A REMARK ON MAZUR-ORLICZ'S NORM

By

Shôzô KOSHI

1. Let Ω be a measure space with finite measure μ, and let $M(u, v)$ be a real valued function which is defined on $[0, \infty)\times\Omega$ such that

(M. 1) $0 \leqq M(u, v) \leqq +\infty$ for $(u, v)\in[0, \infty)\times\Omega$ with $M(0, v)=0$ a.e. $v\in\Omega$;

(M. 2) $M(u, v)$ is an increasing function and left-hand continuous with $\lim_{u\to 0} M(u, v) < +\infty$ a.e. $v\in\Omega$;

(M. 3) $M(u, v)$ is a measurable function of u for a fixed $v\in[0, \infty)$;

(M. 4) $\lim_{u\to\infty} M(u, v) > \lim_{u\to 0} M(u, v)$ a.e. $v\in\Omega$.

Now, we shall consider the function space $L_{M(u,v)}$ whose element f is as follows:

$$\rho(\alpha f)=\int_\Omega M(\alpha|f(v)|, v)d\mu < +\infty \quad \text{for some } \alpha>0. \tag{1}$$

If we identify f and g when $f(v)=g(v)$ except a measure zero set, then we can consider $L_{M(u,v)}$ as a conditionally complete vector lattice with a functional ρ[1]:

$$\rho(f)=\int_\Omega M(|f(v)|, v)d\mu. \tag{2}$$

In the case that $M(u, v)=M(u)$ for every $v\in\Omega$, and $\lim_{u\to 0} M(u)=0$, Mazur and Orlicz has considered in his paper [2], the quasi-norm such that

$$\|f\|=\inf\left\{\varepsilon;\ \rho\left(\frac{f}{\varepsilon}\right)<\varepsilon\right\}. \tag{3}$$

$\|\cdot\|$ has the following properties:

(F. 1) $\|f+g\| \leqq \|f\|+\|g\|$ for $f, g\in L_{M(u)}$;

(F. 2) $\alpha\to 0$, then $\|\alpha f\|\to 0$ for each $f\in L_{M(u)}$;

(F. 3) $\|f\|\to 0$, then $\|\alpha f\|\to 0$ for every real number α;

(F. 4) $0\leqq f\leqq g$, then $\|f\|\leqq\|g\|$;

(F. 5) $0\leqq f_1\leqq f_2\leqq\cdots$, $\sup_n \|f_n\|<+\infty$, then $\bigcup_{n=1}^\infty f_n\in L_{M(u)}$ and $\|\bigcup_{n=1}^\infty f_n\| = \sup_n \|f_n\|$;

---

1) This space is an example of quasi-modular spaces. cf. [3].

(F. 6) $\|\cdot\|$ is complete.

The fact that $\|\cdot\|$ is complete is considered as a generalization of Riesz-Fisher's theorem concerning the completeness of norms. (cf. [1])

In the case of $L_{M(u,v)}$, we define ρ^* and $\|\cdot\|$ with

$$\rho^*(f)=\rho(f)-\lim_{\alpha\to 0}\rho(\alpha f) \qquad \text{for } f\in L_{M(u,v)} \tag{4}$$

and

$$\|f\|=\inf\left\{\varepsilon;\ \rho^*\left(\frac{f}{\varepsilon}\right)<\varepsilon\right\} \qquad \text{for } f\in L_M(u,v). \tag{5}$$

Then, $\|\cdot\|$ has the properties (F. 1), (F. 2), (F. 3), (F. 4). But, $\|\cdot\|$ is not complete in general cases.

We have the following theorem.

Theorem 1. $\|\cdot\|$ *is complete, if and only if* $\lim_{u\to 0} M(u,v)=M(v)$ *is an integralable function of* Ω *with respect to* μ.

Moreover, we have

Theorem 2. $L_{M(u,v)}$ *has a complete quasi-norm if and only if* $\lim_{u\to 0} M(u,v)=M(v)$ *is an integrable function of* Ω *with respect to* μ.

2. From the definition of ρ and ρ^*, we have

$$\rho(|f|)=\rho(f), \qquad \rho^*(|f|)=\rho^*(f) \tag{6}$$

and

$$\rho(\alpha f+\beta g)\leqq\rho(f)+\rho(g), \qquad \rho^*(\alpha f+\beta g)\leqq\rho^*(f)+\rho^*(g) \qquad \text{for } \alpha+\beta=1;\ \alpha,\beta\geqq 0. \tag{7}$$

Hence, for $f_1, f_2, \cdots, f_n\in L_{M(u,v)}$,

$$\rho^*(|f_1|+|f_2|+\cdots+|f_n|)\leqq\rho^*(2f_1)+\cdots+\rho^*(2^nf_n). \tag{8}$$

Proof of Theorem 1. It is easily proved that if $M(v)$ is an integrable function of Ω, it follows

$$\lim_{\alpha\to 0}\rho(\alpha f)<+\infty \qquad \text{for each } f\in L_{M(u,v)}; \tag{9}$$

especially

$$\lim_{\alpha\to 0}\rho(\alpha 1)<+\infty \qquad \text{where 1 is a characteristic function of } \Omega. \tag{10}$$

Let $\{f_n\}$ be a sequence of elements of $L_{M(u,v)}$ with $\|f_n\|\leqq 1/2^n$. From the definition of $\|\cdot\|$, $\|f_n\|\leqq 1/2^n$ implies

$$\rho^*(2^nf_n)\leqq\frac{1}{2^n} \qquad n=1,2,\cdots. \tag{11}$$

Hence, $f=\sum_{n=1}^{\infty}|f_n|$ is an element of $L_{M(u,v)}$ because of

$$\rho(|f_1|+\cdots+|f_n|)\leqq\lim_{\alpha\to0}\rho(\alpha 1)+\rho^*(|f_1|+\cdots+|f_n|)\tag{12}$$
$$\leqq\lim_{\alpha\to0}\rho(\alpha 1)+\sum_i\rho^*(2^if_i)<+\infty\,;$$

i.e. $\rho(f)<+\infty$.

Since $L_{M(u,v)}$ is a conditionally complete (in order sence), $\sum_{n=1}^{\infty}f_n$ exists in $L_{M(u,v)}$ (in order sense).

Let $\{g_n\}$ $(n=1,2,\cdots)$ be a Cauchy sequence of $L_{M(u,v)}$. There exists a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ with

$$||g_{n_i}-g_{n_{i+1}}||\leqq\frac{1}{2^i}.\tag{12}$$

Hence

$$h=g_{n_1}+(g_{n_2}-g_{n_1})+\cdots+(g_{n_\nu}-g_{n_{\nu-1}})+\cdots$$

is an element of $L_{M(u,v)}$, and

$$||h-g_{n_i}||\leqq\frac{1}{2^{i-1}}\,;\tag{13}$$

i.e. g_{n_i} is convergent to h in norm's sense. This shows that $\{g_n\}$ is a convergent sequence.

Now, we shall prove the converse. We shall assume

$$\int_\Omega M(v)d\mu=+\infty.\tag{14}$$

Putting

$$\inf\{M(v),\,n\}=M_n(v),\tag{15}$$

we have

$$\int_\Omega M_n(v)d\mu<+\infty.\tag{16}$$

Now, we define the sets as follows:

$$A_n=\{v\in\Omega\quad;\quad M_n(v)\neq0\}\tag{17}$$

and

$$A=\bigcup_{n=1}^{\infty}A_n.\tag{18}$$

Moreover, if we put

$$B_n=A_n-A_{n-1}\qquad(A_0=\phi),\tag{19}$$

then

$$A = \sum_{n=1}^{\infty} B_n. \tag{20}$$

We can choose the function f_n with

$$\rho^*(f_n) \leqq \frac{1}{2^n} \tag{21}$$

and

$$B_n = \{v;\ f_n(v) \neq 0\}. \tag{22}$$

If we put $f_n^0 = \frac{1}{2^n} f_n$, then we have

$$\|f_n^0\| \leqq \frac{1}{2^n}. \tag{23}$$

Hence, $g_m = \sum_{n=1}^{m} f_n^0$ $(m=1, 2, \cdots)$ is a Cauchy sequence. But $\sum_{n=1}^{\infty} f_n^0 \bar{\in} L_{M(u,v)}$. This shows that $\|\cdot\|$ is not complete. Q. E. D.

The proof of Theorem 2 is quite similar to that of Theorem 1. Because, if $\int_\Omega M(v) d\mu = +\infty$, then there exists a sequence f_n $(n=1, 2, \cdots)$ satisfying (21), (22), (23) for a quasi-norm $\|\cdot\|^*$ defined on $L_{M(u,v)}$ which is not necessary equal to (5). Similarly to the proof of Theorem 1, $\{f_n\}$ is a Cauchy sequence which is not convergent.

Theorem 1 is essentially equal to Theorem 3.2 in [4]. But the proof here is more simpler than that of Theorem 3.2.

References

[1] I. Amemiya: A generalization of Riesz-Fischer's theorem, Jour. of Math. Soc. of Japan, Vol. 5 (1953), pp. 353–354.

[2] S. Mazur-W. Orlicz: On some classes of linear metric spaces, Studia Math. 17 (1958), pp. 97–119.

[3] S. Koshi-T. Shimogaki: On quasi-modular spaces, Studia Math. 21 (1961), pp. 15–35.

[4] S. Koshi-T. Shimogaki: On F-norms of Quasi-modular spaces, Jour. of the Fac. of Sci. Hokkaido University, Series 1, Vol. 15 (1961), pp. 202–218.

Department of Mathematics,
Hokkaido University

(Received May 7, 1962)