Hokkaido Mathematical Journal Vol. 44 (2015) p. 101–164

On the variational problem associated with standard differential systems

Noboru Tanaka

(Received December 24, 2012)

1. Preliminaries

1.1. Left invariant differential systems on Lie groups, and conditions $(C_0), (C)$ and (C')

At the outset we consider a new condition on differential systems (M, D) as follows:

(C_0) any two points p and g of M can be connected by a piece-wise regular integral curve of (M, D).

Clearly condition (C_0) implies condition (C). Suppose that M is connected, and $\mathrm{rank}(D) > 0$. Then it is known that condition (C') implies condition (C_0): (see Appendix in [5]). Furthermore it can be shown that if (M, D) is real analytic, the three conditions $(C_0), (C)$ and (C') are mutually equivalent. The proof of this fact is based on the fact above and Nagano's theorem ([4]) on real analytic involutive differential systems possibly with singularities.

Now, let G be a connected Lie groups and $\mathfrak{g}$ its Lie algebra. In the present paper $\mathfrak{g}$ is defined to be the tangent space $T_e(G)$, e being the identity element $\mathfrak{g}$, equipped with the natural Lie algebra structure. We denote by exp the exponential mapping of $\mathfrak{g}$ to G, and by L_a the left translation of G corresponding to an element of G.

Let ϑ be a subspace of $\mathfrak{g}$. We denote by D the left invariant differential system on G induced by the subspace ϑ: (i) $\mathrm{D_e} = \vartheta$, and (ii) $D_a b = dL_a(D_b)$, $a, b \in G$. It is clear that the differential system (M, D) satisfies condition (C'), if and only if the Lie algebra $\mathfrak{g}$ is generated by ϑ. For completeness we shall prove the following

---

Professor emeritus Noboru Tanaka has passed away on March 4 in 2011. The present manuscript was completed after his retirement from Hokkaido University and later communicated to one of the editors (Keizo Yamaguchi) by his family. We add the table of contents, some notes and references to the original manuscript. – Editor's note – May 20, 2013.

Proposition 1.1 *Assume that* $\dim \vartheta > 0$. *Then Conditions* $(C_0), (C)$ *and* (C') *are mutually equivalent.*

We shall only prove the equivalence $(C') \Leftrightarrow (C_0)$. The equivalence $(C') \Leftrightarrow (C)$ can be similarly dealt with.

For $a \in G$ we denote by $\Omega_0(a)$ the set of all piece-wise regular integral curves $\alpha(t)$ $(t \in [0,1])$ of (G, D) from e to a, and set

$$H = \{a \in G \mid \Omega_0(a) \neq \phi\}.$$

We first assert that $H \neq \phi$. Indeed, $x \in \vartheta - \{0\}$. Clearly the curve $\alpha(t) = \exp(tx)$ $(t \in \mathbb{R})$ is a regular integral curve of (G, D). Hence $\alpha(t)$ is in H for each $t \in \mathbb{R}$, proving our assertion. We next assert that H is a subgroup of G. Indeed, let $a, b \in H$, and take $\alpha \in \Omega_0(a)$, $\beta \in \Omega_0(b)$. Then we define a curve $\gamma(t)$ $(t \in [0,1])$ of G as follows:

$$\gamma(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \dfrac{1}{2}, \\ a\,\beta(2t-1), & \dfrac{1}{2} < t \leq 1. \end{cases}$$

Then we have $\gamma \in \Omega_0(ab)$ and hence $ab \in H$. We now define a curve $\gamma(t)$ $(t \in [0,1])$ of G by

$$\gamma(t) = a^{-1}\gamma(1-t), \quad 0 \leq t \leq 1.$$

Then we have $\gamma \in \Omega_0(a^{-1})$ and hence $a^{-1} \in H$, proving our assertion. Furthermore it is clear that if $a \in H$ and $\omega \in \Omega_0(a), \omega(t)$ is in H for each $t \in [0,1]$. Therefore it follows that H becomes a connected Lie subgroup of G (see Appendix 4 [2]).

Now, we denote by $\mathfrak{g}(\vartheta)$ the subalgebra of $\mathfrak{g}$ generated by the subspace ϑ of $\mathfrak{g}$, and by $G(\vartheta)$ the connected Lie subgroup of G generated by $\mathfrak{g}(\vartheta)$. Then we show that the two Lie subgroups H and $G(\vartheta)$ coincide, from which follows immediately the equivalence $(C') \Leftrightarrow (C_0)$. First we have $G(\vartheta) \subset H$. Indeed, take any $X \in \vartheta - \{0\}$. As we have seen, $\exp(tx)$ is in H for each $t \in \mathbb{R}$, meaning that X is in the Lie algebra $\mathfrak{h}$ of H. We have therefore shown that $\vartheta \subset \mathfrak{h}$ and hence $\mathfrak{g}(\vartheta) \subset \mathfrak{h}$. This means that $G(\vartheta) \subset H$, proving our assertion. Next we have $H \subset G(\vartheta)$. Indeed, we denote by $\hat{D}$ the

left invariant differential system on G induced by the subalgebra $\mathfrak{g}(\vartheta)$ of $\mathfrak{g}$. Then $\hat{D}$ is completely integrable, and $G(\vartheta)$ is the maximal connected integral manifold of $\hat{D}$ through e. Now, let $a \in H$, and take $\omega \in \Omega_0(a)$. Since $D \subset \hat{D}$, ω is a piece-wise integral curve of $\hat{D}$. It follows that $\omega(t)$ is in $G(\vartheta)$ for each $t \in [0,1]$. We have therefore shown that $a = \omega(1) \in G(\vartheta)$ and hence $H \subset G(\vartheta)$.

1.2. Fundamental graded Lie algebras and standard differential systems

Let $\mathfrak{g}$ be a Lie algebra (over the field $\mathbb{R}$ of real numbers), and $(\mathfrak{g}_p)_{p\in\mathbb{Z}}$ a family of subspaces of $\mathfrak{g}$, where $\mathbb{Z}$ denotes the additive group of integers. Let us consider the following conditions on the pair $(\mathfrak{g}, (\mathfrak{g}_p))$:

$$\mathfrak{g} = \sum_p \mathfrak{g}_p \text{ (direct sum)}, \tag{GLA.1}$$

$$\dim \mathfrak{g}_p < \infty, \tag{GLA.2}$$

$$[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}. \tag{GLA.3}$$

Under these conditions the pair $(\mathfrak{g}, (\mathfrak{g}_p))$ or the direct sum $\mathfrak{g} = \sum_p \mathfrak{g}_p$ is called a graded Lie algebra.

Then a graded Lie algebra $\mathfrak{g} = \sum_p \mathfrak{g}_p$ is called a *fundamental graded Lie algebra* or briefly a *FGLA*, if the following conditions are satisfied:

(FGLA.1) $\dim \mathfrak{g} < \infty$,
(FGLA.2) $\mathfrak{g}_1 \neq \{0\}$, and the Lie algebra $\mathfrak{g}$ is generated by $\mathfrak{g}_1$.

Let $\mathfrak{g} = \sum_p \mathfrak{g}_p$ be a FGLA. Then we see that $\mathfrak{g}_p = \{0\}$ for $p \leq 0$ and $\mathfrak{g}_{p+1} = [\mathfrak{g}_1, \mathfrak{g}_p]$ for $p \geq 1$. It follows that there is a positive integer k such that $\mathfrak{g}_p \neq \{0\}$ for $1 \leq p \leq k$ and $\mathfrak{g}_p = \{0\}$ for $p > k$. We also note that $\mathfrak{g}$ becomes a nilpotent Lie algebra.

Let μ be a positive number. Then the FGLA, is called of the μ-th kind, if $k = \mu$, furthermore the FGLA is called a *euclidean FGLA*, if there is given an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}_1$.

In the following we shall be concerned with a fixed FGLA,

$$\mathfrak{g} = \sum_p \mathfrak{g}_p = \sum_{p=1}^{\mu} \mathfrak{g}_p,$$

of the μ-th kind. Let G be a simply connected Lie group whose Lie algebra is $\mathfrak{g}$. Since $\mathfrak{g}$ is a nilpotent Lie algebra, the exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism (onto). We then denote by D the left invariant differential system on G induced by the subspace $\mathfrak{g}_1$ of G. Since the Lie algebra $\mathfrak{g}$ is generated by $\mathfrak{g}_1$, we know from Proposition 1.1 that the differential system (G, D) satisfies conditions $(C_0), (C)$ and (C'). The differential system (G, D) thus obtained is called the *standard differential system associated with the FGLA*, or simply a *standard differential system of the μ-th kind*.

Now, assume that $\mathfrak{g} = \sum_p \mathfrak{g}_p$ is a euclidean FGLA. We denote by g the left invariant inner product of D induced by the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}_1$:

$$\text{(i)}\quad g_e(X, Y) = \langle X, Y\rangle, \quad X, Y \in \mathfrak{g}_1,$$

and

$$\text{(ii)}\quad g_{af}(dL_a(X), dL_a(Y)) = g_f(X, Y), \quad X, Y \in D_f, a, f \in G.$$

The riemannian differential system (G, D, g) thus obtained is called the *standard riemannian differential system associated with the euclidean FGLA* or simply a *standard riemannian differential system of the μ-th kind*. Concerning the standard riemannian differential system (G, D, g) we may consider the spaces $\Omega(G, D, a, f)$ $(a, f \in G)$, the energy functionals $E : \Omega(G, D, a, f) \to \mathbb{R}$, and the distance function $d(a, f)$. Finally we extend, once for all, the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}_1$, to an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, so that any two of the subspaces $\mathfrak{g}_p$ are mutually or orthogonal. Then we denote by $\hat{g}$ the left invariant riemannian metric on G induced by the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, and by $\hat{d}(a, b)$ the associated distance function on G. Clearly both the distance functions d and $\hat{d}$ are left invariant: $d(ca, cb) = d(a, b)$ for $a, b, c \in G$, and the same for $\hat{d}$.

1.3. The distance functions d and $\hat{d}$

In this and the subsequent paragraphs we are concerned with a euclidean FGLA of the μ-th kind, and preserve the notations in the previous paragraph.

We take any vector X of $\mathfrak{g}$, and set $X = \sum_{p=1}^{\mu} X_p$, where $X_p \in \mathfrak{g}_p$. Then we define a function φ on $\mathfrak{g}$ by

$$\varphi(X) = \sum_{p=1}^{\mu} |X_p|^{1/p}.$$

We also have the function $d(e, \exp(X))$ on $\mathfrak{g}$.

Proposition 1.2 *The two functions $\varphi(X)$ and $d(e, \exp(X))$ are equivalent in the following sense: There are positive constants C_1 and C_2 such that*

$$C_1\varphi(X) \leq d(e, \exp(X)) \leq C_2\varphi(X), \quad X \in \mathfrak{g}$$

Proof. Let $\mathbb{R}_+$ be the multiplicative group of positive numbers. For $\lambda \in \mathbb{R}_+$ we define an automorphism $\tilde{\lambda}$ the Lie algebra $\mathfrak{g}$ by

$$\tilde{\lambda}(X) = \lambda^p X, \quad X \in \mathfrak{g}_p, \quad 1 \leq p \leq \mu.$$

Clearly the assignment $\lambda \to \tilde{\lambda}$ gives an injective homomorphism of the group $\mathbb{R}_+$ into the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. We then denote by $\hat{\lambda}$ the automorphism of the Lie group G generated by the automorphism $\tilde{\lambda}$ of $\mathfrak{g}$: $d\hat{\lambda}(X) = \tilde{\lambda}(X)$ and $\hat{\lambda}(\exp(X)) = \exp(\tilde{\lambda}(X))$, where $X \in \mathfrak{g}$.

Lemma 1 *For any $a, b \in G$ and $\lambda \in \mathbb{R}_+$ the following equality holds*:

$$d(\hat{\lambda}(a), \hat{\lambda}(b)) = \lambda d(a, b).$$

Proof. This fact follows immediately from the following equality

$$|d\tilde{\lambda}(X)| = \lambda|X|, \quad X \in D_c,\ c \in G,$$

which is proved as follows: We take $Y \in \mathfrak{g}$, such that $X = dL_c(Y)$. If we put $\bar{c} = \hat{\lambda}(c)c^{-1}$, we have $\hat{\lambda} \circ L_c = L_{\bar{c}} \circ L_c \circ \hat{\lambda}$. It follows that $d\hat{\lambda}(dL_c(Y)) = \lambda dL_{\bar{c}}(dL_c(Y))$ and hence $d\hat{\lambda}(X) = \lambda dL_{\bar{c}}(X)$. Therefore we obtain $|d\hat{\lambda}(X)| = \lambda|X|$, proving the desired equality.

Lemma 2 *There are a neighborhood U of e and a positive constant M such that*

$$d(e, a) \leq M, \quad a \in U$$

Proof. For each $1 \leq p \leq \mu$ we set $n_p = \dim \mathfrak{g}_p$, and take a basis $(e_\alpha^{(p)})_{1\leq\alpha\leq n_p}$ of $\mathfrak{g}_p$. We then define a mapping of $\mathbb{R}^{n_p}$ to G by

$$\sigma^{(p)}(x^{(p)}) = \exp\big(x_1^{(p)}e_1^{(p)}\big)\cdots\exp\big(x_{n_p}^{(p)}e_{n_p}^{(p)}\big),$$

where $x^{(p)} = \big(x_1^{(p)},\dots,x_{n_p}^{(p)}\big) \in \mathbb{R}^{n_p}$. We further define a mapping $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_p}$ to G by

$$\sigma(x) = \sigma^{(1)}(x^{(1)})\dots\sigma^{(\mu)}(x^{(\mu)}),$$

where $x = (x^{(1)},\dots,x^{(\mu)}) \in \mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_\mu}$. Then σ gives a diffeomorphism of $\mathbb{R}_X^{n_1}\cdots X\mathbb{R}^{n_\mu}$ onto G. We now set

$$U = \big\{\sigma(x) \mid |x_\alpha^{(p)}| < 1,\ 1\leq\alpha\leq n_p,\ 1\leq p\leq\mu\big\},$$

being an open neighborhood of e and take any point $a = \sigma(x)$ of U. Since the distance function d is left invariant, we have

$$d(e,a) \leq \sum_{p=1}^{\mu}\sum_{\alpha=1}^{n_p} d\big(e,\exp(x_\alpha^{(p)}e_\alpha^{(p)})\big).$$

Here, we notice that each $x_\alpha^{(p)}$ may be described as follows: $x_\alpha^{(p)} = \varepsilon_\alpha^p(\lambda_\alpha^{(p)})^p$, where $\varepsilon_\alpha^{(p)}$ is 1 or -1 or 0, and $\sigma < \lambda_\alpha^{(p)} < 1$. Then we have $\tilde{\lambda}_\alpha^{(p)}(\varepsilon_\alpha^{(p)}e_\alpha^{(p)}) = x_\alpha^{(p)}e_\alpha^p$, and hence $\exp(x_\alpha^{(p)}e_\alpha^{(p)}) = \hat{\lambda}_\alpha^{(p)}(\exp(\varepsilon_\alpha^{(p)}e_\alpha^{(p)}))$. Therefore we see from Lemma 1 that

$$\begin{aligned} d\big(e,\exp(x_\alpha^{(p)}e_\alpha^{(p)})\big) &= \lambda_\alpha^{(p)} d\big(e,\exp(\varepsilon_\alpha^{(p)}e_\alpha^{(p)}\big) \\ &\leq d\big(e,\exp(\varepsilon_\alpha^{(p)}e_\alpha^{(p)}\big) \\ &\leq d\big(e,\exp(e_\alpha^{(p)})\big) + d\big(e,\exp(-e_\alpha^{(p)})\big). \end{aligned}$$

Consequently if we put

$$M = \sum_{p=1}^{\mu}\sum_{\alpha=1}^{n_p}\big\{d\big(e,\exp(e_\alpha^{(p)})\big) + d\big(e,\exp(-e_\alpha^{(p)})\big)\big\},$$

we obtain $d(e,a) \leq M$, proving Lemma 2.

We are now in a position to prove Proposition 1.2. Take any $X \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$. Since $\varphi(\tilde{\lambda}(X)) = \lambda\varphi(X)$, we see from Lemma 1 that

$$d(e, \exp(x))\varphi(\tilde{\lambda}(X)) = d(e, \exp(\tilde{\lambda}X))\varphi(X). \qquad (*)$$

Now, S_δ denotes the sphere of radius δ centered at $0 \in \mathfrak{g}$. U being as in Lemma 2, we choose a sufficiently small δ so that $S_\delta \subset U$. By Lemma 2 we then have $d(e, \exp(X)) \leq M$ for any $X \in S_\delta$. Let m a positive constant such that $\varphi(X) \geq m$ for any $X \in S_\delta$. If we put $C_2 = M/m$, we see from $(*)$ that

$$d(e, \exp(\tilde{\lambda}(X))) \leq C_2\varphi(\tilde{\lambda}(X)), \quad \lambda \in \mathbb{R}_+,\ X \in S_\delta.$$

Furthermore there are positive constants M' and m' such that $\varphi(X) \leq M'$ and

$$d(e, \exp(X)) \geq \hat{d}(e, \exp(X)) \geq m'$$

for any S_δ. If we put $C_1 = m'/M'$, it follows from $(*)$ that

$$C_1\varphi(\tilde{\lambda}(X)) \leq d(e, \exp(\tilde{\lambda}(X))), \quad \lambda \in \mathbb{R}_+,\ X \in S_\delta.$$

As is easily verified, we have

$$\mathfrak{g} - \{0\} = \{\tilde{\lambda}(X) \mid \lambda \in \mathbb{R}_+,\ X \in S_\delta\}.$$

We have therefore shown that

$$C_1\varphi(X) \leq d(e, \exp(X)) \leq C_2\varphi(X), \quad X \in \mathfrak{g},$$

which proves Proposition 1.2.

By Proposition 1.2 we have the following two corollaries:

Corollary 1 (cf. Corollary to Theorem 5, Appendix in [5]) *Let K be a compact set of G. Then there is a positive constant C such that*

$$\hat{d}(a,b) \leq d(a,b) \leq C\hat{d}(a,b)^{1/\mu}, \quad a, b \in K.$$

In particular it follows that the topology of G defined by the distance function d coincides with the topology of the manifold G.

Corollary 2 *The distance function d is complete, that is, any Cauchy sequence in G with respect to d converges to a point of G.*

Proof of Corollary 1. There are positive constants ε' and C' such that

$$|\varphi(X)| \leq C'|X|^{1/\mu},$$

provided $|X| < \varepsilon'$. Furthermore there are positive constants ε'' and C'' such that

$$|X| \leq C''\hat{d}(e, \exp(X)),$$

provided $\hat{d}(e, \exp(X)) < \varepsilon''$. Therefore it follows from Proposition 1.2 that there are positive constants ε and C such that

$$d(e, a) \leq C\hat{d}(e, a)^{1/\mu},$$

provided $\hat{d}(e, a) < \varepsilon$. Since both d and $\hat{d}$ are left invariant, we have shown that

$$d(a, b) \leq C\hat{d}(a, b)^{1/\mu},$$

provided $\hat{d}(a, b) < \varepsilon$, from which follows easily Corollary 1.

Proof of Corollary 2. For any positive number R we define a subset K of $\mathfrak{g}$ by

$$K = \{X \in \mathfrak{g} \mid d(e, \exp(X)) < R\}.$$

Then it suffices to show that K is a compact set of $\mathfrak{g}$. By Proposition 1.2 we see that K is bounded in $\mathfrak{g}$, with respect to the norm $|\cdot|$. Furthermore K is closed in $\mathfrak{g}$, because the function $X \to d(e, \exp(X))$ is continuous by Corollary 2. We have thus shown that K is compact, proving the corollary.

1.4. An expression of the distance function d in terms of the energy functionals

For simplicity we set $\Omega(a,b) = \Omega(G,D,a,b)$.

Proposition 1.3 *In terms of the energy functionals the distance function d may be described as follows*

$$d(a,b)^2 = \underset{\omega\in\Omega(a,b)}{Inf}\, E(\omega), \quad a,b \in G.$$

Proof. This fact is clear in the case where $\mu = 1$ i.e., the euclidean FGLA is reduced to a euclidean vector space. Accordingly we may assume that $\mu \geq 2$, implying that $\dim \mathfrak{g}_1 \geq 2$.

For any $a, b \in G$ we denote by $\Omega_0(a,b)$ the set of all piece-wise regular integral curves $\omega(t)$ $(t \in [0,1])$ of (G,D). Since the differential system (G,D) satisfies condition (C_0), we have $\Omega_0(a,b) \neq \phi$. This being said, we set

$$d_0(a,b) = \underset{\omega\in\Omega_0(a,b)}{\mathrm{Inf}}\, L(\omega)\,.$$

Since $d(a,b) \leq d_0(a,b)$, $d_0(a,b)$ becomes a distance function on G. Here, we notice that Proposition 1.2. together with its corollary remains true when d is replaced by d_0. Especially it follows that the topology of G defined by d_0 coincides with the topology of the manifold G.

Now, let $\eta(a,b)$ denote the right hand side of the equality in the proposition. Then we have $d(a,b)^2 \leq \eta(a,b)$, because $L(\omega)^2 \leq E(\omega)$ for any $\omega \in \Omega(a,b)$. Let ε be any positive number. Then we can find $\omega \in \Omega(a,b)$ such that $L(\omega) < d(a,b) + \varepsilon/2$. By Lemma below we can also find $\omega' \in \Omega_0(a,b)$ such that $L(\omega') < L(\omega) + \varepsilon/2$. Hence we obtain $L(\omega') < d(a,b) + \varepsilon$. ω' being a piecewise regular curve, we may assume that $|d\omega'/dt|$ is constant. Therefore it follows that $\eta(a,b) \leq E(\omega') = L(\omega')^2 < (d(a,b) + \varepsilon)^2$, whence $\eta(a,b) \leq d(a,b)^2$. We have thus proved Proposition 1.3.

Lemma *Let ω be any path in $\Omega(a,b)$ and ε any positive number. Then there is $\omega' \in \Omega_0(a,b)$ such that*

$$|L(\omega') - L(\omega)| < \varepsilon.$$

Proof. Clearly we may assume that ω is smooth. We set

$$X(t) = \omega(t)^{-1}\frac{d\omega}{dt}(t), \quad t \in [0,1]$$

which defines a smooth curves of $\mathfrak{g}_1$. For any $A \in \mathfrak{g}$, we now set $X_A(t) = X(t) - A$, $(t \in [0,1])$, and consider the unique curve $\omega_A(t)$ $(t \in [0,1])$ of G such that $\omega_A(0) = e$ and

$$\omega_A(t)^{-1}\frac{d\omega_A}{dt}(t) = X_A(t), \quad t \in [0,1],$$

which is an integral curve of (G, D). Since $\dim \mathfrak{g}_1 \geq 2$, it follows from the Sard theorem that we can find a sequence $(A_i)_{i\geq 1}$ of vectors $\in \mathfrak{g}$, such that $A_i \notin X([0,1])$ for any $i \geq 1$ and A_i converges to 0, as i tends to $+\infty$. Then we have $X_{A_i}(t) \neq 0$ for any i and $t \in [0,1]$, and hence ω_{A_i} becomes a regular curve. Now, $\omega_A(1)$ is continuous with respect to the parameter A, from which follows that $d_0(\omega_{A_i}(1), b)$ converges to 0 as i tends to $+\infty$. Furthermore, since $|X_{A_i}(t) - X(t)| = |A_i|$, we see that $|L(\omega_{A_i}) - L(\omega)| \leq |A_i|$. Therefore it follows that for any positive number ε there is i such that

$$|L(\omega_{A_i}) - L(\omega)| < \frac{1}{2}\varepsilon,$$

$$d_0(\omega_{A_1}(1), b) < \frac{1}{2}\varepsilon.$$

By this last equality we can find $\theta \in \Omega_0(\omega_{A_i}(1), b)$ such that $L(\theta) < (1/2)\varepsilon$. Now, define a path $\omega' \in \Omega(a, b)$ as follows:

$$\omega'(t) = \omega_{A_i}(2t)\left(0 \leq t \leq \frac{1}{2}\right)$$

$$= \theta(2t-1)\left(\frac{1}{2} < t \leq 1\right).$$

Then we have $\omega' \in \Omega_0(a, b)$, and $|L(\omega') - L(\omega)| < \varepsilon$, proving the lemma.

2. The calculus of variations for the energy functionals

2.1. Euclidean FGLA of the 2nd kind

Here after a FGLA will always mean, that of the 2nd kind. Given a euclidean vector space V, $\mathrm{Skew}(V)$ will denote the vector space of all skew

symmetric endomorphisms.

Now, let $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ be a euclidean FGLA, and let $\mathfrak{g}_2^*$ be the dual space of $\mathfrak{g}_2$. For any $\theta \in \mathfrak{g}_2^*$ we define a skew symmetric endomorphism Λ_θ of $\mathfrak{g}_1$ by

$$\langle [x, y], \theta \rangle = \langle x, \Lambda_\theta y \rangle, \quad x, y \in \mathfrak{g}_1,$$

where the parenthesis $\langle\ ,\ \rangle$ in the left hand side stands for the duality between $\mathfrak{g}_2$ and $\mathfrak{g}_2^*$. Since $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1]$, we see that the assignment $\theta \to \Lambda_\theta$ gives an injective linear mapping of $\mathfrak{g}_2^*$ to $\mathrm{Skew}(\mathfrak{g}_1)$. Then we denote by $\mathscr{A}$ the image of $\mathfrak{g}_2^*$ by this linear mapping:

$$\mathscr{A} = \{\Lambda_\theta \mid \theta \in \mathfrak{g}_2^*\},$$

The subspace $\mathscr{A}$ of $\mathrm{Skew}(\mathfrak{g}_1)$ or the pair $(\mathfrak{g}_1, \mathscr{A})$ thus obtained will be called associated with the euclidean FGLA, $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$.

Conversely let V be a euclidean vector space, and $\mathscr{A}$ a subspace of $\mathrm{Skew}(V)$. Assuming that $\dim \mathscr{A} > 0$, we set

$$\mathfrak{g}_1 = V,\ \mathfrak{g}_2 = V^*,\ \text{and } \mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2,$$

and define a bracket operation $[\ ,\ ]$ in $\mathfrak{g}$ as follows:

(i) $[x_1, y_2] = [x_2, y_1] = [x_2, y_2] = 0$,

(ii) $[x_1, y_1] \in \mathfrak{g}_2$, and $\langle A, [x_1, y_1] \rangle = \langle x_1, A y_1 \rangle,\ A \in \mathscr{A}$,

where $x_i, y_i \in \mathfrak{g}_i$ $(i = 1, 2)$. Then it is clear that $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ becomes a euclidean FGLA, which is called associated with the subspace $\mathscr{A}$ of $\mathrm{Skew}(V)$ or the pair $(V, \mathscr{A})$.

In this way we have seen that there is a natural one-to-one correspondence between the euclidean FGLA, $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ and the pairs $(V, \mathscr{A})$ of euclidean vector spaces V and subspaces $\mathscr{A}$ of $\mathrm{Skew}(V)$ (up to the respective isomorphisms).

Example 1 Let $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ be a euclidean FGLA, and $\mathscr{A}$ the associated subspace of $\mathrm{Skew}(\mathfrak{g}_1)$. If we set $n = \dim \mathfrak{g}_1$, we have $\dim \mathfrak{g}_2 = \dim \mathscr{A} \leq \dim \mathrm{Skew}(\mathfrak{g}_1) = (1/2)n(n-1)$. Given an integer $n \geq 2$, we now denote by $\mathscr{E}_n$ (resp. by $\hat{\mathscr{E}}_n$) the class of all euclidean FGLA, $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, with

$\dim \mathfrak{g}_1 = n$ (resp. with $\dim \mathfrak{g}_1 = n$ and $\dim \mathfrak{g}_2 = (1/2)n(n-1)$). Then we have the following.

(i) To every euclidean FGLA, $\mathfrak{g}$ in the class $\mathscr{E}_n$ there is naturally associated a euclidean FGLA, $\hat{\mathfrak{g}}$, in the class $\hat{\mathscr{E}}_n$ together with a homomorphism of $\hat{\mathfrak{g}}$ onto $\mathfrak{g}$

(ii) Any two euclidean FGLA in the class $\hat{\mathscr{E}}_n$ are mutually isomorphic.

Indeed let V be a euclidean vector space and $\mathscr{A}$ a subspace of $\mathrm{Skew}(V)$. Let $\mathfrak{g}$ be the euclidean FGLA associated with the pair $(V, \mathscr{A})$. Furthermore setting $\hat{\mathscr{A}} = \mathrm{Skew}(\mathfrak{g}_1)$, let $\hat{\mathfrak{g}}$ be the euclidean FGLA associated with the pair $(V, \hat{\mathscr{A}})$. Then the identity mapping of V and the natural linear mapping of $\hat{\mathscr{A}}^*$ onto $\mathscr{A}^*$ give rise to a homomorphism of $\hat{\mathfrak{g}}$ onto $\mathfrak{g}$, from which follows immediately our assertions. In view of the facts above a euclidean FGLA in the class $\hat{\mathscr{E}}_n$ will be called universal.

Example 2 A FGLA, $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, is called a *strongly pseudo-convex FGLA*, if $\dim \mathfrak{g}_2 = 1$, and if the subspace $\mathfrak{g}_1$ is equipped with a complex structure I satisfying the following conditions:

(i) $[Ix, Iy] = [x, y], \quad x, y, \in \mathfrak{g}_1$,

(ii) $[Ix, x] \neq 0$ for any nonzero $x \in \mathfrak{g}_1$.

Now, let $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ be a strongly pseudo-convex FGLA. We fix a basis θ of $\mathfrak{g}_2^*$, and set

$$\langle x, y \rangle = \langle [Ix, y], \theta \rangle, \quad x, y \in \mathfrak{g}_1,$$

which gives a definite symmetric bilinear form on $\mathfrak{g}_1$. Accordingly we may assume that $\langle x, y \rangle$ is positive definite by replacing θ with $-\theta$ if necessary. We have thus seen that the strongly pseudo-convex FGLA becomes naturally a euclidean FGLA. Clearly the space $\mathscr{A}$ associated with the euclidean FGLA is spanned by I. Incidentally let (G, D) be the standard differential system associated with the underlying FGLA. Then we notice that D is a contact structure, because the condition "$x \in \mathfrak{g}_1$, $[x, \mathfrak{g}_1] = \{0\}$" implies $x = 0$. Furthermore let us denote by the same letter I the left invariant complex structure of D (as a vector bundle) induced by I. Then we remark that the triplet (G, D, I) gives a pseudo-complex manifold or a CR manifold, which is called the *standard strongly pseudo-convex manifold associated with the strongly pseudo-convex FGLA*.

2.2. Reduction of the energy functionals

For the rest of the present paper we are concerned with a fixed euclidean FGLA $\mathfrak{g}_0 = \mathfrak{g}_1 + \mathfrak{g}_2$ and present the notations in paragraph 1.2. Hereafter we identity the Lie group G with the product manifold $\mathfrak{g}_1 \times \mathfrak{g}_2$ through the diffeomorphism $(a_1, a_2) \to \exp(a_1 + a_2)$ of $\mathfrak{g}_1 \times \mathfrak{g}_2$ onto G. In terms of the product manifold the group multiplication of G is given by

$$(a_1, a_2) \cdot (b_1, b_2) = \left(a_1 + b_1, a_2 + b_2 + \frac{1}{2}[a_1, b_1] \right),$$

where $(a_1, a_2), (b_1, b_2) \in G$. Note that $e = (0, 0)$, e being the identity element of G. We denote by π the projection of G onto $\mathfrak{g}_1$. Then G is a fibred manifold over the euclidean space $\mathfrak{g}_1$ with projection π, and the differential system D defines a connection in the fibred manifold in a generalized sense so that the differential $d\pi$ of π, restricted to the system D, preserves the inner products.

In the present section we develop the calculus of variations for the energy functionals $E : \Omega(G, D, a, b) \to \mathbb{R}$. For this purpose it clearly suffices to deal with the energy functionals $E : \Omega(G, D, e, a) \to \mathbb{R}$. For simplicity the space $\Omega(G, D, e, a)$ will be denoted by $\Omega(G, D, a)$. Our task from now on is to show that through the projection $\pi : G \to \mathfrak{g}_1$ the variational problem for the energy functionals $E : \Omega(G, D, a) \to \mathbb{R}$ can be reduced to the variational problem with suitable additional conditions for the euclidean energy functionals.

Let $\omega(t)$ $(t \in I)$ be a smooth curve of G, I being an open interval, and set $\omega(t) = (\omega_1(t), \omega_2(t))$. Then we have

$$\omega(t)^{-1} \frac{d\omega}{dt}(t) = \frac{d\omega_1}{dt}(t) + \left(\frac{d\omega_2}{dt}(t) - \frac{1}{2} \left[\omega_1(t), \frac{d\omega_1}{dt}(t) \right] \right).$$

It follows that ω is an integral curve of (G, D) (or a horizontal curve of the connection defined by D), if and only if

$$\frac{d\omega_2}{dt} = \frac{1}{2} \left[\omega_1, \frac{d\omega_1}{dt} \right],$$

and further that if ω is an integral curve of (G, D),

$$\left|\frac{d\omega}{dt}(t)\right| = \left|\frac{d\omega_1}{dt}(t)\right|.$$

Now, let a be any point of G, and set $a = (a_1, a_2)$. Then we denote by $\Omega_1(\mathfrak{g}_1, a_1)$ the set of all piece-wise smooth paths $\gamma(t)$ $(t \in [0,1])$ of $\mathfrak{g}_1$ from 0 to a_1 and by E_1 the euclidean energy functional on $\Omega_1(\mathfrak{g}_1, a_1)$:

$$E_1(\gamma) = \int_0^1 \left|\frac{d\gamma}{dt}\right|^2 dt, \quad \gamma \in \Omega_1(\mathfrak{g}_1, a_1).$$

We also define a functional $F : \Omega_1(\mathfrak{g}_1, a_1) \to \mathfrak{g}_2$ by

$$F(\gamma) = \frac{1}{2}\int_0^1 \left[\gamma, \frac{d\gamma}{dt}\right] dt, \quad \gamma \in \Omega_1(\mathfrak{g}_1, a_1).$$

We then set

$$\tilde{\Omega}_1(\mathfrak{g}_1, a) = \{\gamma \in \Omega_1(\mathfrak{g}_1, a_1) \mid F(\gamma) = a_2\},$$

and denote by $\tilde{E}_1$ the restriction of E_1 to $\tilde{\Omega}_1(\mathfrak{g}_1, a)$.

From the discussion above it follows that (i) if $\omega \in \Omega(G, D, a)$, the image $\gamma = \pi \circ \omega$ of ω by π (or the $\mathfrak{g}_1$-component ω_1 of ω) is in $\tilde{\Omega}_1(\mathfrak{g}_1, a)$, (ii) the assignment $\omega \to \gamma$ gives a bijection of $\Omega(G, D, a)$ to $\tilde{\Omega}_1(\mathfrak{g}_1, a)$, and (iii)

$$E(\omega) = \tilde{E}_1(\gamma).$$

We have therefore shown that the variational problem for the energy functionals $E : \Omega(G, D, a) \to \mathbb{R}$ is reduced to that for the energy functionals $\tilde{E}_1 : \tilde{\Omega}_1(\mathfrak{g}_1, a) \to \mathbb{R}$.

2.3. The space $\tilde{\Omega}(\mathfrak{g}_1, a)$

For simplicity we set $\Omega_1 = \Omega_1(\mathfrak{g}_1, a)$ and $\tilde{\Omega}_1 = \tilde{\Omega}_1(\mathfrak{g}_1, a)$. As usual we shall think of Ω_1 as something like "an infinite dimensional manifold." In terms of a basis of $\mathfrak{g}_2$ let us express the functional F as a system (F_λ) of $\mathbb{R}$-valued functionals and similarly the vector a_2 as a system (a_2^λ) of constants. Then $\tilde{\Omega}_1$ is defined by the finite system of equations: $F^\lambda = a_2^\lambda$ and hence we may think of $\tilde{\Omega}_1$ as something like "a subvariety (possibly with singular points) of finite codimension of Ω_1".

Let γ be a path in Ω_1. By definition a variation of γ in Ω_1 is a mapping

$\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega_1$, ε being a small positive number, which satisfies the following conditions:

(a) $\bar{\alpha}(0) = \gamma$

(b) There is a subdivision $t_0 = 0 < t < \cdots < t_k = 1$ of the interval $[0, 1]$ such that the mapping $\alpha : (-\varepsilon, \varepsilon) \times [0, 1] \to \mathfrak{g}_1$ defined by $\alpha(s, t) = \bar{\alpha}(s)(t)$ is smooth on each strip $(-\varepsilon, \varepsilon) \times [t_{i-1}, t_i]$.

Let $\bar{\alpha}$ be a variation of γ in Ω_1. Then we define a mapping or rather vector field $X : [0, 1] \to \mathfrak{g}_1$ by

$$X(t) = \frac{\partial}{\partial t}\alpha(s, t)\bigg|_{s=0},$$

which is (continuous and) piece-wise smooth, and satisfies the condition:

$$X(0) = X(1) = 0.$$

The vector field X thus obtained may be naturally regarded as a vector field along the path γ, and will be called *induced* by the variation $\bar{\alpha}$ of γ.

This being said, we denote by $T_\gamma(\Omega_1)$ the vector space of all piece-wise smooth vector fields $X : [0, 1] \to \mathfrak{g}_1$ satisfying the condition: $X(0) = X(1) = 0$. Clearly every vector field X in $T_\gamma(\Omega_1)$ is integrable in Ω_1, that is, it is induced by some variation $\bar{\alpha}$ of γ in Ω_1. The vector space $T_\gamma(\Omega_1)$ thus defined is called the tangent vector space to Ω_1 at γ.

Now, suppose that γ is in $\tilde{\Omega}_1$. Then a variation $\bar{\alpha}$ of γ in Ω_1 is called a variation of γ in $\tilde{\Omega}_1$, if $\bar{\alpha}(s) \in \tilde{\Omega}_1$ for all s. Furthermore a vector field X in $T_\gamma(\Omega_1)$ is called *integrable* in $\tilde{\Omega}_1$, if it is induced by some variation $\bar{\alpha}$ of γ in $\tilde{\Omega}_1$.

We denote by $\Sigma(\gamma)$ the subspace of $\mathfrak{g}_1$ spanned by the vectors $d\gamma/dt(t)$ $(t \in [0, 1])$, which may be characterized as the smallest subspace of $\mathfrak{g}_1$ containing the entire path γ, because $\gamma(0) = 0$. Then we define a subspace $\mathscr{A}(\gamma)$ of $\mathscr{A}$ by

$$\begin{aligned}\mathscr{A}(\gamma) &= \left\{ A \in \mathscr{A} \,\middle|\, A\frac{d\gamma}{dt} = 0 \right\} \\ &= \{A \in \mathscr{A} \mid A\Sigma(\gamma) = \{0\}\}.\end{aligned}$$

These being prepared, we have the following two propositions.

Proposition 2.0 *Let γ be a path in $\tilde{\Omega}_1$, and $\bar{\alpha}$ a variation of γ in Ω_1. Then $\bar{\alpha}$ is a variation of γ in $\tilde{\Omega}_1$, if and only if*

$$\int_0^1 \left\langle \frac{\partial \alpha}{\partial s}, A\frac{\partial \alpha}{\partial t} \right\rangle dt = 0, \quad A \in \mathscr{A}.$$

Proof. We have

$$F(\bar{\alpha}(s)) = \frac{1}{2}\int_0^1 \left[\alpha, \frac{\partial \alpha}{\partial t}\right] dt,$$

whence

$$\frac{d}{ds}F(\bar{\alpha}(s)) = \frac{1}{2}\int_0^1 \left(\left[\frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t}\right] + \left[\alpha, \frac{\partial^2 \alpha}{\partial s \partial t}\right] \right) dt.$$

Since $\alpha(s,0) = 0$ and $\alpha(s,1) = a_1$, and

$$\frac{\partial}{\partial t}\left[\alpha, \frac{\partial \alpha}{\partial s}\right] = \left[\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial s}\right] + \left[\alpha, \frac{\partial^2 \alpha}{\partial t \partial s}\right],$$

it follows that

$$\frac{d}{ds}F(\bar{\alpha}(s)) = \int_0^1 \left[\frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t}\right] dt.$$

Clearly this means that

$$\left\langle \frac{d}{ds}F(\bar{\alpha}(s)), \theta \right\rangle = \int_0^1 \left\langle \frac{\partial \alpha}{\partial s}, \Lambda_\theta \frac{\partial \alpha}{\partial t} \right\rangle dt, \quad \theta \in \mathfrak{g}_2^*,$$

from which follows immediately Proposition 2.0.

Remark Incidentally we see from the equality just above that

$$\langle (dF)_\gamma(X), \theta \rangle = \int_0^1 \left\langle X, \Lambda_\theta \frac{d\gamma}{dt} \right\rangle dt, \quad X \in T_\gamma(\Omega_1).$$

Here, $(dF)_\gamma$ is the differential of F at γ, and is defined by

$$(dF)_\gamma(X) = \frac{d}{ds}F(\bar{\alpha}(s))\Big|_{s=0}, \quad X \in T_\gamma(\Omega_1),$$

$\bar{\alpha}$ being some variation of γ in Ω_1 which induces X.

Proposition 2.1 *Let γ be a path in $\tilde{\Omega}_1$, and X a vector field in $T_\gamma(\Omega_1)$. If X is integrable in $\tilde{\Omega}_1$, then it satisfies the following equations*:

$$(\text{ i }) \quad \int_0^1 \left\langle X, A\frac{d\gamma}{dt} \right\rangle dt = 0, \quad A \in \mathscr{A},$$

$$(\text{ ii }) \quad \int_0^1 \left\langle X, A\frac{dX}{dt} \right\rangle dt = 0, \quad A \in \mathscr{A}(\gamma).$$

Proof. Being integrable in $\tilde{\Omega}_1$, X is induced by some variation $\bar{\alpha}$ of γ in $\tilde{\Omega}_1$. Therefore (i) follows immediately from Proposition 2.0. By the same proposition we have

$$\int_0^1 \left(\left\langle \frac{\partial^2\alpha}{\partial s^2}, A\frac{\partial\alpha}{\partial t} \right\rangle + \left\langle \frac{\partial\alpha}{\partial s}, A\frac{\partial^2\alpha}{\partial s\partial t} \right\rangle \right) dt = 0, \quad A \in \mathscr{A}.$$

If we set

$$X_2(t) = \frac{\partial^2\alpha}{\partial s^2}(0,t),$$

it follows that

$$\int_0^1 \left(\left\langle X_2, A\frac{d\gamma}{dt} \right\rangle + \left\langle X, A\frac{dX}{dt} \right\rangle \right) dt = 0, \quad A \in \mathscr{A},$$

from which follows immediately (ii).

Let γ be a path in $\tilde{\Omega}_1$. Then we define the tangent space $T_\gamma(\tilde{\Omega}_1)$ to the subvariety $\tilde{\Omega}_1$ at γ to be the subset of $T_\gamma(\Omega_1)$ consisting of all vector fields $X \in T_\gamma(\Omega_1)$ which satisfy the equations (i) and (ii) in Proposition 2.1, which is a cone in the tangent vector space $T_\gamma(\Omega_1)$. We also define the ideal tangent space $IT_\gamma(\tilde{\Omega}_1)$ to $\tilde{\Omega}_1$ at γ to be the subset of $T_\gamma(\Omega_1)$ consisting of all vector fields $X \in T_\gamma(\Omega_1)$ which are integrable in $\tilde{\Omega}_1$. Then Proposition 2.1 means that $IT_\gamma(\tilde{\Omega}_1) \subset T_\gamma(\tilde{\Omega}_1)$.

The path γ is called a *non-singular path*, if $\mathscr{A}(\gamma) = \{0\}$, and a *singular path* otherwise. By the remark below Proposition 2.0 we know that γ is

a non-singular path, if and only if the differentials $(dF^\lambda)_\gamma$ of F^λ at γ are linearly independent. Note that if γ is a non-singular path, $T_\gamma(\tilde{\Omega}_1)$ is a subspace of the tangent vector space $T_\gamma(\Omega_1)$.

Now, we shall prove the following.

Theorem 2.2 *If γ is a non-singular path in $\tilde{\Omega}_1$, then every vector field X in $T_\gamma(\tilde{\Omega}_1)$ is integrable in $\tilde{\Omega}_1$: $IT_\gamma(\tilde{\Omega}_1) = T_\gamma(\tilde{\Omega}_1)$.*

In general let γ be a path in $\tilde{\Omega}_1$. We choose a complimentary subspace $\mathscr{A}'$ of $\mathscr{A}(\gamma)$ in $\mathscr{A}$. In the following the indices i, j, k range over the integers $1, 2, \ldots, \dim \mathscr{A}'$. We take a basis (A_i) of $\mathscr{A}'$ and a smooth function χ on [0,1] such that $\chi(t) \geq 0$ for all $t \in [0, 1]$ and the zeros of χ consist of $0, 1$ and the discontinuous points of $d\gamma/dt$. Then we define vector fields $H_i \in T_\gamma(\tilde{\Omega}_1)$ by

$$H_i = \chi A_i \frac{d\gamma}{dt}.$$

If we put

$$\rho_{ij} = \int_0^1 \left\langle H_i, A_j \frac{d\gamma}{dt} \right\rangle dt = \int_0^1 \chi \left\langle A_i \frac{d\gamma}{dt}, A_j \frac{d\gamma}{dt} \right\rangle dt,$$

we see that the matrix (ρ_{ij}) is symmetric and positive definite. Accordingly we may assume that $\rho_{ij} = \delta_{ij}$.

Now, take any vector field $X \in T_\gamma(\tilde{\Omega}_1)$, and consider a mapping $\bar{\alpha}$: $(-\varepsilon, \varepsilon) \to \Omega_1$, ε being a small positive number, of the following form:

$$\bar{\alpha}(s)(t) = \alpha(s, t) = \gamma(t) + sX(t) + \sum_i f_i(s) H_i(t),$$

where $s \in (-\varepsilon, \varepsilon)$, $t \in [0, 1]$, and $f_i(s)$ are smooth functions on $(-\varepsilon, \varepsilon)$ satisfying the initial condition

$$f_i(0) = \frac{df_i}{ds}(0) = 0.$$

Clearly $\bar{\alpha}$ is a variation of γ in Ω_1, and X is induced by $\bar{\alpha}$. By Proposition 2.0, $\bar{\alpha}$ is a variation of γ in $\tilde{\Omega}_1$, if and only if

$$\int_0^1 \left\langle \frac{\partial\alpha}{\partial s}, A_k \frac{\partial\alpha}{\partial t} \right\rangle dt = 0, \quad \int_0^1 \left\langle \frac{\partial\alpha}{\partial s}, A \frac{\partial\alpha}{\partial t} \right\rangle dt = 0, \quad A \in \mathscr{A}(\gamma).$$

The explanation above will be also available in the next paragraph.

Let us now proceed to the proof Theorem 2.2. Accordingly we assume that γ is a non-singular path, meaning that $\mathscr{A}(\gamma) = \{0\}$ or $\mathscr{A}' = \mathscr{A}$. We have

$$\left\langle \frac{\partial\alpha}{\partial s}, A_k \frac{\partial\alpha}{\partial t} \right\rangle = \left\langle X + \sum_i \frac{df_i}{ds} H_i, A_k \frac{d\gamma}{dt} + S A_k \frac{dX}{dt} + \sum_j f_j A_k \frac{dH_j}{dt} \right\rangle$$

and

$$\int_0^1 \left\langle X, A_k \frac{d\gamma}{dt} \right\rangle dt = 0.$$

Hence it follows that $\bar{\alpha}$ is a variation of γ in $\tilde{\Omega}_1$, if and only if (f_i) satisfies the following differential equation:

$$\sum_i \frac{df_i}{ds} \left(\delta_{ik} + a_{ik} s + \sum_j a_{ikj} f_j \right) + \left(b_k s + \sum_j b_{kj} f_j \right) = 0,$$

where a_{ik}, a_{ikj}, etc are given by

$$a_{ik} = \int_0^1 \left\langle H_i, A_k \frac{dX}{dt} \right\rangle dt, \quad a_{ikj} = \int_0^1 \left\langle H_i, A_k \frac{dH_j}{dt} \right\rangle dt,$$

$$b_k = \int_0^1 \left\langle X, A_k \frac{dX}{dt} \right\rangle dt, \qquad b_{kj} = \int_0^1 \left\langle X, A_k \frac{dH_j}{dt} \right\rangle dt.$$

This being said, we solve differential equation $(*)$ under the initial condition $f_i(0) = 0$, and denote by (f_i) its unique solution, which clearly satisfies $(df_i/ds)(0) = 0$. Therefore if we define the mapping $\bar{\alpha}$ by the use of the solution (f_i), we know that $\bar{\alpha}$ is a variation of γ in $\tilde{\Omega}_1$, completing the proof of Theorem 2.2.

2.4. Critical paths

Let us consider the energy functional $E_1 : \Omega_1 \to \mathbb{R}$.For any path γ in

Ω_1 we denote by $(dE_1)_\gamma$ the differential of E_1 at γ, which is the linear form on $T_\gamma(\tilde{\Omega}_1)$ defined by

$$(dE_1)_\gamma(X) = \frac{d}{ds}E_1(\bar{\alpha}(s))\bigg|_{s=0}, \quad X \in T_\gamma(\tilde{\Omega}_1),$$

where $\bar{\alpha}$ is some variation of γ in Ω_1 which induces X. As is well known, $(dE_1)_\gamma(X)$ may be described as follows:

$$\frac{1}{2}(dE_1)_\gamma(X) = -\int_0^1 \left\langle X, \frac{d^2\gamma}{dt^2} \right\rangle dt + \sum_{0<t<1} \left\langle X(t), \Delta_t \frac{d\gamma}{dt} \right\rangle$$

(see [3, Section 12]). Here, $\Delta_t(d\gamma/dt)$ stands for the discontinuity of $d\gamma/dt$ at t:

$$\Delta_t \frac{d\gamma}{dt} = \frac{d\gamma}{dt}(t-) - \frac{d\gamma}{dt}(t+).$$

Let us new consider the energy functional $\tilde{E}_1 : \tilde{\Omega}_1 \to \mathbb{R}$, being the restriction of E_1 to $\tilde{\Omega}_1$. For any path γ in $\tilde{\Omega}_1$ we denote by $(d\tilde{E}_1)_\gamma$ the restriction of $(dE_1)_\gamma$ to $IT_\gamma(\tilde{\Omega}_1)$, which is called the differential of $\tilde{E}_1$ at γ. Now, the path γ is called a *critical path* for $\tilde{E}_1$, if $(d\tilde{E}_1)_\gamma = 0$.

The path γ is called a *reduced geodesic path* if it is smooth, and satisfies a differential equation of the following form:

$$\frac{d^2\gamma}{dt^2} = A\frac{d\gamma}{dt},$$

Note that a reduced geodesic path γ is non-singular, if and only if A is uniquely determined by γ where $A \in \mathscr{A}$. Then we assert that a geodesic path γ is a critical path for $\tilde{E}_1$. Indeed we have

$$(d\tilde{E}_1)_\gamma(X) = -\int_0^1 \left\langle X, A\frac{d\gamma}{dt} \right\rangle dt = 0, \quad X \in IT_\gamma(\tilde{\Omega}_1),$$

because $IT_\gamma(\tilde{\Omega}_1) \subset T_\gamma(\tilde{\Omega}_1)$.

Theorem 2.3 *Let γ be a non-singular path in $\tilde{\Omega}_1$. Then γ is a critical path for the energy functional $\tilde{E}_1$, if and only if it is a reduced geodesic path.*

Proof. Assume that γ is a critical path for $\tilde{E}_1$. We take any $Y \in T_\gamma(\Omega_1)$, and set

$$\tilde{Y} = Y - \sum_i \left(\int_0^1 \left\langle Y, A_i \frac{d\gamma}{dt} \right\rangle dt \right) H_i.$$

Clearly we have

$$\int_0^1 \left\langle \tilde{Y}, A_i \frac{d\gamma}{dt} \right\rangle dt = 0,$$

meaning that $\tilde{Y}$ is in $T_\gamma(\tilde{\Omega}_1)$. Since $IT_\gamma(\tilde{\Omega}_1) = T_\gamma(\tilde{\Omega}_1)$ by Theorem 2.2, it follows that

$$\frac{1}{2}(d\tilde{E}_1)_\gamma(\tilde{Y}) = -\int_0^1 \left\langle \tilde{Y}, \frac{d^2\gamma}{dt^2} \right\rangle dt + \sum_t \left\langle \tilde{Y}, \Delta_t \frac{d\gamma}{dt} \right\rangle = 0.$$

We have

$$\int_0^1 \left\langle \tilde{Y}, \frac{d^2\gamma}{dt^2} \right\rangle dt = \int_0^1 \left\langle Y, \frac{d^2\gamma}{dt^2} - A\frac{d\gamma}{dt} \right\rangle dt,$$

where

$$A = \sum_i d_i A_i \quad \text{with} \quad d_i = \int_0^1 \left\langle H_i, \frac{d^2\gamma}{dt^2} \right\rangle dt.$$

We have

$$\sum_t \left\langle \tilde{Y}, \Delta_t \frac{d\gamma}{dt} \right\rangle = \sum_t \left\langle Y, \Delta_t \frac{d\gamma}{dt} \right\rangle.$$

Accordingly we have shown that

$$-\int_0^1 \left\langle Y, \frac{d^2\gamma}{dt^2} - A\frac{d\gamma}{dt} \right\rangle dt + \sum_t \left\langle Y, \Delta_t \frac{d^2\gamma}{dt} \right\rangle = 0.$$

As is well known, it follows that

$$\frac{d^2\gamma}{dt^2} = A\frac{d\gamma}{dt}, \quad \Delta_t \frac{d\gamma}{dt} = 0,$$

from which follows especially that γ is smooth. We have thus proved Theorem 2.3.

2.5. Supplements to Theorems 2.2 and 2.3

In Theorems 2.2 and 2.3 we have assumed that the path γ in $\tilde{\Omega}_1$ is a non-singular path. There naturally arises the problem of whether the theorems remain true or not in the case when γ is a singular path. First of all we study this problem in the case where $\mathscr{A}$ is abelian, that is, $AB = BA$ for all $A, B \in \mathscr{A}$. Clearly this condition is satisfied, if $\dim \mathfrak{g}_2 = 1$. It should be noted that if $\dim \mathfrak{g}_2 \geq 2$, the condition depends on the choice of the inner product on $\mathfrak{g}_1$.

Proposition 2.4 *Assume that $\mathscr{A}$ is abelian, and let γ be a path in $\tilde{\Omega}_1$.*

(1) *Every vector field X in $T_\gamma(\tilde{\Omega}_1)$ is integrable in $\tilde{\Omega}_1$: $IT_\gamma(\tilde{\Omega}_1) = T_\gamma(\tilde{\Omega}_1)$.*
(2) *γ is a critical path for $\tilde{E}_1$, if and only if it is a reduced geodesic path.*

Proof. We shall use the notations as in the proof of Theorem 2.2. Since $\mathscr{A}$ is abelian we have $AH_i = 0$ for all $A \in \mathscr{A}(\gamma)$.

(1) We take any $X \in T_\gamma(\tilde{\Omega}_1)$ and consider the mapping $\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega_1$ in that proof which is determined by X and arbitrary functions $f_i(s)$ with $f_i(0) = (df_i/ds)(0) = 0$. Then we have

$$\int_0^1 \left\langle \frac{\partial \alpha}{\partial s}, A\frac{\partial \alpha}{\partial t} \right\rangle dt = 0, \quad A \in \mathscr{A}(\gamma)$$

because $AH_i = 0$, and

$$\int_0^1 \left\langle X, A\frac{dX}{dt} \right\rangle dt = 0.$$

Hence it follows that $\bar{\alpha}$ is a variation of γ in $\tilde{\Omega}_1$, if and only if

$$\int_0^1 \left\langle \frac{\partial \alpha}{\partial s}, A_k\frac{\partial \alpha}{\partial t} \right\rangle dt = 0.$$

Therefore, reasoning in the same manner as before, we find that X is inte-

grable in $\tilde{\Omega}_1$, proving the first assertion.

(2) Assume that γ is a critical path for $\tilde{E}_1$. We denote by $C_\gamma(\Omega_1)$ the cone of $T_\gamma(\Omega_1)$ consisting of all $Y \in T_\gamma(\Omega_1)$ such that

$$\int_0^1 \left\langle Y, A\frac{dY}{dt} \right\rangle dt = 0, \quad A \in \mathscr{A}(\gamma).$$

We take any $Y \in C_\gamma(\Omega_1)$, and set

$$\tilde{Y} = Y - \sum_i \left(\int_0^1 \left\langle Y, A_i \frac{d\gamma}{dt} \right\rangle dt \right) H_i.$$

Then we have

$$\int_0^1 \left\langle \tilde{Y}, A_i \frac{d\gamma}{dt} \right\rangle dt = 0, \quad \int_0^1 \left\langle \tilde{Y}, A\frac{d\tilde{Y}}{dt} \right\rangle dt - 0, \quad A \in A(\gamma),$$

meaning that $\tilde{Y} \in T_\gamma(\tilde{\Omega}_1)$.

Furthermore we have $IT_\gamma(\tilde{\Omega}_1) = T_\gamma(\tilde{\Omega}_1)$, as we have just seen. Therefore, reasoning in the same manner as in the proof of Theorem 2.3, we obtain

$$-\int_0^1 \left\langle Y, \frac{d^2\gamma}{dt^2} - A\frac{d\gamma}{dt} \right\rangle dt + \sum_t \left\langle Y, \Delta_t \frac{d\gamma}{dt} \right\rangle = 0,$$

where $Y \in C_\gamma(\Omega_1)$, and A is given by

$$A = \sum_i d_i A_i \quad \text{with} \quad d_i = \int_0^1 \left\langle H_i, \frac{d^2\gamma}{dt^2} \right\rangle dt.$$

Now, let f be any piece-wise smooth function on $[0,1]$ such that $f(0) = f(1) = 0$, and let x be any vector of $\mathfrak{g}_1$. Clearly the vector field fx is in $C_\gamma(\Omega_1)$ and $T_\gamma(\Omega_1)$ is spanned by the vector fields of this form. Accordingly the equality above for $Y \in C_\gamma(\Omega_1)$ holds likewise for any $Y \in T_\gamma(\Omega_1)$. Hence it follows that

$$\frac{d^2\gamma}{dt^2} = A\frac{d\gamma}{dt}, \quad \Delta_t \frac{d\gamma}{dt} = 0,$$

proving the second assertion.

We shall now explain another result in this paragraph, which is due to Mr. Moriyuki Honma who was a postgraduate student of Hokkaido University.

Let γ be a path in $\tilde{\Omega}_1$. We denote by Σ_γ the subspace of $T_\gamma(\Omega_1)$ consisting of all vector fields X in $T_\gamma(\Omega_1)$ which take values in $\Sigma(\gamma)$. Clearly the intersection $\Sigma_\gamma \cap T_\gamma(\tilde{\Omega}_1)$ is composed of all $X \in \Sigma_\gamma$ which satisfy the equation

$$\int_0^1 \left\langle X, A\frac{d\gamma}{dt} \right\rangle dt = 0, \quad A \in \mathscr{A}.$$

Remark As we have remarked, $\Sigma(\gamma)$ in the smallest subspace of $\mathfrak{g}_1$ containing the path γ. This being said, we denote by $\Omega_1(\gamma)$ the subset of Ω_1 consisting of all paths $\gamma' \in \Omega_1$ which are contained in $\Sigma(\gamma)$. Then $\Omega_1(\gamma)$ may be regarded as an infinite dimensional submanifold of Ω_1 and Σ_γ as the tangent vector space to $\Omega_1(\gamma)$ at γ. Honma's basic idea of the problem is solely to consider variations of γ in $\Omega_1(\gamma) \cap \tilde{\Omega}_1$.

For any $A \in \mathscr{A}$ we now define a skew-symmetric endomorphism $\bar{A}$ of $\Sigma(\gamma)$ by

$$\langle \bar{A}x, y\rangle = \langle Ax, y\rangle, \quad x, y \in \Sigma(\gamma).$$

It is clear that if γ satisfies a differential equation of the form as in Theorem 2.3, A leaves the subspace $\Sigma(\gamma)$ of $\mathfrak{g}_1$ invariant, γ satisfies the following differential equation:

$$\frac{d^2\gamma}{dt^2} = \bar{A}\frac{d\gamma}{dt}.$$

These being prepared, Honma's result may be stated as follows, which above all assumes the regularity of a singular critical path.

Proposition 2.5 (1) *Every vector filed X in $\Sigma_\gamma \cap T_\gamma(\tilde{\Omega}_1)$ is integrable in $\tilde{\Omega}_1$.*

(2) *If γ is a critical path for $\tilde{E}_1$, then γ is smooth and satisfies a differential equation of the following form*:

$$\frac{d^2\gamma}{dt^2} = \bar{A}\frac{d\gamma}{dt},$$

where $A \in \mathscr{A}$.

Corollary *Assume that the euclidean FGLA,* $\bar{\mathfrak{g}} = \mathfrak{g}_1 + \mathfrak{g}_2$, *is universal and let* γ *be any path in* $\tilde{\Omega}_1$. *Then* γ *is a critical path for* $\tilde{E}_1$, *if and only if it is a reduced geodesic path.*

Indeed the assumption means that $\mathscr{A} = \mathrm{Skew}(\mathfrak{g}_1)$. Hence it follows that if $A \in \mathscr{A}$, $\bar{A}$ is likewise in $\mathscr{A}$. Here $\bar{A}$ should be naturally extended to a skew-symmetric endomorphism of $\mathfrak{g}_1$. Therefore the corollary is an immediate consequence of the proposition. Let us now make some remarks on the proof of Proposition 3.5. We denote by $\mathscr{A}_0$ the kerned of the linear mapping $A \to \bar{A}$ which consists of all $A \in \mathscr{A}$ sending $\Sigma(\gamma)$ to its orthogonal complement in $\mathfrak{g}_1$ so that it vanished on the orthogonal complement of $\Sigma(\gamma)$ in $\mathfrak{g}_1$. We then choose a complementary subspace $\mathscr{A}_0'$ in $\mathscr{A}$. In the following the indices i, j range over the integers $1, 2, \ldots, \dim \mathscr{A}_0'$. Then we take a basis (A_i) of $\mathscr{A}_0'$ and define vector fields $\bar{H}_i \in T_\gamma(\Omega_1)$ by

$$\bar{H}_i = \chi \bar{A}_i \frac{d\gamma}{dt}.$$

If we put

$$\bar{\rho}_{ij} = \int_0 \left\langle \bar{H}_i, A_j \frac{d\gamma}{dt} \right\rangle = \int_0^1 \chi \left\langle \bar{A}_i \frac{d\gamma}{dt}, \bar{A}_j \frac{d\gamma}{dt} \right\rangle dt,$$

we see that the matrix $(\bar{\rho}_{ij})$ is symmetric and positive definite. Hence we may assume that $\bar{\rho}_{ij} = \delta_{ij}$.

Now, the proof of the proposition can be carried out in the same manner as in the proofs of Theorems 2.2 and 2.3 by considering the vector fields $\bar{H}_i$ instead of the vector fields H_i. The details are left to the readers as an exercise.

3. Geodesics

3.1. Geodesics and the exponential mapping

In view of the discussion in paragraph 2.4 we first give the following definition. A smooth curve $\gamma(t)$ $(t \in \mathbb{R})$ of $\mathfrak{g}_1$ is called a *reduced geodesic*

of the standard system (G, D, g) if it satisfies a differential equation of the following form:

$$\frac{d^2\gamma}{dt^2} = A\frac{d\gamma}{dt}, \tag{rg}$$

where $A \in \mathscr{A}$.

The reduced geodesic is called *non-singular* if A is uniquely determined by γ, and *singular* otherwise.

Let us now recall that D defines a connection in the fibred manifold G over $\mathfrak{g}_1$ with projection π. Then a geodesic of the standard system (G, D, g) is simply defined to be a horizontal lift ω of a reduced geodesic γ. Accordingly let $\omega(t)$ $(t \in \mathbb{R})$ be a smooth curve of G, and set $\omega(t) = (\omega_1(t), \omega_2(t))$. Then ω is a geodesic, if and only if it satisfies a system of differential equations of the following form:

$$(g.1) \quad \begin{cases} \dfrac{d^2\omega}{dt^2} = A\dfrac{d\omega_1}{dt}, \\ \dfrac{d\omega_2}{dt} = \dfrac{1}{2}\left[\omega_1, \dfrac{d\omega_1}{dt}\right], \end{cases}$$

where $A \in \mathscr{A}$.

A geodesic ω is called non-singular (resp. singular) if so is the image $\gamma = \pi \circ \omega_1$ of ω by π or the $\mathfrak{g}_1$-component ω_1 of ω, being a reduced geodesic.

Given an element A of $\mathscr{A}$, a geodesic ω will be called *corresponding to* A, if the defining equation $(g, 1)$ for ω is considered with respect to the given A. Furthermore let (x, A) be a point of $\mathfrak{g}_1 \times \mathscr{A}$. Then there is a unique geodesic ω corresponding to A satisfying the following initial condition:

$$\omega(0) = e, \quad \frac{d\omega_1}{dt}(0) = x,$$

which will be called *corresponding to* (x, A).

Remark 1 (1) If ω is a geodesic, $|(d\omega/dt)(t)| = |(d\omega_1/dt)(t)|$ is constant. Indeed we have

$$\frac{d}{dt}\left\langle \frac{d\omega_1}{dt}, \frac{d\omega_1}{dt} \right\rangle = 2\left\langle \frac{d^2\omega_1}{dt^2}, \frac{d\omega_1}{dt} \right\rangle = 2\left\langle A\frac{d\omega_1}{dt}, \frac{d\omega_1}{dt} \right\rangle = 0,$$

because A is a skew-symmetric endomorphism of $\mathfrak{g}_1$.

(2) The geodesics corresponding to a given A are invariant under the left translations of the Lie group G. Indeed let a be an element of G, and $\omega(t)$ $(t \in \mathbb{R})$ be a smooth curve of G. Set $a = (a_1, a_2)$, $\omega(t) = (\omega_1(t), \omega_2(t))$ and $a\omega(t) = (\bar{\omega}_1(t), \bar{\omega}_2(t))$. Then we have

$$\bar{\omega}_1(t) = a_1 + \omega_1(t), \quad \bar{\omega}_2(t) = a_2 + \omega_2(t) + \frac{1}{2}[a_1, \omega_1(t)],$$

from which follows that

$$\frac{d^2\bar{\omega}}{dt^2} - A\frac{d\bar{\omega}_1}{dt} = \frac{d^2\omega}{dt^2} - A\frac{d\omega_1}{dt},$$
$$\frac{d\bar{\omega}_2}{dt} - \frac{1}{2}\left[\bar{\omega}, \frac{d\bar{\omega}}{dt}\right] = \frac{d\omega_2}{dt} - \frac{1}{2}\left[\omega_1, \frac{d\omega_1}{dt}\right].$$

This proves our assertion.

Now, let (x, A) be any point of $\mathfrak{g}_1 \times \mathscr{A}$, and ω the corresponding geodesic. Let us consider the exponential mapping $B \to e^B$ of $\mathrm{Skew}(\mathfrak{g}_1)$ to the orthogonal group $O(\mathfrak{g}_1)$ of $\mathfrak{g}_1$,

$$e^B = \sum_{m=0}^{\infty} \frac{1}{m!} B^m, \quad B \in \mathrm{Skew}(\mathfrak{g}_1).$$

Then we have

$$\frac{d\omega_1}{dt}(t) = e^{tA}x.$$

Hence the curves ω_1 and ω_2 may be described as follows:

$$\omega_1(t) = \frac{e^{tA} - 1}{A}x,$$
$$\omega_2(t) = \frac{1}{2}\int_0^t \left[\frac{e^{\tau A} - 1}{A}x, e^{\tau A}x\right] d\tau.$$

This being said, we define a mapping Φ_1 of $\mathfrak{g}_1 \times A$ to $\mathfrak{g}_1$ by

$$\Phi_1(x, A) = \omega_1(1) - \frac{e^A - 1}{A}x,$$

a mapping Φ_2 of $\mathfrak{g}_1 \times \mathscr{A}$ to $\mathfrak{g}_2$ by

$$\Phi_2(x, A) = \omega_2(1) = \frac{1}{2}\int_0^1 \left[\frac{e^{\tau A} - 1}{A}x, e^{\tau A}x\right] d\tau,$$

and a mapping Φ of $\mathfrak{g}_1 \times A$ to G by

$$\Phi(x, A) = \omega(1) = (\Phi_1(x, A), \phi_2(x, A)).$$

Clearly Φ is a real analytic mapping.

The mapping Φ obtained in this manner is called the exponential mapping of $\mathfrak{g}_1 \times \mathscr{A}$ to G.

Remark 2 (1) For each $(x, A) \in \mathfrak{g}_1 \times \mathscr{A}$ the curve $t \to \Phi(tx, tA)$ of G is a geodesic corresponding to (x, A).

(2) For each $(x, A) \in \mathfrak{g}_1 \times \mathscr{A}$ the following equality holds:

$$\Phi((s + t)x, (s + t)A) = \Phi(sx, sA) \cdot \Phi(te^{sA}x, tA),$$

where $s, t, \in \mathbb{R}$, and the dot "$\cdot$" in the right hand side stands for the group multiplication.

3.2. The geodesic flow

We denote by V the left invariant differential system on G induced by the subspace $\mathfrak{g}_2$ of $\mathfrak{g}$, which may be characterized as the vertical tangent bundle of the fibred manifold G over $\mathfrak{g}_1$ with projection π:

$$V = \{X \in T(G) \mid d\pi(X) = 0.\}$$

Clearly $T(G) = D + V$ (direct sum), and V is completely integrable. Then we denote by V^* the vector bundle dual to V, and by $\tilde{T}$ the Whitney sum $D + V^*$ of D and V^*.

Now the tangent bundle $T(G)$ of G may be identified with the product manifold $G \times \mathfrak{g}$ as vector bundles. Indeed, the mapping which maps energy $(a, X) \in G \times \mathfrak{g}$ to $dL_a X \in T(G)$ gives a diffeomorphism of $G \times \mathfrak{g}$ onto $T(G)$, L_a being the left translation of the Lie group corresponding to a. On the

Lie basis of this identification V^* may identified with the product manifold $G\times\mathfrak{g}_2^*$, D with the product manifold $G\times\mathfrak{g}_1$, and $\tilde{T}$ with the product manifold $G\times(\mathfrak{g}_1+\mathfrak{g}_2^*)$.

Let us now consider a system of differential equations for curves $(\omega(t),\theta(t))$ of V^*:

$$(g.2)\quad \begin{cases} \dfrac{d^2\omega_1}{dt^2} = -\Lambda_\theta \dfrac{d\omega_1}{dt}, \\ \dfrac{d\omega_2}{dt} = \dfrac{1}{2}\left[\omega_1, \dfrac{d\omega_1}{dt}\right], \\ \dfrac{d\theta}{dt} = 0. \end{cases}$$

Here, we set $\omega(t)=(\omega_1(t),\omega_2(t))$, and $Lambda_\theta$ means the function $t\to\Lambda_{\theta(t)}(\in\mathscr{A})$. If we put $\xi = d\omega_1/dt$ in system $(g.2)$, we have the following system of differential equations of the first order:

$$(g.3)\quad \begin{cases} \dfrac{d\omega_1}{dt} = \xi \\ \dfrac{d\omega_2}{dt} = \dfrac{1}{2}[\omega_1,\xi], \\ \dfrac{d\xi}{dt} = -\Lambda_\theta\xi, \\ \dfrac{d\theta}{dt} = 0. \end{cases}$$

If $(\omega(t),\theta(t))$ is a solution of system $(g.2)$, $\theta(t)$ is constant, and $\omega(t)$ is part of a geodesic corresponding to Λ_{θ_0}, where $\theta_0=\theta(t)$. Then we see that given $\theta_0\in\mathfrak{g}_2^*$ the projection of V^* onto G induces a one-to-one correspondence between the maximal solutions $(\omega(t),\theta(t))$ of system $(g.2)$ with $\theta(t)=\theta_0$ and the geodesics $\omega(t)$ corresponding to Λ_{θ_0}. Furthermore a solution $(\omega(t),\xi(t),\theta(t))$ of system $(g.3)$ may be regarded as a curve of $\tilde{T}$ through the identification $(\omega(t),\xi(t),\theta(t)) = (\omega(t),\xi(t)+\theta(t))$. Then we see that the projection of $\tilde{T}$ onto V^* induces a one-to-one correspondence between the maximal solutions $(\omega(t),\xi(t)+\theta(t))$ of system $(g.3)$ and the maximal solutions $(\omega(t),\theta(t))$ of system $(g.2)$.

For each $t \in \mathbb{R}$ we now define a mapping Φ_t of $\tilde{T}$ to itself as follows: Take any $(a, \tau_0) \in \tilde{T}$ and set $\tau_0 = \xi_0 + \theta_0$, where $\xi_0 \in \mathfrak{g}_1$, and $\theta_0 \in \mathfrak{g}_2^*$. Let $(\omega(t), \xi(t) + \theta(t))$ be the unique maximal solution of system $(g.3)$ which satisfies the initial condition:

$$\omega(0) = a, \quad \xi(0) = \xi_0, \quad \text{and} \quad \theta(0) = \theta_0,$$

Then $\Phi_t(a, \tau_0)$ is defined by

$$\Phi_t(a, \tau_0) = (\omega(t), \xi(t) + \theta(t)).$$

Clearly the family Φ_t gives a one-parameter group of transformations of $\tilde{T}$, which is called the geodesic flow associated with the riemannian differential system (G, D, g). We denote by $\mathscr{X}$ the vector field on $\tilde{T}$ induced by the geodesic flow, which is called the *spray* associated with the system (G, D, g). We shall give a concrete description of the spray in terms of a canonical coordinate system of $\tilde{T}$.

For this purpose we first prepare some notations, which will be also utilized in the next paragraph. In the following the indices i, j, k range over the integers $1, \ldots \dim \mathfrak{g}_1$, and the indices α, β over the integers $1, \ldots \dim \mathfrak{g}_2$. Let (e_i) (resp. (f_α)) be a basis of $\mathfrak{g}_1$ (resp of $\mathfrak{g}_2$), and (e^i) (resp of (f^α)) its dual basis. Set

$$[e_i, e_j] = \sum_\alpha C_{ij}^\alpha f_\alpha.$$

(a) We denote by (x^i) (resp by (y^α)) the coordinate system of $\mathfrak{g}_1$ (resp of $\mathfrak{g}_2$) associated with the basis (e_i) (resp. (f_α)). Then (x^i, y^α) gives a coordinate system of G.

(b) We define 1-forms η^α on G by

$$\eta^\alpha = dy^\alpha - \frac{1}{2} \sum_{i,j} C_{ij}^\alpha x^i dx^j.$$

Then (dx^i, η^α) gives a basis of the space $\mathscr{L}^*(G)$ of Maurer- Cartan forms on G which corresponds to the basis (e^i, f^α) of $\mathfrak{g}^*$. Let us now consider the cotangent bundle $T^*(G)$ of G. Since $T(G) = D + V$ (direct sum) we have $T^*(G) = D^* + V^*$ (direct sum). Clearly (dx^i) (resp. (η^α)) gives a global

moving frame of D^* (resp. of V^*).

(c) We define vector fields X_i on G by

$$X_i = \frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{j,\alpha} C^\alpha_{ji} x_j \frac{\partial}{\partial y^\alpha}.$$

Then $(X_i, \partial/\partial y^\alpha)$ gives a basis of the Lie algebra $\mathscr{L}(G)$ of left invariant vector fields on G which corresponds to the basis (e_i, f_α) of $\mathfrak{g}$. Clearly (X_i) (resp. $(\partial/\partial y^\alpha)$) give a global moving frame of D (resp. of V).

(d) We denote by $(x^i, y^\alpha, u^j, v_\beta)$ the coordinate system of $\tilde{T}$ associated with the coordinate system (x^i, y^α) of G and the global moving frame (X_i, η^α) of $\tilde{T}$:

$$\tilde{d} = \sum_i u^i(\tilde{d})(X_i)_a + \sum_\alpha v_\alpha(\tilde{d})(\eta^\alpha)_a, \quad \tilde{d} \in \tilde{T},$$

where $a = \rho(\tilde{d})$, ρ being the projection of $\tilde{T}$ onto G.

We are now in a position to describe the spray in terms of the canonical coordinate system. Let us consider the dependent variables ω_1, ω_2, ξ and θ in system $(g, 3)$, and set

$$\omega_1 = \sum_i x^i e_i, \quad \omega_2 = \sum_\alpha y^\alpha f_\alpha,$$
$$\xi = \sum_i u^i e_i, \qquad \theta = \sum_\alpha v_\alpha f^\alpha.$$

Then system $(g.3)$ may be rewritten as follows:

$$\begin{cases} \dfrac{dx^i}{dt} = u^i, \\ \dfrac{dy^\alpha}{dt} = \dfrac{1}{2}\displaystyle\sum_{i,j} C^\alpha_{ij} x^i u^j, \\ \dfrac{du^i}{dt} = -\displaystyle\sum_{\alpha,j} C^\alpha_{ij} v_\alpha u^j, \\ \dfrac{dv_\alpha}{dt} = 0 \end{cases}$$

It follows immediately that

$$(S.1)\quad \begin{cases} \mathscr{X}x^i = u^i \\ \mathscr{X}y^\alpha = \dfrac{1}{2}\displaystyle\sum_{i,j} C_{ij}^\alpha x^i u^j, \\ \mathscr{X}u^i = -\displaystyle\sum_{\alpha,j} C_{ij}^\alpha v_\alpha u^j \\ \mathscr{X}v_\alpha = 0, \end{cases}$$

giving the desired description of the spray $\mathscr{X}$.

3.3. The spray and the hamiltonian mechanics

Let us consider the vector bundle $\tilde{T} = D + V^*$ and the cotangent bundle $T^*(G) = D^* + V^*$.

First of all the inner product $\mathfrak{g}$ of D naturally gives rise to an isomorphism φ of D onto D^*:

$$\langle d, \varphi(\delta)\rangle = g(d, \delta), \quad d, \delta \in D_a,\ a \in G.$$

Then the isomorphism φ together with the identity mapping naturally gives rise to an isomorphism of $\tilde{T}$ onto $T^*(G)$, which is denoted by the same letter φ.

Next, we denote by Θ the canonical symplectic form on $T^*(G)$:

$$\Theta = -d\psi.$$

Here, ψ is the canonical 1-form on $T^*(G)$, and is defined by

$$\psi(X) = \langle d\varpi(X), d^*\rangle, \quad X \in T_{d^*}(T^*(G)), \quad d^* \in T^*(G),$$

ϖ being the projection of $T^*(G)$ onto G. For any smooth function f on $T^*(G)$ there is a unique vector field $\mathscr{X}_f$ on $T^*(G)$ such that

$$\frac{1}{2}df = \mathscr{X}_f \rfloor \Theta,$$

that is,

$$\frac{1}{2}df(Y) = \Theta(\mathscr{X}_f, Y), \quad Y \in T_{d^*}(T^*(G)), \quad d^* \in T^*(G),$$

which is called the *Hamiltonian vector field* corresponding to f.

Finally we define the energy function E on D^* by

$$E(d^*) = |d^*|^2,$$

where $|d^*|$ stands for the norm of d^* with respect to the inner product g of D. The function E may be naturally confounded with a function on $T^*(G)$ through the projection of $T^*(G)$ onto D^*, which is also called the energy function. These being prepared, we shall prove the following.

Proposition 3.1 *The differential $d\varphi$ of the isomorphism φ of $\tilde{T}$ onto $T^*(G)$ sends the spray $\mathscr{X}$ to the Hamiltonian vector field $\mathscr{X}_E$ corresponding to the energy function E on $T^*(G)$.*

Let $(x^i, y^\alpha, u_j, v_\beta)$ be the coordinate system of $T^*(G)$ associated with the coordinate system (x^i, y^α) of G and the moving frame (dx^i, η^α) of $T^*(G)$:

$$d^* = \sum_i u_i(d^*)(dx^i)_a + \sum_\alpha v_\alpha(d^*)(\eta^\alpha)_a, \quad d^* \in T^*(G),$$

where $a = \varpi(d^*)$. Then we have

$$E = \sum_i u_i^2,$$

$$\psi = \sum_i u_i dx^i + \sum_\alpha v_\alpha \eta^\alpha,$$

where η^α should be confounded with 1-forms on $T^*(G)$ through the projection ϖ of $T^*(G)$ onto G. Hence we obtain

$$\frac{1}{2}dE = \sum_i u_i du_i,$$

$$\Theta = \sum_i dx^i \wedge du_i + \sum_\alpha \eta^\alpha \wedge dv_\alpha + \frac{1}{2}\sum C_{ij}^\alpha v_\alpha dx^i \wedge dx^j.$$

Then we have

$$\mathscr{X}_E \rfloor \Theta = E\mathscr{X}_E x^i du_i - \sum_i \mathscr{X}_E u_i dx^i + \sum_\alpha \eta^\alpha(\mathscr{X}_E) dv_\alpha$$
$$- \sum \mathscr{X}_E v_\alpha \eta^\alpha + \sum_{\alpha,i,j} C_{ij}^\alpha v_\alpha \mathscr{X}_E x^i dx^j.$$

Since $(1/2)dE = \mathscr{X}_E \rfloor \Theta$, it follows that

$$(S.2) \quad \begin{cases} \mathscr{X}_E x^i = u_i \\ \mathscr{X}_E y^\alpha \dfrac{1}{2} \displaystyle\sum_{i,j} C_{ij}^\alpha x^i u_j, \\ \mathscr{X}_E u_i = -\displaystyle\sum_{\alpha,j} C_{ij}^\alpha v_\alpha u_j, \\ \mathscr{X}_E v_\alpha = 0. \end{cases}$$

Now in terms of the coordinate systems of $\tilde{T}$ and $\tilde{T}(G)$, the isomorphism $\varphi : \tilde{T} \to T^*(G)$ may be defined by

$$x^i \circ \varphi = x^i, \quad y^\alpha \circ \varphi = y^\alpha, \quad u_i \circ \varphi = u^i, \quad u_\alpha \circ \varphi = v_\alpha.$$

Consequently we conclude from systems (S.1) and (S.2) that $d\varphi(\mathscr{X}) = \mathscr{X}_E$, proving Proposition 3.1.

4. Minimizing geodesic segments and the exponential mapping

4.1. Preliminaries

Let V be a euclidean vector space and A a skew-symmetric endomorphism of V.

(A) The canonical decomposition of V with respect to A.

We denote by V^c the complexification of V. The inner product of V^c as a complex vector space, and A an endomorphism of V^c which will be denoted by the same letter A. Note that $\tilde{A} = (1/\sqrt{-1})A$ is a hermitian endomorphism of $\mathfrak{g}_1^c$.

We denote by $\lambda_i(A)$ $(1 \leq i \leq k(A))$ the distinct positive eigenvalues of $\tilde{A}$, where we promise that

$$\lambda_1(A) > \cdots > \lambda_{k(A)}(A).$$

Then $\pm\lambda_i(A)$ $(1 \leq i \leq k(A))$ are the distinct nonzero eigenvalues of $\tilde{A}$ and for each i the eigenvalues $\lambda(A)$ and $-\lambda_i(A)$ have the same multiplicity. For each $1 \leq i \leq k(A)$ we now denote by $U_i(A)$ the eigenspace of $\tilde{A}$ corresponding to the eigenvalue $\lambda_i(A)$:

$$U_i(A) = \{x \in V^c \mid \tilde{A}x = \lambda_i(A)x\}.$$

Then the conjugate $\overline{U_i(A)}$ of $U_i(A)$ with respect to the real form V of V^c is the eigenspace of $\tilde{A}$ corresponding to the eigenvalue $-\lambda_i(A)$, and V^c is orthogonally decomposed as follows:

$$V^c = U_0(A) + \sum_{i=1}^{k(A)} U_i(A) + \sum_{i=1}^{k(A)} \overline{U_i(A)},$$

where $U_0(A)$ denotes the kernel of $\tilde{A}$.

For each $1 \leq i \leq k(A)$ we denote by $V_i(A)$ the real part of $U_i(A)+\overline{U_i(A)}$:

$$V_i(A)^C = U_i(A) + \overline{U_i(A)}.$$

Then V is orthogonally decomposed as follows:

$$V = \sum_{i=0}^{k(A)} V_i(A),$$

where $V_0(A)$ denotes the kernel of the skew-symmetric endomorphism A of $\mathfrak{g}_1$: $V_0(A) = V \cap U_0(A)$. For each $1 \leq i \leq k(A)$ we then denote by $I_i(A)$ the natural complex structure on $V_i(A)$:

$$U_i(A) = \left\{x \in V_i(A)^c \mid I_i(A)x = \sqrt{-1}x\right\}.$$

Then we have

$$Ax = \lambda_i(A)I_i(A)x, \quad x \in V_i(A).$$

According to the decomposition above of V every vector $x \in V$ is decomposed as follows:

$$x = \sum_{i=0}^{k(A)} x_i(A) \quad \text{with} \quad x_i(A) \in V_i(A).$$

Furthermore, if we set

$$r(A) = \frac{1}{2}\operatorname{rank}(A),$$
$$r_i(A) = \frac{1}{2}\dim V_i(A)\ (= \dim U_i(A)), \quad 1 \leq i \leq k(A).$$

We have

$$r(A) = \sum_{i=1}^{k(A)} r_{i(A)}.$$

(B) The quadratic form H_A.

We denote by $\mathfrak{T}(V)$ the vector space of all piece-wise smooth vector fields $X : [0,1] \to V$ such that $X(0) = X(1) = 0$. We then define a quadratic form H_A on $\mathfrak{T}(V)$ by

$$H_A(X) = \int_0^1 \left|\frac{dX}{dt}\right|^2 dt + \int_0^1 \left\langle X, A\frac{dX}{dt}\right\rangle dt, \quad X \in \mathfrak{T}(V).$$

We also denote by $\|A\|$ the operator norm $\|A\|$ of the skew-symmetric endomorphism A of $\mathfrak{g}_1$:

$$\|A\| = \operatorname*{Max}_{x \neq 0} \frac{|Ax|}{|x|},$$

which is equal to the maximum of the positive eigenvalues of the hermitian endomorphism $\tilde{A}$. Hence we have

$$\|A\| = \lambda_1(A).$$

These being prepared, we shall prove the following.

Proposition 4.0 (1) *If $\|A\| < 2\pi$, then H_A is positive define.*

(2) *If $\|A\| = 2\pi$, then H_A is positive semi-definite, and its null space consists of all vector fields $X \in \mathfrak{T}(V)$ of the following form*:

$$X = e^{2\pi t I_1(A)} c - c,$$

where $c \in V_1(A)$.

(3) *If* $\|A\| > 2\pi$, *then* H_A *is indefinite.*

For simplicity we set $k = k(A)$, $\lambda_i = \lambda_i(A)$ and $I_i = I_i(A)$. Now, let X be a vector field in $\mathfrak{X}(V)$. For each $0 \leq i \leq k$ we denote by X_i the $V_i(A)$-component of X with respect to the decomposition $V = \sum_{i=0}^{k} V_i(A)$. Then for each $1 \leq i \leq k$ X_i can be expanded to a Fourier series as follows:

$$X_i = \sum_n e^{2n\pi t I_i} c_{i,n},$$

where $c_{i,n} \in V_i(A)$.

Lemma 4.0 *In terms of the Fourier series of* X_i $(1 \leq i \leq k)$ $H_A(X)$ *may be described as follows*:

$$H_A(X) = \int_0^1 \left| \frac{dX_0}{dt} \right|^2 dt + 2\pi \sum_{i=1}^{k} \sum_n (2n^2\pi - n\lambda_i)|c_{i,n}|^2.$$

Proof. We have

$$H_A(X) = \sum_{i=0}^{k} \int_0^1 \left| \frac{dX_i}{dt} \right|^2 dt + \sum_{i-1}^{k} \lambda_i \int_0^1 \left\langle X_i, I_i \frac{dX}{dt} \right\rangle dt.$$

As is well known, the derivatives dX_i/dt of X_i $(1 \leq i \leq k)$ can be expanded to a Fourier series as follows:

$$\frac{dX_i}{dt} = \sum_n 2n\pi I_i e^{2n\pi t I_i} c_{i,n}.$$

For each $1 < i \leq k$ we therefore obtain

$$\int_0^1 \left| \frac{dX_i}{dt} \right|^2 dt = 4\pi^2 \sum_n n^2 |c_{i,n}|^2,$$

$$\int_0^1 \left\langle X_i, I_i \frac{dX_i}{dt} \right\rangle dt = -2\pi \sum_n n|c_{i,n}|^2.$$

From these equalities follows immediately Lemma 4.0.

We are now in a position to prove Proposition 4.0. Let X be as above, and assume that $\|A\| \leq 2\pi$. Then we have $\lambda_i \leq 2\pi$ for any $1 \leq i \leq k$. Hence we obtain $2n^2\pi - n\lambda_i \geq 0$ for any $1 \leq i \leq k$ and any n. Therefore it follows from Lemma 4.0 that $H_A(X) \geq 0$. Now, assume that $H_A(X) = 0$. We first consider the case where $\|A\| < 2\pi$. Then we have $\lambda_i < 2\pi$ for any $1 \leq i \leq k$. Hence we obtain $2n^2\pi - n\lambda_i > 0$ for any $1 \leq i \leq k$ and any $n \neq 0$. Therefore it follows from Lemma 4.0 that $dX_0/dt = 0$, and $c_{i,n} = 0$ $(1 \leq i \leq k,\ n \neq 0)$. Since $X_i(0) = X_i(1) = 0$ $(0 \leq i \leq k)$, we have thus shown that $X_i = 0$ $(0 \leq i \leq k)$, that is, $X = 0$. Let us now consider the case where $\|A\| = 2\pi$. Then we have $\lambda_1 = 2\pi$ and $\lambda_i < 2\pi$ for any $2 \leq i \leq k$. Hence we obtain $2n^2\pi - n\lambda_i > 0$ for any $2 \leq i \leq k$ and any $n \neq 0$ and $2n^2\pi - n\lambda_1 > 0$ for any $n \neq 0$. Since $X_i(0) = X_i(1) = 0$ $(0 \leq i \leq k)$, it follows from Lemma 4.0 that $X_0 = X_i = 0$ $(2 \leq i \leq k)$, and

$$X_1(t) = e^{2\pi t I_1} c_{1,1} + c_{1,0} = e^{2\pi t I_1} c_{1,1} - c_{1,1}.$$

Putting $c = c_{1,1}$, we therefore obtain

$$X(t) = e^{2\pi t I_1} c - c, \quad c \in V_1(A).$$

Conversely it is clear that if $\|A\| = 2\pi$, a vector field of this form is in the null space of H_A. We have thus completed the proof of (1) and (2) of Proposition 4.0. Finally the proof of (3) is left to the readers as an exercise.

4.2. Minimizing geodesic segments

Let (x, A) be a point of $\mathfrak{g}_1 \times \mathscr{A}$, and ω the corresponding geodesic segment, by which we mean the restriction to the interval $[0, 1]$ of the geodesic corresponding to (x, A). In other words $\omega(t) = \Phi(tx, tA)$ $(t \in [0, 1])$. Let γ be the $\mathfrak{g}_1$-component of ω. We set $a = \omega(1) = \Phi(x, A)$, and consider the energy functionals $E : \Omega(G, D, a) \to \mathbb{R}$ and $\tilde{E}_1 : \tilde{\Omega}_1(\mathfrak{g}_1, a) \to \mathbb{R}$. Then ω gives a path in $\Omega(G, D, a)$, and γ a critical path for $\tilde{E}_1$ as a reduced geodesic path. Furthermore we have

$$L(\omega)^2 = E(\omega) = \tilde{E}_1(\gamma) = |x|^2.$$

In the discussion below we shall apply the notations explained in the previous paragraph to the pair $(V, A) = (\mathfrak{g}, A)$. (The symbols $V_i(A)$ will be preserved,

so that the canonical decomposition of $\mathfrak{g}_1$ will take the following form: $\mathfrak{g}_1 = \sum_i V_i(A)$.)

Theorem 4.1 *Assume that* $\|A\| \leq 2\pi$.

(1) ω *minimizes the energy functional* E :

$$E(\omega) \leq E(\omega'), \quad \omega \in \Omega(G, D, a).$$

(2) *Assume that* $E(\omega) = E(\omega')$ *for some* $\omega' \in \Omega(G, D, a)$. *If* $\|A\| < 2\pi$, *then* $\omega' = \omega$. *If* $\|A\| = 2\pi$, *then there is* $x' \in \mathfrak{g}_1$ *such that* $|x'| = |x|$, $x' - x \in V_1(A)$, *and* ω' *is the geodesic segment corresponding to* (x', A).

A geodesic segment $\theta(t)$ $(t \in [t_1, t_2])$ is called a *minimizing geodesic segment*, if $L(\theta) = d(\theta(t_1), \theta(t_2))$. By virtue of Proposition 1.3 we have the following.

Corollary 1 *If* $\|A\| < 2\pi$, *then* ω *is a unique minimizing geodesic segment. If* $\|A\| = 2\pi$ *then* ω *is a minimizing geodesic segment.*

Corollary 2 *Let* (x, A) *be a point of* $\mathfrak{g}_1 \times \mathscr{A}$. *If* $\|A\| \leq 2\pi$, *then*

$$d(e, \Phi(x, A)) = |x|.$$

Before proceeding to the proof of Theorem 4.1 we make a general consideration on the energy functional $\tilde{E}_1$. (For the time being we do not assume that $\|A\| \leq 2\pi$.)

For simplicity we set $\Omega_1 = \Omega_1(\mathfrak{g}_1, a_1)$ and $\tilde{\Omega}_1 = \tilde{\Omega}_1(\mathfrak{g}_1, a)$, a_1 being the $\mathfrak{g}_1$-component of a. If $X \in \mathfrak{T}(\mathfrak{g}_1)$ $(= T_\gamma(\tilde{\Omega}_1))$, $\gamma + X$ gives a path in Ω_1, and the assignment $X \to \gamma + X$ a bijection of $\mathfrak{T}(\mathfrak{g}_1)$ to Ω_1. Let us consider the quadratic form H_A on $\mathfrak{T}(\mathfrak{g}_1)$.

Lemma *Let* X *be a vector field in* $\mathfrak{T}(\mathfrak{g}_1)$.

(1) $\gamma + X$ *is in* $\tilde{\Omega}_1$ *if and only if*

$$2\int_0^1 \left\langle X,\ B\frac{d\gamma}{dt} \right\rangle dt + \int_0^1 \left\langle X,\ B\frac{dX}{dt} \right\rangle dt = 0, \quad B \in \mathscr{A}.$$

(2) *If* $\gamma + X$ *is in* $\tilde{\Omega}_1$, *then*

$$\tilde{E}_1(\gamma + X) - \tilde{E}_1(\gamma) = H_A(X).$$

Proof. We have

$$2(F(\gamma + X) - F(\gamma)) = \int_0^1 \left[\gamma, \frac{dX}{dt}\right] dt + \int_0^1 \left[\gamma, \frac{dX}{dt}\right] dt + \int_0^1 \left[X, \frac{dX}{dt}\right] dt.$$

Since $X(0) = X(1) = 0$, and

$$\frac{d}{dt}[\gamma, X] = \left[\frac{d\gamma}{dt},\ X\right] + \left[\gamma, \frac{dX}{dt}\right],$$

it follows that

$$2(F(\gamma + X) - F(\gamma)) = 2\int_0^1 \left[X, \frac{d\gamma}{dt}\right] dt + \int_0^1 \left[X, \frac{dX}{dt}\right] dt.$$

Hence we see that $\gamma + X$ is in $\tilde{\Omega}_1$, if and only if

$$2\int_0^1 \left[X, \frac{d\gamma}{dt}\right] dt + \int_0^1 \left[X, \frac{dX}{dt}\right] dt = 0$$

or equivalently

$$2\int_0^1 \left\langle X,\ B\frac{d\gamma}{dt}\right\rangle dt + \int_0^1 \left\langle X,\ B\frac{dX}{dt}\right\rangle dt = 0, \quad B \in \mathscr{A},$$

proving the first assertion. Next, we have

$$\tilde{E}_1(\gamma + X) - \tilde{E}_1(\gamma) = \int_0^1 \left|\frac{dX}{dt}\right|^2 dt + 2\int_0^1 \left\langle \frac{dX}{dt}, \frac{d\gamma}{dt}\right\rangle dt.$$

Accordingly we must prove the equality

$$2\int_0^1 \left\langle \frac{dX}{dt}, \frac{d\gamma}{dt}\right\rangle dt = \int_0^1 \left\langle X, A\frac{dX}{dt}\right\rangle dt.$$

First of all we see from the first assertion that

$$2\int_0^1 \left\langle X, A\frac{d\gamma}{dt}\right\rangle dt + \int_0^1 \left\langle X, A\frac{dX}{dt}\right\rangle dt = 0.$$

Furthermore we have

$$\frac{d}{dt}\left\langle X, \frac{d\gamma}{dt}\right\rangle = \left\langle \frac{dX}{dt}, \frac{d\gamma}{dt}\right\rangle + \left\langle X, A\frac{d\gamma}{dt}\right\rangle,$$

because $d^2\gamma/dt^2 = A(d\gamma/dt)$. Since $X(0) = X(1) = 0$, it follows that

$$\int_0^1 \left\langle \frac{dX}{dt}, \frac{d\gamma}{dt}\right\rangle + \int_0^1 \left\langle X, A\frac{d\gamma}{dt}\right\rangle dt = 0.$$

We have therefore established the desired equality, proving the second assertion.

Remark Let Y be any vector field in $IT_\gamma(\tilde{\Omega}_1)$, and take a variation $\bar{\alpha} : (-\varepsilon, \varepsilon) \to \tilde{\Omega}_1$ of γ in $\tilde{\Omega}_1$ which induces Y. For each $|s| < \varepsilon$ set $X(s) = \bar{\alpha}(s) - \gamma$. By the lemma we then have

$$\tilde{E}_1(\bar{\alpha}(s)) - \tilde{E}_1(\gamma) = H_A(X(s)).$$

Since

$$X(0) = 0, \quad \frac{d}{ds}X(s)\bigg|_{s=0} = Y,$$

it follows that

$$\frac{1}{2}\frac{d^2}{ds^2}\tilde{E}_1(\bar{\alpha}(s))\bigg|_{s=0} = H_A(Y).$$

Accordingly the restriction of H_A to $IT_\gamma(\tilde{\Omega}_1)$ is denoted by $\tilde{H}_\gamma$, and is called the *Hessian* of $\tilde{E}_1$ at the critical path.

We are now in a position to prove Theorem 4.1. Let ω' be a path in $\Omega(G, D, a)$, and γ' its $\mathfrak{g}_1$-component. Set $X = \gamma' - \gamma$ $(\in \mathfrak{T}(\mathfrak{g}_1))$. Then it follows from the lemma that

$$E(\omega') - E(\omega) = \tilde{E}_1(\gamma + X) - \tilde{E}_1(\gamma) = H_A(X).$$

Therefore the first assertion as well as the first half of the second assertion follows immediately from Proposition 4.0. Let us prove the second half. By the same proposition we then have $X = e^{2\pi t I_1}c - c$ with some $c \in V_1(A)$, where $I_1 = I_1(A)$. Hence we obtain

$$\gamma'(t) = \gamma(t) + e^{2\pi t I_1}c - c.$$

Since $d^2\gamma/dt^2 = A(d\gamma/dt)$, it follows that

$$\frac{d^2\gamma'}{dt^2} = A\frac{d\gamma'}{dt}.$$

Then we have

$$\frac{d\gamma'}{dt}(0) = x + 2\pi I_1 c.$$

If we set $x' = x + 2\pi I_1 c$, we have therefore seen that $x' - x \in V_1(A)$, and ω' is the geodesic segment corresponding to (x', A). Furthermore we have $|x'|^2 = E(\omega') = E(\omega) = |x|^2$, proving the second half. We have thus completed the proof of Theorem 4.1.

4.3. Injectivity and regularity for the exponential mapping

Let (x, A) be a point of $\mathfrak{g}_1 \times \mathscr{A}$. Let ω be the corresponding geodesic, and γ its $\mathfrak{g}_1$-component. Then we denote by $\Sigma(x, A)$ the subspace of $\mathfrak{g}_1$ spanned by $(d\gamma/dt)(t)$ $(t \in \mathbb{R})$, and define a subspace $\mathscr{A}(x, A)$ of $\mathscr{A}$ by

$$\begin{aligned}\mathscr{A}(x, A) &= \left\{ B \in \mathscr{A} \,\middle|\, B\frac{d\gamma}{dt} = 0 \right\} \\ &= \{B \in \mathscr{A} \mid B\Sigma(x, A) = \{0\}\}\end{aligned}$$

(cf. the subspace $\Sigma(\gamma)$ of $\mathfrak{g}_1$ and the subspace $\mathscr{A}(\gamma)$ of $\mathscr{A}$ defined in paragraph 2.3). Since $(d\gamma/dt)(t) = e^{tA}x$, $\Sigma(x, A)$ is spanned by the vectors $x, Ax, A^2x, \ldots$. The point (x, A) is called *nonsingular* or preferably *regular*, if $\mathscr{A}(x, A) = \{0\}$, and *singular* otherwise. Clearly the point (x, A) is regular if and only is so is the corresponding geodesic ω.

Now, let (x, A) and (x', A') be two points of $\mathfrak{g}_1 \times \mathscr{A}$. Let ω and ω' be the corresponding geodesics and let γ and γ' be their $\mathfrak{g}_1$-components. Then (x', A') is said to be *equivalent* to (x, A), if $\omega = \omega'$ or equivalently

$\gamma = \gamma'$. We can easily verify that $(x'A')$ is equivalent to (x, A), is and only if $x = x'$ and $A' - A \in \mathscr{A}(x, A)$. Finally assume that (x, A) and (x', A') are equivalent. Then we make an obvious remark that

(∗) For example a point (x, A) of $\mathfrak{g}_1 \times \mathscr{A}$ is a singular point, if either $x = 0$ or $A \neq 0$ and $x \in V_0(A)$.

(i) $\sum(x, A) = \sum(x', A')$, (ii) $\mathscr{A}(x, A) = \mathscr{A}(x', A')$, and (iii) $\Phi(x, A) = \Phi(x', A')$, and (iv) $(x, A) = (x', A')$, if (x, A) is a regular point.

Proposition 4.2 *Let (x, A) and (x', A') be two points of $\mathfrak{g}_1 \times \mathscr{A}$, and assume that $\|A\|, \|A'\| \leq 2\pi$.*

(1) *The case where $\|A\| < 2\pi$. Then $\Phi(x, A) = \Phi(x', A')$, if and only if (x, A) and (x', A') are equivalent.*
(2) *The case where $\|A\| = 2\pi$. If (x, A) and (x', A') are equivalent, then $\Phi(x', A') = \Phi(x', A')$, then $|x'| = |x|$, $x' - x \in V_1(A)$, and (x', A) and (x', A') are equivalent.*

Corollary *Let (x, A) and (x', A') be two points of $\mathfrak{g}_1 \times \mathscr{A}$. Assume that $\|A\| < 2\pi$ and $\|A'\| \leq 2\pi$, and that (x, A) is a regular point. Then $\Phi(x, A) = \Phi(x', A')$ if and only if $(x, A) = (x', A')$.*

Proof of Proposition 4.2. Let ω and ω' be the geodesic segments corresponding to (x, A) and (x', A'). Assume that $\Phi(x, A) = \Phi(x', A')$. Then it follows from (1) of Theorem 4.1 that $E(\omega) = E(\omega')$.

(1) The case where $\|A\| < 2\pi$. By (2) of Theorem 4.1 we have $\omega = \omega'$, meaning that (x, A) and (x', A') are equivalent.

(2) The case where $\|A\| = 2\pi$. By (2) of Theorem 4.1, these is $x'' \in \mathfrak{g}_1$ such that $|x''| = |x|$, $x'' - x \in V_1(A)$, and ω' is the geodesic segment correspondent to (x'', A). Then it follows that (x', A') and (x'', A) are equivalent, implying that $x'' = x$. We have thus proved the second assertion.

Now, let (x, A) be a point of $\mathfrak{g}_1 \times \mathscr{A}$, and let us consider the differential $(d\Phi)_{(x,A)}$ of Φ at (x, A). Then we assert that the kernel of $(d\Phi)_{(x,A)}$ contains the subspace $0 \times \mathscr{A}(x, A)$ of $\mathfrak{g}_1 \times \mathscr{A}$. Indeed, we have $\Phi(x, A) = \Phi(x, A+tB)$ for any $B \in \mathscr{A}(x, A)$ and $t \in \mathbb{R}$, because (x, A) and $(x, A+tB)$ are equivalent. Hence we obtain $(d\Phi)_{(x,A)}(0, B) = 0$, proving our assertion.

Proposition 4.3 *Let (x, A) be a point of $\mathfrak{g} \times \mathscr{A}$, and assume that $\|A\| \leq 2\pi$.*

(1) *If* $\|A\| < 2\pi$, *then the kernel of* $(d\Phi)_{(x,A)}$ *coincides with the subspace* $0 \times \mathscr{A}(x, A)$ *of* $\mathfrak{g} \times \mathscr{A}$.
(2) *If* $\|A\| = 2\pi$, *then the kernel of* $(d\Phi)_{(x,A)}$ *contains the subspace* $0 \times \mathscr{A}(x, A)$ *of* $\mathfrak{g}_1 \times \mathscr{A}$, *and is contained in the subspace* $V_1(x, A) \times \mathscr{A}(x, A)$ *of* $\mathfrak{g}_1 \times \mathscr{A}$, *where* $V_1(x, A) = \{z \in V_1(A) \mid \langle z,\ x_1(A)\rangle = 0\}$.

Corollary *Let* (x, A) *be a point of* $\mathfrak{g}_1 \times \mathscr{A}$, *and assume that* $\|A\| < 2\pi$. *Then* Φ *is regular at* (x, A), *if and only if* (x, A) *is a regular point.*

Before proceeding to the proof of the proposition we make a general consideration on infinitesimal variations (by geodesics) of a geodesics ω.

Let (x, A) be any point of $\mathfrak{g}_1 \times A$, and ω the corresponding geodesic. Set $\omega(t) = (\gamma(t), \delta(t))$. Let (z, B) be any vector of $\mathfrak{g}_1 \times \mathscr{A}$, and take a smooth curve $(x_s, A_s)(|s| < \varepsilon)$ of $\mathfrak{g}_1 \times \mathscr{A}$, ε being a small positive number, which satisfies the following conditions:

$$x_0 = x, \quad \left.\frac{dx_s}{ds}\right|_{s=0} = z,$$

$$A_0 = A \quad \left.\frac{dA_s}{ds}\right|_{s=0} = B.$$

For each s let ω_s be the geodesic corresponding to the point (x_s, A_s), and set $\omega_s(t) = (\gamma_s(t), \delta_s(t))$. Then $\omega_0 = \omega$ or $\gamma_0 = \gamma$, $\delta_0 = \delta$, and ω_s satisfies the following system of differential equations as well as the following initial condition:

$$\begin{cases} \dfrac{d^2\gamma_s}{dt^2} = A_s \dfrac{d\gamma_s}{dt}, \\ \dfrac{d\delta_s}{dt} = \dfrac{1}{2}\left[\gamma_s, \frac{d\gamma_s}{dt}\right], \\ \gamma_s(0) = 0,\ \dfrac{d\gamma_s}{dt}(0) = x_s,\ \delta_s(0) = 0. \end{cases}$$

Now, we define vector fields $J, K : \mathbb{R} \to \mathfrak{g}_1$ as follows:

$$J(t) = \left.\frac{\partial}{\partial s}\gamma_s(t)\right|_{s=0}, \quad K(t) = \left.\frac{\partial}{\partial s}\delta_s(t)\right|_{s=0}.$$

Then we have

$$(a) \quad \frac{d^2J}{dt^2} = A\frac{dJ}{dt} + B\frac{d\gamma}{dt},$$

$$(b) \quad \frac{dK}{dt} = \frac{1}{2}\left[J, \frac{d\gamma}{dt}\right] + \frac{1}{2}\left[\gamma, \frac{dJ}{dt}\right]$$

$$(c) \quad J(0) = 0, \ \frac{dJ}{dt}(0) = z, \ K(0) = 0.$$

Here, we remark that the pair (J, K) can be characterized as a unique solution of the system of equations (a) and (b) satisfying the initial condition (c), which is therefore uniquely determined by the vector (z, B). This being said, the pair (J, K) will be called the *Jacobi field* along the geodesic ω corresponding to the vector (z, B).

Since $\Phi(x_s, A_s) = \omega_s(1) = (\gamma_s(1), \delta_s(1))$, we have

$$(d\Phi)_{(x,A)}(z, B) = (J(1), K(1)).$$

Let us consider the quadratic form H_A on $\mathfrak{T}(\mathfrak{g}_1)$, and denote by $\bar{J}$ the restriction of J to the interval $[0, 1]$.

Lemma (1) *(z, B) is in the kernel of $(d\Phi)_{(x,A)}$, if and only if $J(1) = 0$, and*

$$\int_0^1 \left\langle J, \ C\frac{d\gamma}{dt}\right\rangle dt = 0, \quad C \in \mathscr{A}.$$

(2) *If (z, B) is in the kernel of $(d\Phi)_{(x,A)}$, then $\bar{J}$ satisfies the following equation*:

$$H_A(\bar{J}) = \int_0^1 \left|\frac{dJ}{dt}\right|^2 dt + \int_0^1 \left\langle J, \ A\frac{dJ}{dt}\right\rangle dt = 0.$$

Proof. Equation (b) can be rewritten as follows:

$$\frac{dK}{dt} = \left[J, \ \frac{d\gamma}{dt}\right] + \frac{1}{2}\frac{d}{dt}[\gamma, J].$$

Since $J(0) = K(0) = 0$, it follows that

$$K(1) = \int_0^1 \left[J, \frac{d\gamma}{dt}\right] dt + \frac{1}{2}[\gamma(1), J(1)].$$

Hence we see that (z, B) is in the kernel of $(d\Phi)_{(x,A)}$, is and only if $J(1) = 0$, and

$$\int_0^1 \left[J, \frac{d\gamma}{dt}\right] dt = 0,$$

or equivalently

$$\int_0^1 \left\langle J,\ C\frac{d\gamma}{dt} \right\rangle dt = 0, \quad C \in \mathscr{A},$$

proving the first assertion. Especially we have

$$\int_0^1 \left\langle J, B\frac{d\gamma}{dt} \right\rangle dt = 0.$$

Therefore it follows from equation (a) that

$$\int_0^1 \left\langle J, \frac{d^2J}{dt^2} \right\rangle dt = \int_0^1 \left\langle J,\ A\frac{dJ}{dt} \right\rangle dt.$$

Since $J(0) = J(1) = 0$, and

$$\frac{d}{dt}\left\langle J, \frac{dJ}{dt} \right\rangle = \left|\frac{dJ}{dt}\right|^2 + \left\langle J, \frac{d^2J}{dt^2} \right\rangle,$$

we have thus shown that $H_A(\bar{J}) = 0$, proving the second assertion.

We are now in a position to prove Proposition 4.3. For this purpose it suffices to show that the notations being as above, $(z, B) \in 0 \times \mathscr{A}(x, A)$ or $(z, B) \in \|V_1(x, A)X\mathscr{A}(x, A)$, according as $\|A\| < 2\pi$ or $\|A\| = 2\pi$. We first consider the case where $\|A\| < 2\pi$. Since $H_A(\bar{J}) = 0$ by the lemma above, it follows from Proposition 4.0 that $J = 0$. Hence we have $z = (dJ/dt)(0) = 0$. Furthermore we see from equation (a) that $B(d\gamma/dt) = 0$, meaning that $B \in \mathscr{A}(x, A)$. Let us now consider the case where $\|A\| = 2\pi$. Since $H_A(\bar{J}) = 0$, it follows from the proposition cited above that

$$J = e^{2\pi t I_1} c - c,$$

where $c \in V_1(A)$, and $I_1 = I_1(A)$. Hence we have $z = 2\pi I_1 c$, implying that

$$J = \frac{e^{2\pi t I_1} - 1}{2\pi I_1} z.$$

By the lemma above we have

$$\int_0^1 \left\langle J,\ A\frac{d\gamma}{dt} \right\rangle dt = 0.$$

Since $d\gamma/dt = e^{tA}x$, it therefore follows that

$$\int_0^1 \left\langle (e^{2\pi t I_1})z, e^{2\pi t I_1} x_1(A) \right\rangle dt = 0.$$

As is easily verified, this means $\langle z,\ x_1(A)\rangle = 0$, that is, $z \in V_1(x, A)$. Furthermore we have

$$\frac{d^2 J}{dt^2} = A\frac{dJ}{dt}.$$

Consequently it follows from equation (a) that $B(d\gamma/dt) = 0$, meaning that $B \in \mathscr{A}(x, A)$. We have thus completed the proof of Proposition 4.3.

5. The singular points of $\mathfrak{g}_1 \times \mathscr{A}$ and the cut locusoid

5.1. The singular points of $\mathfrak{g}_1 \times \mathscr{A}$

Let V be a (real) algebraic set of $\mathbb{R}^n$. Then the dimension of V may be defined to be the maximum of the dimensions of all submanifolds of $\mathbb{R}^n$ which are open sets of V, where V as well as the submanifolds is equipped with the relative topology. For the standard definition of the dimension, see the book [1] of R. Benedetti and J. J. Risler. Let V' be the complement of V in $\mathbb{R}^n$, and O a connected open subset of $\mathbb{R}^n$. Then it can be shown that if codim $V \geq 2$, the intersection on V' is a connected open dense subset of O.

Now, we denote by $\mathscr{R}$ (resp.by $\mathscr{S}$) the set of all regular (resp. singular) points of $\mathfrak{g}_1 \times \mathscr{A}$. Then we shall prove the following.

Theorem 5.1 *$\mathscr{S}$ is an algebraic set of $\mathfrak{g}_1 \times \mathscr{A}$, and codim $\mathscr{S} \geq 2$. Ac-*

cordingly if O is a connected open subset of $\mathfrak{g}_1 \times \mathscr{A}$, then the intersection $O \cap \mathscr{R}$ is a connected open dense subset of O.

The proof is preceded by several lemmas.

Lemma 1 *$\mathscr{S}$ is an algebraic set of $\mathfrak{g}_1 \times \mathscr{A}$.*

Proof. Set $n = \dim \mathfrak{g}_1$. By the Cayley-Hamilton theorem we know that if $A \in \mathscr{A}$, A^n may be described as a linear combination of the n endomorphisms $1, A, \ldots, A^{n-1}$. It follows that if $(x, A) \in \mathfrak{g}_1 \times \mathscr{A}$, the subspace $\Sigma(x, A)$ of $\mathfrak{g}_1$ is spanned by the n vectors $x, Ax, \ldots, A^{n-1}x$. This being said, for each $(x, A) \in \mathfrak{g}_1 \times \mathscr{A}$ we define a linear mapping $\varphi_{(x,A)}$ of $\mathscr{A}$ to $\mathfrak{g}_1^n$, the product of n copies of $\mathfrak{g}_1$, by

$$\varphi_{(x,A)}(B) = (Bx, BAx, \ldots, BA^{n-1}x), \quad B \in \mathscr{A}.$$

Then we see that $(x, A) \in \mathscr{S}$, if and only if $\operatorname{rank} \varphi_{(x,A)} < \dim \mathscr{A}$, from which follows immediately the lemma.

Lemma 2 *If $(x, A) \in \mathfrak{g}_1 \times \mathscr{A}$, then $\Sigma(x, A)$ is spanned by the $2k(A) + 1$ vectors*

$$x_0(A),\ x_i(A),\ I_i(A)x_i(A) \quad (1 \leq i \leq k(A)).$$

Proof. For simplicity we set $k = k(A)$, $\lambda_i = \lambda_i(A)$, $I_i = I_i(A)$ and $x_i = x_i(A)$. Then we have

$$x = x_0 + \sum_{i=1}^{k} x_i,$$

and

$$(-1)^j A^{2j-1}x = \sum_{i=1}^{k} \lambda_i^{2j-1} I_i x_i,$$

$$(-1)^j A^{2j}x = \sum_{i=1}^{k} \lambda_i^{2j} x_i.$$

Since $\det(\lambda_i^{2j})_{1 \leq i,j \leq k} \neq 0$, the lemma follows immediately from these equalities.

Now, let N be a submanifold of $\mathfrak{g}_1 \times \mathscr{A}$ which is an open set of the algebraic set $\mathscr{S}$. We shall show that codim $N \geq 1$. For our purpose N may be clearly replaced by its open submanifold.

First of all we may assume that $A \neq 0$ for any $(x, A) \in N$. Indeed suppose that $\dim \mathscr{A} = 1$. Then we have $\mathscr{S} = V_0(A_0) \times \mathscr{A}$, where A_0 is a basis of $\mathscr{A}$ (see paragraph 5.3). Hence we have $\operatorname{codim} \mathscr{S} \geq 2\gamma(A_0) \geq 2$. Now, suppose that $\dim \mathscr{A} \geq 2$ and $A = 0$ for any $(x, A) \in N$. Then we have codim $N \geq \operatorname{codim}(\mathfrak{g}_1 \times 0) \geq 2$. Clearly these considerations justify our assumption.

Let us now denote by N_* the largest open set of N on which both the functions $(x, A) \in N \to \gamma(A) \in \mathbb{Z}$ and $(x, A) \in N \to k(A) \in \mathbb{Z}$ are locally constant. Since these functions are lower semi-continuous, we know that N_* is a dense subset of N, implying that $N_* \neq \phi$. Furthermore we know that the functions $(x, A) \in N_* \to \lambda_i(A)$ $(1 \leq i \leq k(A))$ are continuous, and hence the functions $(x, A) \in N_* \to r_i(A) \in \mathbb{Z}$ $(1 \leq i \leq k(A))$ are upper semi-continuous. Since $r(A) = \sum r_i(A)$ for any $(x, A) \in N_*$, it follows that the functions $(x, A) \in N_* \to r_i(A)$ are locally constant.

Therefore we may further assume that being restricted to N, the functions $(x, A) \to r(A)$, $(x, A) \to k(A)$ and $(x, A) \to r_i(A)$ are all constant. Then we set

$$\gamma = \gamma(A),\ k = k(A) \quad \text{and} \quad \gamma_i = \gamma_i(A) \quad \text{for } (x, A) \in N.$$

Lemma 3 *For any $(x, A) \in N$ at least one of the components $x_i(A)$ $(1 \leq i \leq k)$ of x vanishes.*

Proof. Assume that there is $(x, A) \in N$ such that $x_i(A) \neq 0$ for any $1 \leq i \leq k$, (x, A) being a singular point, we can take a nonzero element B of $\mathscr{A}(x, A)$. Since (x, A) and $(x, A + tB)$ are equivalent for any $t \in \mathbb{R}$, we see that $(x, A + tB)$ is a singular point: $(x, A + tB) \in \mathscr{S}$. Since N is an open set of $\mathscr{S}$, there is a positive number ε such that $(x, A + tB) \in N$ for any $|t| < \varepsilon$. Then we assert that $\lambda_i(A + tB) = \lambda_i(A)$ for any $1 \leq i \leq k$ and $|t| < \varepsilon$. Indeed, we have $Bx_i(A) = BI_i(A)x_i(A) = 0$ by Lemma 2. If we put $y_i = x_i(A) - \sqrt{-1}I_i(A)x_i(A)$, it follows that

$$\frac{1}{\sqrt{-1}}(A + tB)y_i = \frac{1}{\sqrt{-1}}Ay_i = \lambda_i(A)y_i.$$

Since $y_i \neq 0$ $(1 \leq i \leq k)$, we see that $\lambda_i(A)$ $(1 \leq i \leq k)$ are the positive eigenvalues of $A + tB$ for any $|t| < \varepsilon$, proving our assertion.

Let us now consider the characteristic polynomials χ_A and $\chi_{(A+tB)}$ of the skew-symmetric endomorphism A and $A + tB$. Then we have shown that

$$\begin{aligned}\chi_{(A+tB)}(\lambda) &= \chi_A(\chi) \\ &= \lambda^{n-2\gamma}(\lambda^2 + \lambda_1^2)^{\gamma_1} \cdots (\lambda^2 + \lambda_k^2)^{\gamma_k}, \quad |t| < \varepsilon,\end{aligned}$$

where $n = \dim \mathfrak{g}_1$ and $\lambda_i = \lambda_i(A)$. Since $\chi_{(A+tB)}(\lambda)$ is a polynomial of the two variables t and λ, this equality is valid for any $t \in \mathbb{R}$. In particular it follows that

$$\|A + tB\| = \lambda_1(A) = \|A\|, \quad t \in \mathbb{R}.$$

Hence we obtain

$$|t|\|B\| \leq \|A + tB\| + \|A\| = 2\|A\|, \quad t \in \mathbb{R},$$

which contradicts to the fact that $\|B\| \neq 0$. We have thus proved the lemma.

We are now in a position to prove Theorem 5.1. Let ϖ denote the projection of N to $\mathscr{A}$: $\varpi(x, A) = A$ for any $(x, A) \in N$. Now, we take any point A of $\varpi(N)$, and set

$$\hat{V}_j(A) = \sum_{i \in I_j} V_i(A), \quad 1 \leq j \leq k,$$

$$\hat{V}(A) = \bigcup_{j=1}^{k} \hat{V}_j(A),$$

where $I_j = \{i \in \mathbb{Z} \mid 0 \leq i \leq k,\ i \neq j\}$. Then we see from Lemma 2 that, being regarded as a subspace of $\mathfrak{g}_1$, $\varpi^{-1}(A)$ is contained in the algebraic set $\hat{V}(A)$ of $\mathfrak{g}_1$. Furthermore since ϖ is smooth, we may assume that (i) ϖ is of constant rank, and hence $\varpi^{-1}(A)$ is a submanifold of N for each $A \in \varpi(N)$, (ii) $\varpi(N)$ is a submanifold of $\mathscr{A}$, and (iii) $\varpi^{-1}(A)$ is connected for each $A \in \varpi(N)$. In particular it follows that for each $A \in \varpi(N)$, $\varpi^{-1}(A)$ is contained in some $\hat{V}_j(A)$. Since $\operatorname{codim} \hat{V}_j(A) = 2\gamma_j \geq 2$, it is now clear that $\operatorname{codim} N \geq 2$, completing the proof Theorem 5.1.

Finally we add the following

Proposition 5.2 *Let (x, A) be a singular point of $\mathfrak{g}_1 \times A$. If $\|A\| < 2\pi$, there is $A' \in \mathscr{A}$ such that (x, A) and (x, A') are equivalent, and $\|A'\| = 2\pi$.*

Proof. (x, A) being a singular point, we can take a nonzero element B of $\mathscr{A}(x, A)$. Let us consider the function $f(t) = \|A + tB\|$ on the straight line. Since $|t|\|B\| - \|A\| \leq \|A+tB\|$ and $\|B\| \neq 0$, $f(t)$ tends to $+\infty$, as t tends to $+\infty$. Since $f(t)$ is continuous, and $f(0) = \|A\| < 2\pi$, it follows that there is $t_0 \in \mathbb{R}$ such that $f(t_0) = 2\pi$. If we put $A' = A + t_0 B$, we have thus seen that (x, A) and (x, A') are equivalent, and $\|A'\| = 2\pi$, proving the proposition.

5.2. The domain Δ and the cutlocusoid Γ

Using the operator norm $\| \ \|$ in the space $\mathscr{A}$, we define an open neighborhood of the origin of $\mathfrak{g}_1 \times \mathscr{A}$ by

$$\mathscr{V} = \{(x, A) \in \mathfrak{g}_1 \times \mathscr{A} \mid \|A\| < 2\pi\},$$

and denote by $\partial\mathscr{V}$ the boundary of $\mathscr{V}$ in $\mathfrak{g}_1 \times \mathscr{A}$:

$$\partial\mathscr{V} = \{(x, A) \in \mathfrak{g}_1 \times \mathscr{A} \mid \|A\| = 2\pi\}.$$

We then define subsets Δ and Γ of G as follows:

$$\Delta = \Phi(\mathscr{V} \cap \mathfrak{R}), \quad \Gamma = \Phi(\partial\mathscr{V}),$$

Φ being the exponential mapping of $\mathfrak{g}_1 \times \mathscr{A}$ to G.

By the use of most results above we shall prove the following.

Theorem 5.3 (1) *$\mathscr{V} \cap \mathfrak{R}$ is a (star-shaped) connected open dense subset of $\mathscr{V}$, Δ is an open set of G, and the exponential mapping Φ gives a diffeomorphism of $\mathscr{V} \cap \mathfrak{R}$ onto Δ. Furthermore Φ maps $\mathscr{V} \cap \mathscr{S}$ into Γ.*

(2) *Every point of Δ can be connected to the identity element e of G by a unique minimizing geodesic segment, and every point of Γ by a minimizing geodesic segment.*

(3) *Γ coincides with the boundary $\partial\Delta$ of Δ in G.*

It should be to be remarked that Γ is an unbounded continuum through the identity element e and hence Δ is not a neighborhood of e. (Since the origin of $\mathfrak{g}_1 \times \mathscr{A}$ is in $\mathscr{V} \cap \mathscr{S}$, e is in Γ. To verify the unboundedness we reduce the problem to the case where $\dim \mathfrak{g}_2 = 1$, and use Proposition 5.4 in

the next paragraph.)

We now proceed to the proof of Theorem 5.3. The first assertion of (1) follows from the corollary to Proposition 4.2, the corollary to Proposition 4.3 and Theorem 5.1, and the second assertion from Proposition 5.2. (2) follows immediately from Corollary 1 to Theorem 4.1. Let us now prove (3). Since $\mathscr{V} \cap \mathfrak{R}$ is dense in $\mathscr{V}$, and since $\Delta \cap \Gamma = \emptyset$ by the corollary to Proposition 4.2, we see that $\Gamma \subset \partial\Delta$. Conversely we assert that $\partial\Delta \subset \Gamma$. Indeed, take any point a of $\partial\Delta$. Then there is a sequence $\{(x_\alpha, A_\alpha)\}_{\alpha \geq 1}$ of points of $\mathscr{V} \cap \mathfrak{R}$ such that

$$\lim_{\alpha \to \infty} \Phi(x_\alpha, A_\alpha) = a.$$

Since $d(e, \Phi(x_\alpha, A_\alpha)) = |x_\alpha|$ and since $\|A_\alpha\| < 2\pi$, we may assume that the sequence converges to a point, say (x, A), of the closure $\overline{\mathscr{V}}$ of $\mathscr{V}$ in $\mathfrak{g}_1 \times \mathscr{A}$. Then we have $\Phi(x, A) = a$. Furthermore we see from (1) that (x, A) is in the union $(\mathscr{V} \cap \mathscr{S}) \cup \partial\mathscr{V}$, and in turn a is in Γ, proving our assertion. We have thus completed the proof of Theorem 5.3.

Corollary *Let X be the complement of Γ in G, being an open set of G. Then Δ is a connected component of X whose boundary in G coincides with Γ. Accordingly if X is connected, then the closure $\overline{\Delta}(= \Delta \cup \Gamma)$ of Δ in G coincides with the whole of G, and hence any two points of G can be connected by a minimizing geodesic segment.*

By Theorem 5.3 *we have known that Γ is something like the cut locus at a point of a complete riemannian manifold. Accordingly it will be called the cut locusoid of the standard riemannian differential system (G, D, g) (at the identity element e).*

5.3. Examples

In the present paragraph we shall treat with the domain Δ, the closed domain $\overline{\Delta}$ and the cut locusoid Γ in the special case where $\dim \mathfrak{g}_2 = 1$ or $\mathscr{A}$ is abelian or $\dim \mathfrak{g}_2 = 3$.

(A) The case where $\dim \mathfrak{g}_2 = 1$. Let A_0 be a fixed basis of $\mathscr{A}$ with $\|A_0\| = 1$. For simplicity we set $k = k(A_0)$, $\sigma_i = \lambda_i(A_0)$ $(1 \leq i \leq k)$, $I_i = I_i(A_0)$ $(1 \leq i \leq k)$, and $y_i = y_i(A_0)$ $(0 \leq i \leq k)$ for any $y \in \mathfrak{g}_1$. Note that $\sigma_1 = \|A_0\| = 1$, and hence $0 < \sigma_i < 1$ $(2 \leq i \leq k)$. Furthermore we identify $\mathfrak{g}_2^*$ with $\mathscr{A}$ through the natural isomorphism of $\mathfrak{g}_2^*$ onto $\mathscr{A}$. First

of all let $(x,\ \lambda A_0)$ be any point of $\mathfrak{g}_1 \times \mathscr{A}$, λ being a real number. We set $\mu_i = \lambda\sigma_i$ $(1 \leq i \leq k)$. Then a direct calculation shows that the components $\Phi_1(x, \lambda A_0)$ and $\Phi_2(x, \lambda A_0)$ of $\Phi(x, \lambda A_0)$ may be described as follows:

$$\Phi_1(x, \lambda A_0) = x_0 + \sum_{i=1}^{k} \frac{1}{\mu_i}(\sin \mu_i \cdot x_i + (1 - \cos \mu_i) I_i x_i)$$

$$\langle \Phi_2(x, \lambda A_0),\ A_0 \rangle = \frac{-1}{2\lambda} \sum_{i=1}^{k} \left(1 - \frac{\sin \mu_i}{\mu_i}\right) |x_i|^2.$$

Now, we set

$$c_i = \frac{\tau_i(\tau_i - \sin \tau_i)}{8\pi(1 - \cos \tau_i)}, \quad 2 \leq i \leq k,$$

where $\tau_i = 2\pi\sigma_i$. Clearly c_i are positive numbers. Then we shall prove the following.

Proposition 5.4 (1) *The cut locusoid* Γ *may be described as the semi-algebraic set of* G *which consists of all* $(y, z) \in G$ *satisfying the following equations*:

$$y_1 = 0, \quad |\langle z,\ A_0 \rangle| \geq \sum_{i=2}^{k} c_i |y_i|^2.$$

(2) *The closed domain* $\overline{\Delta}$ *coincides with the whole of* G.

Note that if $k = 1$, the semi-algebraic set is reduced to the linear subspace $V_0(A_0) \times \mathfrak{g}_2$ of $G = \mathfrak{g}_1 \times \mathfrak{g}_2$.

Now, let $(x, \lambda A_0)$ be any point of $\partial\mathscr{V}$. Since $|\lambda| = 2\pi$, we have

$$\Phi_1(x, \lambda A_0) = x_0 + \sum_{i=2}^{k} \frac{1}{\mu_i}\{\sin \mu_i x_i + (1 - \cos \mu_i) I_i x_i\},$$

$$\langle \Phi_2(x, \lambda A_0),\ A_0 \rangle = \frac{1}{2\lambda}\left\{|x_1|^2 + \sum_{i=2}^{k} \left(1 - \frac{\sin \mu_i}{\mu_i}\right) |x_i|^2\right\}.$$

If we set $y = \Phi_1(x, \lambda A_0)$ and $z = \Phi_2(x, \lambda A)$, we easily see that

$$x_0 = y_0, \quad y_1 = 0,$$

$$x_i = \frac{\mu_i}{2(1-\cos\mu_i)}\{\sin\mu_i y_i - (1-\cos\mu_i)Iy_i\cdot\}, \quad 2 \leq i \leq k,$$

$$\langle z,\ A_0\rangle = \frac{-1}{2\lambda}\left\{|x_1|^2 + \sum_{i=2}^{k}\frac{\mu_i(\mu_i-\sin\mu_i)}{2(1-\cos\mu_i)}|y_i|^2\right\}.$$

Since $\tau_i = |\mu_i|$, we have

$$4\pi c_i = \frac{\mu_i(\mu_i-\sin\mu_i)}{2(1-\cos\mu_i)}.$$

Therefore it follows that

$$|\langle z, A_0\rangle| \geq \sum_{i=2}^{k} c_i|y_i|^2.$$

Let us now denote by Γ_0 the semi-algebraic set stated in Proposition 5.4. Then we have shown that $\Gamma \subset \Gamma_0$. Conversely we can easily verify that $\Gamma \supset \Gamma_0$. Hence we obtain $\Gamma = \Gamma_0$, proving the first assertion. If we set

$$\hat{V}_1(A_0) = V_0(A_0) + \sum_{i=2}^{k} V_i(A_0),$$

we have $\Gamma \subset \hat{V}_1(A_0) \times \mathfrak{g}_2$, and $\operatorname{codim}(\hat{V}_1(\mathfrak{g}_0) \times \mathfrak{g}_2) = 2\gamma_1(A_0) \geq 2$. It follows immediately that the complement X of Γ is connected. Consequently we know from Corollary to Theorem 5.3 that $\overline{\Delta} = G$, proving the second assertion.

Now, we remark that the set $\mathscr{S}$ of singular points of $\mathfrak{g}_1 \times \mathscr{A}$ may be described as follows:

$$\mathscr{S} = V_0(A_0) \times \mathscr{A},$$

and the image $\Phi(\mathscr{S})$ of $\mathscr{S}$ by the exponential mapping Φ as follows:

$$\Phi(\mathscr{S}) = V_0(A_0) \times 0 (\subset \Gamma).$$

(Note that if (x, A) is a singular point, the corresponding (singular) geodesic

ω is of the following form: $\omega(t) = (tx,\ 0)$, where $x \in V_0(A_0)$.

Furthermore we add the following

Proposition 5.5 (cf. Proposition 4.2 and 4.3)

(1) *Let* (x, A) *and* (x', A') *be two points of* $\partial\mathscr{V}$. (a) *Assume that* (x, A) *is a regular point. Then* $\Phi(x, A) = \Phi(x', A')$, *if and only if* $|x'| = |x|$, $x' - x \in V_1(A_0)$, *and* $A = A'$. (b) *Assume that* (x, A) *is a singular point. Then* $\phi(x, A) = \Phi(x', A')$, *if and only if* $x = x'$.

(2) *Let* (x, A) *be a point of* $\partial\mathscr{V}$. (a) *Assume that* (x, A) *is a regular point. Then* $\mathrm{Ker}(d\Phi)_{(x,A)} = V_1(x, A_0) \times 0$. (b) *Assume that* (x, A) *is a singular point. Then* $\mathrm{Ker}(d\Phi)_{(x,A)} = V_1(A_0) \times \mathscr{A}$.

Here, $\mathrm{Ker}(d\Phi)_{(x,A)}$ *denotes the kernel of the differential* $(d\Phi)_{(x,A)}$ *of the exponential mapping* Φ, *and we also recall that* $V_1(x, A_0) = \{z \in V_1(A_0) \mid \langle z, x_1(A_0)\rangle = 0\}$. *The proof of this proposition is left to the readers as an exercise.*

Remark Let us consider the special case where $k(A_0) = 1$ and $2\gamma(A_0) = \dim \mathfrak{g}_1$, which means that $\mathscr{A}$ admits a basis I giving a complex structure on $\mathfrak{g}_1$. Clearly the FGLA, $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, together with the complex structure I gives a strongly pseudo-convex FGLA, and the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}_1$ is naturally associated with it. Accordingly the inner product g of D is likewise naturally associated with the standard strongly pseudo-convex manifold (G, D, I), and D is a contact structure. Now, we have

$$\Gamma = 0 \times \mathfrak{g}_2.$$

Let us identify the Lie group $G(= \mathfrak{g}_1 \times \mathfrak{g}_2)$ with a euclidean space in a natural fashion, and each fibre of the contact structure D with a hyperplane of G. Then we know that the fibre D_a of D at any point $a \in \Gamma$ is parallel to the hyperplane $D_e = \mathfrak{g}_1 \times 0$, and that every geodesic issuing from the identity element e describes a spiral in G so that the projection of it to the euclidean space $\mathfrak{g}_1$ describes a circle through the origin. This fact reminds us of the navigation of an airplane. We mention that the variational problem in the present case has been settled more than twenty years ago in our unpublished paper, and our present study may be regarded as its generalization.

(B) The case where $\mathscr{A}$ is abelian. We denote by V_0 the null space of $\mathscr{A}$:

$$V_0 = \{x \in \mathfrak{g}_1 \mid Ax = 0 \ \text{ for any } \ A \in \mathscr{A}\}.$$

Then there are a positive integer k and k subspaces V_i of $\mathfrak{g}_i$ equipped with complex structures $I_i (1 \leq i \leq k)$ and k nonzero linear forms ρ_i on $V_i (1 \leq i \leq k)$ which satisfy the following conditions:

(i) $\mathfrak{g}_1 = \sum_{i=0}^{k} V_i$ (direct sum),
(ii) $Ax = \rho_i(A) I_i x$, $A \in \mathscr{A}$, $x \in V_i$, $1 \leq i \leq k$,
(iii) The $2k$ linear forms $\pm\rho_i$ are distinct one another.

Note that the $k+1$ subspaces V_i of $\mathfrak{g}_1$ are mutually orthogonal, and the k linear forms ρ_i span the dual space $\mathscr{A}^*$ of $\mathscr{A}$.

For any $x \in \mathfrak{g}_1$ and $0 \leq i \leq k$ let us denote by x_i the V_i-component of x with respect to the decomposition $\mathfrak{g}_1 = \sum_i V_i$. Then we have

$$\Phi_1(x, A) = x_0 + \sum_{i=1}^{k} \frac{1}{\rho_i(A)} \{\sin \rho_i(A) x_i + (1 - \cos \rho_i(A)) I_i x_i\}$$

$$\Phi_2(x, A) = \frac{-1}{2} \sum_{i=1}^{k} \frac{1}{\rho_i(A)} \left(1 - \frac{\sin \rho_i(A)}{\rho_i(A)}\right) |x_i|^2 \rho_i,$$

where $(x, A) \in \mathfrak{g}_1 \times \mathscr{A}$, and $B \in \mathscr{A}^*$, and $\mathfrak{g}_2^*$ should be naturally identified with $\mathscr{A}$.

Here we shall prove the following proposition only. (The problem of determining the cut locusoid Γ seems to be rather complicated, which is therefore left to the readers as an exercise.)

Proposition 5.6 *The closed domain $\overline{\Delta}$ coincides with the whole of G.*

We set

$$\hat{V}_j = \sum_{i \in I_j} V_i, \quad 1 \leq j \leq k,$$

$$\hat{V} = \bigcup_{j=1}^{k} \hat{V}_j,$$

where $I_j = \{i \in \mathbb{Z} \mid 0 \leq i \leq k, \ i \neq j\}$. We also define a subset $\hat{\Gamma}$ of G by

$$\hat{\Gamma} = \hat{V} \times \mathfrak{g}_2.$$

Since

$$\|A\| = \max_i |\rho_i(A)|, \quad A \in \mathscr{A},$$

it follows from the expression above for $\Phi_1(x, A)$ that $\Gamma \subset \hat{\Gamma}$. Since $\dim V_i \geq 2$ $(1 \leq i \leq k)$, $\hat{\Gamma}$ is an algebraic set of $\mathfrak{g}_1 \times \mathscr{A}$, and $\operatorname{codim} \hat{\Gamma} \geq 2$. Hence we see that the complement of $\hat{\Gamma}$ in G is a connected open dense subset of G, from which follows that the complement X of Γ in G is connected. Therefore we have shown that $\bar{\Delta} = G$, proving Proposition 5.6.

(C_1) The case where $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_2 = 3$. Let $\mathfrak{k}$ be a compact simple Lie algebra of dimension 3. Hereafter the notations $[\ ,\ ]$ and ad will be exclusively used to denote the bracket operation in the Lie algebra $\mathfrak{k}$ and the adjoint representation of the Lie algebra $\mathfrak{k}$ respectively, while the notation $[[\ ,\ ]]$ to denote the bracket operation in the Lie algebra $\mathfrak{g}$. We denote by $\langle\ ,\ \rangle$ the inner product on $\mathfrak{k}$ defined by the Killing form B of $\mathfrak{k}$: $\langle x, y\rangle = -B(x, y)$ for any $x, y \in \mathfrak{k}$.

Now, the adjoint algebra $\operatorname{ad}(\mathfrak{k})$ of $\mathfrak{k}$ coincides with the space $\operatorname{Skew}(\mathfrak{k})$. Since the given euclidean FGLA, $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, is universal and $\dim \mathfrak{g}_1 = 3$, it follows that the euclidean FGLA may be regarded as associated with the pair $(\mathfrak{k}, \operatorname{ad}(\mathfrak{k}))$ and hence we have the following:

(i) $\mathfrak{g}_1 = \mathfrak{k}$ as euclidean vector spaces,
(ii) $\mathscr{A} = \operatorname{ad}(\mathfrak{k})$,
(iii) $\mathfrak{g}_2 = \mathscr{A}^*$,
(iv) $\langle [[x, y]],\ \operatorname{ad}(z)\rangle = \langle x,\ \operatorname{ad}(z)y\rangle, \quad x, y, z, \in \mathfrak{k}.$

Remark We define an isomorphism φ of $\mathfrak{k}$ onto $\mathfrak{g}_2$ by

$$\langle \varphi(x), \operatorname{ad}(z)\rangle = -\langle x, z\rangle, \quad x, z \in \mathfrak{k}.$$

Then we have

$$[[x, y]] = \varphi([x, y]), \quad x, y \in \mathfrak{k}.$$

The spaces $\mathfrak{g}_2$ and $\mathfrak{k}$ will be soon identified through the isomorphism φ.

First of all we shall give a concrete description of the exponential mapping Φ.

Let z be a nonzero vector of $\mathfrak{k}$. Then there are an orthonormal basis

(e_0, e_1, e_2) of $\mathfrak{k}$ such that

$$[e_0, e_1] = e_2, \quad [e_1, e_2] = e_0, \quad [e_2, e_0] = e_1,$$
$$z = \lambda e_0 \quad \text{with} \quad \lambda = |z|.$$

Hence we obtain

$$\mathrm{ad}(z)e_0 = 0, \quad \mathrm{ad}(z)e_1 = \lambda e_2, \quad \mathrm{ad}(z)e_2 = -\lambda e_1.$$

$\mathscr{A}$ being identified with $\mathrm{ad}(\mathfrak{k})$, $\mathrm{ad}(z)$ gives an element of $\mathscr{A}$. If we put $A = \mathrm{ad}(z)$, we therefore have the following : (i) $k(A) = 1$, (ii) $\lambda_1(A) = \lambda$, (iii) $V_0(A)$ is spanned by e_0, and $V_1(A)$ is spanned by e_1 and e_2 and (iv) $I_1(A)$ is defined by $I_1(A)e_1 = e_2$ and $I_1(A)e_2 = -e_1$. For simplicity we set $I = I_1(A)$. Let us now take another vector x of $\mathfrak{k}$ and consider the point $(x, \mathrm{ad}(z))$ of $\mathfrak{g}_1 \times \mathscr{A}$. Then we set

$$x = \xi_0 e_0 + \xi_1 e_1 + \xi_2 e_2,$$
$$x_0 = \xi_0 e_0, \quad x_1 = \xi_1 e_1 + \xi_2 e_2.$$

We have

$$Ix_1 = -\xi_2 e_1 + \xi_1 e_2, \qquad [x_1, Ix_1] = (\xi_1^2 + \xi_2^2)e_0$$
$$[x_0, x_1] = -\xi_0\xi_2 e_1 + \xi_0\xi_1 e_2, \quad [x_0, Ix_1] = -\xi_0\xi_1 e_1 - \xi_0\xi_2 e_2.$$

In the following we identify $\mathfrak{g}_2$ with $\mathfrak{k}$ through the isomorphism φ. Hence we have $[[x, y]] = [x, y]$ for any $x, y \in \mathfrak{k}$, and $G = \mathfrak{k} \times \mathfrak{k}$. Then a direct calculation proves the following:

$$\Phi_1(x, \mathrm{ad}(z)) = x_0 + \frac{\sin\lambda}{\lambda}x_1 + \frac{1 - \cos\lambda}{\lambda}Ix_1,$$
$$\Phi_2(x, \mathrm{ad}(z)) = \frac{1}{2}\left(\frac{1}{\lambda} - \frac{\sin\lambda}{\lambda^2}\right)[x_1, Ix_1] + \frac{1}{2}\left(\frac{\sin\lambda}{\lambda} + \frac{2(\cos\lambda - 1)}{\lambda^2}\right)[x_0, x_1]$$
$$- \frac{1}{2}\left(\frac{1 + \cos\lambda}{\lambda} - \frac{2\sin\lambda}{\lambda^2}\right)[x_0, Ix_1].$$

Now, we define polynomial functions f_1 and f_{-1} on G as follows:

$$f_\varepsilon(a) = |a_1|^2|a_2|^2 - \langle a_1, a_2\rangle^2 - \frac{\varepsilon}{\pi}|a_1|^3\langle a_1, a_2\rangle,$$

where $\varepsilon \in \{1, -1\}$, and $a = (a_1, a_2) \in G$. We also define open sets X_0, X_1 and X_{-1} of G as follows:

$$X_0 = \{a \in G \mid f_1(a) > 0,\ f_{-1}(a) > 0\},$$
$$X_\varepsilon = \{a \in G \mid f_\varepsilon(a) < 0\},$$

where $\varepsilon \in \{1, -1\}$. Then we shall prove the following.

Proposition 5.7 (1) *The cut locusoid* Γ *may be described as the algebraic set of* G *defined by the equation* $f_1 \cdot f_{-1} = 0$.

(2) *The open sets* X_0, X_1 *and* X_{-1} *are the connected components of the complement* X *of* Γ *in* G.

(3) *The domain* Δ *coincides with the component* X_0.

Unlike the preceding two examples, this proposition indicates that the subset $\overline{\Delta}$ of G does not coincides with the whole of G.

Now, let $(x, \mathrm{ad}(z))$ be any point of $\partial\mathscr{V}$. Since $\|\mathrm{ad}(z)\| = \lambda = 2\pi$, we have

$$\Phi_1(x, \mathrm{ad}(z)) = \xi_0 e_0,$$
$$\Phi_2(x, \mathrm{ad}(z)) = \frac{\xi_1^2 + \xi_2^2}{4\pi}e_0 + \frac{\xi_0\xi_1}{2\pi}e_1 + \frac{\xi_0\xi_2}{2\pi}e_2.$$

If we set $a_1 = \Phi_1(x, \mathrm{ad}(z))$ and $a_2 = \Phi_2(x, \mathrm{ad}(z))$, we easily see that

$$|a_1|^2|a_2|^2 - \langle a_1, a_2\rangle^2 = \frac{\varepsilon}{\pi}|a_1|^3\langle a_1, a_2\rangle,$$

where ε is 1 or -1, and is determined by $\xi_0 = \varepsilon|a_1|$, provided $a \neq 0$. Let us now denote by Γ_0 the algebraic set stated in Proposition 5.7. Then we have shown that $\Gamma \subset \Gamma_0$. Conversely it can be easily verified that $\Gamma \supset \Gamma_0$. Hence we obtain $\Gamma = \Gamma_0$ proving the first assertion.

Let us now consider a moving point $a = (a_1, a_2)$ of G with $a_1 \neq 0$ if we put $\lambda = |a_1|$, $b_1 = a_1/|a_1|$ and $b_2 = a_2/|a_1|^2$, $f_\varepsilon(a)$ may be described as follows:

$$f_\varepsilon(a) = \lambda^6\left(|b_2|^2 - \langle b_1, b_2\rangle^2 - \frac{\varepsilon}{\pi}\langle b_1, b_2\rangle\right).$$

Furthermore λ and b_1 being fixed, b_2 may be described as follows:

$$b_2 = x + yb_1,$$

where $x \in \mathfrak{k}$, $y \in \mathbb{R}$, and $\langle x, b_1\rangle = 0$, and hence $f_\varepsilon(a)$ as follows:

$$f_\varepsilon(a) = \lambda^6\left(|x|^2 - \frac{\varepsilon}{\pi}y\right).$$

From this discussion we easily deduce the second assertion.

It is clear that X_0 is a unique connected component of X whose boundary in G coincides with Γ. Therefore we know from Corollary to Theorem 5.3 that $\Delta = X_0$, proving the third assertion.

Remark We have calculated the kernel of the differential $(d\Phi)_{(x,A)}$ of the exponential mapping Φ at any point (x, A) of G. The result indicates that the behavior in the large of the geodesics issuing from the identity element e of G is, to a great extent, complicated.

Finally we remark that the set $\mathscr{S}$ of singular points of $\mathfrak{g}_1 \times \mathscr{A}$ consists of all $(x, \mathrm{ad}(z)) \in \mathfrak{g}_1 \times \mathscr{A}$ such that x and z are linearly dependent, and that the image $\Phi(\mathscr{S})$ of $\mathscr{S}$ by Φ may be described as follows:

$$\Phi(\mathscr{S}) = \mathfrak{g}_1 \times 0 (\subset \Gamma).$$

(C_2) The case where $\dim \mathfrak{g}_1 = 3$ and $\dim \mathfrak{g}_2 = 2$. Let $\mathfrak{k}$ be as in the preceding example. The notations $[\ ,\ ]$, ad and $\langle\ ,\ \rangle$ will be used in the same meaning as there. The notation $[[\ ,\ ]]'$ will used to denote the bracket operation in the present euclidean FGLA.

Let $\mathfrak{m}$ be a 2-dimensional subspace of $\mathfrak{k}$. Then $\mathrm{ad}(\mathfrak{m})$ gives a 2-dimensional subspace of $\mathrm{Skew}(\mathfrak{k})(= \mathrm{ad}(\mathfrak{k}))$. Since the adjoint group of $\mathfrak{k}$ acts transitively on the Grassmann manifold of all 2-dimensional subspaces of $\mathfrak{k}$, the euclidean FGLA, $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, may be regarded as associated with the pair $(\mathfrak{k}, \mathrm{ad}(\mathfrak{m}))$. Hence we have the following

(i) $\mathfrak{g} = \mathfrak{k}$ as euclidean vector spaces,
(ii) $\mathscr{A} = \mathrm{ad}(\mathfrak{m})$,

(iii) $\mathfrak{g}_2 = \mathscr{A}^*$
(iv) $\langle [[x,y]]', \mathrm{ad}(z)\rangle = \langle x, \mathrm{ad}(z)y\rangle,\ x, y \in \mathfrak{k},\ z \in \mathfrak{m}.$

Let $\mathfrak{m}^\perp$ be the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$. For $x \in \mathfrak{m}$ we denote by $x_\mathfrak{m}$ the $\mathfrak{m}$-component of x with respect to the decomposition: $\mathfrak{k} = \mathfrak{m} + \mathfrak{m}^\perp$. We now define an isomorphism of $\mathfrak{m}$ onto $\mathfrak{g}_2$ by

$$\langle \varphi'(x), \mathrm{ad}(z)\rangle = -\langle x, z\rangle, \quad x,\ z \in \mathfrak{m}.$$

Then we have

$$[[x,y]]' = \varphi'([x,y]_\mathfrak{m}), \quad x, y \in \mathfrak{m}.$$

In the following we identify $\mathfrak{g}_2$ with $\mathfrak{m}$ through the isomorphism φ. Hence we have $[[x,y]]' = [x,y]_\mathfrak{m}$ for any $x, y, \in \mathfrak{k}$ and $G = \mathfrak{k} \times \mathfrak{m}$. Let us now denote by $\hat{\Phi}$ the exponential mapping "Φ" in the preceding paragraph, which maps $\mathfrak{k} \times \mathrm{ad}(\mathfrak{k})$ to $\mathfrak{k} \times \mathfrak{k}$. As is clear from the definition of the exponential mapping, we then have the following equalities:

$$\Phi_1(x, \mathrm{ad}(z)) = \hat{\Phi}_1(x, \mathrm{ad}(z)),$$
$$\Phi_2(x, \mathrm{ad}(z)) = (\hat{\Phi}_2(x, \mathrm{ad}(z)))_\mathfrak{m},$$

where $(x, \mathrm{ad}(z)) \in \mathfrak{g} \times \mathscr{A}$.

Note that the reasoning above is essentially based on the universality for the euclidean FGLA in the preceding example, which can be easily generalized to any euclidean FGLA, $\mathfrak{g}$ and the associated universal euclidean FGLA, $\hat{\mathfrak{g}}$.

Now, we denote by f_ε the restriction to $\mathfrak{m} \times \mathfrak{m}$ of the polynomial functions f_ε on $\mathfrak{k} \times \mathfrak{k}$ defined in the preceding example. Then we shall prove the following.

Proposition 5.8 (1) *The cut locusoid* Γ *may be described as the semi-algebraic set of* $\mathfrak{m} \times \mathfrak{m}$ *which consists of all* $a \in G$ *such that* $f_1'(a) \leq 0$ *or* $f_{-1}'(a) \leq 0$.
(2) *The closed domain* $\overline{\Delta}$ *coincides with the whole of* G.

We fix a unit vector e_2 of $\mathfrak{m}^\perp$. Let $(x, \mathrm{ad}(z))$ be any point of $\mathfrak{g}_1 \times \mathscr{A} (= \mathfrak{k} \times \mathrm{ad}(\mathfrak{m}))$ with $z \neq 0$. Then there is a unique orthonormal basis (e_0, e_1) of $\mathfrak{m}$ such that

$$[e_0, e_1] = e_2, \quad [e_1, e_2] = e_0, \quad [e_2, e_0] = e_1,$$
$$z = \lambda e_0 \quad \text{with} \quad \lambda = |z|.$$

We set

$$x = \xi_0 e_0 + \xi_1 e_1 + \xi_2 e_2.$$

Suppose that $(x, \mathrm{ad}(z))$ is in $\partial\mathscr{V}$. In view of the equalities above for $\Phi_1(x, \mathrm{ad}(z))$ and $\Phi_2(x, \mathrm{ad}(z))$ we then have the following:

$$\Phi_1(x, \mathrm{ad}(z)) = \xi_0 e_0,$$
$$\Phi_2(x, \mathrm{ad}(z)) = \frac{\xi_1^2 + \xi_2^2}{4\pi} e_0 + \frac{\xi_0 \xi_1}{2\pi} e_1.$$

If we set $a_1 = \Phi_1(x, \mathrm{ad}(z))$ and $a_2 = \Phi_2(x, \mathrm{ad}(z))$, we easily see that $(a_1, a_2) \in \mathfrak{m} \times \mathfrak{m}$, and

$$|a_1|^2 |a_2|^2 - \langle a_1, a_2 \rangle^2 \geq \frac{\varepsilon}{\pi} |a_1|^3 \langle a_1, a_2 \rangle,$$

where $\varepsilon = 1$ or -1, and is determined by $\xi_0 = \varepsilon |a_1|$, provided $a_1 \neq 0$. Let us now denote by Γ_0 the semi-algebraic set stated in the proposition. Then we have shown that $\Gamma \subset \Gamma_0$. Conversely it can be easily verified that $\Gamma \supset \Gamma_0$. Hence we obtain $\Gamma = \Gamma_0$, proving the first assertion. Since Γ is a proper semi-algebraic set of $\mathfrak{m} \times \mathfrak{m} (= \mathfrak{k} \times \mathfrak{m})$ it is clear that the complement X of Γ in G is connected. Therefore we have shown that $\overline{\Delta} = G$ proving Proposition 5.8.

Finally we remark that the set $\mathscr{S}$ of singular points of $\mathfrak{g}_1 \times \mathscr{A}$ consists of all $(x, \mathrm{ad}(z)) \in \mathfrak{g}_1 \times \mathscr{A}$ such that $x \in \mathfrak{m}$, and x and z are linearly dependent, and that the image $\Phi(\mathscr{S})$ of Φ by the exponential mapping Φ may be described as follows:

$$\Phi(\mathscr{S}) = \mathfrak{m} \times 0 (\subset \Gamma).$$

Notes Added by the Editors

Let us recall, for the convenience of the readers, the basic notions in the theory of differential systems.

Let M be a manifold. Then a subbundle D of the tangent bundle TM is called a *differential systems* on M. For the sheaf of sections $\underline{D}$ to D, and for $\ell > 0$, we define the (weak) derived system $\underline{D}^\ell$ of $\underline{D}$ by $\underline{D}^1 = \underline{D}$ and

$$\underline{D}^\ell = [\underline{D}, \underline{D}^{\ell-1}] + \underline{D}^{\ell-1}.$$

Set $\underline{D}' = \cup_\ell \underline{D}^\ell$ (see Appendix of [5]).

For a differential system (M, D), the condition (C_0) introduced in the sub-section 1.1 is compared with the following conditions (C) and (C'):

(C) any two points p and q can be connected by a piece-wise smooth integral curve of (M, D).

(C') Chow's condition or the bracket generating condition: $\underline{D}' = \underline{TM}$, the total sheaf of vector fields over M.

For related works on distances and geodesics in nilpotent groups, see [8], [9] for instance. The general theory on sub-Riemann geometry can be seen in [6], [13], [10], [11].

This paper has the origin in Tanaka's original theory on differential system. See for instance [12]. Moreover, to understand the works by Noboru Tanaka we recommend to see also [7].

References

[1] Benedetti R. and Risler J. J., *Real Algebraic and Semi-Algebraic Sets*, Hermann (1990).

[2] Kobayashi S. and Nomizu K., *Foundations of differential geometry I*, Interscience Publishers, a division of John Wiley & Sons, New York, London (1963).

[3] Milnor J., *Morse Theory*, Ann. Math. Studies, **51**, Princeton Univ. Press (1963).

[4] Nagano T., *Linear differential systems with singularities and an application to transitive Lie algebras*. J. Math. Soc. Japan. **118** (1996), 398–404.

[5] Tanaka N., *A differential geometric study on strongly pseudo-convex manifolds*, Lectures in Mathematics, Dept. Math. Kyoto Univ., **9**, Kinokuniya, Tokyo, (1975).

References Added by Editors:

[6] Agrachev A. A., Sarychev A. V., *Control Theory from Geometric Viewpoint*, Encyclopaedia of Mathematical Sciences, **87**, Springer-Verlag,

(2004).

[7] Ezhov V., McLaughlin B., Schmalz G., *From Cartan to Tanaka: Getting Real in the Complex World*. Notices of the Amer. Math. Soc. **58(1)** (2011), 20–27.

[8] Karidi R., *Geometry of Balls in nilpotent Lie groups*. Duke Math. J. **74(2)** (1994), 301–317.

[9] Karidi R., *A note on Carnot geodesics in nilpotent Lie groups*. J. Daynam. Control Systems. **1(4)** (1995), 535–549.

[10] Liu W., Sussman H. J., *Shortest paths for sub-Riemannian metrics on rank-two distributions*, Memoirs of Amer. Math. Soc., **118–564**, Amer. Math. Soc., (1995).

[11] Montgomery R., *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, Mathematical Surveys and Monographs, **91**, Amer. Math. Soc. (2002).

[12] Tanaka N., *On differential systems, graded Lie algebras and pseudo-groups*. J. Math. Kyoto Univ. **10** (1970), 1–82.

[13] Vershik A. M., Gershkovich V. Ya., *Nonholonomic dynamical systems, geometry of distributions and variational problems*, in Dynamical Systems VII, ed. by V. I. Arnol'd, S. P. Novikov, Encyclopaedia of Math. Sci., **16**, Springer-Verlag (1994), pp. 1–81.