us comment on these conditions. (**e 1**) is nothing but (**q 1**) and requires that the value of the functional $J(f)$ can be calculated from the integral of truncated functions. (**e 2**) requires that $\mathfrak{R}$ is something like an *ideal* in a vector lattice, though the class of all functions satisfying (**e 1**) is not a vector lattice under the usual definition of the order. If (**e 2**) is changed as follows: g is in $\mathfrak{R}$ whenever $|g| \leqslant |f|$ for some $f\in\mathfrak{R}$, then the class $\mathfrak{R}$ necessarily coincide with the minimum class $\mathfrak{L}$.

Proof of Theorem 2. It suffices to prove that every function in $\mathfrak{R}$ obeys the requirement (**q 2**). Suppose, contrarily, that (**q 2**) is not valid for some $f\in\mathfrak{R}$, then there are a sequence $\{\gamma_n\}$ and a constant $\varepsilon>0$ such that

$$0<\gamma_n<\frac{1}{4}\gamma_{n+1} \quad \text{and} \quad \gamma_n m(|f|\geqslant\gamma_n)>\varepsilon \qquad n=1,2,\cdots$$

We can take a sequence $\{\delta_n\}$ for which $\frac{1}{2}<\delta_n<1$ and $\delta_n\gamma_n m(|f|\geqslant\gamma_n)$ does not converge. Every set $E_n=\{\gamma_{n+1}>|f|\geqslant\gamma_n\}$ can be written as the sum of three mutually disjoint sets F_n, G_n and H_n such that

$$m(F_n) = \frac{1}{2}m(E_n) \quad \text{and} \quad m(G_n) = m(H_n) = \frac{1}{4}m(E_n).$$

If the functions g and h are defined by

$$g = \sum \gamma_n\{c_{F_n} - c_{G_n} - c_{H_n}\} \quad \text{and} \quad h = \sum\gamma_n\{c_{G_n} - c_{H_n}\}$$

where c_E denotes the characteristic function of E, we have

$$|g|\leqslant|f|,\ |h|\leqslant|f| \quad \text{and} \quad \int g^{(\gamma)}\, dm = \int h^{(\gamma)}\, dm = 0 \qquad \text{for all } \gamma>0,$$

hence by (**e 2**) both g and h, and their sum $k=g+h$ are in $\mathfrak{K}$, however, we have without difficulty

$$\int k^{(\delta_n\gamma_n)}\,dm = \frac{1}{4}\delta_n\gamma_n\, m(|f|\geqslant\gamma_n)$$

which does not converge by assumption, contradicting the requirement (**e 1**).

Originally Kunugi [3] obtained the class of all (E.R.) integrable functions (or the class $\mathfrak{Q}$) by the procedure of completion with respect to the *rank* which is something between a general uniform topology and a metric topology. A detailed discussion of ranked spaces and their relation to the (E.R.) integral can be found in [5]. It is natural to ask whether the class $\mathfrak{Q}$ can be obtained as the completion of the class $\mathfrak{L}$ with respect to some reasonable metric, just as the latter is the completion of the class of all bounded functions with respect to the usual L_1-norm. In this respect, the conditions (**q 1**) and (**q 2**) already suggest how to define a norm-like functional.

Lemma. *The functional defined by the formula*

$$\|f\|_* = \sup_{\gamma>0}\left|\int f^{(\gamma)}\,dm\right| + \sup_{\gamma>0}\gamma\, m(|f|\geqslant\gamma)$$

has the following properties:

(**n 1**) $0\leqslant\|f\|_*\leqslant\infty$ $\quad(=0$ *if and only if* $f=0)$

(**n 2**) $\|f\|_*<\infty$ *if* $f\in\mathfrak{Q}$,

(**n 3**) $\|\alpha f\|_* = |\alpha|\,\|f\|_*$,

(**n 4**) $\|f+g\|_*\leqslant 6\{\|f\|_*+\|g\|_*\}$

(**n 5**) $\left|(Q)\int f\,dm\right|\leqslant\|f\|_*\leqslant 2\int|f|\,dm$ if $f\in\mathfrak{Q}$.

Proof. (**n 1**) and (**n 2**) are trivial. (**n 3**) follows from the relation $(\alpha f)^{(\gamma)} =\alpha f^{\left(\frac{\gamma}{\alpha}\right)}$ or $=\alpha f^{\left(\frac{-\gamma}{\alpha}\right)}$ according as $\alpha>0$ or <0. Since

$$\{|f+g|\geqslant 2\gamma\}\subset\{|f|\geqslant\gamma\}\cup\{|g|\geqslant\gamma\}$$

and

$$|(f+g)^{(2\gamma)}(x)-f^{(\gamma)}(x)-g^{(\gamma)}(x)| \begin{cases} =0 & \text{if } |f(x)|\leqslant\gamma \text{ and } |g(x)|\leqslant\gamma \\ \leqslant 4\gamma & \text{otherwise,}\end{cases}$$

we have

$$\left|\int(f+g)^{(2\gamma)}\,dm-\int f^{(\gamma)}\,dm-\int g^{(\gamma)}\,dm\right|\leqslant 4\gamma\, m(|f|\geqslant\gamma)+4\gamma\, m(|g|\geqslant\gamma),$$

hence

$$\left|\int (f+g)^{(2\gamma)}\,dm\right| + 2\gamma\, m(|f+g| \geqslant 2\gamma)$$
$$\leqslant \left|\int f^{(\gamma)}\,dm\right| + \left|\int g^{(\gamma)}\,dm\right| + 6\gamma\, m(|f| \geqslant \gamma) + 6\gamma\, m(|g| \geqslant \gamma)$$

and (**n** 4) follows. The left hand inequality of (**n** 5) follows immediately from the definition of the Q integral, and finally the right hand one from the following:

$$\left|\int f^{(\gamma)}\,dm\right| \leqslant \int |f|\,dm \quad \text{and} \quad \gamma\, m(|f| \geqslant \gamma) \leqslant \int |f|\,dm\,.$$

The properties (**n** 1) to (**n** 4) guarantee that the system of the sets $\left\{f\colon\ \|f\|_* \leqslant \frac{1}{n}\right\}$ $n=1, 2, \cdots$ forms a basis of the linear metric topology on the class $\mathfrak{Q}$ (cf. [1] p. 13) and the Q integral becomes a continuous linear functional by (**n** 5). The functional $\|f\|^*$ will be called the *quasi-norm.*

Theorem 5. *The class $\mathfrak{Q}$ is complete with respect to the quasi-norm and contains the class $\mathfrak{L}$ as a dense subclass. In other words, $\mathfrak{Q}$ is considered as the completion of $\mathfrak{L}$ with respect to the quasi-norm.*

Proof. Let $\{f_n\}$ be a Cauchy sequence, then it is fundamental in measure, in fact, for every $\gamma > 0$

$$m(|f_n - f_m| \geqslant \gamma) \leqslant \frac{1}{\gamma}\|f_n - f_m\|_* \to 0 \quad \text{as } n, m \to \infty\,,$$

hence f_n converges in measure to some function f (cf. [2] p. 91), we have obviously

$$m(|f - f_n| > \gamma) \leqslant \sup_{k \geqslant n} m(|f_k - f_n| \geqslant \gamma)$$

and

$$\left|\int (f - f_n)^{(\gamma)}\,dm\right| \leqslant \sup_{k \geqslant n} \left|\int (f_k - f_n)^{(\gamma)}\,dm\right|$$

so that

$$\|f - f_n\|_* \leqslant 2 \cdot \sup_{k \geqslant n} \|f_k - f_n\|_* \to 0$$

showing that f_n converges to f with respect to the quasi-norm. We have to prove that f is contained in $\mathfrak{Q}$. Using quite a similar procedere as in the proof of Lemma, we have

$$\left|\int f^{(2\gamma)}\,dm - \int f_n^{(\gamma)}\,dm\right| \leqslant 4\,\|f - f_n\|_* + 4\gamma\, m(|f_n| \geqslant \gamma)$$

and

$$\gamma\, m(|f|\geqslant 2\gamma)\leqslant \|f-f_n\|_* + \gamma\, m(|f_n|\geqslant \gamma)$$

consequently by (**q 1**) and (**q 2**) for f_n

$$\lim\cdot\sup_{\gamma\to\infty}\left|\int f^{(2\gamma)}\,dm-(Q)\int f_n\,dm\right|\leqslant 4\|f-f_n\|_*$$

and

$$\lim_{\gamma\to\infty}\gamma\, m(|f|\geqslant\gamma)\leqslant 4\|f-f_n\|_*\to 0.$$

On the other hand, since by (**n 5**)

$$\left|(Q)\int f_n\,dm-(Q)\int f_k\,dm\right|\leqslant \|f_n-f_k\|_*\to 0 \quad \text{as} \quad n,k\to\infty\,,$$

we can conclude from the above that (**q 1**) and (**q 2**) are satisfied for f.

In order to see that $\mathfrak{L}$ is dense in $\mathfrak{Q}$, it suffices to prove that for every $g\in\mathfrak{Q}$ $g^{(n)}$ converges to g. Given $\varepsilon>0$, there is a positive number γ_0 such that

$$\gamma\, m(|g|\geqslant\gamma)\leqslant\varepsilon \quad \text{and} \quad \left|\int g^{(\gamma)}\,dm-\int g^{(\gamma_0)}\,dm\right|\leqslant\varepsilon \qquad \text{for} \quad \gamma\geqslant\gamma_0.$$

Since, as is seen without difficulty,

$$m(|g-g^{(n)}|\geqslant\gamma)=m(|g|\geqslant\gamma+n)$$

and

$$\int\{g-g^{(n)}\}^{(\gamma)}\,dm=\int g^{(\gamma+n)}\,dm-\int g^{(n)}\,dm,$$

we have for every $n\geqslant\gamma_0$

$$\begin{aligned}\|g-g^{(n)}\|_*&\leqslant\sup_{\gamma>0}\left|\int g^{(\gamma+n)}\,dm-\int g^{(n)}\,dm\right|+\sup_{\gamma>0}\gamma\, m(|g|\geqslant\gamma+n)\\&\leqslant 2\varepsilon+\frac{\gamma_0\varepsilon}{\gamma_0+n}\leqslant 3\varepsilon\,.\end{aligned}$$

This completes the proof.

As the quasi-norm $\|f\|_*$ has properties similar to a norm, it is interesting to ask whether the topology it induces is equivalent to that induced by some norm or, more generally, is equivalent to some locally convex topology (cf. [1] p. 12). One of the advantages of local convexity is that it guarantees the existence of plenty of different continuous linear functionals. However, in our case, the Q integral turns out the only continuous linear functional up to scalar.

Theorem 4. *Every continuous linear functional on Q is a scalar multiple of the Q integral.*

Proof. Let $J(f)$ be a continuous linear functional, then when restricted to the subclass $\mathfrak{L}$, it is continuous even with respect to the L_1-norm, for the continuity combined with (**n 3**) and (**n 5**) implies

$$|J(f)| \leqslant \alpha \|f\|_* \leqslant 2\alpha \int |f|\, dm \qquad \text{for some} \quad \alpha > 0.$$

As is known (cf. [1] p. 30), there is the uniquely determined bounded function h such that

$$J(g) = \int h \cdot g\, dm \qquad \text{for all} \quad g \in \mathfrak{L}$$

Since

$$\left| J(f) - J(f^{(n)}) \right| \leqslant \alpha \| f - f^{(n)} \|_*$$

and the right hand side converges to 0 as in the proof of Theorem 3, we can conclude

$$J(f) = \lim_{n \to \infty} \int h \cdot f^{(n)}\, dm$$

Now if h is a constant function, the assertion follows from the definition of the Q integral. Suppose that h is not constant, then there are two real numbers $\beta < \rho$ for which

$$m(h > \rho) \neq 0 \quad \text{and} \quad m(h < \beta) \neq 0 .$$

Without loss of generality we may assume that $\frac{1}{2}(\beta + \rho) = 0$.

Take measurable sets E and F such that

$$E \subset \{h > \rho\}\ , \quad F \subset \{h < \beta\} \quad \text{and} \quad m(E) = m(F) \neq 0$$

and write E (respectively F) as the sum of mutually disjoint subsets E_n (respectively F_n) such that

$$m(E_n) = m(F_n) = \frac{m(E)}{2^n} \qquad n = 1, 2, \cdots$$

Now for the function g defined by

$$g = \sum \frac{2^n}{n} \{c_{E_n} - c_{F_n}\}$$

where c_E is the characteristic function of E, we have obviously

$$\int g^{(\gamma)}dm = 0 \qquad\qquad \text{for all} \quad \gamma > 0\,\cdot$$

and

$$\gamma\, m(|g| \geqslant \gamma) = \gamma \sum_{j \geqslant n} \{m(E_j) + m(F_j)\}$$
$$\leqslant \frac{\gamma\, m(E)}{2^{n-2}} \leqslant \frac{4m(E)}{n+1} \to 0$$

where n is determined by the relation

$$\frac{2^n}{n} \leqslant \gamma < \frac{2^{n+1}}{n+1}\,,$$

hence g is a function in $\mathfrak{Q}$. On the other hand, from the definition of g, it follows

$$h \cdot g^{(\gamma)} = |h \cdot g^{(\gamma)}| \qquad \text{and} \qquad J(g^{(\gamma)}) = \int |h \cdot g^{(\gamma)}|\, dm$$

and the last term diverges as $\gamma \to \infty$, in fact,

$$\int |h \cdot g^{(\gamma)})|\, dm \geqslant (\rho - \beta) m(E) \left\{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right\}$$

leading to the contradiction. Thus h must be a constant function.

4. Conjugate functions. In this section, let X be the interval $(-\pi, \pi)$ and m be the Lebesgue measure. The *conjugate function* $\tilde{f}(x)$ of an integrable function $f(x)$ is defined by

$$\tilde{f}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cot \frac{1}{2}(x-t)\, dt$$

where the integral is taken in the sense of Cauchy's principal value, i. e.

$$\lim_{\varepsilon \to 0} \left\{ \int_{-\pi}^{x-\varepsilon} + \int_{x+\varepsilon}^{\pi} \right\}.$$

It is known that $\tilde{f}(x)$ is defined almost everywhere ([10], p. 131), but not necessarily integrable. Titchmarsh [6] proved, in essense, that the conjugate function has both the properties (**q 1**) and (**q 2**). Careful inspection of his proof leads to the following statement.

Theorem 5. *The conjugate function of any integrable function is Q integrable, and the correspondence* $\mathfrak{L} \ni f \to \tilde{f} \in \mathfrak{Q}$ *is continuous, when* $\mathfrak{L}$ *and* $\mathfrak{Q}$ *are provided with the* L_1*-norm and the quasi-norm* $\|f\|_*$ *respectively.*

Let the Fourier series of $f(x)$ be

$$\frac{1}{2}a_0 + \sum (a_n \cos nx + b_n \sin nx) .$$

Roughly speaking, the Fourier series of the conjugate function $f(x)$ *is*

$$\sum (b_n \cos nx - a_n \sin nx) ,$$

in fact, the latter is summable $(C, 1)$ to the conjugate function almost everywhere ([10], p. 92). The notion of the Q integral permits a more precise formulation, which was basically obtained by Titchmarsh [6] and was explicity mentioned by Ulyanov [8]

Theorem 6. *For every n, both $\tilde{f}(x) \sin nx$ and $\tilde{f}(x) \cos nx$ are Q integrable and*

$$(Q)\int_{-\pi}^{\pi} \tilde{f}(x) \sin nx dx = -\int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$(Q)\int_{-\pi}^{\pi} \tilde{f}(x) \cos nx dx = \int_{-\pi}^{\pi} f(x) \sin nx dx .$$

Many interesting applications of the Q integral to Fourier series can be found in a series of papers [7], [8], [9].

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(Received September 17, 1964)