

Integral identities for Bi-Laplacian problems and their application to vibrating plates

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Abstract. In this paper we derive boundary integral identities for the bi-Laplacian eigenvalue problems under Dirichlet, Navier and simply-supported boundary conditions. By using these integral identities, we prove that the first eigenvalue of the eigenvalue problem under the simply-supported boundary conditions strictly increases with Poisson's ratio. In addition, we establish the boundary integral expressions for the strain energy calculation of the vibrating plates under the three types of boundary conditions.

Key words: Bi-Laplacian eigenvalue problems, Dirichlet boundary conditions, Simply-supported boundary conditions, Rellich's identity, Pohozaev's identity, Vibrating plates, Poisson's ratio, Rayleigh's conjecture.

1. Introduction

Resonance problems are a major concern in engineering analysis and design. The natural frequencies of a solid structure are the dominant factor which affects the structural dynamic behavior. When designing a structure consisting of a thin plate or thin plate components, it is critical to avoid having a driving frequency which is too close to a natural frequency of the structure. In practice, it is common to assure that the system frequency is much lower than the first natural frequency of the structure. Therefore, the first resonance of a plate is often more important than other resonances.

The bi-Laplacian eigenvalue problems with Dirichlet, Navier and simply-supported boundary conditions are classical problems in solid mechanics. Mechanical behavior of vibrating thin plates with these boundary conditions can be described by the three types of eigenvalue problems. In this paper we first convert the traditional simply-supported boundary condition into a simpler new expression, and then give the problem statements for the three eigenvalue problems in Section 2.

F. Rellich introduced a new test function and applied it to the Laplace operator in 1940 [1]. His idea has been used to establish the Rellich-type

integral identities for polyharmonic equations in R^n [2], [3]. These integral identities have been extended to study the acoustic and elastic Helmholtz problems and the elliptic eigenvalue problem [4], [5], [6], [7], [8]. In addition, the integral identity was used to obtain boundary energy estimates for electromagnetic problems [9]. Recently, a conjecture regarding the theory of electromagnetic resonators has been resolved by means of the integral identities for the Laplacian eigenvalue problems [10].

In Section II, we give three Bi-Laplace eigenvalue problems and show a new form of the boundary condition for the third eigenvalue problem, which greatly simplifies the associated problem. This new form is derived in the appendix of the paper. Following Rellich's idea, in Section 3, we first use an elementary method to derive three boundary integral identities as a theorem for the Dirichlet, Navier and simply-supported bi-Laplacian problems. We then apply the theorem and its corollaries to vibrating-plate problems in Section 4. In the first part of this section we show the dependence of the eigenvalue of the supported problem on Poisson's ratio. In the second part of this section we derive three boundary integral expressions for the strain energies of the vibrating plates. Finally, we will finish this paper by providing a brief conclusion and discussion.

2. Bi-Laplacian Eigenvalue Problems

2.1. Dirichlet and Navier Problems

Assume that $\Omega \subset R^n$, $n \geq 2$, is a bounded domain having a $C^{4,\beta}$ boundary $\partial\Omega$ ($0 < \beta < 1$). Let Λ be an eigenvalue for which the Dirichlet eigenvalue problem,

$$\begin{aligned} \Delta\Delta U &= \Lambda U && \text{in } \Omega, \\ U &= |\nabla U| = 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

has a nontrivial solution U , and let λ be an eigenvalue for which the Navier eigenvalue problem,

$$\begin{aligned} \Delta\Delta V &= \lambda V && \text{in } \Omega, \\ V &= \Delta V = 0 && \text{on } \partial\Omega, \end{aligned} \tag{2}$$

has a nontrivial solution V , where $\Delta\Delta = \Delta^2$ denotes the n -dimensional bi-Laplacian.

2.2. Simply-supported Boundary Conditions

A plate (short for an elastic homogeneous and isotropic thin plate) is a particular two-dimensional representation of a three-dimensional solid occupying a two-dimensional region Ω , which has a much smaller thickness in comparison with the in-plane dimensions [11]. The supported boundary conditions (short for simply-supported boundary conditions) are obtained from their mechanical property and implementation. These conditions are natural, based on physical grounds, and are used in bi-Laplacian eigenvalue problems [11].

We consider a plate under the supported boundary conditions

$$W|_{\partial\Omega} = 0,$$

$$M_\nu|_{\partial\Omega} = 0,$$

where W is the deflection of the plate in the vertical direction, ν is the normal unit vector directed outward from Ω and M_ν is the bending moment about the ν direction. These boundary conditions in two dimensions are expressed as (See [11, p. 94] and [4]),

$$M_\nu|_{\partial\Omega} = \mu\Delta W|_{\partial\Omega} + (1 - \mu) \left[\cos^2 \theta \frac{\partial^2 W}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 W}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 W}{\partial y^2} \right] \Big|_{\partial\Omega} = 0,$$

where μ is Poisson's ratio, ($0 < \mu < 1$), and θ is the angle between the normal ν and x -axis ($\cos \theta = \cos(x, \nu) = \nu_x$). This traditional expression has been used for decades. Payne reduced this supported condition to another expression (See [12, p. 112]). It is shown in the appendix that the factor in the brackets of the above equality is equal to $\partial^2 W / \partial \nu^2$ on $\partial\Omega$. That is

$$\cos^2 \theta \frac{\partial^2 W}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 W}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 W}{\partial y^2} = \frac{\partial^2 W}{\partial \nu^2} \quad \text{on } \partial\Omega.$$

Thus, we obtain

$$M_\nu|_{\partial\Omega} = \mu\Delta W|_{\partial\Omega} + (1 - \mu) \frac{\partial^2 W}{\partial \nu^2} \Big|_{\partial\Omega} = 0.$$

Using the above definition and the relation $\Delta W|_{\partial\Omega} = (\partial^2 W/\partial\nu^2)|_{\partial\Omega} + (n - 1)\kappa(\partial W/\partial\nu)|_{\partial\Omega}$, which is derived in the appendix, and setting $n = 2$, we obtain

$$\Delta W|_{\partial\Omega} = (1 - \mu)\kappa \frac{\partial W}{\partial\nu} \Big|_{\partial\Omega} = c_0 \frac{\partial W}{\partial\nu} \Big|_{\partial\Omega}, \tag{*}$$

where κ is the curvature of $\partial\Omega$ and $c_0 = (1 - \mu)\kappa \geq 0$ for convex domains.

2.3. The Supported Problem and Classical Identities

We now define the supported bi-Laplacian eigenvalue problem, which is called the Steklov eigenvalue problem [3]. Let $\Omega \subset R^2$ be a convex domain defined in Subsection 2.1 and let γ be an eigenvalue for which the bi-Laplacian eigenvalue problem under the supported boundary conditions,

$$\begin{aligned} \Delta\Delta W &= \gamma W && \text{in } \Omega, \\ W &= 0 && \text{on } \partial\Omega, \\ \Delta W &= c_0 \frac{\partial W}{\partial\nu} && \text{on } \partial\Omega, \end{aligned} \tag{3}$$

has a nontrivial solution W . Elliptic regularity theorems ensure that problems (1), (2) and (3) will have nontrivial solutions U , V and $W \in C^4(\bar{\Omega})$, respectively [13]. Assuming that α and u are the eigenvalue and the corresponding eigenfunction of problem (1) or (2) or (3), the Green’s identity shows that

$$\begin{aligned} \int_{\Omega} (\alpha u)ud\Omega &= \int_{\Omega} (\Delta\Delta u)ud\Omega \\ &= \int_{\partial\Omega} \frac{\partial\Delta u}{\partial\nu}udS - \int_{\Omega} \nabla\Delta u \cdot \nabla u d\Omega \\ &= \int_{\partial\Omega} \frac{\partial\Delta u}{\partial\nu}udS - \int_{\partial\Omega} \frac{\partial u}{\partial\nu}\Delta u dS + \int_{\Omega} (\Delta u)^2 d\Omega. \end{aligned} \tag{4}$$

The conditions, $U|_{\partial\Omega} = V|_{\partial\Omega} = W|_{\partial\Omega} = 0$, imply that the tangential derivatives of U , V and W vanish on $\partial\Omega$. By breaking down the gradient into the normal and tangential components, we get $|\nabla U|_{\partial\Omega} = (\partial U/\partial\nu)|_{\partial\Omega}$, $|\nabla V|_{\partial\Omega} = (\partial V/\partial\nu)|_{\partial\Omega}$ and $|\nabla W|_{\partial\Omega} = (\partial W/\partial\nu)|_{\partial\Omega}$. For problems (1) and (2), both of the surface integrals of (4) vanish on $\partial\Omega$. Therefore, the

eigenvalues of the two problems can be expressed by the same formula

$$\Lambda \text{ or } \lambda = \frac{\int_{\Omega} (\Delta u)^2 d\Omega}{\int_{\Omega} u^2 d\Omega}, \tag{5}$$

where u can be either U or V . For problem (3), (4) leads to

$$\gamma = \frac{\int_{\Omega} (\Delta W)^2 d\Omega - \int_{\partial\Omega} c_0 \left(\frac{\partial W}{\partial \nu}\right)^2 dS}{\int_{\Omega} W^2 d\Omega}. \tag{6}$$

The positiveness of the eigenvalues comes from the positivity of the energy functionals and the variational theory of eigenvalues [4], [14].

3. Bi-Laplacian Boundary Integral Identities

Theorem 3.1 *Let Ω be defined as in Section 2.1.*

(i) *Let U be a nontrivial solution of problem (1). Then,*

$$\Lambda = \frac{\int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial^2 U}{\partial \nu^2}\right)^2 dS}{4 \int_{\Omega} U^2 d\Omega}.$$

(ii) *Let V be a nontrivial solution of problem (2). Then,*

$$\lambda = \frac{- \int_{\partial\Omega} (x \cdot \nabla V) \frac{\partial \Delta V}{\partial \nu} dS}{2 \int_{\Omega} V^2 d\Omega}.$$

(iii) *Let $\Omega \subset R^2$ as defined and let W be a nontrivial solution of problem (3). Then,*

$$\begin{aligned} \gamma = & \frac{- \int_{\partial\Omega} (x \cdot \nu) \left(c_0 \frac{\partial W}{\partial \nu}\right)^2 dS + 2 \int_{\partial\Omega} c_0 \frac{\partial W}{\partial \nu} \left(\sum_{i,j}^2 x_i \nu_j \frac{\partial^2 W}{\partial x_i \partial x_j}\right) dS}{4 \int_{\Omega} W^2 d\Omega} \\ & - \frac{2 \int_{\partial\Omega} (x \cdot \nabla W) \frac{\partial}{\partial \nu} \left(c_0 \frac{\partial W}{\partial \nu}\right) ds}{4 \int_{\Omega} W^2 d\Omega}. \end{aligned}$$

Proof of Theorem 3.1. Following Rellich [1], we first multiply both sides of the bi-Laplacian eigenequation of problem (1), (2) and (3) by the Rellich

test function $(x \cdot \nabla u)$, and integrate over Ω to obtain

$$\int_{\Omega} (\Delta \Delta u)(x \cdot \nabla u) d\Omega = \int_{\Omega} \alpha u(x \cdot \nabla u) d\Omega. \quad (7)$$

Applying the Green's identity to (7), the right-hand side of (7) becomes

$$\begin{aligned} \int_{\Omega} \alpha u \sum_{k=1}^n x_k \frac{\partial u}{\partial x_k} d\Omega &= \frac{\alpha}{2} \int_{\Omega} \sum_{k=1}^n x_k \frac{\partial u^2}{\partial x_k} d\Omega \\ &= -\frac{n\alpha}{2} \int_{\Omega} u^2 d\Omega + \frac{\alpha}{2} \int_{\partial\Omega} (x \cdot \nu) u^2 dS, \end{aligned} \quad (8)$$

and the left-hand side of (7) becomes

$$\int_{\Omega} (\Delta \Delta u)(x \cdot \nabla u) d\Omega = - \int_{\Omega} \nabla \Delta u \cdot \nabla (x \cdot \nabla u) d\Omega + \int_{\partial\Omega} \frac{\partial \Delta u}{\partial \nu} (x \cdot \nabla u) dS. \quad (9)$$

The volume integral term of the right-hand side of (9) can be written as

$$\begin{aligned} - \int_{\Omega} \nabla \Delta u \cdot \nabla (x \cdot \nabla u) d\Omega &= - \int_{\Omega} \sum_{j=1}^n \frac{\partial \Delta u}{\partial x_j} \frac{\partial (\sum_{i=1}^n x_i \frac{\partial u}{\partial x_i})}{\partial x_j} d\Omega \\ &= - \int_{\Omega} \sum_{j=1}^n \frac{\partial \Delta u}{\partial x_j} \left(\frac{\partial u}{\partial x_j} + \sum_{i=1}^n x_i \frac{\partial^2 u}{\partial x_i \partial x_j} \right) d\Omega \\ &= \int_{\Omega} \sum_{j=1}^n \Delta u \frac{\partial}{\partial x_j} \left(\frac{\partial u}{\partial x_j} + \sum_{i=1}^n x_i \frac{\partial^2 u}{\partial x_i \partial x_j} \right) d\Omega \\ &\quad - \int_{\partial\Omega} \sum_{j=1}^n \Delta u \left(\frac{\partial u}{\partial x_j} + \sum_{i=1}^n x_i \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \nu_j dS, \end{aligned}$$

where $\nu_j = \cos(x_j, \nu)$ and (x_j, ν) is the angle between the x_j -axis and the normal ν . The volume integral term of the right-hand side of the above equality can be further written as

$$\int_{\Omega} \Delta u \left(2 \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} + \sum_{i,j=1}^n x_i \frac{\partial^3 u}{\partial x_i \partial x_j^2} \right) d\Omega$$

$$\begin{aligned}
 &= \int_{\Omega} \left(2(\Delta u)^2 + \frac{1}{2} \sum_{i=1}^n x_i \frac{\partial(\Delta u)^2}{\partial x_i} \right) d\Omega \\
 &= 2 \int_{\Omega} (\Delta u)^2 d\Omega - \frac{n}{2} \int_{\Omega} (\Delta u)^2 d\Omega + \frac{1}{2} \int_{\partial\Omega} (\Delta u)^2 (x \cdot \nu) dS. \tag{10}
 \end{aligned}$$

From (4), we have

$$\alpha \int_{\Omega} u^2 d\Omega = \int_{\partial\Omega} \frac{\partial\Delta u}{\partial\nu} u dS - \int_{\partial\Omega} \frac{\partial u}{\partial\nu} \Delta u dS + \int_{\Omega} (\Delta u)^2 d\Omega.$$

Expressing $\int_{\Omega} (\Delta u)^2 d\Omega$ in terms of the above equality, (10) becomes

$$\left(2 - \frac{n}{2} \right) \left(\alpha \int_{\Omega} u^2 d\Omega - \int_{\partial\Omega} \frac{\partial\Delta u}{\partial\nu} u dS + \int_{\partial\Omega} \frac{\partial u}{\partial\nu} \Delta u dS \right) + \frac{1}{2} \int_{\partial\Omega} (\Delta u)^2 (x \cdot \nu) dS. \tag{11}$$

Substituting (11) into (9) and equating the result to (7), the following identity results

$$\begin{aligned}
 &\left(2 - \frac{n}{2} \right) \left(\alpha \int_{\Omega} u^2 d\Omega - \int_{\partial\Omega} \frac{\partial\Delta u}{\partial\nu} u dS + \int_{\partial\Omega} \frac{\partial u}{\partial\nu} \Delta u dS \right) \\
 &+ \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) (\Delta u)^2 dS - \int_{\partial\Omega} \Delta u \left(\frac{\partial u}{\partial\nu} + \sum_{i,j=1}^n x_i \nu_j \frac{\partial^2 u}{\partial x_i \partial x_j} \right) dS \\
 &+ \int_{\partial\Omega} \frac{\partial\Delta u}{\partial\nu} (x \cdot \nabla u) dS \\
 &= -\frac{n\alpha}{2} \int_{\Omega} u^2 d\Omega + \frac{\alpha}{2} \int_{\partial\Omega} (x \cdot \nu) u^2 dS,
 \end{aligned}$$

where the relation

$$\begin{aligned}
 &\int_{\partial\Omega} \Delta u \sum_{j=1}^n \left(\frac{\partial u}{\partial x_j} + \sum_{i=1}^n x_i \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \nu_j dS \\
 &= \int_{\partial\Omega} \Delta u \left(\frac{\partial u}{\partial\nu} + \sum_{i,j=1}^n x_i \nu_j \frac{\partial^2 u}{\partial x_i \partial x_j} \right) dS
 \end{aligned}$$

is used. After cancellation, the above identity can be written as

$$\begin{aligned}
 2\alpha \int_{\Omega} u^2 d\Omega &= -\frac{1}{2} \int_{\partial\Omega} (x \cdot \nu)(\Delta u)^2 dS + \int_{\partial\Omega} \Delta u \left(\frac{\partial u}{\partial \nu} + \sum_{i,j=1}^n x_i \nu_j \frac{\partial^2 u}{\partial x_i \partial x_j} \right) dS \\
 &\quad - \int_{\partial\Omega} \frac{\partial \Delta u}{\partial \nu} (x \cdot \nabla u) dS + \frac{\alpha}{2} \int_{\partial\Omega} (x \cdot \nu) u^2 dS \\
 &\quad + \left(2 - \frac{n}{2} \right) \int_{\partial\Omega} \left(\frac{\partial \Delta u}{\partial \nu} u - \Delta u \frac{\partial u}{\partial \nu} \right) dS, \tag{12}
 \end{aligned}$$

which is called Pohozaev's or Rellich's identity. Given this identity, we first consider the Dirichlet boundary conditions of problem (1). Let $u = U$, $\alpha = \Lambda$ and $U|_{\partial\Omega} = |\nabla U|_{\partial\Omega} = 0$, then (12) becomes

$$2\Lambda \int_{\Omega} U^2 d\Omega = -\frac{1}{2} \int_{\partial\Omega} (\Delta U)^2 (x \cdot \nu) dS + \int_{\partial\Omega} \Delta U \left(\sum_{i,j=1}^n x_i \nu_j \frac{\partial^2 U}{\partial x_i \partial x_j} \right) dS.$$

Solving for Λ using $(\partial^2 u / \partial x_i \partial x_j)|_{\partial\Omega} = \nu_i \nu_j (\partial^2 u / \partial \nu^2)|_{\partial\Omega}$ (See the appendix for its derivation), then

$$\begin{aligned}
 \Lambda &= \frac{-\int_{\partial\Omega} (x \cdot \nu)(\Delta U)^2 dS + 2 \int_{\partial\Omega} \Delta U \left(\sum_{i,j=1}^n x_i \nu_j \frac{\partial^2 U}{\partial x_i \partial x_j} \right) dS}{4 \int_{\Omega} U^2 d\Omega} \\
 &= \frac{-\int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial^2 U}{\partial \nu^2} \right)^2 dS + 2 \int_{\partial\Omega} \frac{\partial^2 U}{\partial \nu^2} \left((x \cdot \nu) \frac{\partial^2 U}{\partial \nu^2} \right) dS}{4 \int_{\Omega} U^2 d\Omega} \\
 &= \frac{\int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial^2 U}{\partial \nu^2} \right)^2 dS}{4 \int_{\Omega} U^2 d\Omega}.
 \end{aligned}$$

Secondly, we impose the Navier boundary conditions of problem (2). Let $u = V$, $\alpha = \lambda$ and $V|_{\partial\Omega} = \Delta V|_{\partial\Omega} = 0$, then (12) becomes

$$2\lambda \int_{\Omega} V^2 d\Omega = - \int_{\partial\Omega} \frac{\partial \Delta V}{\partial \nu} (x \cdot \nabla V) dS.$$

Solving for λ , we obtain

$$\lambda = \frac{-\int_{\partial\Omega}(x \cdot \nabla V) \frac{\partial \Delta V}{\partial \nu} dS}{2 \int_{\Omega} V^2 d\Omega} = \frac{-\int_{\partial\Omega}(x \cdot \nu) \frac{\partial V}{\partial \nu} \frac{\partial \Delta V}{\partial \nu} dS}{2 \int_{\Omega} V^2 d\Omega},$$

where $\nabla V = (\partial V / \partial \nu) \cdot \nu$ on $\partial\Omega$ is used. Thirdly, we impose the supported boundary conditions of problem (3). Letting $u = W$, $\alpha = \gamma$, and inserting $W|_{\partial\Omega} = 0$ and $\Delta W|_{\partial\Omega} = c_0(\partial W / \partial \nu)|_{\partial\Omega}$ into (12), we obtain

$$\begin{aligned} 2\gamma \int_{\Omega} W^2 d\Omega &= -\frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \left(c_0 \frac{\partial W}{\partial \nu} \right)^2 dS \\ &+ \int_{\partial\Omega} c_0 \frac{\partial W}{\partial \nu} \left(\sum_{i,j=1}^2 x_i \nu_j \frac{\partial^2 W}{\partial x_i \partial x_j} \right) dS \\ &- \int_{\partial\Omega} \frac{\partial}{\partial \nu} \left(c_0 \frac{\partial W}{\partial \nu} \right) (x \cdot \nabla W) dS. \end{aligned}$$

Solving for γ , we obtain

$$\begin{aligned} \gamma &= \frac{-\int_{\partial\Omega} (x \cdot \nu) \left(c_0 \frac{\partial W}{\partial \nu} \right)^2 dS + 2 \int_{\partial\Omega} c_0 \frac{\partial W}{\partial \nu} \left(\sum_{i,j}^2 x_i \nu_j \frac{\partial^2 W}{\partial x_i \partial x_j} \right) dS}{4 \int_{\Omega} W^2 d\Omega} \\ &- \frac{2 \int_{\partial\Omega} (x \cdot \nabla W) \frac{\partial}{\partial \nu} \left(c_0 \frac{\partial W}{\partial \nu} \right) dS}{4 \int_{\Omega} W^2 d\Omega}. \end{aligned} \quad \square$$

Corollary 3.2

- (i) If U is a solution of problem (1) with $\Lambda > 0$ and if U , in addition, satisfies $(\partial^2 U / \partial \nu^2)|_{\partial\Omega} = 0$, then $U \equiv 0$. In other words, if U is a nontrivial eigenfunction of problems (1) with $\Lambda > 0$, then $\partial^2 U / \partial \nu^2$ cannot be identically zero on $\partial\Omega$.
- (ii) If V is a solution of problem (2) with $\lambda > 0$ and if V , in addition, satisfies $(\partial V / \partial \nu)|_{\partial\Omega} = 0$, then $V \equiv 0$. In other words, if V is a nontrivial eigenfunction of problem (2) with $\lambda > 0$, then $\partial \Delta V / \partial \nu$ or $\partial V / \partial \nu$ cannot be identically zero on $\partial\Omega$.
- (iii) If W is a solution of problem (3) with $\gamma > 0$ and if W , in addition, satisfies $(\partial W / \partial \nu)|_{\partial\Omega} = 0$, then $W \equiv 0$. In other words, if W is a nontrivial eigenfunction of problems (3) with $\gamma > 0$, then $\partial W / \partial \nu$ cannot be identically zero on $\partial\Omega$.

It is seen that a nontrivial eigenfunction of problems (2) or (3) demands $(\partial U/\partial\nu)|_{\partial\Omega} \neq 0$, but a nontrivial eigenfunction of problems (1) requires $(\partial W/\partial\nu)|_{\partial\Omega} = 0$. Therefore, we obtain

Corollary 3.3 *A nontrivial eigenfunction cannot simultaneously be an eigenfunction of the Dirichlet and supported (or Navier) plate of the same shape.*

4. Application to Solid Mechanics

4.1. Effect of Poisson's Ratio on the Eigenvalue

Theorem 4.1 *Let $\Omega \subset R^2$ be a convex domain as defined and let $W(x, \mu) \in C_1^{4,\beta}(\bar{\Omega}, (0, 1))$, i.e., $W(x, \cdot) \in C^{4,\beta}(\bar{\Omega})$ and $W(\cdot, \mu) \in C^1(0, 1)$, be the first nontrivial eigenfunction of the supported eigenvalue problem with the associated eigenvalue γ . Then,*

$$\frac{\partial\gamma}{\partial\mu} > 0.$$

Proof of Theorem 4.1. Since W is the first nontrivial eigenfunction of problem (3), $\partial W/\partial\mu$ is continuously differentiable [15]. By using $\Delta\Delta W = \gamma W$ and differentiating the identity with respect to μ , we obtain $(\partial\Delta\Delta W)/\partial\mu = \gamma(\partial W/\partial\mu) + W(\partial\gamma/\partial\mu)$ and $(\partial\Delta\Delta W)/\partial\mu = \Delta\Delta(\partial W/\partial\mu) = \gamma(\partial W/\partial\mu) + W(\partial\gamma/\partial\mu)$. Multiplying both sides of the second equality by W and integrating in Ω we have

$$\int_{\Omega} W\Delta\Delta\frac{\partial W}{\partial\mu} d\Omega = \int_{\Omega} \left(\gamma W\frac{\partial W}{\partial\mu} + W^2\frac{\partial\gamma}{\partial\mu} \right) d\Omega. \quad (13)$$

Applying the Green's identity, the left-hand side of the above identity becomes

$$\begin{aligned} & \int_{\Omega} W\Delta\Delta\frac{\partial W}{\partial\mu} d\Omega \\ &= \int_{\partial\Omega} W\frac{\partial}{\partial\nu}\Delta\frac{\partial W}{\partial\mu} dS - \int_{\partial\Omega} \frac{\partial W}{\partial\nu}\Delta\frac{\partial W}{\partial\mu} dS + \int_{\partial\Omega} \frac{\partial^2 W}{\partial\mu\partial\nu}\Delta W dS \\ & \quad - \int_{\partial\Omega} \frac{\partial\Delta W}{\partial\nu}\frac{\partial W}{\partial\mu} dS + \int_{\Omega} \Delta\Delta W\frac{\partial W}{\partial\mu} d\Omega. \end{aligned}$$

Using $\Delta\Delta W = \gamma W$ and (13), and imposing the supported boundary conditions $W|_{\partial\Omega} = 0$ and $\Delta W|_{\partial\Omega} = c_0(\partial W/\partial\nu)|_{\partial\Omega}$, where $c_0 = (1 - \mu)\kappa$, we obtain

$$\begin{aligned} \int_{\Omega} W\Delta\Delta \frac{\partial W}{\partial\mu} d\Omega &= - \int_{\partial\Omega} \frac{\partial W}{\partial\nu} \left(-\kappa \frac{\partial W}{\partial\nu} + c_0 \frac{\partial^2 W}{\partial\nu\partial\mu} \right) dS \\ &\quad + \int_{\partial\Omega} c_0 \frac{\partial W}{\partial\nu} \frac{\partial^2 W}{\partial\nu\partial\mu} dS + \int_{\Omega} \gamma W \frac{\partial W}{\partial\mu} d\Omega \\ &= \int_{\partial\Omega} \kappa \left| \frac{\partial W}{\partial\nu} \right|^2 dS + \int_{\Omega} \gamma W \frac{\partial W}{\partial\mu} d\Omega. \end{aligned}$$

By using (13), the above identity can be written as

$$\int_{\partial\Omega} \kappa \left| \frac{\partial W}{\partial\nu} \right|^2 dS = \int_{\Omega} W^2 \frac{\partial\gamma}{\partial\mu} d\Omega.$$

Hence,

$$\frac{\partial\gamma}{\partial\mu} = \frac{\int_{\partial\Omega} \kappa \left| \frac{\partial W}{\partial\nu} \right|^2 dS}{\int_{\Omega} W^2 d\Omega} \geq 0, \tag{14}$$

which shows that γ is a non-decreasing function of μ . Since $\kappa > 0$ on $\partial\Omega$, (14) implies that if $\partial\gamma/\partial\mu = 0$, then $\partial W/\partial\nu = 0$ on $\partial\Omega$. But, Corollary 3.2 (iii) implies $W \equiv 0$ in Ω . This contradicts the assumption that W is a nontrivial solution of problem (3). Hence, $\partial\gamma/\partial\mu \neq 0$. Therefore,

$$\frac{\partial\gamma}{\partial\mu} > 0. \tag{□}$$

Corollary 4.2 *The principal eigenvalue of problem (3) with $\kappa > 0$ is strictly monotonic with respect to Poisson’s ratio μ .*

It is known that a natural frequency ω of a plate is given by $\omega = \sqrt{\gamma(D/\bar{m})}$, where \bar{m} denotes the mass per unit area of the plate, $D = (1/12)(Eh^3/(1 - \mu^2))$, E denotes the modulus of elasticity and h is the thickness of the plate [11]. Therefore, we obtain

Corollary 4.3 *The first natural frequency of a simply-supported plate with*

$\kappa > 0$ increases strictly with Poisson's ratio.

4.2. Boundary Integral Expressions of the Strain Energies

Strain energies of mechanical structures provide a good measure for exceeded stresses or strains in the structures. Their distributions are often used, with certain failure criteria of materials, to evaluate if a structure under certain loads is in a safe condition.

For the Dirichlet problem (1), applying Theorem 3.1 (i) and (5), we have

$$\int_{\Omega} (\Delta U)^2 d\Omega = \frac{1}{4} \int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial^2 U}{\partial \nu^2} \right)^2 dS. \quad (15)$$

For the Navier problem (2), applying Theorem 3.1 (ii) and (5), we have

$$\int_{\Omega} (\Delta V)^2 d\Omega = -\frac{1}{2} \int_{\partial\Omega} (x \cdot \nabla V) \frac{\partial \Delta V}{\partial \nu} dS. \quad (16)$$

For the supported problem (3), applying Theorem 3.1 (iii) and (6), we have

$$\begin{aligned} \int_{\Omega} (\Delta W)^2 d\Omega &= -\frac{1}{4} \int_{\partial\Omega} (x \cdot \nu) \left(c_0 \frac{\partial W}{\partial \nu} \right)^2 dS + \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \frac{\partial c_0}{\partial \nu} \left(\frac{\partial W}{\partial \nu} \right)^2 dS \\ &\quad + \int_{\partial\Omega} c_0 \left(\frac{\partial W}{\partial \nu} \right)^2 dS. \end{aligned} \quad (17)$$

The general expression for the strain energy of a bent thin plate is given by (See [11, p. 95])

$$E_s = \frac{1}{2} D \int_{\Omega} \left\{ \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)^2 - 2(1 - \mu) \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 \right] \right\} d\Omega, \quad (18)$$

where u is the deflection of the plate in the vertical direction and the integration is over the entire surface of the plate. Here u can be the solutions of problems (1) or (2) or (3). In the derivation of the strain energy expressions the following identity is used (See [16, p. 87])

$$2 \int_{\Omega} \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 \right] d\Omega = \int_{\partial\Omega} \left[2 \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial s^2} + \kappa \left(\frac{\partial u}{\partial \nu} \right)^2 + \kappa \left(\frac{\partial u}{\partial s} \right)^2 \right] dS.$$

Under the Dirichlet boundary conditions this identity reduces to

$$\int_{\Omega} \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 \right] d\Omega = 0. \quad (19)$$

Under the Navier or supported boundary conditions this identity reduces to

$$\int_{\Omega} \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 \right] d\Omega = \frac{1}{2} \int_{\partial\Omega} \kappa \left(\frac{\partial u}{\partial \nu} \right)^2 dS. \quad (20)$$

For the Dirichlet problem, substituting (15) and (19) into (18) we obtain

$$E_s^D = \frac{1}{8} D \int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial^2 U}{\partial \nu^2} \right)^2 dS,$$

where E_s^D is the strain energy of the vibrating plate under the Dirichlet boundary conditions. For the Navier problem substituting (16) and (20) into (18) we obtain

$$E_s^N = -\frac{D}{2} \int_{\partial\Omega} \left[\frac{1}{2} (x \cdot \nu) \frac{\partial V}{\partial \nu} \frac{\partial \Delta V}{\partial \nu} + (1 - \mu) \kappa \left(\frac{\partial V}{\partial \nu} \right)^2 \right] dS,$$

where E_s^N is the strain energy of the vibrating plate under the Navier boundary conditions. For the supported problem, substituting (17) and (20) into (18) we obtain

$$\begin{aligned} E_s^S &= -\frac{D}{8} \int_{\partial\Omega} (x \cdot \nu) \left(c_0 \frac{\partial W}{\partial \nu} \right)^2 dS - \frac{D}{4} \int_{\partial\Omega} (x \cdot \nu) \frac{\partial c_0}{\partial \nu} \left(\frac{\partial W}{\partial \nu} \right)^2 dS \\ &\quad + \frac{D}{2} \int_{\partial\Omega} c_0 \left(\frac{\partial W}{\partial \nu} \right)^2 dS - \frac{D}{2} (1 - \mu) \int_{\partial\Omega} \kappa \left(\frac{\partial W}{\partial \nu} \right)^2 dS, \end{aligned}$$

where E_s^S is the strain energy of the vibrating plate under the supported boundary conditions. Notice that E_s^D , E_s^N and E_s^S hold only when $\Omega \subset R^2$

Remark 4.4 It is straightforward to calculate the strain energies at resonance using the three formulas when boundary value data are available.

5. Conclusion and Discussion

The Rellich test function used in this paper allows us to develop boundary-integral representations for the eigenvalues of the three bi-Laplacian operators. Based on these integral identities we have revealed the property of the eigenfunctions of the bi-Laplacian operators, as stated by Corollaries 3.2 or 3.3. Using Corollary 3.2 and the reduced expression (*) of the supported boundary condition, we have proven that the first natural frequency of a simply-supported plate with $\kappa > 0$ increases strictly with Poisson's ratio. This result is of considerable significance in some practical problems. We have also applied Theorem 3.1 to obtain the surface-integral expressions of the strain energies of the vibrating plates. These expressions provide simplified formulas for calculating the strain energies.

It is seen that problem (3) can be reduced to problem (2) if $\partial\Omega$ consists of straight sides (with the Lipschitz corners). It will be of practical significance to show how the principal eigenvalue of problem (3) is affected by changing μ when $\partial\Omega$ is convex, but contains one or more Lipschitz corners.

Rayleigh's conjecture of the clamped vibrating plate was posed more than one hundred years ago and has been solved by Nadirashvili when $n = 2$ [17] and by Ashbaugh and Benguria when $n = 3$ [18]. Much of the efforts has been devoted to prove this conjecture in R^n [19]. In a related vein, we pose a conjecture for the simply-supported principal eigenvalue in terms of the notation used in this paper: Rayleigh's conjecture for the simply-supported vibrating plate

$$\gamma(\Omega) \leq \gamma(\Omega^*) \quad \text{for } \Omega \subset R^2$$

where Ω^* is a disk having the same area as that of the domain Ω . This inequality holds strictly if and only if Ω is not a disk.

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6. Appendix

We shall prove that on $\partial\Omega$ under the condition $(\partial u/\partial\nu)|_{\partial\Omega} = 0$, $\cos^2\theta(\partial^2 u/\partial x^2) + 2\sin\theta\cos\theta(\partial^2 u/\partial x\partial y) + \sin^2\theta(\partial^2 u/\partial y^2) = \partial^2 u/\partial\nu^2$, $\Delta u = (\partial^2 u/\partial\nu^2) + (n-1)\kappa(\partial u/\partial\nu)$ and under the Dirichlet conditions,

$$(\partial^2 u / \partial x_i \partial x_j) = \nu_i \nu_j (\partial^2 u / \partial \nu^2).$$

Let (s_1, \dots, s_{n-1}) be a local coordinates on $\partial\Omega$, i.e., $\partial\Omega$ is locally represented by

$$\tilde{X} : R^{n-1} \rightarrow \partial\Omega.$$

Let

$$\nu(s_1, \dots, s_{n-1})$$

be its unit outer normal of $\partial\Omega$ at $\tilde{X}(s_1, \dots, s_{n-1})$. We also define a local coordinate system near $\partial\Omega$ by

$$X(s_1, \dots, s_{n-1}, t) = \tilde{X}(s_1, \dots, s_{n-1}) - t\nu(s_1, \dots, s_{n-1}).$$

For any u defined near $\partial\Omega$, we have

$$\begin{aligned} \frac{\partial}{\partial x_i} \frac{\partial u}{\partial x_j} &= \frac{\partial}{\partial x_i} \left(\sum_{k=1}^{n-1} \frac{\partial u}{\partial s_k} \frac{\partial s_k}{\partial x_j} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x_j} \right) \\ &= \left(\sum_{k,l=1}^{n-1} \frac{\partial^2 u}{\partial s_l \partial s_k} \frac{\partial s_l}{\partial x_i} + \sum_{k=1}^{n-1} \frac{\partial^2 u}{\partial t \partial s_k} \frac{\partial t}{\partial x_i} \right) \frac{\partial s_k}{\partial x_j} + \sum_{k=1}^{n-1} \frac{\partial u}{\partial s_k} \frac{\partial^2 s_k}{\partial x_i \partial x_j} \\ &\quad + \left(\sum_{l=1}^{n-1} \frac{\partial^2 u}{\partial t \partial s_l} \frac{\partial s_l}{\partial x_i} + \frac{\partial^2 u}{\partial t^2} \frac{\partial t}{\partial x_i} \right) \frac{\partial t}{\partial x_j} + \frac{\partial u}{\partial t} \frac{\partial^2 t}{\partial x_i \partial x_j}. \end{aligned}$$

On $\partial\Omega$, we have $\partial u / \partial s_k = 0$ and $\partial^2 u / \partial s_l \partial s_k = 0$. Hence,

$$\begin{aligned} \frac{\partial^2 u}{\partial x_i \partial x_j} &= \sum_{k=1}^{n-1} \frac{\partial^2 u}{\partial t \partial s_k} \left(\frac{\partial t}{\partial x_i} \frac{\partial s_k}{\partial x_j} + \frac{\partial t}{\partial x_j} \frac{\partial s_k}{\partial x_i} \right) \\ &\quad + \frac{\partial^2 u}{\partial t^2} \frac{\partial t}{\partial x_i} \frac{\partial t}{\partial x_j} + \frac{\partial u}{\partial t} \frac{\partial^2 t}{\partial x_i \partial x_j}. \end{aligned} \tag{21}$$

We also have $\nabla t = -\nu$, $\nabla s \cdot \nabla t = 0$, $\partial^2 u / \partial \nu^2 = \partial^2 u / \partial t^2$, $\partial u / \partial \nu = -\partial u / \partial t$ on $\partial\Omega$. Using (21), we obtain

$$\begin{aligned}
\Delta u &= \sum_{i=j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^n \sum_{k=1}^{n-1} \frac{\partial^2 u}{\partial t \partial s_k} \left(\frac{\partial t}{\partial x_i} \frac{\partial s_k}{\partial x_i} + \frac{\partial t}{\partial x_i} \frac{\partial s_k}{\partial x_i} \right) \\
&\quad + \sum_{i=1}^n \left(\frac{\partial^2 u}{\partial t^2} \frac{\partial t}{\partial x_i} \frac{\partial t}{\partial x_i} + \frac{\partial u}{\partial t} \frac{\partial^2 t}{\partial x_i \partial x_i} \right) \\
&= 2 \sum_{k=1}^{n-1} \sum_{i=1}^n \frac{\partial^2 u}{\partial t \partial s_k} \frac{\partial t}{\partial x_i} \frac{\partial s_k}{\partial x_i} + \frac{\partial^2 u}{\partial t^2} \sum_{i=1}^n \frac{\partial t}{\partial x_i} \frac{\partial t}{\partial x_i} + \frac{\partial u}{\partial t} \sum_{i=1}^n \frac{\partial^2 t}{\partial x_i \partial x_i} \\
&= 2 \sum_{k=1}^{n-1} \frac{\partial^2 u}{\partial t \partial s_k} \nabla t \cdot \nabla s_k + \frac{\partial^2 u}{\partial t^2} (-\nu)^2 + \frac{\partial u}{\partial t} \Delta t \\
&= \frac{\partial^2 u}{\partial \nu^2} + (n-1)\kappa \frac{\partial u}{\partial \nu} \quad \text{on } \partial\Omega,
\end{aligned}$$

where, from differential geometry, $\Delta t = -(n-1)\kappa$ is used. For the Dirichlet conditions, since $u|_{\partial\Omega} = (\partial u / \partial \nu)|_{\partial\Omega} = 0$ implies $(\partial^2 u / \partial t \partial s_k)|_{\partial\Omega} = 0$, based on (21),

$$\begin{aligned}
\frac{\partial^2 u}{\partial x_i \partial x_j} &= \sum_{k=1}^{n-1} \frac{\partial^2 u}{\partial t \partial s_k} \left(-\nu_i \frac{\partial s_k}{\partial x_j} - \nu_j \frac{\partial s_k}{\partial x_i} \right) + \nu_i \nu_j \frac{\partial^2 u}{\partial \nu^2} - \frac{\partial u}{\partial \nu} \frac{\partial^2 t}{\partial x_i \partial x_j} \\
&= \nu_i \nu_j \frac{\partial^2 u}{\partial \nu^2} \quad \text{on } \partial\Omega.
\end{aligned}$$

In a two dimensional case, for any u defined near $\partial\Omega$, we choose the coordinate system (s, t) , such that t is the distance from a point x in Ω to $\partial\Omega$ and s is the arc-length parameter on $\partial\Omega$ in the counter-clockwise direction. We write $\nu = (\nu_x, \nu_y)$ and have $\nabla s = (-\nu_y, \nu_x)$. Hence, based on (21), on $\partial\Omega$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t \partial s} \left(\frac{\partial t}{\partial x} \frac{\partial s}{\partial x} + \frac{\partial t}{\partial x} \frac{\partial s}{\partial x} \right) + \frac{\partial^2 u}{\partial t^2} \frac{\partial t}{\partial x} \frac{\partial t}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial^2 t}{\partial x^2} \\
&= 2\nu_x \nu_y \frac{\partial^2 u}{\partial t \partial s} + \nu_x^2 \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 t}{\partial x^2} \frac{\partial u}{\partial t},
\end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= -2\nu_x \nu_y \frac{\partial^2 u}{\partial t \partial s} + \nu_y^2 \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 t}{\partial y^2} \frac{\partial u}{\partial t}, \\ \frac{\partial^2 u}{\partial x \partial y} &= (-\nu_x^2 + \nu_y^2) \frac{\partial^2 u}{\partial t \partial s} + \nu_x \nu_y \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 t}{\partial x \partial y} \frac{\partial u}{\partial t}. \end{aligned} \tag{22}$$

Let $\cos \theta = \cos(x, \nu) = \nu_x$, $\sin \theta = \cos(y, \nu) = \nu_y$ and define

$$Gu = \frac{\partial^2 u}{\partial x^2} \cos^2 \theta + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \theta \sin \theta + \frac{\partial^2 u}{\partial y^2} \sin^2 \theta$$

Substituting (22) into the above expression for G , then

$$\begin{aligned} Gu &= \frac{\partial^2 u}{\partial t \partial s_k} (2\nu_x^3 \nu_y - 2\nu_x^3 \nu_y + 2\nu_x \nu_y^3 - 2\nu_x \nu_y^3) \\ &\quad + \frac{\partial^2 u}{\partial t^2} (\nu_x^4 + 2\nu_x^2 \nu_y^2 + \nu_y^4) + \frac{\partial u}{\partial t} \left(\nu_x^2 \frac{\partial^2 t}{\partial x^2} + 2\nu_x \nu_y \frac{\partial^2 t}{\partial x \partial y} + \nu_y^2 \frac{\partial^2 t}{\partial y^2} \right) \\ &= \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \left(\nu_x^2 \frac{\partial^2 t}{\partial x^2} + 2\nu_x \nu_y \frac{\partial^2 t}{\partial x \partial y} + \nu_y^2 \frac{\partial^2 t}{\partial y^2} \right) \\ &= \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} Gt = \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \nu^2} \quad \text{on } \partial\Omega, \end{aligned}$$

where we have used the fact that

$$\begin{aligned} 0 &= \frac{\partial^2 t}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial t}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial t}{\partial y} \frac{\partial y}{\partial t} \right) \\ &= \frac{\partial^2 t}{\partial x^2} \left(\frac{\partial x}{\partial t} \right)^2 + 2 \frac{\partial^2 t}{\partial x \partial y} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 t}{\partial y^2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{\partial t}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial t}{\partial y} \frac{\partial^2 y}{\partial t^2} \\ &= \left(\nu_x^2 \frac{\partial^2 t}{\partial x^2} + 2\nu_x \nu_y \frac{\partial^2 t}{\partial x \partial y} + \nu_y^2 \frac{\partial^2 t}{\partial y^2} \right) = Gt \quad \text{on } \partial\Omega, \end{aligned}$$

since $(\partial^2 x / \partial t^2)|_{\partial\Omega} = (\partial^2 y / \partial t^2)|_{\partial\Omega} = 0$. □

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