ON THE CONTINUITY AND THE MONOTONOUSNESS OF NORMS

By

Tetsuya SHIMOGAKI

§ 1. Let R be a *universally continuous semi-ordered linear space*[1] (i.e. *conditionally complete vector lattice* in Birkhoff's sense) and $\|\cdot\|$ be a norm on R satisfying the following conditions throughout this paper:

(N. 1) $|x|\leqq|y|$ $(x, y\in R)$ *implies* $\|x\|\leqq\|y\|$;

(N. 2) $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda}x$ *implies* $\|x\|=\sup_{\lambda\in\Lambda}\|x_\lambda\|$[2].

A norm $\|\cdot\|$ on R is called *continuous*, if

$$\inf_{\nu=1,2,\cdots}\|x_\nu\|=0 \qquad \text{for any } x_\nu\downarrow_{\nu=1}^{\infty}0\text{[3]}. \tag{1.1}$$

The continuity of norms on R plays an important rôle in the theory of semi-ordered linear spaces. In fact, it is well known [8, 9; § 31] that *every norm-continuous linear functional f on R is (order-) universally continuous*, i.e.

$$\inf_{\lambda\in\Lambda}|f(x_\lambda)|=0 \qquad \text{for any } x_\lambda\downarrow_{\lambda\in\Lambda}0, \tag{1.2}$$

and R becomes *superuniversally continuous*[4] as a space in this case.

It is clear that if a norm $\|\cdot\|$ on R is continuous, the another norm $\|\cdot\|_1$ which is equivalent to $\|\cdot\|$ is also continuous. As for the conditions under which norms $\|\cdot\|$ on R are continuous, there are the detailed investigations by T. Andô [3, 4].

A norm $\|\cdot\|$ on R is called *monotone* [8], if

$$|x|\lneqq|y| \ (x, y\in R) \qquad \textit{implies } \|x\|\lneqq\|y\|, \tag{1.3}$$

and is called *uniformly monotone* [8, 9; § 30], if

1) This terminology is due to H. Nakano [9]. We use mainly notation and terminology of [9] here.

2) A norm satisfying (N. 1) and (N. 2) is called *semi-continuous* in [10]. A norm on $\|\cdot\|$ satisfying (N. 1) is called *monotone* in [7]. On the other hand, (N. 1) is assumed for any norm of normed lattices in [6].

3) This means $x_1\geqq x_2\geqq\cdots\geqq0$ and $\bigcap_{\nu=1}^{\infty}x_\nu=0$.

4) R is called *superuniversally continuous*, if for any $0\leqq x_{\lambda,\ \lambda\in\Lambda}\leqq a$ there exists $\{x_{\lambda_\nu}\}_{\nu=1}^{\infty}\leqq\{x_\lambda\}_{\lambda\in\Lambda}$ such that $\bigcup_{\nu=1}^{\infty}x_{\lambda_\nu}=\bigcup_{\lambda\in\Lambda}x_\lambda$.

(1.4) *for any* $\gamma, \varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$x \frown y = 0,\ \|x\| \leqq \gamma \text{ and } \|y\| \geqq \varepsilon \quad \text{imply } \|x+y\| \geqq \|x\| + \delta.$$

When a norm $\|\cdot\|$ is *of* L_p*-type*[5] ($1 \leqq p < +\infty$), it is both continuous and monotone (uniformly monotone), and when $\|\cdot\|$ is *of* L_∞*-type*, it is neither continuous nor monotone. This fact suggests that there may be some correlation between the continuity and the monotonousness of norms on R, in spite of the existence of a continuous norm which is not monotone.

In § 2 we shall study this relation and show consequently that if a norm $\|\cdot\|$ on R is continuous there exists a monotone norm $\|\cdot\|_1$ which is equivalent to $\|\cdot\|$ (Theorem 3).

In § 3 we shall show a sufficient condition for the continuity of the associated norm of $\|\cdot\|$ and a necessary and sufficient condition under which the second conjugate norm $\|\bar{\bar{x}}\|$ ($\bar{\bar{x}} \in \bar{\bar{R}}''$) is continuous on $\bar{\bar{R}}''$.

In the earlier paper [10] the author defined a property of a norm called *finitely monotone*[6] which is stronger than the continuity. In § 4 we shall prove that a norm $\|\cdot\|$ on R is finitely monotone, if and only if there exists an equivalent norm $\|\cdot\|_1$ which is at the same time a lower semi-p-norm for some $1 \leqq p < +\infty$ (i.e. $x \frown y = 0$ implies $\|x+y\|_1^p \geqq \|x\|_1^p + \|y\|_1^p$) (Theorem 6). Since a lower semi-p-norm is uniformly monotone, we see that a norm $\|\cdot\|$ is finitely monotone if and only if we may define an equivalent norm $\|\cdot\|_1$ which is uniformly monotone. At last some notes on finitely monotone norms shall be made.

In the sequel we denote by $\tilde{R}''$ the *norm associated space* of R (i.e. the totality of all norm-continuous linear functionals on R) and by $\bar{R}''$ the *norm conjugate space* of R (i.e. the totality of all universally continuous linear functionals on R which is norm-continuous too)[7]. The completeness of $\|\cdot\|$ on R shall not be assumed, unless otherwise provided.

§ 2. Let $\|\cdot\|$ be an arbitrary norm on R satisfying (N. 1) and (N. 2) in the sequel.

Definition 1. An element $a \in R$ is said to be a *continuous element*

5) A norm is called to be *of* L_p*-type* [1], if $\|x+y\|^p = \|x\|^p + \|y\|^p$ for any $x, y \in R$ with $x \frown y = 0$ in the case $1 \leqq p < +\infty$, and $\|x+y\| = \max(\|x\|, \|y\|)$ in the case $p = +\infty$.

6) For the definition of the finitely monotone norm see § 4. It was discussed first in [2].

7) $\bar{R}$ denotes the totality of all universally continuous linear functionals on R. When $\|\cdot\|$ is complete, $\bar{R}'' = \bar{R}$ holds. But $\bar{R}'' \subseteqq \bar{R}$ in general.

(*with respect to* $\|\cdot\|$), if $|a|\geqq a_\nu \downarrow_{\nu=1}^{\infty} 0$ implies $\lim_{\nu\to\infty}\|a_\nu\|=0$.

It is easily verified that $a\in R$ is a continuous element if and only if $\inf_{\nu=1,2,\cdots}\|[p_\nu]a\|=0$ for any $[p_\nu]\downarrow_{\nu=1}^{\infty}0$[8] (cf. [9; Th. 30.8]). If a is a continuous element and $\alpha|a|\geqq b$, $b\in R$, then b is also a continuous element by the definition. Hence we see that the totality of all continuous elements of R constitutes a semi-normal manifold[9] of R and we denote it by R_c. A norm $\|\cdot\|$ on R is continuous if and only if $R_c=R$. We call $\|\cdot\|$ to be *almost continuous*, if R_c is a complete semi-normal manifold of R. Being stated above, R with a continuous norm is always *semi-regular*[10]. But this fact remains to be true by replacing the continuity by the almost continuity, that is, R with an almost continuous norm is semi-regular.

Let M be a linear manifold of R. A linear functional $\bar{a}\in\bar{\bar{R}}''$ is called *complete* on M if $|\bar{a}|(b)=0$, $b\in M$ implies $b=0$. It is shown [9; § 20] that for any $0\neq\bar{a}\in\bar{\bar{R}}''$ there exists a normal manifold N on which $\bar{a}$ is complete. We denote by B_a ($a\in R$) the semi-normal manifold consisting of all elements $x\in R$ such that $|x|\leqq\alpha a$ for some real α (depending on x).

Now we have

Lemma 1. *If* $0\leqq a\in R$ *is a continuous element with respect to* $\|\cdot\|$, *there exists a universally continuous linear functional* $\bar{a}\in\bar{\bar{B}}_a''$ *which is complete on* B_a.

Proof. Since B_a is a semi-normal manifold, B_a is a normed semi-ordered linear space with $\|\cdot\|$ by itself and the norm $\|\cdot\|$ is continuous on B_a from the definition. Being stated above, B_a is semi-regular and hence there exists the system of elements $\{\bar{a}_\lambda\}_{(\lambda\in\Lambda)}$ $(0\leqq\bar{a}_\lambda\in\bar{\bar{B}}_a'', \lambda\in\Lambda)$ with $\bigcup_{\lambda\in\Lambda}[\bar{a}_\lambda]^{B_a}=1$[11] and $\|\bar{a}_\lambda\|=1$ $(\lambda\in\Lambda)$. Since B_a is superuniversally continuous, we can find a subsequence $\{\bar{a}_{\lambda_\nu}\}_{(\nu=1,2,\ldots)}$ of $\{\bar{a}_\lambda\}_{(\lambda\in\Lambda)}$ with $\bigcup_{\nu=1}^{\infty}[\bar{a}_{\lambda_\nu}]^{B_a}=1$. Now, as $\bar{\bar{B}}_a''$ is complete (with respect to the norm),

$$\bar{a}_0=\sum_{\nu=1}^{\infty}\frac{1}{2^\nu}\bar{a}_{\lambda_\nu}\in\bar{\bar{B}}_a''$$

8) For $p\in R$, $[p]$ denotes the projector by p. i.e. $[p]x=\bigcup_{n=1}^{\infty}(n|p|\frown x)$ for $x\geqq 0$.

9) A linear lattice manifold $M\subseteqq R$ is called a *semi-normal manifold*, if $a\in M$, $|a|\geqq|b|$ implies $b\in M$. A semi-normal manifold M is called complete, if $\{M^{\perp}\}^{\perp}=\{0\}$.

10) R is called *semi-regular*, if $\bar{a}(a)=0$ for all $\bar{a}\in\bar{R}$ implies $a=0$.

11) $[\bar{a}]^{B_a}$ is a projector on B_a such that $\bar{b}([\bar{a}]^{B_a}x)=[\bar{a}]\bar{b}(x)$ holds for every $\bar{b}\in\bar{B}_a$ and $x\in B_a$ [9; § 22].

and $\bar{a}_0$ is complete on B_a obviously. Q. E. D.

Lemma 2. *Let $0 \leqq a \in R$ be a continuous element. For any $\varepsilon > 0$, there exists a positive integer $n = n(a, \varepsilon)$ such that $a = \sum_{i=1}^{\nu} a_i$, $a_i \geqq 0$ and $\|a_i\| \geqq \varepsilon$ $(i = 1, 2, \cdots, \nu)$ imply $\nu \leqq n$.*

Proof. There exists a complete linear functional $0 \leqq \bar{a} \in \bar{\bar{B}}_a$ on B_a with $\bar{a}(a) = 1$ by virtue of Lemma 1. If the conclusion of Lemma 2 is not true, we can find a sequence of elements of R^+:[12] $\{a_{\nu,\mu};\ \mu = 1, 2, \cdots, \kappa_\nu;\ \nu = 1, 2, \cdots\}$ with $a = \sum_{\mu=1}^{\kappa_\nu} a_{\nu,\mu}$, $2^\nu \leqq \kappa_\nu$ and $\|a_{\nu,\mu}\| \geqq \varepsilon$ for each $1 \leqq \nu$ and $1 \leqq \mu \leqq \kappa_\nu$. Since $\bar{a}$ is linear, there exists a_{ν,μ_ν} with $\bar{a}(a_{\nu,\mu_\nu}) \leqq 1/2^\nu$ for each $1 \leqq \nu$. Putting $b_i = \bigcup_{\nu \geqq i} a_{\nu,\mu_\nu} \leqq a$ we obtain $0 \leqq b_i \downarrow_{i=1}^{\infty}$ and $b_0 = \bigcap_{i=1}^{\infty} b_i \in B_a$. Since $\bar{a}(b_i) \leqq 1/2^{i-1}$ and $\bar{a} \geqq 0$, we have $\bar{a}(b_0) = 0$ and a fortiori $b_i \downarrow_{i=1}^{\infty} b_0 = 0$, because $\bar{a}$ is complete on B_a.

On the other hand, the fact that $\|b_i\| \geqq \|a_{i,\mu_i}\| \geqq \varepsilon$ for all $i \geqq 1$ is inconsistent with the assumption, which establishes the proof. Q. E. D.

Definition 2. $a \in R^+$ is called a *purely monotone element* (with respect to $\|\cdot\|$), if for any $\varepsilon > 0$ there exists $\delta = \delta(a, \varepsilon) > 0$ such that $a \geqq b \geqq 0$ and $\|b\| \geqq \varepsilon$ imply $\|a - b\| \leqq \|a\| - \delta$.

Now we obtain

Theorem 1. *If $a \in R^+$ is a purely monotone element, a is a continuous one.*

Proof. If $a \geqq a_\nu \downarrow_{\nu=1}^{\infty} 0$ and $\|a_\nu\| \geqq \varepsilon$ $(\nu = 1, 2, \cdots)$, we have for some $\delta > 0$

$$\|a_1 - a_\nu\| \leqq \|a\| - \delta \qquad (\nu = 1, 2, \cdots),$$

since $a \in R^+$ is a purely monotone element. This contradicts (N. 2) in § 1. Therefore a is a continuous element by the definition. Q. E. D.

Theorem 2. *For any norm $\|\cdot\|$ on R, there exists a norm $\|\cdot\|_1$ equivalent to $\|\cdot\|$ such that every continuous element a with respect to $\|\cdot\|$ is purely monotone one with respect to $\|\cdot\|_1$.*

Proof. We define $\|\cdot\|_1$ by the formula:

$$\|x\|_1 = \|x\| + \sup_{|x| = \sum_{\nu=1}^{\infty} y_\nu,\ y_\nu \geqq 0} \left\{ \sum_{\nu=1}^{\infty} \frac{\|y_\nu\|}{2^\nu} \right\} \qquad (x \in R).$$

From the definition of $\|\cdot\|_1$, it is clear that $\|\cdot\|_1$ is a norm on R

12) R^+ denotes the set of all positive elements of R.

satisfying (N. 1), (N. 2) and

$$||x|| \leqq ||x||_1 \leqq 2||x|| \qquad \text{for every } x \in R.$$

Let $a \in R^+$ be a continuous element with respect to $||\cdot||$. By virtue of Lemma 2, for any $\varepsilon > 0$ there exists an integer $n_0 = n_0(a, \varepsilon/2)$ such that $a = \sum_{\nu=1}^{n} a_\nu$, $a_\nu \in R^+$ and $||a_\nu|| \geqq \varepsilon/2$ $(\nu = 1, 2, \cdots, n)$ imply $n \leqq n_0$. Suppose $0 \leqq b \leqq a$ and $||b|| \geqq \varepsilon$, then for any $\{b_\nu\}_{\nu=1}^{\infty} \subset R^+$ with $a - b = \sum_{\nu=1}^{\infty} b_\nu$ and $||b_1|| \geqq ||b_2|| \geqq \cdots$ we have

$$||a-b|| + \sum_{\nu=1}^{\infty} \frac{||b_\nu||}{2^\nu} = ||a-b|| + \sum_{\nu=1}^{n_0} \frac{||b_\nu||}{2^\nu} + \frac{||b_{n_0+1}||}{2^{n_0+1}} + \sum_{\nu=n_0+2}^{\infty} \frac{||b_\nu||}{2^\nu}.$$

Here we have on account of $||b_{n_0+1}|| < \varepsilon/2$

$$\frac{||b_{n_0+1}||}{2^{n_0+1}} \leqq \frac{\varepsilon}{2^{n_0+2}} \leqq \frac{||b||}{2^{n_0+1}} - \frac{\varepsilon}{2^{n_0+2}},$$

which implies

$$||a-b|| + \sum_{\nu=1}^{\infty} \frac{||b_\nu||}{2^\nu} \leqq ||a-b|| + \sum_{\nu=1}^{n_0} \frac{||b_\nu||}{2^\nu} + \frac{||b||}{2^{n_0+1}} + \sum_{\nu=n_0+2}^{\infty} \frac{||b_\nu||}{2^\nu} - \frac{\varepsilon}{2^{n_0+2}}$$
$$\leqq ||a||_1 - \frac{\varepsilon}{2^{n_0+2}},$$

because of $\sum_{\nu=1}^{n_0} b_\nu + b + \sum_{\nu=n_0+2}^{\infty} b_\nu \leqq a$. Therefore, we obtain

$$||a-b||_1 \leqq ||a||_1 - \frac{\varepsilon}{2^{n_0+2}},$$

which shows that a is a purely monotone element. Q. E. D.

Remark 1. The norm $||\cdot||_1$ constructed in the above theorem has the following property: *for any continuous element $a \in R$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $a \in R^+$, $||b||_1 \geqq \varepsilon$ implies $||a+b||_1 \geqq ||a||_1 + \delta$.*

From Theorem 2 we obtain immediately

Corollary 1. *If a norm $||\cdot||$ on R is almost continuous, then there exists a norm $||\cdot||_1$ equivalent to $||\cdot||$ such that the set of all purely monotone elements coincides with a complete semi-normal manifold R_c*[13].

If each $a \in R^+$ is a purely monotone element with respect to a norm $||\cdot||$, $||\cdot||$ is monotone. Therefore, we have

Theorem 3. *If a norm $||\cdot||$ on R is continuous, there exists a monotone norm $||\cdot||_1$ which is equivalent to $||\cdot||$*[14].

13) R_c is the same for all equivalent norms.

14) Theorems 2 and 3 hold to be true for any norm $||\cdot||$ on R satisfying (N. 1) only.

Remark 2. On the other hand, as an easy example shows, there exists a monotone norm $||\cdot||$ which has no continuous norm equivalent to it.

§ 3. In this section we shall give some notes on the continuity of the norm on R. First we shall show a simple sufficient condition[15] for the continuity of the associated norm on $\tilde{R}''$. We shall consider the following condition (*) for norms $||\cdot||$ on R:

$$(*) \qquad ||x_\nu|| \leqq 1, \; x_\nu \in R^+ \quad (\nu = 1, 2, \cdots) \qquad \text{implies } \lim_{n\to\infty} \frac{||\bigcup_{\nu=1}^{n} x_\nu||}{n} = 0.$$

Now we have

Theorem 4. *Suppose that a norm $||\cdot||$ on R satisfy (*). Then the associate norm $||\cdot||$ on $\tilde{R}''$ (or the conjugate norm on $\bar{\bar{R}}''$) is continuous.*

Proof. If the associated norm $||\cdot||$ on $\tilde{R}''$ is not continuous, we can find a positive number $\varepsilon > 0$ and a sequence of elements $0 \leqq \tilde{x}_\nu \in \tilde{R}''$ ($\nu = 1, 2, \cdots$) such that $||\tilde{x}_\nu|| \geqq 2\varepsilon$ and $||\sum_{\nu=1}^{n} \tilde{x}_\nu|| \leqq 1$ for $n \geqq 1$. Now we can find also a sequence of elements $\{x_\nu\}_{(\nu \geqq 1)} \subset R^+$ such that $\tilde{x}_\nu(x_\nu) \geqq \varepsilon$ and $||x_\nu|| \leqq 1$ for each $\nu \geqq 1$. Putting $y_n = \bigcup_{\nu=1}^{n} x_\nu$, we obtain for any $n \geqq 1$

$$||y_n|| \geqq (\sum_{\nu=1}^{n} \tilde{x}_\nu) y_n \geqq \sum_{\nu=1}^{n} \tilde{x}_\nu(x_\nu) \geqq n\varepsilon,$$

hence

$$\lim_{n\to\infty} \frac{||y_n||}{n} = \lim_{n\to\infty} \frac{||\bigcup_{\nu=1}^{n} x_\nu||}{n} \geqq \varepsilon > 0.$$

This contradicts the condition (*). Q. E. D.

Since there exists a semi-ordered linear space R with a continuous norm which has, however, the second conjugate space $\bar{\bar{R}}''$[16] whose norm is continuous no longer, we see that in Theorem 4 we can not exchange (*) for the continuity of a norm of R without failing to hold the validity.

Here we give a necessary and sufficient condition for the continuity of the second conjugate norm $||X||$ ($X \in \bar{\bar{R}}''$) of a norm $||\cdot||$ on R.

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15) In [5] T. Andô gave a necessary condition [5; § 4 Lemma 4.2].

16) The conjugate space of $\bar{R}''$ is denoted by $\bar{\bar{R}}'$. If R is semi-regular, R can be considered as a complete semi-normal manifold of $\bar{\bar{R}}''$ [9].

Theorem 5. *In order that the second conjugate norm on* $\overline{\overline{R}}''$ *of* $\|\cdot\|$ *be continuous, it is necessary and sufficient that the norm* $\|\cdot\|$ *on* R *satisfies the following condition:*

$$(\sharp) \qquad \sup_{n\geq 1}\|\sum_{\nu=1}^{n} x_\nu\| < +\infty, \qquad x_\nu \in R^+ \ (\nu=1,2,\cdots)$$

implies $\lim_{\nu\to\infty}\|x_\nu\|=0$.

Proof. Necessity. Since $R\subseteq\overline{\overline{R}}''$, $\|x\|=\|f_x\|$[17] $=\sup_{\bar{x}\in\overline{R}',\ \|\bar{x}\|\leq 1}|\bar{x}(x)|$ for all $x\in R$ and $\overline{\overline{R}}''$ is monotone complete[18] [9], $\sup_{n\geq 1}\|\sum_{\nu=1}^{n} f_{x_\nu}\|<+\infty$ $(x_\nu\in R)$ implies $X_0=\sum_{\nu=1}^{\infty} f_{x_\nu}\in\overline{\overline{R}}''$. Then $Y_n=\sum_{\nu=n}^{\infty} f_{x_\nu}\in\overline{\overline{R}}''$, $Y_n\downarrow_{n=1}^{\infty}0$, and hence

$$\lim_{n\to\infty}\|x_n\|=\lim_{n\to\infty}\|f_{x_n}\|=\lim_{n\to\infty}\|Y_n\|=0,$$

on account of the continuity of the second conjugate norm $\|X\|$ $(X\in\overline{\overline{R}}'')$.

Sufficiency. We shall first show that the norm $\|\cdot\|$ on R is continuous in this case. Indeed, let $R\ni[p_\nu]a\downarrow_{\nu=1}^{\infty}0$. From the assumption $\{[p_\nu]a\}_{\nu=1}^{\infty}$ is a Cauchy sequence of R, whence we have $\lim_{\nu\to\infty}\|[p_\nu]a\|=0$ by virtue of the semi-continuity of $\|\cdot\|$.

Now if $\|X\|$ $(X\in\overline{\overline{R}}'')$ is not continuous, we may find a sequence of elements $\{X_\nu\}_{\nu=1}^{\infty}$ of $\overline{\overline{R}}''$ such that $\|X_\nu\|\geq 1$ and $\|\sum_{\nu=1}^{n} X_\nu\|\leq\gamma$ $(n,\nu=1,2,\cdots)$ for some $\gamma\geq 1$. Since the norm $\|\cdot\|$ on R is continuous, R is semi-regular, and hence R is a complete semi-normal manifold of $\overline{\overline{R}}''$. Thus there exists a sequeuce of elements $\{x_\nu\}_{\nu=1}^{\infty}$ such that $X_\nu\geq f_{x_\nu}$, $\|x_\nu\|\geq 1/2$[19] $(\nu=1,2,\cdots)$. This contradicts ($\sharp$), because of $\sup_{n\geq 1}\|\sum_{\nu=1}^{n} x_\nu\|=\sup_{n\geq 1}\|\sum_{\nu=1}^{n} f_{x_\nu}\|\leq\gamma$.

Corollary 2. *In order that a norm* $\|\cdot\|$ *on* R *be monotone complete and continuous, it is necessary and sufficient that* $\|\cdot\|$ *is complete*[20] *and satisfies the condition* ($\sharp$).

§ 4. A norm $\|\cdot\|$ on R is called *finitely monotone* [10], if it satisfies the following:

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17) f_x $(x\in R)$ denotes an element of $\overline{\overline{R}}''$ for which $f_x(\bar{x})=\bar{x}(x)$ holds for each $\bar{x}\in\overline{R}'$.

18) R is called *monotone complete*, if $0\leq a_\lambda\uparrow_{\lambda\in\Lambda}$ and $\sup_{\lambda\in\Lambda}\|a_\lambda\|<+\infty$ implies $\bigcup_{\lambda\in\Lambda} a_\lambda\in R$.

19) When R is semi-regular. A norm satisfying (N. 1) and (N. 2) is reflexive, i.e. $\|x\|=\sup_{\|\bar{x}\|\leq 1,\ \bar{x}\in\overline{\overline{R}}''}|\bar{x}(x)|$ [11].

20) If a norm $\|\cdot\|$ on R is monotone complete, it is complete [10; § 30]. The converse of this is not true in general. Cf. Corollary 2 with Theorem 2.1 of [12] in modular spaces.

(4.1) *for any* $0<\varepsilon\leqq 1$ *there exists a natural number* $N(\varepsilon)$ *such that* $\|x_i\|\geqq\varepsilon$, $x_i\frown x_j=0$, $i\neq j$ $(i,j=1,2,\cdots,n)$ *and* $\|\sum_{i=1}^{n}x_i\|\leqq 1$ *imply* $n\leqq N(\varepsilon)$.

It is clear that *every finitely monotone norm is continuous and any norm* $\|\cdot\|$ *which is equivalent to a finitely monotone norm* $\|\cdot\|$ *is also such a one.* This topologically invariant property of a finitely monotone norm is important and may be utilized. Here we shall characterize a finitely monotone norm by showing the possibility of conversion of it into the another norm of the more familar and simpler form.

A norm $\|\cdot\|$ on R is called a *lower semi-p-norm* (*upper semi-p-norm*) if for any $x\frown y=0$. $x,y\in R$, $\|x+y\|^p\geqq\|x\|^p+\|y\|^p$ (resp. $\|x+y\|^p\leqq\|x\|^p+\|y\|^p$) holds, where p is a real number with $1\leqq p\leqq+\infty$ [1].

Being well known [8], the lower semi-p-norm and the upper semi-q-norm are *of conjugate type*[21], where $1/p+1/q=1$ and the former is uniformly monotone in the case $p<+\infty$, and hence finitely monotone.

At first we shall prove an auxiliary lemma:

Lemma 3. *Let* $\|\cdot\|$ *be a finitely monotone norm and* p *be a real number such that* $2^p=N(1/2)+1$[22] *holds. Suppose also that* $\varepsilon/2\leqq\|x_\nu\|<\varepsilon$ $(\nu=1,2,\cdots,m)$ *dnd* $x_\nu\frown x_\mu=0$ *for* $\nu\neq\mu$. *If* l *is a natural number such that* $0\leqq m-l2^p<2^p$ *holds, there exist mutually orthogonal elements* y_μ $(\mu=1,2,\cdots,l)$ *such that* $\varepsilon\leqq\|y_\mu\|$ $(\mu=1,2,\cdots,l)$ *and* $\sum_{\mu=1}^{l}y_\mu\leqq\sum_{\nu=1}^{n}x_\nu$.

Proof. If $\{x_{\nu_i}\}_{i=1}^{2^p}$ is arbitrary subsequence of $\{x_\nu\}_{\nu=1}^{n}$, it follows that $\|(1/\varepsilon)x_{\nu_i}\|\geqq 1/2$ $(i=1,2,\cdots,2^p)$ and $2^p>N(1/2)$. This implies $\|1/\varepsilon\sum_{i=1}^{2^p}x_{\nu_i}\|\geqq 1$, whence we have $\|\sum_{i=1}^{2^p}x_{\nu_i}\|\geqq\varepsilon$. From this we see that we can find $\{y_\mu\}_{\mu=1}^{l}$ which satisfies the above condition.

Now we have

Theorem 6. *A norm* $\|\cdot\|$ *on* R *is finitely monotone if and only if there exists a lower semi-p-norm* $\|\cdot\|_1$ *equivalent to* $\|\cdot\|$, *where* $1\leqq p<+\infty$.

Proof. Since a lower semi-p-norm is finitely monotone and the finite monotonousness is topological invariant, it suffices to prove the necessity of the theorem.

Let $\|\cdot\|$ be finitely monotone and p be a real number satisfying $2^p=N(1/2)+1$. We put now

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21) That is, if $\|\cdot\|$ is a lower semi-p-norm (upper semi-q-norm), the conjugate norm is an upper semi-q-norm (resp. lower semi-p-norm).

22) $N(1/2)$ is a natural number which appears in (4.1) for $\varepsilon=1/2$ with respect to the finitely monotone norm $\|\cdot\|$.

$$\|x\|_1 = \sup \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} \text{[23]} \qquad (x \in R).$$

$$x = \bigoplus_{i=1}^{n} x_i, \quad n = 1, 2, \cdots$$

From this definition it is clear that $\|x+y\|_1^p \geqq \|x\|_1^p + \|y\|_1^p$ holds for each $x, y \in R$ with $x \frown y = 0$. Furthermore it is also evident that $\|x\| \leqq \|x\|_1$ and $\|\alpha x\|_1 = |\alpha| \, \|x\|_1$ hold for each α real and $x \in R$. The sub-additivity of $\|\cdot\|_1$ follows from Minkowski's inequality and $\|\cdot\|_1$ satisfies (N. 1) and (N. 2), because $\|\cdot\|$ does. Therefore, it is sufficient to prove that we can find $\kappa > 0$ for which $\|x\|_1 \leqq \kappa \|x\|$ holds for each $x \in R$. Let $\{x_\nu\}_{\nu=1}^n$ be a mutually orthogonal sequence of positive elements of R such that $\|\sum_{\nu=1}^{n} x_\nu\| = 1$ and k_i be the number of x_ν with $1/2^i < \|x_\nu\| \leqq 1/2^{i-1}$ $(i = 1, 2, \cdots)$. Then

$$n = k_1 + k_2 + \cdots + k_m, \qquad k_m \neq 0$$

holds for some $m \geqq 1$. If $n = k_1$, $(\sum_{\nu=1}^{n} \|x_\nu\|^p)^{1/p} \leqq (N(1/2))^{1/p} \leqq 2$ holds. Thus we shall assume $n \neq k_1$ in the argument below.

Now we can find an integer $0 \leqq l_m$ such that

$$0 \leqq k_m - l_m 2^p < 2^p$$

holds. This implies

$$\sum_{x_\nu \in A_m} \|x_\nu\|^p \leqq k_m \frac{1}{2^{(m-1)p}} \leqq \frac{l_m + 1}{2^{p(m-2)}},$$

where $A_j = \{x_\nu : x_\nu \in \{x_i\}_{i=1}^n,\ 1/2^j < \|x_\nu\| \leqq 1/2^{j-1}\}$ for every $1 \leqq j \leqq m$. We note here that there exists a sequence of mutually orthogonal elements $\{y_\mu\}_{\mu=1}^{l_m}$ such that $\sum_{\mu=1}^{l_m} y_\mu \leqq \sum_{x_\nu \in A_m} x_\nu$ and $\|y_\mu\| \geqq 1/2^{m-1}$ $(\mu = 1, 2, \cdots, l_m)$ hold, if $l_m \neq 0$, in virtue of Lemma 3.

Next we choose an integer $0 \leqq l_{m-1}$ for which

$$0 \leqq k_{m-1} + l_m - l_{m-1} 2^p < 2^p$$

holds. Now this yields

$$\sum_{x_\nu \in A_m \cup A_{m-1}} \|x_\nu\|^p \leqq \frac{l_m + 1}{2^{p(m-2)}} + \sum_{x_\nu \in A_{m-1}} \|x_\nu\|^p \leqq \frac{l_{m-1}}{2^{p(m-3)}} + \frac{1}{2^{p(m-2)}} + \frac{1}{2^{p(m-3)}},$$

and there exists also a mutually orthogonal sequence $\{z_\mu\}_{\mu=1}^{l_{m-1}}$ such that

$$\sum_{\mu=1}^{l_{m-1}} z_\mu \leqq \sum_{x_\nu \in A_m \cup A_{m-1}} x_\nu \quad \text{and} \quad \|z_\mu\| \geqq \frac{1}{2^{m-2}}$$

hold for all $1 \leqq \mu \leqq l_{m-1}$. Let l_{m-2} be similarly defined as before and pro-

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23) $x = \bigoplus_{i=1}^{n} x_i$ means that $x = \sum_{i=1}^{n} x_i$ and $x_i \frown x_j = 0$ for $i \neq j$.

ceeding this process we obtain finally an integer l_2 such that

$$0 \leqq k_2 + l_3 - l_2 2^p < 2^p$$

and

$$\sum_{x_\nu \in A_m \cup A_{m-1} \cup \cdots \cup A_2} ||x_\nu||^p \leqq l_2 + \frac{1}{2^{p(m-2)}} + \frac{1}{2^{p(m-3)}} + \cdots + 1$$

holds. Now, in virtue of Lemma 3 there exists also a mutually orthogonal sequence $\{\omega_\mu\}_{\mu=1}^{l_2}$ such that $\sum_{\mu=1}^{l_2} \omega_\mu \leqq \sum_{x_\nu \in A_m \cup A_{m-1} \cup \cdots \cup A_2} x_\nu$ and $||\omega_\mu|| \geqq 1/2$. Therefore we obtain

$$\begin{aligned}\sum_{\nu=1}^{n} ||x_\nu||^p &= \sum_{x_\nu \in A_m \cup A_{m-1} \cup \cdots \cup A_2} ||x_\nu||^p + \sum_{x_\nu \in A_1} ||x_\nu||^p \\ &\leqq l_2 + k_1 + \frac{1}{2^{p(m-2)}} + \frac{1}{2^{p(m-3)}} + \cdots + 1 \\ &\leqq N\left(\frac{1}{2}\right) + 2,\end{aligned}$$

whence $(\sum_{\nu=1}^{n} ||x_\nu||^p)^{1/p} \leqq (N(1/2)+2)^{1/p} \leqq 3$. This shows $||x||_1 = \sup_{x=\bigoplus_{i=1}^{n} x_i,\ n=1,2,\cdots} (\sum_{i=1}^{n} ||x_i||^p)^{1/p}$ $\leqq 3$ for each $x \in R$ with $||x|| \leqq 1$ and establishes the theorem. Q. E. D.

Since a uniformly monotone norm $||\cdot||$ is finitely monotone [10], it follows from above

Corollary 3. *If a norm $||\cdot||$ on R is uniformly monotone, we can define a lower semi-p-norm $||\cdot||_1$ which is equivalent to $||\cdot||$ for some p with $1 \leqq p < +\infty$.*

A norm $||\cdot||$ on R is called *finitely flat* [10], if

(4.2) *for any $\gamma > 0$ there exists $\varepsilon > 0$ such that $||x_i|| \leqq \varepsilon$, $x_i \frown x_j = 0$, $i \neq j$ $(i, j = 1, 2, \cdots, n)$ and $n \leqq \gamma/\varepsilon$ imply $||\sum_{i=1}^{n} x_i|| \leqq 1$.*

It is known that the finitely monotonousness and the finite flatness are of conjugate type [10; Th. 1.4, 1.5]. Thus each upper semi-p-norm $||\cdot||$ $(1 < p)$ is finitely flat.

Since the norm satisfying (N. 1), (N. 2) is reflexive [11][24], we obtain in virtue of Theorem 6

Theorem 7. *Let a norm $||\cdot||$ on R be finitely flat and R be semi-regular. Then we can define an upper semi-p-norm $||\cdot||_1$ which is equivalent to $||\cdot||$, where $1 < p \leqq +\infty$.*

Furthermore we have

24) See the footnote 19) for the definition of *reflexivity*.

Theorem 8. *If a norm $||\cdot||$ on R is both finitely monotone and finitely flat, we can define an upper semi-p and lower semi-q-norm $||\cdot||_1$ which is equivalent to $||\cdot||$, where $1<p\leqq q<+\infty$.*

Proof. Since the norm $||\cdot||$ is finitely monotone, there exists a lower semi-q-norm $||\cdot||_0$ equivalent to $||\cdot||$ for some $1\leqq q<+\infty$ in virtue of Theorem 6. Then the conjugate norm $||\bar{\bar{x}}||_0$ $(\bar{\bar{x}}\in\bar{\bar{R}}'')$ of $||\cdot||_0$ is an upper semi-q'-norm $(1/q'+1/q=1)$ and it is also finitely monotone, because $||\cdot||$ is finitely flat too. Now we define

$$||\bar{\bar{x}}||_1=\sup_{\bar{\bar{x}}=\bigoplus_{i=1}^{n}\bar{\bar{x}}_i,\ n=1,2,\cdots}\left(\sum_{\nu=1}^{n}||x_i||_0^{p'}\right)^{\frac{1}{p'}}\qquad(\bar{\bar{x}}\in\bar{\bar{R}}''),$$

where p' is a real number such that $2^{p'}=N(1/2)+1$[25]. It is clear that $q'\leqq p'$ and $||\bar{\bar{x}}||_1$ is a lower semi-p'-norm and is equivalent to $||\bar{\bar{x}}||_0$ $(\bar{\bar{x}}\in\bar{\bar{R}}'')$, as is shown in Theorem 6.

On the other hand, from Minkowski's inequality it follows that $||\bar{\bar{x}}||_1$ $(\bar{\bar{x}}\in\bar{\bar{R}}'')$ remains still to be an upper semi-q'-norm. Thus, the conjugate norm $||X||$ $(X\in\bar{\bar{R}}'')$ of $||\bar{\bar{x}}||_1$ $(\bar{\bar{x}}\in\bar{\bar{R}}'')$ is the upper semi-p and lower semi-q-norm, where $1/p+1/p'=1$ and $1<p\leqq q<+\infty$.

Since $R\subset\bar{\bar{R}}''$ and the norm $||x||$ $(x\in R)$ is reflexive, we have obtained an upper semi-p- and lower semi-q-norm $||\cdot||_1$ which is equivalent to $||\cdot||$.

Q. E. D.

At last we shall make a note on finitely monotone norms.

Lemma 4. *If a norm $||\cdot||$ is finitely monotone, it satisfies the following:*

(4.3) *for any $1>\varepsilon>0$ there exists a natural number $n(\varepsilon)$ such that $\varepsilon\leqq||x_i||$, $x_i\in R^+$ $(i=1,2,\cdots,n)$ and $||\sum_{i=1}^{n}x_i||\leqq1$ implies $n\leqq n(\varepsilon)$.*

Proof. At first suppose that ε_1 be a real number with $1/2<\varepsilon_1<1$, α be such that $1/2<\alpha<\varepsilon_1$ and $[p_i]=[(x_i-\alpha x)^+]$, where $x=\sum_{i=1}^{n}x_i$, $x_i\geqq0$ and $||x_i||\geqq\varepsilon_1$ $(i=1,2,\cdots,n)$. Now we have $[p_i]\neq0$ for each i with $1\leqq i\leqq n$, because, in the contrary case, we have $x_i\leqq\alpha x$ for some i $(1\leqq i\leqq n)$, and hence $||x_i||\leqq\alpha||x||\leqq\alpha<\varepsilon_1$, which is a contradiction. We have also that $[p_i][p_j]=0$ for $i\neq j$ holds. Because, $[p_i][p_j]\neq0$ implies

$$[p_i][p_j](x_i+x_j)\geqq\alpha[p_i][p_j]x+\alpha[p_i][p_j]x\gneqq[p_i][p_j]x$$

which is inconsistent with the fact that $x=\sum_{i=1}^{n}x_i$ and $x_i\geqq0$ $(i=1,2,\cdots,n)$.

25) Since $||\bar{\bar{x}}||_0$ $(\bar{\bar{x}}\in\bar{\bar{R}}'')$ is finitely monotone, there exists an natural number which satisfies (4.1) for $\varepsilon=1/2$. We denote it by $N(1/2)$ here.

Now putting $y_i=[p_i]x_i$ $(1\leqq i\leqq n)$, we obtain $y_i\frown y_j=0$ $(i\neq j)$, $||y_i||=||[p_i]x_i|| \geqq ||x_i||-||(1-[p_i])x_i||\geqq||x_i||-||\alpha x||\geqq\varepsilon_1-\alpha$ and $||\sum_{i=1}^{n} y_i||\leqq||\sum_{i=1}^{n} x_i||\leqq 1$. As $||\cdot||$ is finitely monotone, there exists a natural number $N(\varepsilon_1-\alpha)$ appeared in (4.1), for which $n\leqq N(\varepsilon_1-\alpha)$[26] holds.

For any $0<\varepsilon<1$ we choose a natural number m such that $0<\varepsilon_1^m<\varepsilon$. Putting $n(\varepsilon)=(N(\varepsilon_1-\alpha))^m$, we can verify evidently that $n(\varepsilon)$ satisfies the condition (4.3). Q. E. D.

Similarly we have

Lemma 5. *If a norm* $||\cdot||$ *is finitely flat, it satisfies the following:*

(4.4) *for any* $\gamma>0$ *there exists* $\varepsilon>0$ *such that* $||a_i||\leqq\varepsilon$, $x_i\in R^+$ $(i=1,2,\cdots,n)$ *and* $n\leqq\gamma/\varepsilon$ *imply* $||\bigcup_{i=1}^{n} x_i||\leqq 1$.

As the proof is quite similar, we omit it.

From these lemmas we obtain a following theorem which enables us to conclude that "finitely monotone" and "finitely flat" are *of dual type* (cf. [10; Th. 1.4, 1.5]), that is,

Theorem 9. *If a norm* $||\cdot||$ *on* R *is finitely monotone (finitely flat), then the associated norm* $||\tilde{x}||$ $(\tilde{x}\in\tilde{R}'')$ *is finitely flat (resp. finitely monotone)*[27].

The proof is obtained by showing that (4.3) and (4.4) are of dual type in the quite same manner as Theorems 1.4 and 1.5 in [10].

References

[1] I. AMEMIYA: On semi-p-modulars, Functional Analysis, Vol. 2, No. 4 (1951), pp. 101-111 (in Japanese).

[2] I. AMEMIYA: A generalization of the theorem of Orlicz and Birnbaum, Jour. Fac. Sci. Hokkaido Univ. 12, No. 2 (1956), pp. 60-64.

[3] T. ANDÔ: On the continuity of norms, Proc. Japan Acad. 33, No. 8 (1957), pp. 429-434.

[4] T. ANDÔ: On the continuity of the norms by a modular. Monograph Series of Inst. App. Electricity. Hokkaidô Univ. No. 7 (1959), pp. 31-44.

[5] T. ANDÔ: Convexity and evenness in modulared semi-ordered linear spaces, Jour. Fac. Sci. Hokkaido Univ. 14, No. 2 (1959), pp. 59-95.

[6] M. M. DAY: Normed linear spaces. Erg. Math. 21, Berlin, 1958.

[7] L.V. KANTROVIČ, B. Z. VULICH and A. G. PINSKER: Functional analysis in partially ordered spaces, Moscow-Leningrad, 1950 (in Russian).

[8] H. NAKANO: Stetige lineare Funktionale auf dem teilweise-geordneten Modul, Jour. Fac. Sci. Imp. Univ. Tokyo, 4 (1942), pp. 201-382.

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26) $N(\varepsilon_1-\alpha)$ depends only on $\varepsilon_1-\alpha$.

27) This theorem was suggested to the author by Dr. T. Andô.

[9] H. NAKANO: Modulared semi-ordered linear spaces, Tokyo, 1950.
[10] T. SHIMOGAKI: On the norms by uniformly finite modulars, Proc. Japan Acad. 33, No. 6 (1957), pp. 304–309.
[11] T. MORI, I. AMEMIYA and H. NAKANO: On the reflexivity of semi-continuous norms, Proc. Japan Acad. 31, No. 10 (1955), pp. 684–685.
]12] S. YAMAMURO: Monotone completeness of normed semi-ordered linear spaces, Pacific J. of Math., Vol. 7, No. 4, (1957), pp. 1715–1725.

Department of Mathematics,
Hokkaido University

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