Compact orientable submanifolds in a Riemannian manifold of constant curvature admitting certain tensor fields

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Introduction.

It has been proved by H. Liebmann [8] that an ovaloid with constant mean curvature in a 3-dimensional Euclidean space is a sphere. The analogous problem for a closed hypersurface in a Riemannian manifold admitting an infinitesimal conformal or homothetic transformation investigated by Y. Katsurada [3], [4] and K. Yano [15]. In [4], Y. Katsurada proved the following theorem by making use of integral formulas.

THEOREM 0.1. (Y. Katsurada) *Let* $\widetilde{M}^{n+1}$ *be an Einstein manifold which admits a vector field* ξ^λ *generating a continuous one-parameter group of conformal transformation in* $\widetilde{M}^{n+1}$ *and* M^n *a closed orientable hypersurface in* $\widetilde{M}^{n+1}$ *such that*

(i) $H_1 = const.$,

(ii) $C_\lambda \xi^\lambda$ *has fixed sign on* M^n.

Then every point of M^n *is umbilic, where* H_1 *and* C_λ *denote the first mean curvature and covariant component of a unit normal vector of* M^n *respectively.*

Certain generalization of Theorem 0.1 for a closed orientable submanifold of codimension greater than 1 in a Riemannian manifold have been studied by Y. Katsurada [5], [6], H. Kôjyô [5] and T. Nagai [6], [11]. They have given the condition that the submanifold be pseudo umbilical by making use of integral formulas.

On the other hand M. Okumura [12] proved the following theorem.

THEOREM 0.2. (M. Okumura) *Let* S^{2n+1} *be an odd dimensional sphere with the natural normal contact structure* $(\phi_\lambda{}^\mu, \xi^\lambda, \eta_\lambda)$ *and* M^{2n-1} *a compact orientable submanifold of codimension* 2 *in* S^{2n+1} *such that*

(i) *the mean curvature vector field* H^λ *of* M^{2n-1} *is parallel with respect to the connection of the normal bundle,*

(ii) $\phi_{\lambda\mu} C^\lambda D^\mu$ *has fixed sign on* M^{2n-1}.

Then M^{2n-1} *is totally umbilical, where* C^λ *and* D^λ *denote mutually orthogonal* *unit normal vector fields of* M^{2n-1}.

To prove the above theorem M. Okumura made use of the fact that the structure tensor $\phi_{\lambda\mu}$ is a conformal Killing tensor of order 2 which has been defined by S. Tachibana [14].

In the previous paper [9], the present author studied on a submanifold of codimension p in a sphere by making use of the existence of a conformal Killing tensor field of order p which has been defined by T. Kashiwada [**2**].

In this paper, the author studies a submanifold of codimension p in a Riemannian manifold of constant curvature admitting a conformal Killing tensor field of order p and proves that the submanifold is totally umbilical under certain assumptions by making use of integral formulas.

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§ 1. Conformal Killing tensors.

Recently S. Tachibana [14] has introduced the notion of conformal Killing tensor of order 2 in a Riemannian manifold and T. Kashiwada [2] has given the definition of conformal Killing tensor of order p in a Riemannian manifold. They discussed such the tensor and obtained some results.

Let $\widetilde{M}^{n+p}$ be a $(n+p)$-dimensional Riemannian manifold with the metric tensor $G_{\lambda\mu}$. We call a skew symmetric tensor $F_{\lambda_1\cdots\lambda_p}$ a conformal Killing tensor of order p if there exists a skew symmetric tensor $f_{\lambda_1\cdots\lambda_{p-1}}$ such that

$$(1.1)\qquad \begin{aligned}\nabla_\lambda F_{\lambda_1\cdots\lambda_p}+\nabla_{\lambda_1} F_{\lambda\lambda_2\cdots\lambda_p} &= 2f_{\lambda_2\cdots\lambda_p}G_{\lambda\lambda_1}\\ &-\sum_{a=2}^{p}(-1)^a(f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p}G_{\lambda\lambda_a}+f_{\lambda\lambda_2\cdots\hat{\lambda}_a\cdots\lambda_p}G_{\lambda_1\lambda_a}),\end{aligned}$$

where $\hat{\lambda}_a$ means that λ_a is omitted. This $f_{\lambda_1\cdots\lambda_{p-1}}$ is called the associated tensor field of $F_{\lambda_1\cdots\lambda_p}$. From (1.1), we have

$$\begin{aligned}&\nabla_\lambda F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}+\nabla_{\lambda_a} F_{\lambda_1\cdots\lambda\cdots\lambda_p}\\ &\quad=(-1)^{a-1}(\nabla_\lambda F_{\lambda_a\lambda_1\cdots\lambda_{a-1}\lambda_{a+1}\cdots\lambda_p}+\nabla_{\lambda_a} F_{\lambda\lambda_1\cdots\lambda_{a-1}\lambda_{a+1}\cdots\lambda_p})\\ &\quad=(-1)^{a-1}\Big\{2f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p}G_{\lambda\lambda_a}-\sum_{b=1}^{a-1}(-1)^{b+1}(f_{\lambda_a\lambda_1\cdots\hat{\lambda}_b\cdots\lambda_{a-1}\lambda_{a+1}\cdots\lambda_p}G_{\lambda\lambda_b}\\ &\qquad+f_{\lambda\lambda_1\cdots\hat{\lambda}_b\cdots\lambda_{a-1}\lambda_{a+1}\cdots\lambda_p}G_{\lambda_a\lambda_b})-\sum_{c=a+1}^{p}(-1)^c(f_{\lambda_a\lambda_1\cdots\lambda_{a-1}\lambda_{a+1}\cdots\hat{\lambda}_c\cdots\lambda_p}G_{\lambda\lambda_c}\\ &\qquad+f_{\lambda\lambda_1\cdots\lambda_{a-1}\lambda_{a+1}\cdots\hat{\lambda}_c\cdots\lambda_p}G_{\lambda_a\lambda_c})\Big\}\\ &\quad=(-1)^{a-1}\Big\{2f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p}G_{\lambda\lambda_a}\end{aligned}$$

$$-\sum_{b=1}^{a-1}(-1)^{b+1}(-1)^{a-2}(f_{\lambda_1\cdots\hat{\lambda}_b\cdots\lambda_a\cdots\lambda_p}G_{\lambda\lambda_b}+f_{\lambda_1\cdots\hat{\lambda}_b\cdots\lambda\cdots\lambda_p}G_{\lambda_a\lambda_b})$$

$$-\sum_{c=a+1}^{p}(-1)^{c}(-1)^{a-1}(f_{\lambda_1\cdots\lambda_a\cdots\hat{\lambda}_c\cdots\lambda_p}G_{\lambda\lambda_c}+f_{\lambda_1\cdots\lambda\cdots\hat{\lambda}_c\cdots\lambda_p}G_{\lambda_a\lambda_c})\Big\},$$

from which we find

$$\begin{aligned}\nabla_\lambda F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}&+\nabla_{\lambda_a}F_{\lambda_1\cdots\lambda\cdots\lambda_p}\\ &=-2(-1)^a f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p}G_{\lambda\lambda_a}\\ &\quad-\sum_{\substack{b=1\\ b\neq a}}^{p}(-1)^b(f_{\lambda_1\cdots\hat{\lambda}_b\cdots\lambda_a\cdots\lambda_p}G_{\lambda\lambda_b}+f_{\lambda_1\cdots\hat{\lambda}_b\cdots\lambda\cdots\lambda_p}G_{\lambda_a\lambda_b}).\end{aligned}\tag{1.2}$$

§ 2. Submanifolds in a Riemannian manifold of constant curvature.

Let $\widetilde{M}^{n+p}$ be a $(n+p)$-dimensional Riemannian manifold of constant curvature with the metric tensor $G_{\lambda\mu}$. Then the curvature tensor $\widetilde{R}_{\lambda\mu\nu\kappa}$ of $\widetilde{M}^{n+p}$ has the form

$$\widetilde{R}_{\lambda\mu\nu\kappa}=k(G_{\lambda\kappa}G_{\mu\nu}-G_{\lambda\nu}G_{\mu\kappa}),\qquad k=\text{const.}.\tag{2.1}$$

Let M^n be an orientable submanifold of codimension p in $\widetilde{M}^{n+p}$. We denote by $\{X^\lambda\}$, $\lambda=1,\cdots,n+p$, the local coordinates of $\widetilde{M}^{n+p}$ and by $\{x^i\}$, $i=1,\cdots,n$, the local coordinates of M^n. Then the submanifold M^n is expressed by the equation

$$X^\lambda=X^\lambda(x^i),\qquad \begin{array}{l}\lambda=1,\cdots,n+p,\\ i=1,\cdots,n.\end{array}\tag{2.2}$$

We put

$$B_i^\lambda=\partial X^\lambda/\partial x^i.\tag{2.3}$$

Then n vectors B_i^λ span the tangent plane of M^n at each point of M^n and the Riemannian metric tensor g_{ji} on M^n induced from $G_{\lambda\mu}$ is given by

$$g_{ji}=G_{\lambda\mu}B_j^\lambda B_i^\mu.\tag{2.4}$$

Next we choose p mutually orthogonal unit normal vectors N_A^λ $(A=n+1,\cdots,n+p)$. Let H_{Aji} $(A=n+1,\cdots,n+p)$ be the second fundamental tensor with respect to N_A^λ and L_{ABj} the third fundamental tensor. Then the Gauss and Weingarten equations are written by

$$\begin{aligned}\nabla_j B_i^\lambda&=\sum_A H_{Aji}N_A^\lambda,\\ \nabla_j N_A^\lambda&=-H_{Aj}{}^hB_h^\lambda+\sum_B L_{ABj}N_B^\lambda.\end{aligned}\tag{2.5}$$

The mean curvature vector field H^λ of M^n is given by

$$H^\lambda = \frac{1}{n}\sum_A H_{At}{}^t N_A{}^\lambda, \tag{2.6}$$

and H^λ is independent of the choice of mutually orthogonal unit normal vectors.

For a normal vector N^λ, if the normal part of $\nabla_j N^\lambda$ vanishes identically along M^n, then we say that N^λ is parallel with respect to the connection of the normal bundle.

When there exists mutually orthogonal unit normal vector fields $N_A{}^\lambda$ $(A=n+1, \cdots, n+p)$ such that $L_{ABj}=0$, we call that the connection of the normal bundle is trivial.

Since the curvature tensor $\tilde{R}_{\lambda\mu\nu\kappa}$ of $\tilde{M}^{n+p}$ has the form (2.1), the equations of Gauss, Codazzi and Ricci are written as

$$R_{kjih} = k(g_{kh}g_{ji} - g_{ki}g_{jh}) + \sum_A (H_{Akh}H_{Aji} - H_{Aki}H_{Ajh}), \tag{2.7}$$

$$\nabla_k H_{Aji} - \nabla_j H_{Aki} + \sum_B (H_{Bji}L_{BAk} - H_{Bki}L_{BAj}) = 0, \tag{2.8}$$

$$\begin{aligned} &H_{Ak}{}^i H_{Bji} - H_{Aj}{}^i H_{Bki} + \nabla_k L_{ABj} - \nabla_j L_{ABk} \\ &\qquad + \sum_D (L_{ADj}L_{DBk} - L_{ADk}L_{DBj}) = 0. \end{aligned} \tag{2.9}$$

The following lemmas are given in the previous paper [9].

LEMMA 2.1. *A necessary and sufficient condition for M^n to be totally umbilical is that the following relations are satisfied*:

$$H_{Aji}H_A{}^{ji} = \frac{1}{n}(H_{At}{}^t)^2, \qquad A=n+1, \cdots, n+p. \tag{2.10}$$

LEMMA 2.2. *In order that the mean curvature vector field H^λ of M^n is parallel with respect to the connection of the normal bundle, it is necessary and sufficient that*

$$\nabla_j H_{At}{}^t = -\sum_B H_{Bt}{}^t L_{BAj}. \tag{2.11}$$

LEMMA 2.3. *If the mean curvature vector field H^λ of M^n is parallel with respect to the connection of the normal bundle, then the mean curvature of M^n is constant.*

LEMMA 2.4. *Assume that the mean curvature vector field H^λ of M^n is parallel with respect to the connection of the normal bundle. Then there exists the following relation*

$$\nabla_j H_{Ak}{}^j = -\sum_B H_{Bk}{}^j L_{BAj}. \tag{2.12}$$

LEMMA 2.5. *The connection of the normal bundle is trivial if and only if that the following relation is satisfied*:

$$H_{Ak}{}^{i}H_{Bji}=H_{Aj}{}^{i}H_{Bki}\,. \tag{2.13}$$

REMARK. When $p=1$, the connection of the normal bundle is trivial under no assumption. When $p=2$, the connection of the normal bundle is trivial under the condition that the mean curvature vector field H^{λ} of M^n is parallel with respect to the connection of the normal bundle.

§ 3. Integral formulas.

Let $\tilde{M}^{n+p}$ be a $(n+p)$-dimensional Riemannian manifold of constant curvature admitting a conformal Killing tensor field $F_{\lambda_1\cdots\lambda_p}$ of order p with the associated tensor field $f_{\lambda_1\cdots\lambda_{p-1}}$ and M^n a compact orientable submanifold of codimension p in $\tilde{M}^{n+p}$. In this section, we assume that the mean curvature vector field H^{λ} of M^n is parallel with respect to the connection of the normal bundle and the connection of the normal bundle is trivial.

Now we put

$$r=F_{\lambda_1\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots N_{n+p}{}^{\lambda_p}\,, \tag{3.1}$$

$$v_i=\sum_{a=1}^{p}H_{n+ai}{}^{h}F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots B_h{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}\,, \tag{3.2}$$

$$w_i=\sum_{a=1}^{p}H_{n+at}{}^{t}F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots B_i{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}\,. \tag{3.3}$$

Then we see that r, v_i and w_i are independent of the choice of mutually orthogonal unit normal vectors in the previous paper [9].

Differentiating (3.2) covariantly and making use of (2.5), we have

$$\begin{aligned}
\nabla_j v_i=&\sum_{a=1}^{p}\nabla_j H_{n+ai}{}^{h}F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots B_h{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}\\
&+\sum_{a=1}^{p}H_{n+ai}{}^{h}B_j{}^{\lambda}\nabla_{\lambda}F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots B_h{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}\\
&+\sum_{a=1}^{p}H_{n+ai}{}^{h}\sum_{\substack{c=1\\ c\neq a}}^{p}F_{\lambda_1\cdots\lambda_c\cdots\lambda_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots\Big(-H_{n+cj}{}^{k}B_k{}^{\lambda_c}+\sum_{B}L_{n+cBj}N_B{}^{\lambda_c}\Big)\cdots\\
&\cdots B_h{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}\\
&+\sum_{a=1}^{p}H_{n+ai}{}^{h}F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots\sum_{B}H_{Bjh}N_B{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}\\
=&\sum_{a=1}^{p}\Big(\nabla_j H_{n+ai}{}^{h}+\sum_{B}H_{Bi}{}^{h}L_{Bn+aj}\Big)F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots B_h{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}
\end{aligned}$$

$$+\sum_{a=1}^{p} H_{n+ai}{}^{h} B_j{}^{\lambda} \nabla_\lambda F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{n+1}{}^{\lambda_1}\cdots B_h{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$

$$-\sum_{\substack{a,c=1\\a\neq c}}^{p} H_{n+ai}{}^{h} H_{n+cj}{}^{k} F_{\lambda_1\cdots\lambda_c\cdots\lambda_a\cdots\lambda_p} N_{n+1}{}^{\lambda_1}\cdots B_k{}^{\lambda_c}\cdots B_h{}^{\lambda_r}\cdots N_{n+p}{}^{\lambda_p}$$

$$+r\sum_{a=1}^{n} H_{n+ai}{}^{h} H_{n+ajh}$$

by virtue of the skew symmetry of $F_{\lambda_1\cdots\lambda_p}$. Therefore we find

$$\nabla^j v_j=\frac{1}{2}\sum_{a=1}^{p} H_{n+a}{}^{ji} B_j{}^{\lambda}(\nabla_\lambda F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}+\nabla_{\lambda_a} F_{\lambda_1\cdots\lambda\cdots\lambda_p}) N_{n+1}{}^{\lambda_1}\cdots B_i{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$
$$+r\sum_A H_{Aji} H_A{}^{ji}$$
$$=\frac{1}{2}\sum_{a=1}^{p} H_{n+a}{}^{ji} B_j{}^{\lambda}\Big\{-2(-1)^a f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p} G_{\lambda\lambda_a}$$
$$-\sum_{\substack{b=1\\b\neq a}}^{p}(-1)^b (f_{\lambda_1\cdots\hat{\lambda}_b\cdots\lambda_a\cdots\lambda_p} G_{\lambda\lambda_b}+f_{\lambda_1\cdots\hat{\lambda}_b\cdots\lambda\cdots\lambda_p} G_{\lambda_a\lambda_b})\Big\} N_{n+1}{}^{\lambda_1}\cdots B_i{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$
$$+r\sum_A H_{Aji} H_A{}^{ji}$$

by virtue of our assumption, (1. 2), (2. 12) and (2. 13), from which we obtain

$$\nabla^j v_j=-\sum_{a=1}^{p}(-1)^a H_{n+at}{}^{t} f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p} N_{n+1}{}^{\lambda_1}\cdots \hat{N}_{n+a}{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$
$$+r\sum_A H_{Aji} H_A{}^{ji},$$

where $\hat{N}_{n+a}{}^{\lambda_a}$ means that $N_{n+a}{}^{\lambda_a}$ is omitted. Consequently we have the following integral formula

$$(3.4)\qquad \int_{M^n}\Big\{r\sum_A H_{Aji} H_A{}^{ji}$$
$$-\sum_{a=1}^{p}(-1)^a H_{n+at}{}^{t} f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p} N_{n+1}{}^{\lambda_1}\cdots \hat{N}_{n+a}{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}\Big\} dM=0$$

by virtue of Green's theorem.

Differentiating (3. 3) covariantly, we have

$$\nabla_j w_i=\sum_{a=1}^{p}\nabla_j H_{n+at}{}^{t} F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{n+1}{}^{\lambda_1}\cdots B_i{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$
$$+\sum_{a=1}^{n} H_{n+at}{}^{t} B_j{}^{\lambda}\nabla_\lambda F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{n+1}{}^{\lambda_1}\cdots B_i{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$
$$+\sum_{a=1}^{p} H_{n+at}{}^{t}\sum_{\substack{c=1\\c\neq a}}^{p} F_{\lambda_1\cdots\lambda_c\cdots\lambda_a\cdots\lambda_p} N_{n+1}{}^{\lambda_1}\cdots(-H_{n+cj}{}^{h} B_h{}^{\lambda_c}$$

$$+\sum_{B} L_{n+cBj}N_B{}^{\lambda_c})\cdots B_i{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$

$$+\sum_{a=1}^{p} H_{n+at}{}^{t}F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots\sum_{B} H_{Bji}N_B{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$

$$=\sum_{a=1}^{p}(\nabla_j H_{n+at}{}^{t}+\sum_{B} H_{Bt}{}^{t}L_{Bn+aj})F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots B_i{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$

$$+\sum_{a=1}^{p} H_{n+at}{}^{t}B_j{}^{\lambda}\nabla_\lambda F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots B_i{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$

$$-\sum_{\substack{a,c=1\\ a\neq c}}^{p} H_{n+at}{}^{t}H_{n+cj}{}^{h}F_{\lambda_1\cdots\lambda_c\cdots\lambda_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots B_h{}^{\lambda_c}\cdots B_i{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$

$$+r\sum_{a=1}^{p} H_{n+at}{}^{t}H_{n+aji}\,,$$

from which we find

$$\nabla^j w_j=\frac{1}{2}\sum_{a=1}^{p} H_{n+at}{}^{t}B^{j\lambda}(\nabla_\lambda F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}+\nabla_{\lambda_a}F_{\lambda_1\cdots\lambda\cdots\lambda_p})N_{n+1}{}^{\lambda_1}\cdots B_j{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$

$$+r\sum_{A}(H_{At}{}^{t})^2$$

$$=\frac{1}{2}\sum_{a=1}^{p} H_{n+at}{}^{t}B^{j\lambda}\Big\{-2(-1)^a f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p}G_{\lambda\lambda_a}$$

$$-\sum_{\substack{b=1\\ b\neq a}}^{p}(-1)^b(f_{\lambda_1\cdots\hat{\lambda}_b\cdots\lambda_a\cdots\lambda_p}G_{\lambda\lambda_b}+f_{\lambda_1\cdots\hat{\lambda}_b\cdots\lambda\cdots\lambda_p}G_{\lambda_a\lambda_b})\Big\}N_{n+1}{}^{\lambda_1}\cdots B_j{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$

$$+r\sum_{A}(H_{At}{}^{t})^2$$

by virtue of our assumption and (1.2). Thus we have

$$\nabla^j w_j=-n\sum_{a=1}^{p}(-1)^a H_{n+at}{}^{t}f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots\hat{N}_{n+a}{}^{\lambda_a}\cdots N_{n+p}{}^{\lambda_p}$$

$$+r\sum_{A}(H_{At}{}^{t})^2\,.$$

Therefore we obtain the following integral formula

$$\int_{M^n}\Big\{r\sum_{A}(H_{At}{}^{t})^2 -n\sum_{a=1}^{p}(-1)^a H_{n+at}{}^{t}f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p}N_{n+1}{}^{\lambda_1}\cdots\hat{N}_{n+a}{}^{\lambda_r}\cdots N_{n+p}{}^{\lambda_l}\Big\}dM=0 \tag{3.5}$$

by means of Green's theorem.

From (3.4) $-\dfrac{1}{n}$ (3.5), we have the following integral formula

$$\int_{M^n} r\sum_{A}\Big\{H_{Aji}H_A{}^{ji}-\frac{1}{n}(H_{At}{}^{t})^2\Big\}dM=0\,. \tag{3.6}$$

THEOREM 3.1. *Let $\tilde{M}^{n+p}$ be a $(n+p)$-dimensional Riemannian manifold of constant curvature admitting a conformal Killing tensor field of order p and M^n a compact orientable submanifold of codimension p in $\tilde{M}^{n+p}$. Assume that the mean curvature vector field H^λ of M^n is parallel with respect to the connection of the normal bundle and the connection of the normal bundle is trivial. If the function r has fixed sign on M^n, then the submanifold M^n is totally umbilical.*

PROOF. Since $H_{Aji}H_A{}^{ji}-\frac{1}{n}(H_{At}{}^t)^2$ is non negative, we find

$$H_{Aji}H_A{}^{ji}-\frac{1}{n}(H_{At}{}^t)^2=0$$

from (3.6) and our assumption. This shows that M^n is totally umbilical by means of Lemma 2.1.

Since a conformal Killing tensor field of order 1 is considered as a conformal Killing vector field, we obtain the following corollaries by virtue of Remark in §2.

COROLLARY 3.2. *Let $\tilde{M}^{n+1}$ be a $(n+1)$-dimensional Riemannian manifold of constant curvature admitting a conformal Killing vector field F_λ and M^n a compact orientable hypersurface in $\tilde{M}^{n+1}$. If the mean curvature of M^n is constant and $F_\lambda C^\lambda$ has fixed sign on M^n, then the hypersurface M^n is umbilical, where C^λ denotes a unit normal vector of M^n.*

The above corollary is included in Theorem 0.1 of Y. Katsurada.

COROLLARY 3.3. *Let $\tilde{M}^{n+2}$ be a $(n+2)$-dimensional Riemannian manifold of constant curvature admitting a conformal Killing tensor field $F_{\lambda\mu}$ of order 2 and M^n a compact orientable submanifold of codimension 2 in $\tilde{M}^{n+2}$. Assume that the mean curvature vector field H^λ of M^n is parallel with respect to the connection of the normal bundle. If $F_{\lambda\mu}C^\lambda D^\mu$ has fixed sign on M^n, then the submanifold M^n is totally umbilical, where C^λ and D^λ denote mutually orthogonal unit normal vectors of M^n.*

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