On transitive permutation groups of degree $3p$ and $4p$

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§ 1. Introduction.

In this paper [7] N. Ito proved that non-solvable transitive permutation groups of degree p have an irreducible complex character whose degree is divisible by p, where p is prime. N. Ito and T. Wada proved that non-solvable transitive permutation groups of degree $2p$ have also the same property.

In this paper we shall prove the following theorems.

THEOREM 1. *Let Ω be the set of symboles* $1. \cdots, 3p$, *where p is an odd prime number* ($\neq 3$). *Let G be a transitive permutation groups on Ω. Then one of the following occurs;*

1) *G has a normal Sylow p-subgroup,*

2) *G has an irreducible complex character whose degree is divisible by p.*

THEOREM 2. *Let Ω be the set of symbols* $1, \cdots, 4p$, *where p is an odd prime number* ($\neq 3$). *Let G be a transitive permutation group on Ω. If G is doubly transitive or imprimitive, then it satisfies the conclusion in Theorem* 1.

§ 2. Proof of Theorem 1.

Let H be the stabilizer of the symbol 1 in G.

(1) The case G is imprimitive on Ω. Let M be a maximal subgroup containing H. Then we have that either $[M:H]=3$ or $[M:H]=p$. At first assume $[M:H]=3$. Let $B=\{\Delta_1, \Delta_2, \cdots, \Delta_p\}$ be the block system of G corresponding to M. Let K be the blockwise stabilizer of B in G. Since the length of $\triangle_i$ is equal to three, K is a solvable $\{2, 3\}$-subgroup and K' is an abelian subgroup. If G/K is non-solvable, then since G/K is a transitive permutation group on B, by [6] G/K and hence G has an irreducible character whose degree is divisible by p.

If G/K is solvable, by [12, p 29], G/K is a Frobenious group. Let Q be a subgroup of G containing K such that Q/K is the Frobenious kernel of G/K. Let P be a p-Sylow subgroup of G. Then $Q=P\cdot K$. If Q is abelian, then P is normal in G. When Q is non-abelian, consider G/K'.

If Q/K' is non-abelian, by [6] the degree of a non-linear irreducible character of Q/K' is divisible by $[Q/K':K/K']=p$, and hence Q has an irreducible character whose degree is divisible by p. Since Q is normal in G by the Theorem of Clifford ([4], p 345), G has an irreducible character whose degree is divisible by p. If Q/K' is abelian, then PK' is normal in G. If PK' is abelian, then P is normal in G, and if PK' is non-abelian, since K' is abelian, it may be proved as above that PK' and hence G has an irreducible character whose degree is divisible by p.

Next assume that $[M:H]=p$. Let $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ be the complete block system of G corresponding to M. Let K be the blockwise stabilizer in G of $\{\Lambda_1, \Lambda_2, \Lambda_3\}$. Since $[G:K]$ is prime to p, K acts transitively on each Λ_i. Let L_1 be the pointwise stabilizer of Λ_1 in K. If K/L_1 is non-solvable, then K/L_1 and hence G has an irreducible character whose degree is divisible by p. Therefore we may assume K/L_1 is solvable and hence it is a Frobenious group. Let Q_1 be a subgroup of K containing L_1 such that Q_1/L_1 is the Frobenious kernel of K/L_1. For a p-Sylow subgroup P_1 of Q_1, $Q_1=P_1L_1$. Since Q_1 is normal in K and $[K:Q_1]$ is prime to p, Q_1 acts transitively on Λ_i $(i=1, 2)$. Let L_2 be the pointwise stabilizer of Λ_2 in Q_1. If Q_1/L_1 is non-solvable, then it may be proved as above that G satisfies 2). Hence we may assume Q_1/L_1 is solvable. Let Q_2 be a subgroup of Q_1 containing L_2 such that Q_2/L_2 is the Frobenious kernel of Q_1/L_2. For a p-Sylow subgroup P_2 of Q_2, $Q_2=P_2\cdot L_2$. Since Q_2 is normal in Q_1, it acts transitively on Λ_3. Let L_3 be the pointwise stabilizer of Λ_3 in Q_2. As above we may assume Q_3/L_3 is solvable. Let Q_3 be a subgroup of Q_2 containing L_3 such that Q_3/L_3 is the Frobenious kernel of Q_2/L_3. We shall show that L_3 is a p-subgroup. Let y be a non-trivial p-'element of L_3, and $|y|=l$. If y is not contained in L_1 or L_2, then y can be written in the form $y=y_1y_2$, where y_1 and y_2 are elements of P_i and L_i, respectively $(i=1$ or $2)$. $yL_i=y_iL_i$ and $y^lL_i=y_i^lL_i$. This is a contradiction. Thus y is an element of $L_1\cap L_2$ and hence $y=1$, this is also a contradiction. Therefore Q_3 is a p-Sylow subgroup of G and normal in G.

(2) The case G is uniprimitive on Ω. By [10] G has an irreducible character of degree p or $2p$ except $p=7$, 19 and 31. Therefore we may assume $p=7$, 19 or 31. Suppose the theorem is false and let G be a minimal counterexample. If the order of G is divisible by p^2, then G contains a p-cycle or the product of two p-cycles. Then by the Theorem of Jordan [12, p 39], G contains the alternating group on Ω. This is a contradiction. We may assume the order of a p-Sylow subgroup of G is p and any p-element $(\neq 1)$ of G is semi-regular on Ω. If G contains a non-trivial normal subgroup

N, then N is transitive on Ω. If N is primitive, then N satisfies 2) by the minimality of G and if N is imprimitive, then N also satisfies the conclution of the Theorem by the first case. Hence G has an irreducible character whose degree is divisible by p, or G must contain a regular normal subgroup and then $p=3$. Therefore we may assume G is a simple group.

Let x be an involution of G. Assume $C_G(x)$ contains a p-element y $(\neq 1)$. Let $\triangle_i$ $(i=1,2,3)$ be orbits of $\langle y\rangle$ on Ω. If x stabilizes each Δ_i, x acts trivially on Ω. This is a contradiction. If x stabilizes Δ_1, and exchanges Δ_2 and Δ_3, then x is written as the product of p transpositions and x is an odd permutation. This contradicts the simplicity of G. Hence we may assume $C_G(x)$ is a p'-subgroup for any involution x, and $C_G(y)$ is a $2'$-subgroup for any p-element y $(\neq 1)$. Let y be a p-element $(\neq 1)$. Since $N_G(\langle y\rangle)/C_G(\langle y\rangle)$ is a cyclic group and $C_G(\langle y\rangle)$ is a $2'$-subgroup, all involutions which invert y are conjugate in $N_G(\langle y\rangle)$. If G has at least two classes of involutions, then by the Theorem of Brauer-Fowler ([1]), G has an irreducible character whose degree is divisible by p, this is a contradiction. Hence we may assume that G has exactly one class of involutions.

Let x be an involution of G such that $xyx^{-1}=y^{-1}$. We denote the number of fixed symbols by an element u of G by $\alpha(u)$. If $\alpha(x)\geqq 4$, then $p=2$, this is a contradiction. If $\alpha(x)=3$, then x is an odd permutation since $(3p-3)/2$ is odd, and it contradicts the simplicity of G. Hence we may assume $\alpha(x)=1$, and $1^x=1$. Let S be a 2-Sylow subgroup of G containing x. Since the stabilizer of any two symbols of Ω is of odd order, S is semiregular on $\Omega-\{1\}$. On the orther hand, by [10], for $p=7$, 19 and 31, the subdegrees of G are (1, 4, 8, 8), (1, 6, 20, 30) and (1, 20, 32, 40) respectively. For $p=19$, the order of S is 2 and it contradicts the simplicity of G. If $p=7$ or 31, then the order of S equals four. By the theorem of Gorenstein-Walter ([5]), G is isomorphic to PSL $(2,q)$, $q>3$, $q\equiv 3$ or 5 (mod 8). But then G has characters whose degrees are $q-1$, q and $q+1$. Hence p divides one of $q-1$, q and $q+1$. Therefore G satisfies 2). This is a contradiction.

(3) The case G is doubly transitive on Ω. Suppose the theorem is false and let G be a minimal counterexample. If G contains the alternating group on Ω, then the degree of the irreducible character of G corresponding to the Yong diagram $[3p-3, 3]$ has degree $(3p-1)/2\cdot p\cdot(3p-5)$. This is a contradiction. Hence likewise in the case (2), we may assume that the order of G is divisible by p only to the first power, and any p-element of G is semiregular on Ω. If G contains a regular normal subgroup, then the degree of G is prime power and it contradicts $p\neq 3$. Hence we may assume that G does not contain a regular normal subgroup.

If the order of the stabilizer of two symbols is odd. then by the theorem of Bender ([2]), G contains a normal subgroup N isomorphic to PSL $(2, q)$, $S_Z(q)$ or $U_3(q)$ as a permutation group, where q is a power of two. Now N is not isomorphic to $S_Z(q)$ since the order of N is divisible by 3. If N is isomorphic to $U_3(q)$, then $q^3+1=3p$ and it contradicts $p\neq 3$. If N is isomorphic to PSL $(2, q)$, then N and hence G has an irreducible character whose degree is $q+1=3p$. This is a contradiction. Hence we may assume that the stabilizer of two symbols in G contains an involution x.

Let P be a p-Sylow subgroup of G and y be a generator of P. If $C_G(x)$ contains a p-element $(\neq 1)$, then likewise in the case (2), it can be shown that x is an odd permutation. Then $[G: G\cap A_{3p}]=2$ and $G\cap A_{3p}$ is doubly transitive on Ω. By the minimality of G, $G\cap A_{3p}$ satisfies 2) and so does G. Hence we may assume $C_G(x)$ is a p'-subgroup and $C_G(y)$ is a $2'$-subgroup. If G contains an involution which is not conjugate to an element of $N_G(P)$, by the theorem of Brauer-Fowler ([1]), G has an irreducible character whose degree is divisible by p. Hence we may assume $xyx=y^{-1}$. If x stabilizes four symbols at least, x must stabilize two symbols in some orbit of P on Ω. It contradicts $p\neq 2$. Hence we may assume $\alpha(u)\leqq 3$ for any involution u of G. Since all involutions are conjugate, then every involution fixes just three symbols of Ω. Then by [9] G is permutation isomorphic to A_7 with $p=5$. Since A_7 has an irreducible character of degree 15, this is a contradiction.

This completes the proof of the Theorem 1.

§ 3. Proof of Theorem 2.

If G is imprimitive, then likewise in § 2, we may prove the Theorem. In the case G is doubly transitive, use [11] and [3], and the theorem can be proved by the same way as in § 2.

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