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# Homology of a certain associative algebra

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**Abstract.** Let $R$ be a commutative ring, and let $A$ be an associative $R$-algebra possessing an $R$-free basis $B$. In this paper, we introduce a homology $H_n(A, B)$ associated to a pair $(A, B)$ under suitable hypotheses. It depends on not only $A$ itself but also a choice of $B$. In order to define $H_n(A, B)$, we make use of a certain submodule of the $(n+1)$-fold tensor product of $A$. We develop a general theory of $H_n(A, B)$. Various examples of a pair $(A, B)$ and $H_n(A, B)$ are also provided.

*Key words*: Homology, $R$-algebra, Tensor product.

## 1. Introduction

Let $G$ be a finite group. A family $\mathcal{H}$ of subgroups of $G$ is regarded as a simplicial complex (the order complex) associated to the poset $\mathcal{H}$ with respect to the inclusion relation $\leq$. A complex $\mathcal{H}$ is called a subgroup complex of $G$. One of the motivations of our earlier works [5], [6] is to pursue the nature of subgroup complexes. An important thing is that a poset $(\mathcal{H}, \leq)$ can be naturally thought as a quiver. From this viewpoint, we studied in [5] representations of path algebras of quivers.

In our subsequent paper [6], we investigated a homology $H_n(Q; R)$ of a quiver $Q$ over a commutative ring $R$. Recall that $H_n(Q; R)$ is defined by a graded $R$-module $\bigoplus_{n\geq 0} C_n(\overline{Q})$ where $C_n(\overline{Q})$ is the $R$-free module generated by the set $\mathsf{P}(\overline{Q})_n$ of paths in a quiver $\overline{Q}$ of length $n$. Here $\overline{Q}$ consists of the same set of vertices of $Q$, and the set of non-trivial paths in $Q$ as arrows. Thus it can be thought that $H_n(Q; R)$ is associated to the path algebra $R[\overline{Q}]$ of $\overline{Q}$ which is an associative $R$-algebra possessing the set $\mathsf{P}(\overline{Q})$ of paths in $\overline{Q}$ as $R$-free basis. In the present paper, we extend this situation to an arbitrary associative $R$-algebra $A$ with an $R$-free basis $B$, and introduce a homology $H_n(A, B)$ $(n \geq 0)$ determined by a pair $(A, B)$ under suitable hypotheses. Note that its structure depends on a choice of $B$, and that our homology contains the notion of $H_n(Q; R)$. We develop a general theory of

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$H_n(A, B)$.

The paper is organized as follows: In Section 2, we recall some basic concept on a quiver $Q$. Among them, a homology $H_n(Q; R)$ of $Q$ is a model case in our investigation. In Section 3, we introduce, under suitable hypotheses, a homology $H_n(A, B)$ $(n \geq 0)$ of an associative $R$-algebra $A$ with respect to an $R$-free basis $B$. Note that $H_n(A, B)$ is defined by a graded $R$-module $\bigoplus_{n\geq 0} A^{[n]}$ where $A^{[n]}$ is the $R$-free module generated by a subset

$$B^{\neq 0}_{\otimes(n+1)} := \{b_0 \otimes \cdots \otimes b_n \mid b_i \in B,\ b_0 \cdots b_n \neq 0\} \subseteq A^{\otimes(n+1)}$$

of the $(n+1)$-fold tensor product $A^{\otimes(n+1)}$ of $A$. In Section 4, we provide a homology $H_n(\mathcal{D})$ of a certain subset $\mathcal{D} \subseteq \bigcup_{n\geq 0} B^{\neq 0}_{\otimes(n+1)}$. If $\mathcal{D}$ is the whole set then $H_n(\mathcal{D}) = H_n(A, B)$ holds. In Section 5, we extend our chain complex to define a homology of degree $-1$. In Section 6, we see that our homology is a natural generalization of $H_n(Q; R)$. In Section 7, we deal with various examples, and present some calculation. In Section 8, we focus on an $R$-algebra defined by a semilattice $L$. In particular, we consider the subgroup lattice of a finite group. Furthermore some relations with the associated order complex of $L$ are also examined. Throughout the paper, let $R$ be a commutative ring with the identity element. For a set $X$, denote by $R[X]$ the $R$-free module with basis $X$. For an $R$-module $M$ and a subset $Y \subseteq M$, the notation $\langle Y \rangle_R$ means an $R$-submodule of $M$ generated by $Y$.

## 2. Preliminaries

In this section, we review some basis concept on a quiver $Q$ (cf. [2, Section III-1], [6, Sections 3 and 5]). In particular, a homology $H_n(Q; R)$ of $Q$ described in Section 2.3 will be fundamental in our consideration.

### 2.1. Quivers and paths

A quiver $Q$ is a quadruple $Q = \big(Q_0, Q_1,\ (s : Q_1 \to Q_0),\ (r : Q_1 \to Q_0)\big)$ where $Q_0$ $(\neq \emptyset)$ and $Q_1$ are sets, and their elements are called vertices and arrows of $Q$ respectively. Furthermore $s$ and $r$ are maps from $Q_1$ to $Q_0$. For an arrow $\alpha \in Q_1$, if $s(\alpha) = a$ and $r(\alpha) = b$ then denote by $\alpha = (a \to b)$ or $a \xrightarrow{\alpha} b$. Elements $s(\alpha)$ and $r(\alpha)$ are called the start and range of $\alpha$ respectively. A path $\Delta$ in Q is either a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ $(k \geq 1)$ of arrows $\alpha_i = (a_{i-1} \to a_i) \in Q_1$ satisfying $r(\alpha_i) = s(\alpha_{i+1})$ for $(1 \leq i \leq k-1)$, or the symbol $e_a$ for $a \in Q_0$ which is called the trivial path. We also write

$$\Delta = \big(a_0 \overset{\alpha_1}{\to} a_1 \overset{\alpha_2}{\to} a_2 \to \cdots \to a_{k-1} \overset{\alpha_k}{\to} a_k\big) \quad \text{and} \quad e_a = (a).$$

A vertex $a$ is identified with $e_a$. Denote by $\mathsf{P}(Q)$ and $\mathsf{P}(Q)^{\mathrm{non}}$ respectively the totality of paths in $Q$, and that of non-trivial paths in $Q$. For $\Delta = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathsf{P}(Q)^{\mathrm{non}}$, define $s(\Delta) := s(\alpha_1)$ and $r(\Delta) := r(\alpha_k)$, and denote by $\ell(\Delta)$ the length $k$ of $\Delta$. On the other hand, for $a \in Q_0$, define $s(e_a) := a$ and $r(e_a) := a$, and set $\ell(e_a) := 0$. The notation $\mathsf{P}(Q)_i$ $(i \geq 0)$ stands for the totality of paths of length $i$. The path algebra $R[Q]$ of $Q$ over $R$ is the $R$-free module with $\mathsf{P}(Q)$ as basis, and a multiplication on $R[Q]$ is defined by extending bilinearly the composition

$$\Delta_1\Delta_2 := \begin{cases} (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m) & \text{if } r(\alpha_k) = s(\beta_1) \\ 0 & \text{otherwise} \end{cases}$$

of paths $\Delta_1 = (\alpha_1, \ldots, \alpha_k)$, $\Delta_2 = (\beta_1, \ldots, \beta_m) \in \mathsf{P}(Q)$. Then $R[Q]$ is an associative $R$-algebra.

### 2.2. The closure of $Q$

For a quiver $Q = (Q_0, Q_1, s, r)$, we extend maps $s$ and $r$ on $\mathsf{P}(Q)^{\mathrm{non}}$ as

$$s : \mathsf{P}(Q)^{\mathrm{non}} \longrightarrow Q_0 \quad \text{by} \quad \Delta \mapsto s(\Delta),$$
$$r : \mathsf{P}(Q)^{\mathrm{non}} \longrightarrow Q_0 \quad \text{by} \quad \Delta \mapsto r(\Delta).$$

Then $\overline{Q} := \big(Q_0, \mathsf{P}(Q)^{\mathrm{non}}, s, r\big)$ forms a quiver which we call the closure of $Q$ (cf. [6, Definition 3.5]). The set of paths in $\overline{Q}$ is expressed as follows:

$$\mathsf{P}(\overline{Q}) = \left\{ \big(x_0 \overset{\Delta_1}{\to} x_1 \to \cdots \to x_{k-1} \overset{\Delta_k}{\to} x_k\big) \;\middle|\; k \geq 0,\ \Delta_i \in \mathsf{P}(Q)^{\mathrm{non}} \right\}.$$

Note that a sequence $(\Delta_1, \ldots, \Delta_n)$ of paths $\Delta_i \in \mathsf{P}(Q)^{\mathrm{non}}$ is a member of $\mathsf{P}(\overline{Q})_n$ if and only if the product $\Delta_1 \cdots \Delta_k$ in $R[Q]$ is non-zero.

### 2.3. Homology of $Q$

Let $Q$ be a quiver. The path algebra $R[\overline{Q}] = \langle \mathsf{P}(\overline{Q}) \rangle_R$ of the closure $\overline{Q}$ is a graded $R$-module $R[\overline{Q}] = \bigoplus_{n \geq 0} C_n(\overline{Q})$ where $C_n(\overline{Q}) := \langle \mathsf{P}(\overline{Q})_n \rangle_R$. Let $\partial_Q : R[\overline{Q}] \longrightarrow R[\overline{Q}]$ be an $R$-endomorphism defined by, for $(\Delta_1, \ldots, \Delta_n) \in$

$C_n(\overline{Q})$ $(n \geq 2)$,

$$\begin{aligned}
&\partial_Q\big(x_0 \overset{\Delta_1}{\to} x_1 \to \cdots \to x_{n-1} \overset{\Delta_n}{\to} x_n\big) \\
&\quad := \big(x_1 \overset{\Delta_2}{\to} x_2 \to \cdots \to x_{n-1} \overset{\Delta_n}{\to} x_n\big) \\
&\qquad + \sum_{i=1}^{n-1}(-1)^i\big(x_0 \overset{\Delta_1}{\to} \cdots \to x_{i-1} \xrightarrow{\Delta_i\Delta_{i+1}} x_{i+1} \to \cdots \overset{\Delta_n}{\to} x_n\big) \\
&\qquad + (-1)^n\big(x_0 \overset{\Delta_1}{\to} x_1 \to \cdots \to x_{n-2} \xrightarrow{\Delta_{n-1}} x_{n-1}\big).
\end{aligned}$$

Furthermore, for $(x_0 \overset{\Delta}{\to} x_1) \in C_1(\overline{Q})$ and $(x) \in C_0(\overline{Q})$, we set $\partial_Q(x_0 \overset{\Delta}{\to} x_1) := (x_1) - (x_0) \in C_0(\overline{Q})$ and $\partial_Q(x) := 0$. Then $(R[\overline{Q}], \partial_Q)$ forms a chain complex, and a homology $H_n(Q;R) := H_n\big(R[\overline{Q}], \partial_Q\big)$ $(n \geq 0)$ of $Q$ over $R$ is defined (cf. [6, Definition 5.10]). Set $(\partial_Q)_n := \partial_Q|_{C_n(\overline{Q})}$ for $n \geq 0$.

## 3. Homology of $(A, B)$

Let $A$ be an $R$-algebra, that is, $A$ is a left $R$-module, and is a ring such that $(ra)b = r(ab) = a(rb)$ for all $r \in R$ and $a, b \in A$. All $R$-algebras are assumed to be associative. Suppose that $A$ is the $R$-free module with $B$ as basis. In this section, we introduce a homology $H_n(A, B)$ of $A$ with respect to $B$ under Hypothesis **(P)** below. This is a natural generalization of a homology of a quiver stated in Section 2.3. A corresponding chain complex is constructed by the tensor product of $A$. So we first prepare the related notations.

**Notation 3.1** *For a non-negative integer $n \geq 0$, denote by*

$$A^{\otimes(n+1)} := A \otimes \cdots \otimes A = \langle a_0 \otimes a_1 \otimes \cdots \otimes a_n \mid a_j \in A\ (0 \leq j \leq n)\rangle_R$$

*the $(n+1)$-fold tensor product of $A$ over $R$. For a subset $D \subseteq A$, we set $D_{\otimes(n+1)} := \{d_0 \otimes \cdots \otimes d_n \mid d_j \in D\ (0 \leq j \leq n)\}$ and $D^{\neq 0}_{\otimes(n+1)} := \{d_0 \otimes \cdots \otimes d_n \in D_{\otimes(n+1)} \mid d_0 \cdots d_n \neq 0\}$. Denote by*

$$A^{[n]} := \big\langle B^{\neq 0}_{\otimes(n+1)}\big\rangle_R = \langle b_0 \otimes \cdots \otimes b_n \in B_{\otimes(n+1)} \mid b_0 \cdots b_n \neq 0\rangle_R$$

*an $R$-submodule of $A^{\otimes(n+1)} = \big\langle B_{\otimes(n+1)}\big\rangle_R$. This is the $R$-free module with*

$B^{\neq 0}_{\otimes(n+1)}$ *as basis. It is a convention that* $A^{[-1]} := \{0\}$.

### 3.1. Standing hypotheses

Let $A$ be an $R$-algebra possessing an $R$-free basis $B = \{b_\lambda\}_{\lambda\in\Lambda} \subseteq A$. We establish the following Hypothesis **(P)** on the product of base elements under which we study throughout this paper.

**Hypothesis (P)** *For* $i, j \in \Lambda$*, we express the product* $b_i b_j$ *in* $A$ *as a unique* $R$*-linear combination*

$$b_i b_j = \sum_{\lambda\in\Lambda} \alpha^{\lambda}_{i,j} b_\lambda \quad \textit{for some } \alpha^{\lambda}_{i,j} \in R.$$

(1) *For any* $b_0 \otimes \cdots \otimes b_n \in B^{\neq 0}_{\otimes(n+1)}$ *and* $0 \leq k \leq n-1$*, we have that* $b_0 \cdots b_{k-1} b_\lambda b_{k+2} \cdots b_n \neq 0$ *whenever* $\alpha^{\lambda}_{k,k+1} \neq 0$ *for* $\lambda \in \Lambda$.

(2) *For any* $i, j \in \Lambda$ *such that* $b_i b_j \neq 0$*, we have that* $\sum_{\lambda\in\Lambda} \alpha^{\lambda}_{i,j} = 1$.

One might think that Hypothesis **(P)** is a little strange at first sight. However this is quite natural for constructing our chain complex described in Section 3.2.

### 3.2. Chain complex and homology of $(A, B)$

In order to introduce a homology $H_*(A, B)$, we define a chain complex depending on an $R$-free basis $B$.

**Definition 3.2** Assume Hypothesis **(P)** (1). For integers $k \geq -1$ and $n \geq 1$, we define an $R$-homomorphism $\mu_{B,k} : A^{[n]} \longrightarrow A^{[n-1]}$ by the formula

$$\mu_{B,k}(b_0 \otimes \cdots \otimes b_n) := \begin{cases} b_1 \otimes \cdots \otimes b_n & \text{if } k = -1 \\ b_0 \otimes \cdots \otimes b_k b_{k+1} \otimes \cdots b_n & \text{if } 0 \leq k \leq n-1 \\ b_0 \otimes \cdots \otimes b_{n-1} & \text{if } k = n \\ 0 & \text{if } k > n \end{cases}$$

for $b_0 \otimes \cdots \otimes b_n \in B^{\neq 0}_{\otimes(n+1)}$. In the case of $n = 0$, a map $\mu_{B,k} : A^{[0]} \longrightarrow A^{[-1]}$ is defined to be the zero map. It should be mentioned that an element

$$\begin{aligned} &b_0 \otimes \cdots \otimes b_k b_{k+1} \otimes \cdots \otimes b_n \\ &\quad = \sum_{\lambda\in\Lambda} \alpha^{\lambda}_{k,k+1} (b_0 \otimes \cdots \otimes b_{k-1} \otimes b_\lambda \otimes b_{k+2} \otimes \cdots \otimes b_n) \end{aligned}$$

where $b_k b_{k+1} = \sum_{\lambda\in\Lambda} \alpha^{\lambda}_{k,k+1} b_\lambda$ is a member of $A^{[n-1]} := \langle B^{\neq 0}_{\otimes n}\rangle_R$ because of Hypothesis **(P)** (1).

**Definition 3.3** Assume Hypothesis **(P)** (1). We define an $R$-endomorphism

$$\partial_B := \sum_{k\geq -1} (-1)^{k+1}\mu_{B,k} : \bigoplus_{n\geq 0} A^{[n]} \longrightarrow \bigoplus_{n\geq 0} A^{[n]}$$

of a graded $R$-module $\bigoplus_{n\geq 0} A^{[n]}$. In other words, $\partial_B$ is defined by

$$\begin{aligned}&\partial_B(b_0\otimes\cdots\otimes b_n)\\ &\quad = (b_1\otimes\cdots\otimes b_n) + \sum_{i=0}^{n-1}(-1)^{i+1}(b_0\otimes\cdots\otimes b_i b_{i+1}\otimes\cdots\otimes b_n)\\ &\qquad + (-1)^{n+1}(b_0\otimes\cdots\otimes b_{n-1})\end{aligned}$$

for $b_0\otimes\cdots\otimes b_n \in B^{\neq 0}_{\otimes(n+1)}$ $(n\geq 1)$, and $\partial_B(b_0) = 0$ for $b_0 \in B^{\neq 0}_{\otimes 1} = B$. In particular, $\partial_B$ is of degree $-1$, that is, $\partial_B(A^{[n]}) \leq A^{[n-1]}$. Set $(\partial_B)_n := \partial_B|_{A^{[n]}}$ for $n\geq 0$.

**Proposition 3.4** *Assume Hypothesis* **(P)**. *Then the equality* $\partial_B\circ\partial_B = 0$ *holds, namely*

$$\cdots \xrightarrow{(\partial_B)_3} A^{[2]} \xrightarrow{(\partial_B)_2} A^{[1]} \xrightarrow{(\partial_B)_1} A^{[0]} = A \xrightarrow{(\partial_B)_0} \{0\}$$

*forms a chain complex.*

*Proof.* We consider $\mathsf{A} := (\partial_B)^2(b_0\otimes\cdots\otimes b_n)$ for $b_0\otimes\cdots\otimes b_n \in B^{\neq 0}_{\otimes(n+1)}$ $(n\geq 2)$. Divide the image $\mathsf{A}$ into $\mathsf{A} = \mathsf{A}_1 + \mathsf{A}_2 + \mathsf{A}_3$ where

$$\begin{aligned}\mathsf{A}_1 &:= \partial_B(b_1\otimes\cdots\otimes b_n) = (b_2\otimes\cdots\otimes b_n)\\ &\quad + \sum_{i=1}^{n-1}(-1)^i(b_1\otimes\cdots\otimes b_i b_{i+1}\otimes\cdots\otimes b_n) + (-1)^n(b_1\otimes\cdots\otimes b_{n-1}),\end{aligned}$$

$$\begin{aligned}\mathsf{A}_2 &:= \partial_B\big((-1)^{n+1} b_0 \otimes \cdots \otimes b_{n-1}\big) = (-1)^{n+1}(b_1 \otimes \cdots \otimes b_{n-1}) \\ &\quad + (-1)^{n+1} \sum_{i=0}^{n-2} (-1)^{i+1}(b_0 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_{n-1}) \\ &\quad - (b_0 \otimes \cdots \otimes b_{n-2}), \\ \mathsf{A}_3 &:= \partial_B\bigg(\sum_{i=0}^{n-1} (-1)^{i+1}(b_0 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n)\bigg).\end{aligned}$$

Then it is straightforward to calculate that

$$\begin{aligned}\mathsf{A}_3 &= \sum_{i=1}^{n-1} (-1)^{i+1}(b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n) \\ &\quad + (-1)^n \sum_{i=0}^{n-2} (-1)^{i+1}(b_0 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_{n-1}) \\ &\quad - \sum_{\lambda \in \Lambda} \alpha_{0,1}^{\lambda}(b_2 \otimes \cdots \otimes b_n) + \sum_{\lambda \in \Lambda} \alpha_{n-1,n}^{\lambda}(b_0 \otimes \cdots \otimes b_{n-2}).\end{aligned}$$

Thus $\mathsf{A} = 0$ if and only if $\sum_{\lambda \in \Lambda} \alpha_{0,1}^{\lambda} = \sum_{\lambda \in \Lambda} \alpha_{n-1,n}^{\lambda} = 1$. So the assertion holds from Hypothesis **(P)** (2). $\square$

**Definition 3.5** Let $A$ be an $R$-algebra possessing an $R$-free basis $B$. Under Hypothesis **(P)**, denote by $H_*(A, B) := \operatorname{Ker} \partial_{\mathrm{B}} / \operatorname{Im} \partial_{\mathrm{B}}$ the factor $R$-module. We call $H_*(A, B)$ a homology $R$-module of $A$ with respect to $B$.

**Remark 3.6** Suppose that $B$ or $B \cup \{0\}$ is a semigroup with respect to a multiplication defined on $A$. Then $B$ satisfies Hypothesis **(P)** since $b_1 b_2 \in B \cup \{0\}$ for any $b_1, b_2 \in B$.

**Remark 3.7** Since a map $\partial_Q$ in Section 2.3 deletes a vertex $x_i$ $(0 \le i \le n)$ in order of index, it is quite natural from a viewpoint of simplicial complexes. On the other hand, by identifying paths $(\Delta_1, \ldots, \Delta_n) \in C_n(\overline{Q})$ with elements $\Delta_1 \otimes \cdots \otimes \Delta_n \in (\mathsf{P}(Q)^{\mathrm{non}})^{\neq 0}_{\otimes n}$ of the tensor product, it is thought that $\partial_B$ in Definition 3.3 is an algebraic generalization of $\partial_Q$ concerning path algebras.

**Remark 3.8** Recall that the standard complex or the bar construction of

$A$ (see [4, page 175]) is obtained from an $R$-homomorphism (bar resolution) $d_n : A^{\otimes(n+1)} \longrightarrow A^{\otimes n}$ defined by

$$d_n(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1}(-1)^i(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$$

for all $a_0 \otimes \cdots \otimes a_n \in A^{\otimes(n+1)}$. This is independent of a choice of $B$.

In this situation, if we add to $d_n$ an operation of cutting out both ends $a_0$ and $a_n$ as in Definition 3.2 then, for example, an element $\mathbf{a} := \alpha_0 a_0 \otimes \alpha_1 a_1 = \alpha_0\alpha_1(a_0 \otimes a_1)$ $(\alpha_i \in R)$ goes to respectively $\alpha_1 a_1 - (\alpha_0\alpha_1)(a_0 a_1) + \alpha_0 a_1$ and $\alpha_0\alpha_1(a_1 - a_0 a_1 + a_0)$ which are different. This implies that the image of $\mathbf{a}$ is not uniquely determined. Furthermore the zero element $0 = 0 \otimes a_1 \otimes a_2$ goes to a non-zero element $(a_1 \otimes a_2) - (0 \otimes a_2) + (0 \otimes a_1 a_2) - (0 \otimes a_1) = a_1 \otimes a_2$, a contradiction.

In order to avoid that trouble, we need to deal with $(\partial_B)_n : A^{[n]} \longrightarrow A^{[n-1]}$ in Definition 3.2 which depends on an $R$-free basis $B^{\neq 0}_{\otimes(n+1)}$.

The following result on the image $\mathrm{Im}(\partial_B)_1$ will be applied in Section 7.4.

**Proposition 3.9** *Assume Hypothesis* **(P)**. *If $A$ contains the identity element $1_A$ and $1_A \in B$ then we have that*

$$\mathrm{Im}(\partial_B)_1 = \langle 1_A, (b - 1_A)(c - 1_A) \mid b, c \in B,\ bc \neq 0\rangle_R.$$

*Proof.* Recall that $(\partial_B)_1 : A^{[1]} \longrightarrow A^{[0]} = A$ where $A^{[1]} = \langle B^{\neq 0}_{\otimes 2}\rangle_R$. For any $b \otimes c \in B^{\neq 0}_{\otimes 2}$, we have that $(\partial_B)_1(b \otimes c) = c - bc + b = 1_A - (b - 1_A)(c - 1_A)$. By our assumption, $1_A \otimes 1_A$ lies in $B^{\neq 0}_{\otimes 2}$. Thus $(\partial_B)_1(1_A \otimes 1_A) = 1_A$. This completes the proof. $\square$

## 3.3. Cohomology of $(A, B)$

By the usual way, we can define a cohomology $H^*(A, B)$ of $(A, B)$ as the dual of $H_*(A, B)$.

**Definition 3.10** Let $A$ be an $R$-algebra possessing an $R$-free basis $B$. Under Hypothesis **(P)**, let

$$d_B : \bigoplus_{n \geq 0} \mathrm{Hom}_R(A^{[n]}, R) \longrightarrow \bigoplus_{n \geq 0} \mathrm{Hom}_R(A^{[n]}, R)$$

be an $R$-endomorphism of a graded $R$-module $\bigoplus_{n\geq 0} \mathrm{Hom}_R(A^{[n]}, R)$ defined by

$$d_B(f) := f \circ (\partial_B)_{n+1} : A^{[n+1]} \xrightarrow{(\partial_B)_{n+1}} A^{[n]} \xrightarrow{f} R$$

for $f \in \mathrm{Hom}_R(A^{[n]}, R)$ $(n \geq 0)$. Then $d_B$ is of degree $+1$ with the property that $d_B \circ d_B = 0$. Denote by $H^*(A, B) := \mathrm{Ker}\, d_B/\mathrm{Im}\, d_B$ the factor $R$-module. We call $H^*(A, B)$ a cohomology $R$-module of $A$ with respect to $B$. Set $(d_B)_n := d_B|_{\mathrm{Hom}_R(A^{[n]},R)}$ for $n \geq 0$.

**Example 3.11** Let $\mathbb{Z}$ be the ring of all rational integers. Let $A := \mathbb{Z}[G]$ be the group algebra of a group $G$ over $\mathbb{Z}$. Since a $\mathbb{Z}$-basis $G$ of $A$ is a group, $G$ satisfies Hypothesis **(P)**. Recall that $A^{[n]} := \left\langle G^{\neq 0}_{\otimes(n+1)} \right\rangle_{\mathbb{Z}} = \langle G_{\otimes(n+1)} \rangle_{\mathbb{Z}}$ $(n \geq 0)$. Then we have the following chain complex

$$\{0\} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A^{[0]}, \mathbb{Z}) \xrightarrow{(d_G)_0} \mathrm{Hom}_{\mathbb{Z}}(A^{[1]}, \mathbb{Z}) \xrightarrow{(d_G)_1} \cdots$$

Here we consider $H^0(A, G) := \mathrm{Ker}(d_G)_0$. For $f \in \mathrm{Ker}(d_G)_0 \subseteq \mathrm{Hom}_{\mathbb{Z}}(A^{[0]}, \mathbb{Z})$, we have that $0 = \big(f \circ (\partial_G)_1\big)(a \otimes b) = f(a + b - ab) = f(a) + f(b) - f(ab)$ for any $a \otimes b \in G_{\otimes 2}$. Denote by $e_G$ the identity element of $G$. Then $f(e_G) = 0$ and $f(gg^{-1}) = f(g) + f(g^{-1})$ for $g \in G$, so that, $f(g^{-1}) = -f(g)$. This yields that $f(G') = \{0\}$ where $G'$ is the commutator subgroup of $G$. Thus $H^0(A, G) = \{0\}$ if $G$ is a perfect group. On the other hand, suppose that $G$ is finite. Take an element $g \in G$ of order $m \geq 1$. Then $0 = f(e_G) = f(g^m) = mf(g)$ and $f(g) = 0$. Thus $H^0(A, G) = \{0\}$ in the finite case too (see also Remark 7.9).

## 4. Homology of a subset $\mathcal{D}$

Let $A$ be an $R$-algebra possessing an $R$-free basis $B$. Assume Hypothesis **(P)**. In this section, we provide a homology $H_n(\mathcal{D})$ of a certain subset $\mathcal{D} \subseteq \bigcup_{n\geq 0} B^{\neq 0}_{\otimes(n+1)}$. This contains the notion of $H_n(A, B)$ discussed in Section 3.

### 4.1. $\partial_B$-invariant subsets

We begin with the definition.

**Definition 4.1** For a subset $\mathcal{D} \subseteq \bigcup_{n\geq 0} B^{\neq 0}_{\otimes(n+1)}$, we say that $\mathcal{D}$ is $\partial_B$-

invariant if $\partial_B(\langle\mathcal{D}\rangle_R) \subseteq \langle\mathcal{D}\rangle_R$.

**Proposition 4.2** *Let $\mathcal{D} \subseteq \bigcup_{n\geq 0} B^{\neq 0}_{\otimes(n+1)}$ be a $\partial_B$-invariant subset. Put $\mathcal{D}_n := \mathcal{D} \cap B^{\neq 0}_{\otimes(n+1)}$ $(n \geq 0)$.*

(1) *The union $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$ is disjoint, and we have a graded $R$-submodule*

$$\langle\mathcal{D}\rangle_R = \bigoplus_{n\geq 0} \langle\mathcal{D}_n\rangle_R \leq \bigoplus_{n\geq 0} A^{[n]}.$$

(2) *We have that $\partial_B(\langle\mathcal{D}_n\rangle_R) = \partial_B(\langle\mathcal{D}\rangle_R \cap A^{[n]}) \subseteq \langle\mathcal{D}\rangle_R \cap A^{[n-1]} = \langle\mathcal{D}_{n-1}\rangle_R$. The restriction*

$$\partial_{B,\mathcal{D}} := \partial_B \big|_{\langle\mathcal{D}\rangle_R} : \bigoplus_{n\geq 0} \langle\mathcal{D}_n\rangle_R \longrightarrow \bigoplus_{n\geq 0} \langle\mathcal{D}_n\rangle_R$$

*is an $R$-endomorphism of $\langle\mathcal{D}\rangle_R$ of degree $-1$ with the property that $\partial_{B,\mathcal{D}} \circ \partial_{B,\mathcal{D}} = 0$. The following is a chain complex:*

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{(\partial_{B,\mathcal{D}})_2} & \langle\mathcal{D}_1\rangle_R & \xrightarrow{(\partial_{B,\mathcal{D}})_1} & \langle\mathcal{D}_0\rangle_R & \xrightarrow{(\partial_{B,\mathcal{D}})_0} & \{0\} \\
\cdots & & \cap & & \cap & & \| \\
\cdots & \xrightarrow{(\partial_B)_2} & A^{[1]} & \xrightarrow{(\partial_B)_1} & A^{[0]} & \xrightarrow{(\partial_B)_0} & \{0\}
\end{array}$$

(3) *Let*

$$d_{B,\mathcal{D}} : \bigoplus_{n\geq 0} \mathrm{Hom}_R(\langle\mathcal{D}_n\rangle_R, R) \longrightarrow \bigoplus_{n\geq 0} \mathrm{Hom}_R(\langle\mathcal{D}_n\rangle_R, R)$$

*be a map defined by $d_{B,\mathcal{D}}(f) := f \circ (\partial_{B,\mathcal{D}})_{n+1}$ for $f \in \mathrm{Hom}_R(\langle\mathcal{D}_n\rangle_R, R)$ $(n \geq 0)$ as in Definition 3.10. Then $d_{B,\mathcal{D}}$ is an $R$-endomorphism of degree $+1$ with the property that $d_{B,\mathcal{D}} \circ d_{B,\mathcal{D}} = 0$.*

*Proof.* Straightforward. □

**Definition 4.3** For a $\partial_B$-invariant subset $\mathcal{D} \subseteq \bigcup_{n\geq 0} B^{\neq 0}_{\otimes(n+1)}$, denote by

$$H_*(\mathcal{D}) := H_*\big((A,B),\mathcal{D}\big) := \mathrm{Ker}\,\partial_{B,\mathcal{D}} \big/ \mathrm{Im}\,\partial_{B,\mathcal{D}},$$

$$H^*(\mathcal{D}) := H^*\big((A,B),\mathcal{D}\big) := \mathrm{Ker}\,d_{B,\mathcal{D}} \big/ \mathrm{Im}\,d_{B,\mathcal{D}}$$

the factor $R$-modules. We call $H_*(\mathcal{D})$ and $H^*(\mathcal{D})$ respectively a homology $R$-module of $\mathcal{D}$, and a cohomology $R$-module of $\mathcal{D}$.

**Remark 4.4** The whole set $\mathcal{D} := \bigcup_{n\geq 0} B^{\neq 0}_{\otimes(n+1)}$ is clearly $\partial_B$-invariant. In this case, $\mathcal{D}_n = B^{\neq 0}_{\otimes(n+1)}$, $\langle \mathcal{D}_n\rangle_R = A^{[n]}$, and $\langle \mathcal{D}\rangle_R = \bigoplus_{n\geq 0} A^{[n]}$. It follows that $H_n(\mathcal{D}) = H_n(A,B)$ and $H^n(\mathcal{D}) = H^n(A,B)$. Thus $H_n(\mathcal{D})$ contains the notion of $H_n(A,B)$.

**Remark 4.5** Suppose that $B$ is a semigroup. If a subset $\mathcal{D} \subseteq \bigcup_{n\geq 0} B^{\neq 0}_{\otimes(n+1)}$ satisfies $\mu_{B,k}(\mathcal{D}) \subseteq \mathcal{D}\cup\{0\}$ for all $k \geq -1$ where $\mu_{B,k}$ is defined in Definition 3.2, then it is clear that $\mathcal{D}$ is $\partial_B$-invariant. We call such $\mathcal{D}$ a "$\mu_B$-invariant subset".

## 4.2. $A$-module actions

Let $V$ be an $A$-module, that is, $V$ is a left $R$-module and a right $A$-module ($A$ considered as a ring) such that $(rv)a = r(va) = v(ra)$ for all $v \in V$, $r \in R$, and $a \in A$. Then there exists the associated algebra homomorphism $\varphi : A \longrightarrow \mathrm{End}_R(V)$ which is a ring homomorphism and an $R$-linear map. This can be extended to an $R$-homomorphism

$$\varphi : A^{\otimes(n+1)} \longrightarrow \mathrm{End}_R(V)^{\otimes(n+1)}\ (n \geq 0),$$

using the same notation $\varphi$, defined by $\varphi(a_0\otimes\cdots\otimes a_n) := \varphi(a_0)\otimes\cdots\otimes\varphi(a_n)$. Recall that $A = \langle B\rangle_R$. Then an $R$-subalgebra $\varphi(A) = \langle\varphi(B)\rangle_R \leq \mathrm{End}_R(V)$ is generated by a set $\varphi(B)$.

**Lemma 4.6** *Suppose that $\varphi(B)$ is an $R$-free basis of $\varphi(A)$.*

(1) *$\varphi(B)$ satisfies Hypothesis* **(P)** *(2).*
(2) *Suppose that $\varphi\big(B^{\neq 0}_{\otimes(n+1)}\big) \subseteq \varphi(B)^{\neq 0}_{\otimes(n+1)}$ $(n \geq 0)$. Then $\varphi(B)$ satisfies Hypothesis* **(P)** *(1).*

*Proof.* (1) Suppose that $0 \neq \varphi(b_i)\varphi(b_j) = \varphi(b_ib_j)$ for some $b_i, b_j \in B$. In particular $b_ib_j \neq 0$. It follows that $\sum_{\lambda\in\Lambda}\alpha^{\lambda}_{i,j} = 1$ by Hypothesis **(P)** (2) on $B$. Since $\varphi(b_ib_j) = \sum_{\lambda\in\Lambda}\alpha^{\lambda}_{i,j}\varphi(b_\lambda)$, the assertion holds.

(2) For $\varphi(b_0)\otimes\cdots\otimes\varphi(b_n) \in \varphi(B)^{\neq 0}_{\otimes(n+1)}$ and $0 \leq k \leq n-1$, we have that $\varphi(b_k)\varphi(b_{k+1}) = \sum_{\lambda\in\Lambda}\alpha^{\lambda}_{k,k+1}\varphi(b_\lambda)$. Since $B$ satisfies Hypothesis **(P)** (1), if $\alpha^{\lambda}_{k,k+1} \neq 0$ then $b_0\otimes\cdots\otimes b_\lambda\otimes\cdots\otimes b_n$ lies in $B^{\neq 0}_{\otimes n}$. Thus

$\varphi(b_0) \otimes \cdots \otimes \varphi(b_\lambda) \otimes \cdots \otimes \varphi(b_n) = \varphi(b_0 \otimes \cdots \otimes b_\lambda \otimes \cdots \otimes b_n)$ belongs to $\varphi\big(B^{\neq 0}_{\otimes n}\big) \subseteq \varphi(B)^{\neq 0}_{\otimes n}$. This completes the proof. □

**Lemma 4.7** *Let $V$ be an $A$-module with the associated algebra homomorphism $\varphi : A \longrightarrow \mathrm{End}_R(V)$. Suppose that $\varphi(B)$ is an $R$-free basis of $\varphi(A)$, and that $\varphi\big(B^{\neq 0}_{\otimes(n+1)}\big) \subseteq \varphi(B)^{\neq 0}_{\otimes(n+1)}$ $(n \geq 0)$. For a $\partial_B$-invariant subset $\mathcal{D} \subseteq \bigcup_{n\geq 0} B^{\neq 0}_{\otimes(n+1)}$, the followings hold:*

(1) *$\varphi(\mathcal{D}) \subseteq \bigcup_{n\geq 0} \varphi(B)^{\neq 0}_{\otimes(n+1)}$ is $\partial_{\varphi(B)}$-invariant, so that, $H_*\big(\varphi(\mathcal{D})\big)$ is defined.*
(2) *$\varphi$ induces an $R$-homomorphism from $H_*(\mathcal{D})$ to $H_*\big(\varphi(\mathcal{D})\big)$.*

*Proof.* Note that $\varphi(B)$ satisfies Hypothesis **(P)** by Lemma 4.6.

(1) For any $n \geq 0$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
A^{[n]} & = & \big\langle B^{\neq 0}_{\otimes(n+1)} \big\rangle & \xrightarrow{\mu_{B,k}} & \big\langle B^{\neq 0}_{\otimes n} \big\rangle & = & A^{[n-1]} \\
\varphi\big\downarrow & & & & & & \big\downarrow\varphi \\
\varphi(A)^{[n]} & = & \big\langle \varphi(B)^{\neq 0}_{\otimes(n+1)} \big\rangle & \xrightarrow[\mu_{\varphi(B),k}]{} & \big\langle \varphi(B)^{\neq 0}_{\otimes n} \big\rangle & = & \varphi(A)^{[n-1]}
\end{array}$$

Since $\varphi\circ\mu_{B,k} = \mu_{\varphi(B),k}\circ\varphi$ for all $k \geq -1$, we have that $\varphi\circ\partial_B = \partial_{\varphi(B)}\circ\varphi$ (see Definition 3.3 ). Thus

$$\partial_{\varphi(B)}\big(\varphi(\langle\mathcal{D}\rangle_R)\big) = \varphi\big(\partial_B(\langle\mathcal{D}\rangle_R)\big) \subseteq \varphi(\langle\mathcal{D}\rangle_R) = \langle\varphi(\mathcal{D})\rangle_R.$$

(2) Straightforward. □

**Example 4.8** ($A_Q$-module action) Let $A_Q := R[Q] = \langle \mathsf{P}(Q)\rangle_R$ be the path algebra of a quiver $Q = (Q_0, Q_1, s, r)$, which is the $R$-free module with basis $B := \mathsf{P}(Q)$. Let $V := R[Q_0]$ be the $R$-free module generated by $Q_0$. First we recall an action of $A_Q$ on $V$ introduced in [5, Section 3.1]. Let $w : Q_1 \longrightarrow R$ be a map which we call a weight function on $Q_1$. Then $w$ can be extended on non-trivial paths by setting $w(\Delta) := \prod_{i=1}^k w(\alpha_i)$ for $\Delta = (\alpha_1, \ldots, \alpha_k) \in \mathsf{P}(Q)$. It is a convention that $w(e_a) := 1$ for $a \in Q_0$. Let $A_Q^\circ$ be the opposite algebra of $A_Q$. Then we regard $V$ as an $A_Q^\circ$-module induced by an algebra homomorphism $\rho_w : A_Q^\circ \longrightarrow \mathrm{End}_R(V)$ which is defined by

$$\rho_w(\Delta) : V \longrightarrow V; \quad a \mapsto a\Delta := w(\Delta)\delta_{a,s(\Delta)} r(\Delta)$$

for $\Delta \in \mathsf{P}(Q)$ and $a \in Q_0$. Note that $(\rho_w(\Delta_1)\rho_w(\Delta_2))(a) = (a\Delta_2)\Delta_1 = a(\Delta_1\Delta_2) = \rho_w(\Delta_1\Delta_2)(a)$ for $\Delta_1, \Delta_2 \in \mathsf{P}(Q)$ and $a \in V$. Here we assume that $\rho_w(B) = \{\rho_w(\Delta) \mid \Delta \in \mathsf{P}(Q)\}$ is an $R$-free basis of $\rho_w(A_Q^\circ) = \langle \rho_w(B) \rangle_R$. If $w \equiv 1$ namely $w(\alpha) = 1$ for any $\alpha \in Q_1$, then this assumption always holds. Now we have the following.

- Since $B \cup \{0\}$ is a semigroup, $B$ satisfies Hypothesis **(P)**.
- For $\Delta_0 \otimes \cdots \otimes \Delta_n \in B_{\otimes(n+1)}$, we have that $\rho_w(\Delta_0 \otimes \cdots \otimes \Delta_n) := \rho_w(\Delta_0) \otimes \cdots \otimes \rho_w(\Delta_n) \in \rho_w(B)_{\otimes(n+1)}$. Recall that $\Delta_0 \cdots \Delta_n \neq 0$ if and only if $\rho_w(\Delta_0) \cdots \rho_w(\Delta_n) = \rho_w(\Delta_0 \cdots \Delta_n) \neq 0$. This implies that $\rho_w\big(B^{\neq 0}_{\otimes(n+1)}\big) = \rho_w(B)^{\neq 0}_{\otimes(n+1)}$.
- Take the whole set $\mathcal{D} := \bigcup_{n \geq 0} B^{\neq 0}_{\otimes(n+1)}$. Then $\mathcal{D}$ is $\partial_B$-invariant (cf. Remark 4.4).

By Lemma 4.7, $\rho_w(\mathcal{D}) = \bigcup_{n\geq 0} \rho_w\big(B^{\neq 0}_{\otimes(n+1)}\big) = \bigcup_{n \geq 0} \rho_w(B)^{\neq 0}_{\otimes(n+1)}$ is $\partial_{\rho_w(B)}$-invariant, and $H_*\big(\rho_w(\mathcal{D})\big)$ is defined. It is clear that $H_*\big(\rho_w(\mathcal{D})\big) = H_*\big(\rho_w(A_Q^\circ), \rho_w(B)\big)$.

**Remark 4.9** (Up-Down algebra) Let $Q = (Q_0, Q_1, s, r)$ be a quiver. For each arrow $\alpha = (a \to b) \in Q_1$, we define the symbol ${}^t\alpha$. Set $Q_1^{\mathrm{ud}} := Q_1 \cup \{{}^t\alpha \mid \alpha \in Q_1\}$. Then

$$Q^{\mathrm{ud}} := \big(Q_0,\ Q_1^{\mathrm{ud}},\ (s : Q_1^{\mathrm{ud}} \to Q_0),\ (r : Q_1^{\mathrm{ud}} \to Q_0)\big)$$

forms a quiver where $s$ and $r$ are extended on $Q_1^{\mathrm{ud}}$ as $s({}^t\alpha) := r(\alpha) = b$ and $r({}^t\alpha) := s(\alpha) = a$ for $\alpha = (a \to b) \in Q_1$ (cf. [6, Definition 3.1]). Thus ${}^t\alpha = (b \to a)$. Note that $\mathsf{P}(Q) \subseteq \mathsf{P}(Q^{\mathrm{ud}}) =: B$.

(1) In Example 4.8, if we just replace $Q$ with a quiver $Q^{\mathrm{ud}} = (Q_0, Q_1^{\mathrm{ud}}, s, r)$ then we have $\rho_w(A^\circ_{Q^{\mathrm{ud}}}) = \langle \rho_w(\Delta) \mid \Delta \in B \rangle_R \leq \mathrm{End}_R(V)$. This $R$-algebra $\rho_w(A^\circ_{Q^{\mathrm{ud}}})$ coincides with the "Up-Down algebra" $\mathrm{UD}(Q, w; R)$ of $Q$ with respect to $w$ over $R$, which is first introduced in [5], and is investigated in it. Thus a homology of $\mathrm{UD}(Q, w; R)$ can be understood in our setting on module action under the assumption that $\rho_w(B)$ is an $R$-free basis of $\rho_w(A^\circ_{Q^{\mathrm{ud}}})$.

(2) Suppose that $Q$ is finite. Suppose further that $\{\Delta \in \mathsf{P}(Q^{\mathrm{ud}}) \mid s(\Delta) = a,\ r(\Delta) = b\} \neq \emptyset$ for any $a, b \in Q_0$, and that $w \equiv 1$. Then, by [5,

Corollary 3.13], $\mathrm{UD}(Q,1;R)$ is isomorphic to the total matrix algebra $M_{|Q_0|}(R)$. In particular, for a quiver $Q$ defined as $a \xrightarrow{\alpha} b$, we have that $\mathrm{UD}(Q,1;\mathbb{C}) \cong M_2(\mathbb{C})$. In Section 7.2, we will consider homology of matrix algebras.

## 5. An extension of a chain complex

Let $A$ be an $R$-algebra possessing an $R$-free basis $B$. Assume Hypothesis **(P)**. In this section, we introduce an extension of a chain complex. This idea will be applied to a realization of a homology of a quiver in the next Section 6.

**Lemma 5.1** *Let $\mathcal{D} \subseteq \bigcup_{n\geq 0} B^{\neq 0}_{\otimes(n+1)}$ be a $\partial_B$-invariant subset, and let*

$$\cdots \xrightarrow{(\partial_{B,\mathcal{D}})_2} \langle \mathcal{D}_1\rangle_R \xrightarrow{(\partial_{B,\mathcal{D}})_1} \langle \mathcal{D}_0\rangle_R \xrightarrow{(\partial_{B,\mathcal{D}})_0} \{0\}$$

*be the corresponding chain complex in Proposition 4.2. Suppose further that there exist an $R$-module $M$ and $R$-homomorphisms $\mathsf{s},\mathsf{r} : \langle \mathcal{D}_0\rangle_R \longrightarrow M$ such that*

- $\mathsf{s}(xy) = \mathsf{s}(x)$ *and* $\mathsf{r}(xy) = \mathsf{r}(y)$ *for all* $x, y \in \mathcal{D}_0 \subseteq B$ *with* $xy \neq 0$.
- $\mathsf{r}(x) = \mathsf{s}(y)$ *for all* $x, y \in \mathcal{D}_0 \subseteq B$ *with* $xy \neq 0$.

*Let $(\partial^M_{B,\mathcal{D}})_0 := \mathsf{r} - \mathsf{s} : \langle \mathcal{D}_0\rangle_R \longrightarrow M$ be an $R$-homomorphism. Then*

$$\cdots \xrightarrow{(\partial^M_{B,\mathcal{D}})_2} \langle \mathcal{D}_1\rangle_R \xrightarrow{(\partial^M_{B,\mathcal{D}})_1} \langle \mathcal{D}_0\rangle_R \xrightarrow{(\partial^M_{B,\mathcal{D}})_0} M \longrightarrow \{0\}$$

*forms a chain complex where $(\partial^M_{B,\mathcal{D}})_n := (\partial_{B,\mathcal{D}})_n$ for $n \geq 1$.*

*Proof.* It is enough to show that $(\partial^M_{B,\mathcal{D}})_0 \circ (\partial^M_{B,\mathcal{D}})_1 = 0$. For any $b_0 \otimes b_1 \in \mathcal{D}_1 \subseteq B^{\neq 0}_{\otimes 2}$, by using our assumptions on $\mathsf{s}$ and $\mathsf{r}$, we have that

$$\begin{aligned}
\big((\partial^M_{B,\mathcal{D}})_0 \circ (\partial^M_{B,\mathcal{D}})_1\big)(b_0 \otimes b_1) &= (\partial^M_{B,\mathcal{D}})_0(b_1 - b_0 b_1 + b_0) \\
&= \big(\mathsf{r}(b_1) - \mathsf{s}(b_1)\big) - \big(\mathsf{r}(b_0 b_1) - \mathsf{s}(b_0 b_1)\big) + \big(\mathsf{r}(b_0) - \mathsf{s}(b_0)\big) \\
&= \big(\mathsf{r}(b_1) - \mathsf{s}(b_1)\big) - \big(\mathsf{r}(b_1) - \mathsf{s}(b_0)\big) + \big(\mathsf{r}(b_0) - \mathsf{s}(b_0)\big) \\
&= \mathsf{r}(b_0) - \mathsf{s}(b_1) = 0.
\end{aligned}$$

This completes the proof. $\square$

**Definition 5.2** Under the situation of Lemma 5.1, there exists an $R$-endomorphism

$$\partial^M_{B,\mathcal{D}} : \langle\mathcal{D}\rangle_R \oplus M \longrightarrow \langle\mathcal{D}\rangle_R \oplus M$$

of a graded $R$-module $\langle\mathcal{D}\rangle_R \oplus M$ of degree $-1$ with the property that $\partial^M_{B,\mathcal{D}} \circ \partial^M_{B,\mathcal{D}} = 0$. Denote by

$$H^M_*(\mathcal{D}) := H^M_*\big((A,B),\mathcal{D}\big) := \operatorname{Ker}\partial^M_{B,\mathcal{D}}/\operatorname{Im}\partial^M_{B,\mathcal{D}}.$$

Note that $H^M_n(\mathcal{D}) = H_n(\mathcal{D})$ for all $n \geq 1$, and that $H^M_0(\mathcal{D})$ and $H_0(\mathcal{D})$ are not necessarily equal. Furthermore $H^M_{-1}(\mathcal{D}) := M/\operatorname{Im}(\partial^M_{B,\mathcal{D}})_0$ of degree $-1$ is newly defined.

**Lemma 5.3** *Suppose that $B$ is a monoid, that is, it is a semigroup having the identity element $1_B$. Suppose further that $1_B$ is contained in $\mathcal{D}_0 \subseteq B$. Then $\mathsf{s} = \mathsf{r}$. In particular, $H^M_{-1}(\mathcal{D}) \cong M$.*

*Proof.* For any $x \in \mathcal{D}_0$, since $x1_B = 1_Bx = x \neq 0$, we have that $\mathsf{r}(x) = \mathsf{s}(1_B)$ and $\mathsf{r}(1_B) = \mathsf{s}(x)$ by the definitions of $\mathsf{s}$ and $\mathsf{r}$. Furthermore $\mathsf{r}(1_B) = \mathsf{s}(1_B)$ since $1_B1_B = 1_B \neq 0$. This implies that $\mathsf{s} = \mathsf{r}$ is a constant map. The proof is complete. $\square$

## 6. A realization of $H_n(Q;R)$

Let $A_Q := R[Q] = \langle\mathsf{P}(Q)\rangle_R$ be the path algebra of a quiver $Q = (Q_0, Q_1, s, r)$. Let $A^\sharp_Q := \langle C\rangle_R$ be a subalgebra of $A_Q$ generated by $C := \mathsf{P}(Q)^{\mathrm{non}}$. Note that $A^\sharp_Q$ is not unital. Since $C \cup \{0\}$ is a semigroup, $C$ satisfies Hypothesis **(P)**. In this section, we see that a homology $H_n(Q;R)$ of $Q$ (see Section 2.3) is realized as a homology of a subalgebra $A^\sharp_Q$ with respect to $C$ as in the following.

**Proposition 6.1** *Let $Q$ be a quiver. Then there exist an $R$-module $M$ and $R$-homomorphisms $\mathsf{s}, \mathsf{r} : A^\sharp_Q \longrightarrow M$ such that $H^M_n(A^\sharp_Q, C) \cong H_{n+1}(Q;R)$ for all $n \geq -1$.*

*Proof.* As in Remark 3.7, $C^{\neq 0}_{\otimes(n+1)}$ can be identified with the set $\mathsf{P}(\overline{Q})_{n+1}$

of all non-trivial paths of length $n+1$ in the closure $\overline{Q}$ of $Q$. Thus we may assume that $C^{\neq 0}_{\otimes(n+1)} = \mathsf{P}(\overline{Q})_{n+1}$ $(n \geq 0)$. A path of length 0 is a trivial path. It follows that $(A^{\sharp}_{Q})^{[n]} := \left\langle C^{\neq 0}_{\otimes(n+1)} \right\rangle_R = \langle \mathsf{P}(\overline{Q})_{n+1} \rangle_R = C_{n+1}(\overline{Q})$. This yields a chain complex

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{(\partial_C)_2} & (A^{\sharp}_{Q})^{[1]} & \xrightarrow{(\partial_C)_1} & (A^{\sharp}_{Q})^{[0]} & \longrightarrow & \{0\} & & \\
\cdots & & \| & & \| & & & & \\
\cdots & \xrightarrow{(\partial_Q)_3} & C_2(\overline{Q}) & \xrightarrow{(\partial_Q)_2} & C_1(\overline{Q}) & \xrightarrow{(\partial_Q)_1} & C_0(\overline{Q}) & \longrightarrow & \{0\}
\end{array}$$

Note that $(\partial_C)_n = (\partial_Q)_{n+1}$ for $n \geq 1$. Therefore $H_n(A^{\sharp}_{Q}, C) = H_{n+1}(Q; R)$ for all $n \geq 1$.

On the other hand, let $M := R[Q_0]$ be the $R$-free module generated by $Q_0$. We identify a vertex $a \in Q_0$ with a corresponding trivial path $e_a \in C_0(\overline{Q})$ in $\overline{Q}$, so that, $M = C_0(\overline{Q})$. Define $R$-homomorphisms

$$\mathsf{s} : (A^{\sharp}_{Q})^{[0]} = A^{\sharp}_{Q} \longrightarrow M \quad \text{by} \quad \Delta \mapsto s(\Delta) \text{ for } \Delta \in C = \mathsf{P}(Q)^{\text{non}},$$

$$\mathsf{r} : (A^{\sharp}_{Q})^{[0]} = A^{\sharp}_{Q} \longrightarrow M \quad \text{by} \quad \Delta \mapsto r(\Delta) \text{ for } \Delta \in C = \mathsf{P}(Q)^{\text{non}}.$$

These maps $\mathsf{s}$ and $\mathsf{r}$ clearly satisfy the two conditions in Lemma 5.1. Thus we have a chain complex

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{(\partial^M_C)_2} & (A^{\sharp}_{Q})^{[1]} & \xrightarrow{(\partial^M_C)_1} & (A^{\sharp}_{Q})^{[0]} & \xrightarrow{(\partial^M_C)_0} & M & \longrightarrow & \{0\} \\
\cdots & & \| & & \| & & \| & & \\
\cdots & \xrightarrow{(\partial_Q)_3} & C_2(\overline{Q}) & \xrightarrow{(\partial_Q)_2} & C_1(\overline{Q}) & \xrightarrow{(\partial_Q)_1} & C_0(\overline{Q}) & \longrightarrow & \{0\}
\end{array}$$

where $(\partial^M_C)_0 := \mathsf{r} - \mathsf{s} : \langle \mathcal{D}_0 \rangle_R \longrightarrow M$ coincides with $(\partial_Q)_1$. It follows that $H^M_0(A^{\sharp}_{Q}, C) = H_1(Q; R)$ and $H^M_{-1}(A^{\sharp}_{Q}, C) = H_0(Q; R)$. The proof is complete. □

## 7. Examples

In this section, we give various examples of a pair $(A, B)$ and $H_n(A, B)$. Denote by $R^m$ the natural $R$-module $\{{}^t(x_1, \ldots, x_m) \mid x_i \in R\}$ with $R$-basis $\{v_i\}_{1 \leq i \leq m}$ where $v_i$ is a vector in $R^m$ whose $i$-th entry is 1, and the other entries are all 0.

### 7.1. Path algebra $A^\sharp_Q$

Let $Q$ be a quiver defined as $a \xrightarrow{\alpha} b$. Then $\mathsf{P}(Q) = \{e_a, e_b, \alpha\}$, $C := \mathsf{P}(Q)^{\mathrm{non}} = \{\alpha\}$, and $A^\sharp_Q := \langle C\rangle_R = \langle \alpha\rangle_R \cong R$. Since $(A^\sharp_Q)^{[n]} := \big\langle C^{\neq 0}_{\otimes(n+1)}\big\rangle_R$ for $n \geq 0$, a corresponding chain complex is as

$$\cdots \longrightarrow \{0\} \longrightarrow \{0\} \xrightarrow{(\partial_C)_1} (A^\sharp_Q)^{[0]} \xrightarrow{(\partial_C)_0} \{0\}.$$

Note that $(A^\sharp_Q)^{[0]} = A^\sharp_Q$. Thus $H_0(A^\sharp_Q, C) = A^\sharp_Q \cong R$. Let $M := R^2$, and define $R$-homomorphisms

$$\mathsf{s} : A^\sharp_Q \longrightarrow M \quad \text{by} \quad \alpha \mapsto v_1,$$

$$\mathsf{r} : A^\sharp_Q \longrightarrow M \quad \text{by} \quad \alpha \mapsto v_2.$$

Then, by Lemma 5.1, we have a chain complex

$$\cdots \longrightarrow \{0\} \xrightarrow{(\partial^M_C)_1} (A^\sharp_Q)^{[0]} \xrightarrow{(\partial^M_C)_0} M \longrightarrow \{0\}$$

where $(\partial^M_C)_0 := \mathsf{r}-\mathsf{s} : A^\sharp_Q \longrightarrow M$ with $\mathrm{Im}(\partial^M_C)_0 = \langle v_2 - v_1\rangle_R \cong R$. It follows that $H^M_0(A^\sharp_Q, C) := \mathrm{Ker}(\partial^M_C)_0 = \{0\}$ and $H^M_{-1}(A^\sharp_Q, C) := M/\mathrm{Im}(\partial^M_C)_0 \cong R$. Summarizing

| $n$ | $n \geq 1$ | $0$ | $-1$ |
|---|---|---|---|
| $H_n(A^\sharp_Q, C)$ | $\{0\}$ | $R$ | $-$ |
| $H^M_n(A^\sharp_Q, C)$ | $\{0\}$ | $\{0\}$ | $R$ |

### 7.2. Matrix algebras

Let $M_m(R)$ be the total matrix $R$-algebra of degree $m$ with $\{e_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq m\}$ as $R$-basis where $e_{ij}$ is a matrix whose $(s,t)$-entry is 1 if $(s,t) = (i,j)$, and 0 otherwise. Let $\Lambda$ be a subset of $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$ such that $(i,k) \in \Lambda$ whenever $(i,j), (j,k) \in \Lambda$. Then

$$A := \bigoplus_{(i,j)\in\Lambda} Re_{i,j}$$

is an $R$-subalgebra of $M_m(R)$ possessing an $R$-free basis $B := \{e_{i,j} \mid (i,j) \in \Lambda\}$. Since $B \cup \{0\}$ is a semigroup, $B$ satisfies Hypothesis **(P)**. Recall that, for $n \geq 0$,

$$\begin{aligned} A^{[n]} &:= \left\langle B^{\neq 0}_{\otimes(n+1)} \right\rangle_R \\ &= \langle e_{i_0 i_1} \otimes e_{i_1 i_2} \otimes \cdots \otimes e_{i_n i_{n+1}} \mid (i_k, i_{k+1}) \in \Lambda,\ 0 \leq k \leq n \rangle_R \end{aligned}$$

Note that $A^{[0]} = \langle B \rangle_R = A$. Then we have a corresponding chain complex

$$\cdots \xrightarrow{(\partial_B)_3} A^{[2]} \xrightarrow{(\partial_B)_2} A^{[1]} \xrightarrow{(\partial_B)_1} A^{[0]} \xrightarrow{(\partial_B)_0} \{0\}.$$

Let $M := R^m$, and define $R$-homomorphisms

$$\mathsf{s} : A \longrightarrow M \quad \text{by} \quad \sum_{(i,j)\in\Lambda} \lambda_{i,j} e_{i,j} \mapsto \sum_{(i,j)\in\Lambda} \lambda_{i,j} v_i,$$

$$\mathsf{r} : A \longrightarrow M \quad \text{by} \quad \sum_{(i,j)\in\Lambda} \mu_{i,j} e_{i,j} \mapsto \sum_{(i,j)\in\Lambda} \mu_{i,j} v_j.$$

Then, for $e_{ij}, e_{jk} \in B$, we have that $\mathsf{s}(e_{ij}e_{jk}) = \mathsf{s}(e_{ik}) = v_i = \mathsf{s}(e_{ij})$, $\mathsf{r}(e_{ij}e_{jk}) = \mathsf{r}(e_{ik}) = v_k = \mathsf{r}(e_{jk})$, and $\mathsf{r}(e_{ij}) = v_j = \mathsf{s}(e_{jk})$. By Lemma 5.1, we have a chain complex

$$\cdots \xrightarrow{(\partial_B^M)_2} A^{[1]} \xrightarrow{(\partial_B^M)_1} A^{[0]} \xrightarrow{(\partial_B^M)_0} M \longrightarrow \{0\}$$

where $(\partial_B^M)_n := (\partial_B)_n$ for $n \geq 1$, and $(\partial_B^M)_0 := \mathsf{r} - \mathsf{s} : A \longrightarrow M$.

**Proposition 7.1** *Suppose that $(1,i) \in \Lambda$ for any $1 \leq i \leq m$. Then*

$$H_n^M(A,B) = \begin{cases} \{0\} & \textit{if } n \geq 0 \\ R & \textit{if } n = -1 \end{cases}$$

*Proof.* Using the assumption that $(1,i) \in \Lambda$ for any $1 \leq i \leq m$, the following $R$-homomorphisms $h_n$ can be defined.

$$h_n : A^{[n]} \longrightarrow A^{[n+1]} \ (n \geq 0)$$
$$\text{by} \quad e_{i_0 i_1} \otimes \cdots \otimes e_{i_n i_{n+1}} \mapsto e_{1 i_0} \otimes e_{i_0 i_1} \otimes \cdots \otimes e_{i_n i_{n+1}},$$

$$h_{-1} : M \longrightarrow A^{[0]} \quad \text{by} \quad v_i \mapsto e_{1i}.$$

Then we can check that $(\partial_B^M)_{n+1} \circ h_n + h_{n-1} \circ (\partial_B^M)_n = \mathrm{Id}_{A^{[n]}}$ for any $n \geq 0$ where $\mathrm{Id}_{A^{[n]}}$ is the identity map on $A^{[n]}$. This shows that $\mathrm{Ker}(\partial_B^M)_n \leq \mathrm{Im}(\partial_B^M)_{n+1}$, and thus $H_n^M(A,B) = \{0\}$ for $n \geq 0$.

On the other hand, we have that $\mathrm{Im}(\partial_B^M)_0 = \langle v_i - v_1 \mid 2 \leq i \leq m \rangle_R$. It follows that $H_{-1}^M(A,B) := M/\mathrm{Im}(\partial_B^M)_0 \cong R$ as desired. □

Let $T_m(R)$ be an $R$-subalgebra consisting of all upper triangular matrices in $M_m(R)$. Then we are able to apply Proposition 7.1 to $R$-algebras $M_m(R)$ and $T_m(R)$. Here it is worth mentioning that we originally calculated homology $H_n^M(A,B)$ for $M_2(R)$ and $T_2(R)$. But the referee pointed out that our earlier results can be generalized as in Proposition 7.1.

**Remark 7.2** Let $Q$ be a quiver defined as $a \xrightarrow{\alpha} b$. Then by Remark 4.9 (1) and (2), we have that $\rho_w(A^\circ_{Q^{\mathrm{ud}}}) = \langle \rho_w(\Delta) \mid \Delta \in \mathsf{P}(Q^{\mathrm{ud}}) \rangle = \mathrm{UD}(Q,w;\mathbb{C}) \cong M_2(\mathbb{C})$ where $w \equiv 1$. So the above result is regarded as the calculation of homology of the Up-Down algebra $\mathrm{UD}(Q,w;\mathbb{C})$, and at the same time, regarded as the calculation of homology of $A_{Q^{\mathrm{ud}}}$-action $\rho_w$ discussed in Section 4.2.

**Remark 7.3** Let $Q$ be a quiver defined as $a \xrightarrow{\alpha} b$, and let $A_Q := \mathbb{C}[Q] = \langle \mathsf{P}(Q) \rangle_{\mathbb{C}}$ be the path algebra of $Q$ where $\mathsf{P}(Q) = \{e_a, e_b, \alpha\}$. Then $\rho_w(A^\circ_Q) = \langle \rho_w(\Delta) \mid \Delta \in \mathsf{P}(Q) \rangle$ appeared in Example 4.8 is isomorphic to $T_2(R)$ where $w \equiv 1$. So the above result is regarded as the calculation of homology of $A_Q$-action $\rho_w$ discussed in Section 4.2.

### 7.3. Group algebras

Let $G$ be a group with the identity element $e_G$. Let $R[G]$ be the group algebra of $G$ over $R$. Since a $R$-basis $G$ of $R[G]$ is a group, $G$ satisfies Hypothesis **(P)**. Thus homology $H_n(R[G],G)$ is defined. Note that, in this case, $R[G]^{[n]} := \langle B^{\neq 0}_{\otimes(n+1)} \rangle_R$ and $G^{\neq 0}_{\otimes(n+1)} = G_{\otimes(n+1)} = \{g_0 \otimes \cdots \otimes g_n \mid g_i \in G\}$ for $n \geq 0$. Then we have a corresponding chain complex

$$\cdots \xrightarrow{(\partial_G)_2} R[G]^{[1]} \xrightarrow{(\partial_G)_1} R[G]^{[0]} \xrightarrow{(\partial_G)_0} \{0\}.$$

Recall that $\mathrm{Im}(\partial_G)_1 = \langle \partial_G(x \otimes y) \mid x,y \in G \rangle_R = \langle x + y - xy \mid x,y \in G \rangle_R$. Let $\mathbb{Q}$ and $\mathbb{Z}$ be respectively the field of all rational numbers and the ring of all rational integers.

**Proposition 7.4** *Suppose that $G$ is finite. Then $H_0(\mathbb{Q}[G], G) = \{0\}$ and $H^M_{-1}(\mathbb{Q}[G], G) \cong M$ for any extension $(\cdots \longrightarrow \mathbb{Q}[G]^{[0]} \longrightarrow M \longrightarrow \{0\})$ defined in Section 5.*

*Proof.* It is enough to show that $(\partial_G)_1$ is surjective. Take any element $g \in G$ of order $m > 1$. Then $\mathrm{Im}(\partial_G)_1$ contains $\sum_{i=1}^{m-1}(\partial_G)_1(g \otimes g^i) = mg - e_G$ and $(\partial_G)_1(e_G \otimes e_G) = e_G$. It follows that $G \subseteq \mathrm{Im}(\partial_G)_1$. Thus $\mathrm{Im}(\partial_G)_1 = \mathbb{Q}[G] = \mathbb{Q}[G]^{[0]}$. Furthermore since $G$ is a group, we have by Lemma 5.3 that $H^M_{-1}(\mathbb{Q}[G], G) \cong M$. □

**Lemma 7.5** *A surjective group homomorphism $h : G \longrightarrow K$ induces a surjective homomorphism $\widehat{h} : H_0(\mathbb{Z}[G], G) \longrightarrow H_0(\mathbb{Z}[K], K)$.*

*Proof.* A map $\widehat{h} : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[K]$ defined by $\widehat{h}(\sum_{g\in G} \alpha_g g) := \sum_{g\in G} \alpha_g h(g)$ gives a surjective homomorphism. Thus it is enough to show that $\widehat{h}\big(\mathrm{Im}(\partial_G)_1\big) \subseteq \mathrm{Im}(\partial_K)_1$. Indeed, for $g_1, g_2 \in G$,

$$\widehat{h}\big((\partial_G)_1(g_1 \otimes g_2)\big) = \widehat{h}(g_1 + g_2 - g_1g_2) = (\partial_K)_1\big(\widehat{h}(g_1) \otimes \widehat{h}(g_2)\big) \in \mathrm{Im}(\partial_K)_1.$$

This completes the proof. □

**Lemma 7.6** *Let $G = \langle x_1\rangle \times \cdots \times \langle x_m\rangle \times \langle h_1\rangle \times \cdots \times \langle h_n\rangle = \langle g_1\rangle \times \cdots \times \langle g_{m+n}\rangle$ be a finitely generated abelian group where $g_i = x_i$ $(1 \le i \le m)$ with $\ell_i := o(x_i) < \infty$, and $g_{m+j} = h_j$ $(1 \le j \le n)$ with $o(h_j) = \infty$. Consider the group algebra $\mathbb{Z}[G]$. Then we have the following.*

(1) $\mathrm{Im}(\partial_G)_1 = \langle \prod_{i=1}^{m+n} g_i^{s_i} - \sum_{i=1}^{m+n} s_i g_i \mid s_i \in \mathbb{Z}\rangle_{\mathbb{Z}} \subseteq \mathbb{Z}[G]$.
(2) $\mathrm{Im}(\partial_G)_1 \cap G = \{e_G\}$.
(3) $H_0(\mathbb{Z}[G], G) \cong G$ *as groups.*

*Proof.* (1) For any $g = \prod_{i=1}^{m+n} g_i^{s_i}$ and $h = \prod_{i=1}^{m+n} g_i^{t_i}$ in $G$ where $s_i, t_i \in \mathbb{Z}$, we have that $(\partial_G)_1(g\otimes h) = g+h-gh = (g-\sum_{i=1}^{m+n} s_ig_i)+(h-\sum_{i=1}^{m+n} t_ig_i)-(gh-\sum_{i=1}^{m+n}(s_i+t_i)g_i)$. Conversely, modulo $\mathrm{Im}(\partial_G)_1 = \langle x+y-xy \mid x, y \in G\rangle_{\mathbb{Z}}$,

$$\overline{\prod_{i=1}^{m+n} g_i^{s_i} - \sum_{i=1}^{m+n} s_i g_i} = \sum_{i=1}^{m+n} s_i\overline{g_i} - \sum_{i=1}^{m+n} s_i\overline{g_i} = \overline{0}.$$

Thus the first assertion holds.

(2) Let $\mathcal{I} := \{(\nu_1, \ldots, v_m, \mu_1, \ldots, \mu_n) \mid 0 \le \nu_i < \ell_i,\ \mu_j \in \mathbb{Z}\}$. For $\Delta = (s_1, \ldots, s_{m+n}) \in \mathcal{I}$, put $g^\Delta := \prod_{i=1}^{m+n} g_i^{s_i} \in G$, $g_\Delta := \sum_{i=1}^{m+n} s_i g_i \in \mathbb{Z}[G]$, and $J_\Delta := \{i \mid 1 \le i \le m+n,\ s_i \ne 0\}$. By the previous result, $\mathrm{Im}(\partial_G)_1 = \langle g^\Delta - g_\Delta \mid \Delta \in \mathcal{I}\rangle_{\mathbb{Z}}$. Let $\mathcal{S}$ be the totality of sequences $\Delta \in \mathcal{I}$ such that $J_\Delta = \{k\}$ for some $1 \le k \le m+n$ and $s_k = 1$. Then, for $\Delta \in \mathcal{I}$, we have that $g^\Delta - g_\Delta = 0$ if and only if $\Delta \in \mathcal{S}$. Thus any element $Y$ in $\mathrm{Im}(\partial_G)_1$ can be expressed as

$$Y = \sum_{\Delta \in \mathcal{S}'} \alpha_\Delta (g^\Delta - g_\Delta) \quad \text{(finite sum)}$$

where $\mathcal{S}' := \mathcal{I} \setminus \mathcal{S}$. Suppose now that $Y = g^{\Delta_0} \in G$ for some $\Delta_0 = (t_1, \ldots, t_{m+n}) \in \mathcal{I}$. If $\Delta_0 \notin \mathcal{S}'$ namely $\Delta_0 \in \mathcal{S}$ then, since $G$ is a $\mathbb{Z}$-free basis, $\alpha_\Delta = 0$ for all $\Delta$. Thus $g^{\Delta_0} = 0$, a contradiction. So $\Delta_0 \in \mathcal{S}'$. Then by the same reason, we have that $g^{\Delta_0} = \alpha_{\Delta_0}(g^{\Delta_0} - g_{\Delta_0})$. This implies that $\alpha_{\Delta_0} = 1$ and $0 = g_{\Delta_0} = \sum_i^{m+n} t_i g_i$. Thus $t_i = 0$ for all $i$, so that, $g^{\Delta_0} = e_G$ as required.

(3) Since $\mathrm{Im}(\partial_G)_1 = \langle x + y - xy \mid x, y \in G\rangle_{\mathbb{Z}}$, we have that $H_0(\mathbb{Z}[G], G) := \mathbb{Z}[G]/\mathrm{Im}(\partial_G)_1 = \langle g_i + \mathrm{Im}(\partial_G)_1 \mid 1 \le i \le m+n\rangle_{\mathbb{Z}}$. Note that $c(g_i + \mathrm{Im}(\partial_G)_1) = (cg_i) + \mathrm{Im}(\partial_G)_1 = (g_i)^c + \mathrm{Im}(\partial_G)_1$ for $c \in \mathbb{Z}$ and $1 \le i \le m+n$. Suppose now that, for some $\Delta_0 = (t_1, \ldots, t_{m+n}) \in \mathcal{I}$,

$$\overline{0} = \sum_{i=1}^{m+n} t_i (g_i + \mathrm{Im}(\partial_G)_1) = g^{\Delta_0} + \mathrm{Im}(\partial_G)_1.$$

Then since $g^{\Delta_0} \in \mathrm{Im}(\partial_G)_1 \cap G = \{e_G\}$, we have that $t_i = 0$ for all $i$. This shows that $\langle g_i + \mathrm{Im}(\partial_G)_1 \mid 1 \le i \le m+n\rangle_{\mathbb{Z}}$ is isomorphic to $G = \langle g_1\rangle \times \cdots \times \langle g_{m+n}\rangle$. The proof is complete. □

**Example 7.7** Let $G$ be a finitely generated group. Then $H_0(\mathbb{Z}[G], G) \cong G/G'$ as groups where $G'$ is the commutator subgroup of $G$.

Indeed, let $\pi : G \longrightarrow G/G'$ be the canonical map. Since $G/G'$ is a finitely generated abelian group, we obtain a surjective homomorphism $\widehat{\pi} : H_0(\mathbb{Z}[G], G) \longrightarrow H_0(\mathbb{Z}[G/G'], G/G') \cong G/G'$ by Lemma 7.5 and 7.6. On the other hand, a map $\kappa : G \longrightarrow H_0(\mathbb{Z}[G], G)$ defined by $\kappa(g) := g + \mathrm{Im}(\partial_G)_1$ for $g \in G$ is a surjective homomorphism such that $\kappa(G') = \{\overline{0}\}$. Thus we have a surjective endomorphism

$$G/G' \xrightarrow{\overline{\kappa}} H_0(\mathbb{Z}[G], G) \xrightarrow{\widehat{\pi}} H_0(\mathbb{Z}[G/G'], G/G) \cong G/G',$$

and this must be an isomorphism in general. It follows that $\overline{\kappa}$ is an isomorphism.

**Remark 7.8** (Relation with group homology) Set $A := R[G]$ and $B := G$. Then we have that $H_n(A, B) = H_{n+1}(G, R)$ for $n \geq 0$ where $H_{n+1}(G, R)$ is the usual group homology (cf. [3, pages 35 and 36]). Indeed, we first recall $H_{n+1}(G, R)$. Let $F_n$ for $n \geq 0$ be the left $R[G]$-free module with basis $\{[g_1|g_2|\cdots|g_n] \mid g_i \in G\}$. Note that $F_0 = R[G][\ ] \cong R[G]$. Define an $R[G]$-homomorphism $\partial_n : F_n \longrightarrow F_{n-1}$ by $\partial_n([g_1|g_2|\cdots|g_n]) = g_1[g_2|\cdots|g_n] + \sum_{i=0}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots g_n] + (-1)^n[g_1|g_2|\cdots|g_{n-1}]$. Then

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} R \longrightarrow \{0\}$$

is a free resolution of the trivial $R[G]$-module $R$. Consider the tensor product $F'_n := R \otimes_{R[G]} F_n$ where $R$ is the right trivial $R[G]$-module. Then $F'_n$ is the $R$-free module with basis $\{[g_1|g_2|\cdots|g_n] \mid g_i \in G\}$, and we have a chain complex

$$F'_\bullet : \quad \cdots \xrightarrow{\partial'_3} F'_2 \xrightarrow{\partial'_2} F'_1 \xrightarrow{\partial'_1} F'_0 \longrightarrow \{0\}$$

where $\partial'_n([g_1|g_2|\cdots|g_n]) = [g_2|\cdots|g_n] + \sum_{i=0}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots g_n] + (-1)^n[g_1|g_2|\cdots|g_{n-1}]$. In particular, $\partial'_1$ is the zero map. Then $H_n(G, R) := H_n(F'_\bullet)$ by definition. Now identifying $g_0 \otimes \cdots \otimes g_n \in A^{[n]}$ with $[g_0|\cdots|g_n] \in F'_{n+1}$, we have the following commutative diagram.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{(\partial_B)_2} & A^{[1]} & \xrightarrow{(\partial_B)_1} & A^{[0]} & & & & \\
\cdots & & \| & & \| & & & & \\
\cdots & \xrightarrow{\partial'_3} & F'_2 & \xrightarrow{\partial'_2} & F'_1 & \xrightarrow{\partial'_1} & F'_0 & \longrightarrow & \{0\}
\end{array}$$

It follows that $H_n(A, B) = H_{n+1}(G, R)$ for $n \geq 0$. From this viewpoint, it is known that $H_0(\mathbb{Z}[G], G) = H_1(G, \mathbb{Z}) \cong G/G'$ for an arbitrary group $G$. Thus, in fact, Example 7.7 holds without the assumption on $G$.

**Remark 7.9** As in Remark 7.8, we have $H^n(R[G], G) = H^{n+1}(G, R)$ for $n \geq 0$ where $H^{n+1}(G, R)$ is the usual group cohomology. It is well known that $H^1(G, R) = \mathrm{Hom}(G, R)$. From this viewpoint, we have that $H^0(\mathbb{Z}[G], G) = H^1(G, \mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Z}) = \{0\}$ if $G/G'$ is finite. This is the conclusion of Example 3.11

### 7.4. The character ring $\mathbb{Q}[\mathbf{NIrr}\,(G)]$ of a finite group

Let $G$ be a finite group, and $\mathrm{Irr}(G)$ be the set of irreducible complex characters of $G$. Denote by $1_G$ and $\rho_G$ respectively the trivial and regular characters of $G$. Let $\mathbb{Z}[\mathrm{Irr}(G)] = \{\sum_{\chi \in \mathrm{Irr}(G)} m_\chi \chi \mid m_\chi \in \mathbb{Z}\}$ be the character ring of $G$ which is a $\mathbb{Z}$-algebra with $\mathrm{Irr}(G)$ as $\mathbb{Z}$-basis. Consider the tensor product $A := \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Irr}(G)]$ over $\mathbb{Z}$ which is a $\mathbb{Q}$-algebra. Define $\widetilde{\chi} := \big(1/\chi(e_G)\big)\chi \in A$ for $\chi \in \mathrm{Irr}(G)$, and put $\mathrm{NIrr}(G) := \{\widetilde{\chi} \mid \chi \in \mathrm{Irr}(G)\} \subseteq A$. Then $A$ possesses $\mathrm{NIrr}(G)$ as $\mathbb{Q}$-basis, that is, $A = \mathbb{Q}[\mathrm{NIrr}(G)]$. Note that $1_G$ is the identity element of $A$. Denote by $\pi(G)$ the set of all primes dividing the order $|G|$ of $G$.

Firstly, we claim that a $\mathbb{Q}$-basis $\mathrm{NIrr}(G)$ satisfies Hypothesis **(P)** (2). Indeed, for $\widetilde{\chi}, \widetilde{\psi} \in \mathrm{NIrr}(G)$, let $\chi\psi = \sum_{\eta \in \mathrm{Irr}(G)} c_\eta \eta$ for some $c_\eta \in \mathbb{Z}$. Then we have that $\widetilde{\chi}\widetilde{\psi} = \sum_{\eta \in \mathrm{Irr}(G)} d_\eta \widetilde{\eta}$ where

$$d_\eta := \frac{c_\eta \times \eta(e_G)}{\chi(e_G) \times \psi(e_G)} \in \mathbb{Q}.$$

Since $\widetilde{\chi}(e_G) = \widetilde{\psi}(e_G) = \widetilde{\eta}(e_G) = 1$, the equality $\sum_{\eta \in \mathrm{Irr}(G)} d_\eta = 1$ holds. Secondly, we claim that a $\mathbb{Q}$-basis $\mathrm{NIrr}(G)$ satisfies Hypothesis **(P)** (1). This is because that for any $\widetilde{\chi_1}, \ldots, \widetilde{\chi_m} \in \mathrm{NIrr}(G)$, we have that $\widetilde{\chi_1} \cdots \widetilde{\chi_m} \neq 0$. In other words, denote by $B := \mathrm{NIrr}(G)$, then $B^{\neq 0}_{\otimes(n+1)} = B_{\otimes(n+1)}$ $(n \geq 0)$ holds. Set $B - 1_G := \{\widetilde{\chi} - 1_G \mid \widetilde{\chi} \in B\}$.

**Proposition 7.10** *Under the above notation, we have the following.*

(1) $\{\varphi \in A \mid \varphi(e_G) = 0\} = \langle B - 1_G \rangle_{\mathbb{Q}}$.
(2) $H_0(A, B) = \{0\}$.

*Proof.* (1) Put $X := \{\varphi \in A \mid \varphi(e_G) = 0\}$. For any $\varphi = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \widetilde{\chi} \in X$, we have that $0 = \varphi(e_G) = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \widetilde{\chi}(e_G) = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi$. This implies that $\varphi = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi (\widetilde{\chi} - 1_G)$ is contained in $\langle B - 1_G \rangle_{\mathbb{Q}}$. The converse is trivial.

(2) It is enough to show that $\mathrm{Im}(\partial_B)_1 = A$. Let

$$\theta := 1_G - \frac{1}{|G|}\rho_G = 1_G - \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(e_G)^2}{|G|}\widetilde{\chi} \in A.$$

Since $\theta(e_G) = 0$, we have that $\theta \in X = \langle B - 1_G\rangle_{\mathbb{Q}}$ and express as $\theta = \sum_{\chi \in \mathrm{Irr}(G)} c_\chi(\widetilde{\chi} - 1_G)$. Furthermore $\varphi\theta = \varphi$ for any $\varphi \in X$ because of $\theta(g) = 1$ for any $g \in G$ with $g \neq e_G$. Thus for any $\widetilde{\psi} - 1_G \in (B - 1_G) \subseteq X$, we have that

$$\begin{aligned} \widetilde{\psi} - 1_G = (\widetilde{\psi} - 1_G)\theta &= \sum_{\chi \in \mathrm{Irr}(G)} c_\chi(\widetilde{\psi} - 1_G)(\widetilde{\chi} - 1_G) \\ &\in \langle(\widetilde{\eta} - 1_G)(\widetilde{\chi} - 1_G) \mid \widetilde{\eta}, \widetilde{\chi} \in B\rangle_{\mathbb{Q}}. \end{aligned}$$

It follows that $\langle 1_G, (\widetilde{\eta}-1_G)(\widetilde{\chi}-1_G) \mid \widetilde{\eta}, \widetilde{\chi} \in B\rangle_{\mathbb{Q}} \supseteq \langle 1_G,\ \widetilde{\psi}-1_G \mid \widetilde{\psi} \in B\rangle_{\mathbb{Q}} = A$, and that those two sets are equal. On the other hand, since $\widetilde{\eta}\widetilde{\chi} \neq 0$ for any $\widetilde{\eta}, \widetilde{\chi} \in B$, we have by Proposition 3.9 that $\mathrm{Im}(\partial_B)_1 = \langle 1_G, (\widetilde{\eta}-1_G)(\widetilde{\chi}-1_G) \mid \widetilde{\eta}, \widetilde{\chi} \in B\rangle_{\mathbb{Q}}$. Thus $\mathrm{Im}(\partial_B)_1 = A$ as wanted. This completes the proof. $\square$

**Remark 7.11** For a positive integer $n \geq 1$, let $\pi(n)$ be the set of all primes dividing $n$. Let $N := N(G)$ be the least common multiple of the degrees $\chi(e_G)$ for all $\chi \in \mathrm{Irr}(G)$. Set $n_\pi := \prod_{p \in \pi(N)} |G|_p$ where $|G|_p$ is the $p$-part of $|G|$, and set $n_{\pi'} := |G|/n_\pi$.

Now we consider a subring $R := \mathbb{Z}[1/N] = \{f(1/N) \mid f(X) \in \mathbb{Z}[X]\}$ of $\mathbb{Q}$ instead of $\mathbb{Q}$ itself, and observe the $R$-free module $R[\mathrm{NIrr}(G)]$ with $B := \mathrm{NIrr}(G)$ as basis. Note that $\pi\big(\chi(e_G)\psi(e_G)\big)$ for any $\chi, \psi \in \mathrm{Irr}(G)$ is a subset of $\pi(N)$. This implies that $d_\eta$ appeared earlier in this Section lives in $R$, and that $R[B]$ is defined as an $R$-algebra. At the same time, we can see that an $R$-basis $B$ satisfies Hypothesis **(P)**.

We follow the notation in the proof of Proposition 7.10. Then, by the same way, we can show that $X = \langle B - 1\rangle_R$. Furthermore let

$$\theta' := n_{\pi'}\left(1_G - \frac{1}{|G|}\rho_G\right) = n_{\pi'}1_G - \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(e_G)^2}{n_\pi}\widetilde{\chi}.$$

Since $\pi(n_\pi) = \pi(N)$, we have that $\theta' \in R[B]$. Then, along the same way as in the proof of Proposition 7.10, it is shown that $\langle 1_G,\ n_{\pi'}(\widetilde{\psi} - 1_G) \mid \widetilde{\psi} \in B\rangle_R = \langle 1_G,\ n_{\pi'}\widetilde{\psi} \mid \widetilde{\psi} \in B\rangle_R$ is contained in $\mathrm{Im}(\partial_B)_1$ by using $\theta'$ instead of $\theta$. It follows that $H_0(R[B], B)$ is a $\big(\pi(G) \setminus \pi(N)\big)$-group. We will see in the next Example 7.12 that it is not always equal to $\{0\}$.

**Example 7.12** Under the situation of Remark 7.11, we give an example. Let $G$ be the dihedral group $D_{10} = \langle (1, 2, 3, 4, 5), (2, 5)(3, 4)\rangle$ of order 10. The character table of $G$ is given as follows:

| | $e_G$ | $(2,5)(3,4)$ | $(1,2,3,4,5)$ | $(1,3,5,2,4)$ |
|---|---|---|---|---|
| $\chi_1 = 1_G$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | $-1$ | 1 | 1 |
| $\chi_3$ | 2 | 0 | $\varepsilon + \varepsilon^4$ | $\varepsilon^2 + \varepsilon^3$ |
| $\chi_4$ | 2 | 0 | $\varepsilon^2 + \varepsilon^3$ | $\varepsilon + \varepsilon^4$ |

where $\varepsilon^5 = 1$ and $\varepsilon \neq 1$. Then $N := N(G) = 2$ and a coefficient ring is $R := \mathbb{Z}\big[\frac{1}{2}\big] \subseteq \mathbb{Q}$. Furthermore $n_\pi = 2$ and $n_{\pi'} = 5$. Now we calculate that $(\partial_B)_1(\widetilde{\chi_1} \otimes \widetilde{\chi_i}) = \widetilde{\chi_1} + \widetilde{\chi_i} - \widetilde{\chi_1}\widetilde{\chi_i} = \widetilde{\chi_1}$ for all $1 \leq i \leq 4$. By the same way, $(\partial_B)_1(\widetilde{\chi_2} \otimes \widetilde{\chi_2}) = 2\widetilde{\chi_2} - \widetilde{\chi_1}$, $(\partial_B)_1(\widetilde{\chi_2} \otimes \widetilde{\chi_3}) = (\partial_B)_1(\widetilde{\chi_2} \otimes \widetilde{\chi_4}) = \widetilde{\chi_2}$, $(\partial_B)_1(\widetilde{\chi_3} \otimes \widetilde{\chi_3}) = 2\widetilde{\chi_3} - \frac{1}{4}\widetilde{\chi_1} - \frac{1}{4}\widetilde{\chi_2} - \frac{1}{2}\widetilde{\chi_4}$, $(\partial_B)_1(\widetilde{\chi_3} \otimes \widetilde{\chi_4}) = \frac{1}{2}\widetilde{\chi_3} + \frac{1}{2}\widetilde{\chi_4}$, $(\partial_B)_1(\widetilde{\chi_4} \otimes \widetilde{\chi_4}) = 2\widetilde{\chi_4} - \frac{1}{4}\widetilde{\chi_1} - \frac{1}{4}\widetilde{\chi_2} - \frac{1}{2}\widetilde{\chi_3}$. Then $\mathrm{Im}(\partial_B)_1$ is generated by those $(\partial_B)_1(\widetilde{\chi_i} \otimes \widetilde{\chi_j})$ over $R$, and we eventually obtain that $\mathrm{Im}(\partial_B)_1 = \langle \widetilde{\chi_1}, \widetilde{\chi_2}, 5\widetilde{\chi_3}, \widetilde{\chi_3} + \widetilde{\chi_4}\rangle_R$. Thus $H_0(R[B], B) = \langle \widetilde{\chi_3} + \mathrm{Im}(\partial_B)_1\rangle_R$ is a cyclic group of order $n_{\pi'} = 5$.

**Example 7.13** Suppose that $G$ is a finite abelian group. Since every irreducible character is linear, we have that $N := N(G) = 1$ and $\mathrm{NIrr}(G) = \mathrm{Irr}(G)$. Furthermore $\mathrm{Irr}(G)$ together with the usual product on characters forms a group which is isomorphic to $G$. Thus $R := \mathbb{Z}[1/N] = \mathbb{Z}$ and $B := \mathrm{NIrr}(G) = \mathrm{Irr}(G) \cong G$. Then by Example 7.7, $H_0(R[B], B) \cong H_0(\mathbb{Z}[G], G) \cong G$.

## 8. Semilattices

Let $(L, \leq)$ be a meet-semilattice, namely this is a poset such that there exists the greatest lower bound $a \wedge b$ for any $a, b \in L$. In this section, we focus on an $R$-algebra $R[L]$. In particular, we consider a subalgebra of $\mathbb{Q}[\mathrm{Sgp}(G)]$

where $\mathrm{Sgp}(G)$ is the subgroup lattice of a finite group $G$. Furthermore some relations with the associated order complex of $L$ are investigated.

**Definition 8.1** For a meet-semilattice $(L, \leq)$, let $R[L]$ be the $R$-free module with basis $L$. A multiplication on $R[L]$ is defined by extending bilinearly the product $ab := a \wedge b \in L$ for $a, b \in L$. Then $R[L]$ is an associative $R$-algebra (cf. [1, page 185]). Note that an $R$-free basis $L$ is a semigroup, so that, $L$ satisfies Hypothesis **(P)**.

### 8.1. Subgroup lattice

Let $G$ be a finite group. Denote by $\mathrm{Sgp}(G)$ the set of all subgroups of $G$. Then $\mathrm{Sgp}(G)$ is a meet-semilattice with respect to the inclusion-relation. Note that $H \wedge K = H \cap K \in \mathrm{Sgp}(G)$ for any $H, K \in \mathrm{Sgp}(G)$. Put $L := \mathrm{Sgp}(G)$ and a $\mathbb{Q}$-algebra $A := \mathbb{Q}[L]$.

**Definition 8.2** For a subgroup $H \in L$, define an element

$$[H] := \frac{1}{|G|} \sum_{g \in G} H^g = \frac{1}{|G:H|} \left( \frac{1}{|H|} \sum_{g \in G} H^g \right) \in \mathbb{Q}[L]$$

where $H^g := g^{-1}Hg$ for $g \in G$. Put $X_G := \{[H] \mid H \in L/_{\sim_G}\}$ where $L/_{\sim_G}$ is a set of representatives of $G$-conjugate classes of $L$. Denote by $\Omega(G) := \mathbb{Q}[X_G]$ the $\mathbb{Q}$-free module generated by $X_G$.

Note that for $[H], [K] \in X_G$, we have that

$$\begin{aligned}
[H][K] &= \frac{1}{|G|^2} \sum_{x,y \in G} H^x \cap K^y = \frac{1}{|G|^2} \sum_{x \in G} \left( \sum_{y \in G} (H \cap K^{yx^{-1}})^x \right) \\
&= \frac{1}{|G|^2} \sum_{x \in G} \left( \sum_{y \in G} (H \cap K^y)^x \right) = \frac{1}{|G|} \sum_{y \in G} \left( \frac{1}{|G|} \sum_{x \in G} (H \cap K^y)^x \right) \\
&= \frac{1}{|G|} \sum_{y \in G} [H \cap K^y] \in \Omega(G).
\end{aligned}$$

This shows that $\Omega(G)$ is a subalgebra of $\mathbb{Q}[L]$ with a $\mathbb{Q}$-basis $X_G$, and also shows that $X_G$ satisfies Hypothesis **(P)**.

**Proposition 8.3** *Under the above notation, we have that* $H_0(\Omega(G), X_G) = \{0\}$.

*Proof.* It suffices to show that $\mathrm{Im}(\partial_{X_G})_1 = \Omega(G)$. Let $E := \{e_G\}$ be the trivial subgroup of $G$. First we note that $(\partial_{X_G})_1([E]\otimes[E]) = 2[E]-[E][E] = 2[E]-[E] = [E]$. Next we take any $[H] \in X_G$ such that $H \neq E$. Then by using the above formula on $[H][K]$,

$$(\partial_{X_G})_1([H]\otimes[H]) = \left(2 - \frac{1}{|G : N_G(H)|}\right)[H] - \frac{1}{|G|}\sum_{y \notin N_G(H)} [H \cap H^y].$$

The coefficient of $[H]$ is non-zero. By induction on the order of a subgroup of $G$, $\mathrm{Im}(\partial_{X_G})_1$ contains $[H]$. Thus $X_G \subseteq \mathrm{Im}(\partial_{X_G})_1$, so that, $\mathrm{Im}(\partial_{X_G})_1 = \Omega(G)$ as desired. $\square$

### 8.2. Associated order complex

For a poset $(L, \leq)$, the order complex $\mathsf{O}(L) = \mathsf{O}(L, \leq)$ of $L$ is the abstract simplicial complex whose $k$-simplices are all inclusion-chains $(x_0 > \cdots > x_k)$ of length $k$ where $x_i \in L$. Denote by $\mathsf{O}(L)_k$ the set of all $k$-simplices of $\mathsf{O}(L)$ for $k \geq 0$. For $s \in L$ and a subset $K \subseteq L$, we define a subposet $K_{<s} := \{a \in K \mid a < s\}$.

**Notation 8.4** *Let $(L, \leq)$ be a meet-semilattice. For $s \in L$ and a subset $K \subseteq L$, denote by*

$$\mathcal{F}(K_{<s})_{n-1} := \big\{b_0 \otimes \cdots \otimes b_{n-1} \mid (b_0 > \cdots > b_{n-1}) \in \mathsf{O}(K_{<s})_{n-1}\big\} \ (n \geq 1).$$

*For $b_0 \otimes \cdots \otimes b_{n-1} \in \mathcal{F}(K_{<s})_{n-1}$, we have the product $b_0 \cdots b_{n-1} = b_{n-1} \neq 0$ in the algebra $R[L]$. Thus $\mathcal{F}(K_{<s})_{n-1} \subseteq L^{\neq 0}_{\otimes n}$. Furthermore denote by*

$$\begin{aligned} s \otimes \mathcal{F}(K_{<s})_{n-1} &:= \big\{s \otimes b_0 \otimes \cdots \otimes b_{n-1} \mid b_0 \otimes \cdots \otimes b_{n-1} \in \mathcal{F}(K_{<s})_{n-1}\big\}, \\ s \otimes \mathcal{F}(K_{<s})_{-1} &:= \{s\}, \\ \mathcal{D}(K_{<s}) &:= \bigcup_{n \geq 0} s \otimes \mathcal{F}(K_{<s})_{n-1} \subseteq \bigcup_{n \geq 0} L^{\neq 0}_{\otimes(n+1)}. \end{aligned}$$

*Recall that $\mathcal{D}(K_{<s})_n := \mathcal{D}(K_{<s}) \cap L^{\neq 0}_{\otimes(n+1)} = s \otimes \mathcal{F}(K_{<s})_{n-1}$ for $n \geq 0$. In particular, $\mathcal{D}(K_{<s})_0 = \{s\}$.*

**Proposition 8.5** *Let $(L, \leq)$ be a meet-semilattice. For $s \in L$ and a subset $K \subseteq L$, we have that $\mathcal{D}(K_{<s})$ is $\partial_L$-invariant, and that $H_n\big(\mathcal{D}(K_{<s})\big) \cong$*

$\widetilde{H}_{n-1}(\mathsf{O}(K_{<s}))$ *for any* $n \geq 0$ *as* $R$*-modules where* $\widetilde{H}_{n-1}(\mathsf{O}(K_{<s}))$ *is the usual reduced homology of a simplicial complex* $\mathsf{O}(K_{<s})$*.*

*Proof.* For $s \otimes b_0 \otimes \cdots \otimes b_{n-1} \in \mathcal{D}(K_{<s})_n$ $(n \geq 1)$, we have that

$$\begin{aligned}
&\partial_L(s \otimes b_0 \otimes \cdots \otimes b_{n-1}) \\
&\quad = (b_0 \otimes \cdots \otimes b_{n-1}) - (sb_0) \otimes b_1 \otimes \cdots \otimes b_{n-1} \\
&\qquad + \sum_{i=0}^{n-2} (-1)^i s \otimes \cdots \otimes (b_i b_{i+1}) \otimes \cdots \otimes b_{n-1} + (-1)^{n-1} s \otimes b_0 \otimes \cdots \otimes b_{n-2} \\
&\quad = \sum_{i=0}^{n-1} (-1)^i s \otimes b_0 \otimes \cdots \otimes \widehat{b_i} \otimes \cdots \otimes b_{n-1} \in \mathcal{D}(K_{<s})_{n-1}.
\end{aligned}$$

Note that $sb_0 = b_0$ and $b_i b_{i+1} = b_{i+1}$ as $s > b_0$ and $b_i > b_{i+1}$ respectively, and that $\widehat{b_i}$ means to delete the element $b_i$. This shows that $\mathcal{D}(K_{<s})$ is $\partial_L$-invariant.

Put $\mathcal{D} := \mathcal{D}(K_{<s})$. An element $s \otimes b_0 \otimes \cdots \otimes b_{n-1} \in \mathcal{D}_n = s \otimes \mathcal{F}(K_{<s})_{n-1}$ $(n \geq 1)$ is identified with an $(n-1)$-simplex $(b_0 > \cdots > b_{n-1}) \in \mathsf{O}(K_{<s})_{n-1}$. Furthermore, for $s \otimes b_0 \in \mathcal{D}_1$, we have that $\partial_L(s \otimes b_0) = s \in \mathcal{D}_0 = \{s\}$. It follows that $(\partial_{L,\mathcal{D}})_1 : \langle \mathcal{D}_1 \rangle_R \longrightarrow \langle \mathcal{D}_0 \rangle_R = \langle s \rangle_R$ gives the augmentation map. Thus we have the following chain complex

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \langle \mathcal{D}_2 \rangle_R & \xrightarrow{(\partial_{L,\mathcal{D}})_2} & \langle \mathcal{D}_1 \rangle_R & \xrightarrow{(\partial_{L,\mathcal{D}})_1} & \langle \mathcal{D}_0 \rangle_R & \longrightarrow & \{0\} \\
 & & \| & & \| & & \| & & \\
 & & \langle \mathsf{O}(K_{<s})_1 \rangle_R & & \langle \mathsf{O}(K_{<s})_0 \rangle_R & & R & &
\end{array}$$

Then the above calculation of $\partial_L$ yields the required isomorphism. $\square$

**Example 8.6** Let $G$ be a finite group. Suppose that $|\pi(G)| \geq 2$. For $p \in \pi(G)$, denote by $\mathcal{S}_p(G)$ the set of all non-trivial $p$-subgroups of $G$. Put $L := \mathcal{S}_p(G) \cup \{\{e_G\}, G\}$. Then $L$ is a meet-semilattice with respect to the inclusion-relation. Note that $P \wedge Q = P \cap Q \in L$ for any $P, Q \in L$ as in Section 8.1. Then for $G \in L$ and a subset $K := \mathcal{S}_p(G) \subseteq L$, we have by Proposition 8.5 that

$$H_n(\mathcal{D}(K_{<G})) \cong \widetilde{H}_{n-1}(\mathsf{O}(K_{<G})) = \widetilde{H}_{n-1}(\mathsf{O}(\mathcal{S}_p(G))).$$

Thus the reduced homology of the Brown complex $\mathsf{O}(\mathcal{S}_p(G))$ can be realized as our homology of an $R$-algebra $R[L]$.

**Theorem 8.7** *For a meet-semilattice $(L, \leq)$, let*

$$\mathcal{D}(L) := \bigcup_{s \in L} \mathcal{D}(L_{<s}) = \bigcup_{n \geq 0} \left( \bigcup_{s \in L} s \otimes \mathcal{F}(L_{<s})_{n-1} \right) \subseteq \bigcup_{n \geq 0} L^{\neq 0}_{\otimes(n+1)}.$$

*Then $\mathcal{D}(L)$ is $\partial_L$-invariant, and*

$$H_n\big(\mathcal{D}(L)\big) \cong \bigoplus_{s \in L} \widetilde{H}_{n-1}\big(\mathsf{O}(L_{<s})\big)$$

*for any $n \geq 0$ as $R$-modules.*

*Proof.* By the definition, $\langle \mathcal{D}(L) \rangle_R = \bigoplus_{s \in L} \langle \mathcal{D}(L_{<s}) \rangle_R$ as $R$-modules. Since $\mathcal{D}(L_{<s})$ is $\partial_L$-invariant by Proposition 8.5, so is $\mathcal{D}(L)$. Furthermore $H_n\big(\mathcal{D}(L_{<s})\big) \cong \widetilde{H}_{n-1}\big(\mathsf{O}(L_{<s})\big)$ for any $n \geq 0$. Thus the assertion clearly holds. $\square$

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