Certain properties of a submanifold in a sphere

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Introduction.

H. Liebmann [8] has proved that an ovaloid with constant mean curvature in a 3-dimensional Euclidean space is a sphere. Y. Katsurada [3], [4] and K. Yano [14] have generalized the above theorem to an m-dimensional hypersurface in a Riemannian manifold admitting an infinitesimal conformal or homothetic transformation. Y. Katsurada [5], [6], H. Kôjyô [5], T. Nagai [6],[10] and K. Yano [15] have given the condition for a submanifold of codimension greater than 1 in a Riemannian manifold to be pseudo umbilical by making use of integral formulas.

On the other hand M. Okumura [11] has given the condition for a submanifold of codimension 2 in an odd dimensional sphere to be totally umbilical by making use of the natural normal contact structure on the sphere.

In this paper, the author studies a submanifold of codimension p in a sphere by making use of a conformal Killing tensor field of degree p on the sphere that has been defined by T. Kashiwada [2] and S. Tachibana [13], and proves that the submanifold is totally umbilical under certain conditions by making use of integral formulas.

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§1. Tensor fields on a sphere induced from an Euclidean space.

Let E^{m+p+1} be a $(m+p+1)$-dimensional Euclidean space with an orthogonal coordinate system X^A $(A=1, 2, \cdots, m+p+1)$. Let S^{m+p} be a $(m+p)$-dimensional sphere of radius 1 in E^{m+p+1}. Then S^{m+p} is represented by the equation

$$\sum_{A=1}^{m+p+1}(X^A)^2=1. \tag{1. 1}$$

We can take a local coordinate x^λ $(\lambda=1, 2, \cdots, m+p)$ of S^{m+p} in such a way that

$$\left\{\begin{array}{ll} X^\lambda=x^\lambda, & \lambda=1, \cdots, m+p, \\ (X^{m+p+1})^2=1-\displaystyle\sum_{\lambda=1}^{m+p}(x^\lambda)^2. & \end{array}\right. \tag{1. 2}$$

If we put

(1.3) $$B_\lambda^A = \partial X^A / \partial x^\lambda ,$$

then we find easily

(1.4) $$B_\lambda^A = \begin{cases} \delta_\lambda^\mu , & A = \mu , \\ -\dfrac{X^\lambda}{X^{m+p+1}} , & A = m+p+1 . \end{cases}$$

The Riemannian metric tensor $G_{\lambda\mu}$ on S^{m+p} induced from E^{m+p+1} and Christoffel's symbol $\left\{ {\tilde{\lambda} \atop \mu\nu} \right\}$ with respect to $G_{\lambda\mu}$ are given by

(1.5) $$G_{\lambda\mu} = \delta_{\lambda\mu} + \frac{X^\lambda X^\mu}{(X^{m+p+1})^2} , \qquad G^{\lambda\mu} = \delta^{\lambda\mu} - X^\lambda X^\mu$$

and

(1.6) $$\left\{ {\tilde{\lambda} \atop \mu\nu} \right\} = X^\lambda G_{\mu\nu}$$

respectively. From (1.5) and (1.6), we find easily that the curvature tensor $\tilde{R}_{\lambda\mu\nu\kappa}$ of S^{m+p} has the form

(1.7) $$\tilde{R}_{\lambda\mu\nu\kappa} = G_{\lambda\kappa} G_{\mu\nu} - G_{\lambda\nu} G_{\mu\kappa} .$$

If we put

(1.8) $$(C^A) = (-X^1, -X^2, \cdots, -X^{m+p+1}) ,$$

then C^A is a unit normal vector field of S^{m+p}.

Now we define the van der Waerden-Bortolotti covariant derivative $\nabla_\lambda B_\mu^A$ and $\nabla_\lambda C^A$ such that

(1.9) $$\nabla_\lambda B_\mu^A = \partial_\lambda B_\mu^A - \left\{ {\tilde{\nu} \atop \lambda\mu} \right\} B_\nu^A ,$$

(1.10) $$\nabla_\lambda C^A = \partial_\lambda C^A ,$$

where ∂_λ denotes $\partial / \partial x^\lambda$. From (1.6), (1.8), (1.9) and (1.10), we obtain easily

(1.11) $$\begin{aligned} &\nabla_\lambda B_\mu^A = -X^A G_{\lambda\mu} = C^A G_{\lambda\mu} , \\ &\nabla_\lambda C^A = -B_\lambda^A . \end{aligned}$$

Now let $\Phi_{A_1 \cdots A_p}$ be a skew symmetric and parallel tensor of degree p on E^{m+p+1}, that is, satisfying

(1.12) $$\begin{aligned} &\Phi_{A_1 \cdots A_a \cdots A_b \cdots A_p} = -\Phi_{A_1 \cdots A_b \cdots A_a \cdots A_p} , \\ &\partial_B \Phi_{A_1 \cdots A_p} = 0 , \end{aligned}$$

where ∂_B means $\partial/\partial X^B$.

We put

$$F_{\lambda_1\cdots\lambda_p}=\Phi_{A_1\cdots A_p}B_{\lambda_1}{}^{A_1}\cdots B_{\lambda_p}{}^{A_p}. \tag{1.13}$$

Then $F_{\lambda_1\cdots\lambda_p}$ is a skew symmetric tensor field of dgree p on S^{m+p}. Differentiating (1.13) covariantly on S^{m+p}, we have

$$\begin{aligned}\nabla_\lambda F_{\lambda_1\cdots\lambda_p} &= B_\lambda{}^A\partial_A\Phi_{A_1\cdots A_p}B_{\lambda_1}{}^{A_1}\cdots B_{\lambda_p}{}^{A_p}\\ &\quad+\sum_{c=1}^{p}\Phi_{A_1\cdots A_c\cdots A_p}B_{\lambda_1}{}^{A_1}\cdots C^{A_c}\cdots B_{\lambda_p}{}^{A_p}G_{\lambda\lambda_c}\end{aligned}$$

by means of (1.11) If we put

$$f_{\lambda_1\cdots\hat{\lambda}_c\cdots\lambda_p}=(-1)^c\Phi_{A_1\cdots A_c\cdots A_p}B_{\lambda_1}{}^{A_1}\cdots C^{A_c}\cdots B_{\lambda_p}{}^{A_p}, \tag{1.14}$$

where $\hat{\lambda}_c$ denotes that λ_c is omitted, then we get

$$\nabla_\lambda F_{\lambda_1\cdots\lambda_p}=\sum_{c=1}^{p}(-1)^c f_{\lambda_1\cdots\hat{\lambda}_c\cdots\lambda_p}G_{\lambda\lambda_c}, \tag{1.15}$$

by virtue of (1.12). Thus $F_{\lambda_1\cdots\lambda_p}$ is a conformal Killing tensor field of degree p on S^{m+p} that has been defined by T. Kashiwada [2] and S. Tachibana [13]. Therefore we find that there exists a conformal Killing tensor field of degree p on S^{m+p}.

§ 2. Submanifold in S^{m+p}.

Let M^m be an orientable submanifold of codimension p in S^{m+p}. In terms of local coordinate $(x^1,\cdots,x^{m+p})$ of S^{m+p} and $(u^1,\cdots,u^m)$ of M^m, M^m is locally expressed by equations

$$x^\lambda=x^\lambda(u^i),\qquad \begin{aligned}\lambda&=1,2,\cdots,m+p,\\ i&=1,2,\cdots,m.\end{aligned} \tag{2.1}$$

If we put

$$B_i{}^\lambda=\partial x^\lambda/\partial u^i, \tag{2.2}$$

then $B_i{}^\lambda$ are m linearly independent local vector field tangent to M^m. The Riemannian metric tensor g_{ji} on M^m induced from the Riemannian metric tensor $G_{\lambda\mu}$ on S^{m+p} is given by

$$g_{ji}=G_{\lambda\mu}B_j{}^\lambda B_i{}^\mu. \tag{2.3}$$

We choose p mutually orthogonal unit normal vectors $N_A{}^\lambda$ $(A=m+1,\cdots,m+p)$, then we find

$$
(2.4)\qquad
\begin{aligned}
&G_{\lambda\mu}B_j^{\ \lambda}N_A^{\ \mu}=0\,, && G_{\lambda\mu}N_A^{\ \lambda}N_B^{\ \mu}=\delta_{AB}\,,\\
&B_i^{\ \lambda}B^h_{\ \lambda}=\delta_i^h\,, && N_A^{\ \lambda}N_{B\lambda}=\delta_{AB}\,,\\
&B_i^{\ \lambda}N_{A\lambda}=0\,, && N_A^{\ \lambda}B^i_{\ \lambda}=0\,,\\
&B_i^{\ \lambda}B^i_{\ \mu}+\sum_A N_A^{\ \lambda}N_{A\mu}=\delta_\mu^\lambda\,, &&
\end{aligned}
$$

where we have put $B^i_{\ \lambda}=G_{\lambda\mu}B_j^{\ \mu}g^{ji}$, $N_{A\lambda}=G_{\lambda\mu}N_A^{\ \mu}$.

Let H_{Aji} $(A=m+1,\cdots,m+p)$ be the second fundamental tensor with respect to $N_A^{\ \lambda}$ and L_{ABj} the third fundamental tensor. Then the Gauss and Weingarten equations are given by

$$
(2.5)\qquad \nabla_j B_i^{\ \lambda}=\sum_A H_{Aji}N_A^{\ \lambda}
$$

and

$$
(2.6)\qquad \nabla_j N_A^{\ \lambda}=-H_{Aj}^{\ \ \ i}B_i^{\ \lambda}+\sum_B L_{ABj}N_B^{\ \lambda}
$$

respectively, where $\nabla_j B_i^{\ \lambda}$ and $\nabla_j N_A^{\ \lambda}$ are defined by

$$
\nabla_j B_i^{\ \lambda}=\partial_j B_i^{\ \lambda}-\left\{ {h \atop j\,i} \right\}B_h^{\ \lambda}+\left\{ {\tilde{\lambda} \atop \mu\nu} \right\}B_j^{\ \mu}B_i^{\ \nu}
$$

and

$$
\nabla_j N_A^{\ \lambda}=\partial_j N_A^{\ \lambda}+\left\{ {\tilde{\lambda} \atop \mu\nu} \right\}B_j^{\ \mu}N_A^{\ \nu}
$$

respectively, $\left\{ {h \atop j\,i} \right\}$ and $\left\{ {\tilde{\lambda} \atop \mu\nu} \right\}$ being the Christoffel's symbols of M^m and S^{m+p}.

We now write the equations of Gauss, Mainardi-Codazzi and Ricci-Kühne:

$$
(2.7)\qquad R_{kjih}=\tilde{R}_{\lambda\mu\nu\kappa}B_k^{\ \lambda}B_j^{\ \mu}B_i^{\ \nu}B_h^{\ \kappa}+\sum_A(H_{Akh}H_{Aji}-H_{Aki}A_{Ajh})\,,
$$

$$
(2.8)\qquad \tilde{R}_{\lambda\mu\nu\kappa}B_k^{\ \lambda}B_j^{\ \mu}B_i^{\ \nu}N_A^{\ \kappa}=\nabla_k H_{Aji}-\nabla_j H_{Aki}+\sum_B(H_{Bji}L_{BAk}-H_{Bki}L_{BAj})\,,
$$

$$
(2.9)\qquad
\begin{aligned}
\tilde{R}_{\lambda\mu\nu\kappa}B_k^{\ \lambda}B_j^{\ \mu}N_A^{\ \nu}N_B^{\ \kappa}=\ &H_{Ak}^{\ \ \ i}H_{Bji}-H_{Aj}^{\ \ \ i}H_{Bki}\\
&+\nabla_k L_{ABj}-\nabla_j L_{ABk}+\sum_D(L_{ADj}L_{DBk}-L_{ADk}L_{DBj})\,,
\end{aligned}
$$

where R_{kjih} and $\tilde{R}_{\lambda\mu\nu\kappa}$ denote the curvature tensor of M^m and S^{m+p} respectively. Since S^{m+p} has the curvature tensor of the form (1.7), the above equations can be written as

$$
(2.10)\qquad R_{kjih}=g_{kh}g_{ji}-g_{ki}g_{jh}+\sum_A(H_{Akh}H_{Aji}-H_{Aki}H_{Ajh})\,,
$$

$$
(2.11)\qquad \nabla_k H_{Aji}-\nabla_j H_{Aki}+\sum_B(H_{Bji}L_{BAk}-H_{Bki}L_{BAj})=0\,,
$$

$$H_{Ak}{}^{t}H_{Bjt}-H_{Aj}{}^{t}H_{Bkt}+\nabla_{k}L_{ABj}-\nabla_{j}L_{ABk}+\sum_{D}(L_{ADj}L_{DBk}-L_{ADk}L_{DBj})=0\,. \tag{2.12}$$

When at each point of M^m the second fundamental tensors H_{Aji} $(A=m+1,\cdots,m+p)$ are proportional to the metric tensor g_{ji}, that is, satisfying the following conditions

$$H_{Aji}=H_{A}g_{ji}\,,\qquad A=m+1,\cdots,m+p,$$

we call M^m a totally umbilical submanifold.

LEMMA 2.1. *A necessary and sufficient condition for M^m to be totally umbilical is that the following equations are satisfied*:

$$H_{Aji}H_{A}{}^{ji}=\frac{1}{m}(H_{At}{}^{t})^{2}\,,\qquad A=m+1,\cdots,m+p. \tag{2.13}$$

PROOF. This follows the identity

$$\left(H_{Aji}-\frac{1}{m}H_{At}{}^{t}g_{ji}\right)\left(H_{A}{}^{ji}-\frac{1}{m}H_{At}{}^{t}g^{ji}\right)=H_{Aji}H_{A}{}^{ji}-\frac{1}{m}(H_{At}{}^{t})^{2}\,,$$

and the positive definiteness of the Riemannian metric g_{ji}.

Next we consider the normal bundle $N(M^m)$ of M^m. For $N^\lambda\in N(M^m)$, a connection $\overset{*}{\nabla}$ on $N(M^m)$ is defined by

$$\overset{*}{\nabla}_{j}N^{\lambda}=(\nabla_{j}N^{\lambda})^{N}\,, \tag{2.14}$$

where $(\nabla_j N^\lambda)^N$ denotes the normal part of $\nabla_j N^\lambda$. When $\overset{*}{\nabla}_j N^\lambda$ vanishes identically along M^m, we say that N^λ is parallel with respect to the connection of the normal bundle $N(M^m)$.

Let H^λ be the mean curvature vector field of M^m. Then H^λ is represented by

$$H^{\lambda}=\frac{1}{m}\sum_{A}H_{At}{}^{t}N_{A}{}^{\lambda}\,, \tag{2.15}$$

and H^λ is independent of the choice of mutually orthogonal unit normal vectors of M^m.

LEMMA 2.2. *In order that the mean curvature vector field H^λ of M^m is parallel with respect to the connection of the normal bundle, it is necessary and sufficient that*

$$\nabla_{j}H_{At}{}^{t}=-\sum_{B}H_{Bt}{}^{t}L_{BAj}\,. \tag{2.16}$$

PROOF. Differentiating (2.15) covariantly, we have

$$\nabla_j H^\lambda = \frac{1}{m}\left\{\sum_A \nabla_j H_{At}{}^t N_A^\lambda + \sum_A H_{At}{}^t(-H_{Aj}{}^h B_h^\lambda + \sum_B L_{ABj} N_B^\lambda)\right\}$$

$$= \frac{1}{m}\left\{-\sum_A H_{At}{}^t H_{Aj}{}^h B_h^\lambda + \sum_A (\nabla_j H_{At}{}^t + \sum_B H_{Bt}{}^t L_{BAj}) N_A^\lambda\right\}$$

from (2.6). Thus we get

$$\overset{*}{\nabla}_j H^\lambda = \frac{1}{m}\sum_A (\nabla_j H_{At}{}^t + \sum_B H_{Bt}{}^t L_{BAj}) N_A^\lambda ,$$

from which we have the assertion of Lemma.

LEMMA 2.3. *Suppose that the mean curvature vector field H^λ of M^m is parallel with respect to the connection of the normal bundle, then the mean curvature h of M^m is constant.*

PROOF. The mean curvature h of M^m is given by

$$h^2 = \frac{1}{m^2}\sum_A (H_{At}{}^t)^2 . \tag{2.17}$$

Differentiating (2.17) covariantly and making use of (2.16), we have

$$\nabla_j h^2 = \frac{2}{m^2}\sum_A H_{At}{}^t \nabla_j H_{As}{}^s$$

$$= -\frac{2}{m^2}\sum_A\sum_B H_{At}{}^t H_{Bs}{}^s L_{BAj} = 0$$

by virtue of $L_{ABj} = -L_{BAj}$. This proves that h is constant.

LEMMA 2.4. *If the mean curvature vector field H^λ of M^m is parallel with respect to the connection of the normal bundle, then we have the following relation*

$$\nabla_j H_{Ak}{}^j = -\sum_B H_{Bk}{}^j L_{BAj} . \tag{2.18}$$

PROOF. By means of (2.11) and (2.16), we get (2.18) easily.

Lemma 2.1, $\cdots$, 2.4 was proved by T. Yamada [16]. But his paper does not appear.

When there exists mutually orthogonal normal vector fields N_A^λ ($A = m+1, \cdots, m+p$) such that $L_{ABj} = 0$, we say that the connection of the normal bundle of M^m is trivial. We obtain the following Lemma by J. Erbacher [1]:

LEMMA 2.5. *The connection of the normal bundle is trivial if and only if that*

$$H_{Ak}{}^t H_{Bji} = H_{Aj}{}^t H_{Bki} . \tag{2.19}$$

REMARK. If $p=1$, the connection of the normal bundle is trivial under no assumption. If $p=2$, the connection of the normal bundle is trivial under the condition that the mean curvature vector field H^λ of M^m is parallel with respect to the connection of the normal bundle.

§ 3. Integral formulas.

In this section, we assume that a submanifold M^m is compact orientable and the mean curvature vector field H^λ of M^m is parallel with respect to the connection of the normal bundle and the connection of the normal bundle is trivial. Let $F_{\lambda_1\cdots\lambda_p}$ be the tensor field of degree p on S^{m+p} that is defined by (1.13) in §1.

Now we put

$$r=F_{\lambda_1\cdots\lambda_p}N_{m+1}{}^{\lambda_1}\cdots N_{m+p}{}^{\lambda_p}\,. \tag{3.1}$$

LEMMA 3.1. *The function r is independent of the choice of mutually orthogonal unit normal vectors.*

PROOF. Let (T_{AB}), $A, B=m+1,\cdots,m+p$, be a orthogonal matrix such that $\det(T_{AB})=1$, that is, satisfying the following conditions

$$\sum_A T_{AB}T_{AC}=\delta_{BC}\,,\qquad \sum_C T_{AC}T_{BC}=\delta_{AB}\,, \tag{3.2}$$
$$\det(T_{AB})=1\,.$$

We put

$$'N_A^\lambda=\sum_B T_{AB}N_B^\lambda\,. \tag{3.3}$$

Then $'N_A^\lambda$ $(A=m+1,\cdots,m+p)$ are mutually orthogonal unit normal vectors. Substituting (3.3) into $'r=F_{\lambda_1\cdots\lambda_p}{}'N_{m+1}{}^{\lambda_1}\cdots{}'N_{m+p}{}^{\lambda_p}$, then we have

$$\begin{aligned}'r&=\sum_{A_1,\cdots,A_p}T_{m+1A_1}\cdots T_{m+pA_p}F_{\lambda_1\cdots\lambda_p}N_{A_1}{}^{\lambda_1}\cdots N_{A_p}{}^{\lambda_p}\\&=\sum_{A_1,\cdots,A_p}\operatorname{sgn}\begin{pmatrix}m+1,\cdots,m+p\\A_1,\cdots,A_p\end{pmatrix}T_{m+1A_1}\cdots T_{m+pA_p}F_{\lambda_1\cdots\lambda_p}N_{m+1}{}^{\lambda_1}\cdots N_{m+p}{}^{\lambda_p}\\&=\det(T_{AB})F_{\lambda_1\cdots\lambda_p}N_{m+1}{}^{\lambda_1}\cdots N_{m+p}{}^{\lambda_p}=r\end{aligned}$$

by virtue of (3.2) and the skew symmetry of $F_{\lambda_1\cdots\lambda_p}$. This proves the assertion of Lemma 3.1.

Differentiating (3.1) covariantly and making use of (1.15) and (2.6), we find

$$\begin{aligned}\nabla_i r&=B_i^\lambda\Big\{\sum_{a=1}^p(-1)^a f_{\lambda_1\cdots\hat\lambda_a\cdots\lambda_p}G_{\lambda\lambda_a}\Big\}N_{m+1}{}^{\lambda_1}\cdots N_{m+p}{}^{\lambda_p}\\&\quad+\sum_{a=1}^p F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{m+1}{}^{\lambda_1}\cdots(-H_{m+ai}{}^hB_h{}^{\lambda_a}+\sum_B L_{m+aBi}N_B^{\lambda_a})\cdots N_{m+p}{}^{\lambda_p}\,,\end{aligned}$$

from which we have

$$(3.4)\qquad \nabla_i r = -\sum_{a=1}^{p} H_{m+ai}{}^{h} F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots B_h{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\,.$$

Differentiating (3.4) covariantly, we have

$$\begin{aligned}
\nabla_j\nabla_i r = &-\sum_{a=1}^{p} \nabla_j H_{m+ai}{}^{h} F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots B_h{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\\
&-\sum_{a=1}^{p} H_{m+ai}{}^{h} B_j{}^{\lambda}\Big\{\sum_{b=1}^{p}(-1)^b f_{\lambda_1\cdots\hat{\lambda}_b\cdots\lambda_p} G_{\lambda\lambda_b}\Big\} N_{m+1}{}^{\lambda_1}\cdots B_h{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\\
&-\sum_{a=1}^{p} H_{m+ai}{}^{h}\sum_{\substack{c=1\\ c\neq a}}^{p} F_{\lambda_1\cdots\lambda_c\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots(-H_{m+cj}{}^{k}B_k{}^{\lambda_c}+\sum_B L_{m+cBj}N_B{}^{\lambda_c})\cdots\\
&\cdots B_h{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\\
&-\sum_{a=1}^{p} H_{m+ai}{}^{h} F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots\sum_A H_{Ajh} N_A{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\\
= &-\sum_{a=1}^{p}(\nabla_j H_{m+ai}{}^{h}+\sum_C H_{Ci}{}^{h} L_{Cm+aj}) F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots B_h{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\\
&+\sum_{\substack{a,c=1\\ a\neq c}}^{p} H_{m+ai}{}^{h} H_{m+cj}{}^{k} F_{\lambda_1\cdots\lambda_c\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots B_k{}^{\lambda_c}\cdots B_h{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\\
&-\sum_{a=1}^{p}(-1)^a H_{m+aji} f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots \hat{N}_{m+a}{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\\
&-r\sum_{a=1}^{p} H_{m+ai}{}^{h} H_{m+ajh}
\end{aligned}$$

by virtue of (1.15), (2.5) and (2.6), where $\hat{N}_{m+a}{}^{\lambda_a}$ means that $N_{m+a}{}^{\lambda_a}$ is omitted. Thus we get

$$\begin{aligned}
\nabla^j\nabla_j r = &-\sum_{a=1}^{p}(\nabla_j H_{m+ah}{}^{j}+\sum_C H_{Ch}{}^{j} L_{Cm+aj}) F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots B^{h\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\\
&+\sum_{\substack{a,c=1\\ a\neq c}}^{p} H_{m+a}{}^{jh} H_{m+cj}{}^{k} F_{\lambda_1\cdots\lambda_c\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots B_k{}^{\lambda_c}\cdots B_h{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\\
&-\sum_{a=1}^{p}(-1)^a H_{m+at}{}^{t} f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots \hat{N}_{m+a}{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\\
&-r\sum_A H_{Aji} H_A{}^{ji}\,,
\end{aligned}$$

from which we have

$$\begin{aligned}
\nabla^j\nabla_j r = &-\sum_{a=1}^{p}(-1)^a H_{m+at}{}^{t} f_{\lambda_1\cdots\hat{\lambda}_a\cdots{}_p} N_{m+1}{}^{\lambda_1}\cdots \hat{N}_{m+a}{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\\
&-r\sum_A H_{Aji} H_A{}^{ji}
\end{aligned}$$

by virtue of our assumptions, Lemma 2.4 and the skew symmetry of $F_{\lambda_1\cdots\lambda_p}$.

Therefore we obtain the following integral formula

$$(3.5)\qquad \int_{M^m}\Big\{r\sum_A H_{Aji}H_A{}^{ji} + \sum_{a=1}^{p}(-1)^a H_{m+at}{}^t f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p}N_{m+1}{}^{\lambda_1}\cdots\hat{N}_{m+a}{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\Big\}dM=0$$

by means of Green's theorem.

Next we put

$$w_i=\sum_{a=1}^{p}H_{m+at}{}^tF_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{m+1}{}^{\lambda_1}\cdots B_i{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}.$$

LEMMA 3.2. *w_i is independent of the choice of mutually orthogonal unit normal vectors. Consequently it defines a vector field on M^m.*

PROOF. We take T_{AB} and $'N_A{}^\lambda$ satisfying (3.2) and (3.3). From (3.2) and (3.3), we find

$$\sum_A T_{AC}{}'N_A{}^\lambda=\sum_A T_{AC}\sum_B T_{AB}N_B{}^\lambda = \sum_B \delta_{CB}N_B{}^\lambda=N_C{}^\lambda,$$

from which we have

$$(3.6)\qquad N_A{}^\lambda=\sum_B T_{BA}{}'N_B{}^\lambda.$$

Let $'H_{Aji}$ be the second fundamental tensor with respect to $'N_A{}^\lambda$. Making use of (3.6), we have

$$\nabla_j B_i{}^\lambda=\sum_A {}'H_{Aji}{}'N_A{}^\lambda=\sum_A H_{Aji}N_A{}^\lambda = \sum_A H_{Aji}\sum_B T_{BA}{}'N_B{}^\lambda=\sum_A\Big(\sum_B H_{Bji}T_{AB}\Big)'N_A{}^\lambda,$$

from which we get

$$'H_{Aji}=\sum_B T_{AB}H_{Bji}.$$

Thus we find

$$(3.7)\qquad 'H_{At}{}^t=\sum_B T_{AB}H_{Bt}{}^t.$$

From (3.3), (3.7) and the skew symmetry of $F_{\lambda_1\cdots\lambda_p}$, we have

$$'w_i=\sum_{a=1}^{p}{}'H_{m+at}{}^tF_{\lambda_1\cdots\lambda_a\cdots\lambda_p}{}'N_{m+1}{}^{\lambda_1}\cdots B_i{}^{\lambda_a}\cdots{}'N_{m+p}{}^{\lambda_p}$$
$$=\sum_{a=1}^{p}\sum_{A_1,\cdots,A_p}H_{A_at}{}^tT_{m+1A_1}\cdots T_{m+aA_a}\cdots T_{m+pA_p}F_{\lambda_1\cdots\lambda_a\cdots\lambda_p}N_{A_1}{}^{\lambda_1}\cdots B_i{}^{\lambda_a}\cdots N_{A_p}{}^{\lambda_p}$$

$$= \sum_{b=1}^{p} H_{m+bt}{}^{t} \sum_{a=1}^{p} \sum_{A_1,\cdots,A_p} T_{m+1A_1} \cdots T_{m+am+b} \cdots T_{m+pA_p} F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} \times$$
$$N_{A_1}{}^{\lambda_1} \cdots B_i{}^{\lambda_a} \cdots N_{A_p}{}^{\lambda_p}$$
$$= \sum_{b=1}^{p} H_{m+bt}{}^{t} \sum_{a,c=1}^{p} (-1)^{a+c} \sum_{A_1,\cdots,A_p} \mathrm{sgn}\begin{pmatrix} m+1, \cdots, \widehat{m+c}, \cdots, m+p \\ A_1, \cdots, \hat{A}_a, \cdots, A_p \end{pmatrix} \times$$
$$T_{m+1A_1} \cdots T_{m+am+b} \cdots T_{m+pA_p} F_{\lambda_1\cdots\lambda_c\cdots\lambda_p} N_{m+1}{}^{\lambda_1} \cdots B_i{}^{\lambda_c} \cdots N_{m+p}{}^{\lambda_p}$$
$$= \sum_{b=1}^{p} H_{m+bt}{}^{t} \sum_{a,c=1}^{p} T_{m+am+b} \tilde{T}_{m+am+c} F_{\lambda_1\cdots\lambda_c\cdots\lambda_p} N_{m+1}{}^{\lambda_1} \cdots B_i{}^{\lambda_c} \cdots N_{m+p}{}^{\lambda_p},$$

where $\tilde{T}_{m+am+c}$ denotes the cofactor of T_{m+am+c}. Making use of the following relation

$$\sum_{a=1}^{p} T_{m+am+b} \tilde{T}_{m+am+c} = \begin{cases} \det(T_{AB}), & b=c, \\ \\ 0, & b \neq c, \end{cases}$$

we obtain

$$'w_i = \det(T_{AB}) \sum_{b=1}^{p} H_{m+bt}{}^{t} F_{\lambda_1\cdots\lambda_b\cdots\lambda_p} N_{m+1}{}^{\lambda_1} \cdots B_i{}^{\lambda_b} \cdots N_{m+p}{}^{\lambda_p}$$
$$= w_i$$

by means of (3.2), which proves the assertion of Lemma.

Differentiating w_i covariantly and making use of (1.15), (2.5) and (2.6), we have

$$\nabla_j w_i = \sum_{a=1}^{p} \nabla_j H_{m+at}{}^{t} F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1} \cdots B_i{}^{\lambda_a} \cdots N_{m+p}{}^{\lambda_p}$$
$$+ \sum_{a=1}^{p} H_{m+at}{}^{t} B_j{}^{\lambda} \left\{ \sum_{b=1}^{p} (-1)^b f_{\lambda_1\cdots\hat{\lambda}_b\cdots\lambda_p} G_{\lambda\lambda_b} \right\} N_{m+1}{}^{\lambda_1} \cdots B_i{}^{\lambda_a} \cdots N_{m+p}{}^{\lambda_p}$$
$$+ \sum_{a=1}^{p} \sum_{\substack{c=1 \\ c \neq a}}^{p} H_{m+at}{}^{t} F_{\lambda_1\cdots\lambda_c\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1} \cdots (-H_{m+cj}{}^{h} B_h{}^{\lambda_c} + \sum_{B} L_{m+cBj} N_B{}^{\lambda_c})$$
$$\cdots B_i{}^{\lambda_a} \cdots N_{m+p}{}^{\lambda_p}$$
$$+ \sum_{a=1}^{p} H_{m+at}{}^{t} F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1} \cdots \sum_{A} H_{Aji} N_A{}^{\lambda_a} \cdots N_{m+p}{}^{\lambda_p}$$
$$= \sum_{a=1}^{p} (\nabla_j H_{m+at}{}^{t} + \sum_{B} H_{Bt}{}^{t} L_{Bm+aj}) F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1} \cdots B_i{}^{\lambda_a} \cdots N_{m+p}{}^{\lambda_p}$$
$$+ \sum_{\substack{a,c=1 \\ a \neq c}}^{p} H_{m+at}{}^{t} H_{m+cj}{}^{h} F_{\lambda_1\cdots\lambda_c\cdots\lambda_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1} \cdots B_h{}^{\lambda_c} \cdots B_i{}^{\lambda_a} \cdots N_{m+p}{}^{\lambda_p}$$
$$+ \sum_{a=1}^{p} (-1)^a H_{m+at}{}^{t} f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1} \cdots \hat{N}_{m+a}{}^{\lambda_a} \cdots N_{m+p}{}^{\lambda_p} g_{ji}$$

$$+ r\sum_{a=1}^{p} H_{m+at}{}^{t}H_{m+aji}\,,$$

from which we get

$$\nabla^j w_j = m\sum_{a=1}^{p}(-1)^a H_{m+at}{}^{t} f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots\hat{N}_{m+a}{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p} + r\sum_A (H_{At}{}^{t})^2$$

by means of our assumptions. Thus we have the following integral formula

$$\begin{aligned}(3.8)\qquad \int_{M^m}\Big\{ & r\sum_A (H_{At}{}^{t})^2 \\ & + m\sum_{a=1}^{p}(-1)^a H_{m+at}{}^{t} f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p} N_{m+1}{}^{\lambda_1}\cdots\hat{N}_{m+a}{}^{\lambda_a}\cdots N_{m+p}{}^{\lambda_p}\Big\} dM = 0\end{aligned}$$

by virtue of Green's theorem.

From (3.5)-(3.8) $\times\dfrac{1}{m}$, we have

$$(3.9)\qquad \int_{M^m} r\sum_A \left\{H_{Aji}H_A{}^{ji} - \frac{1}{m}(H_{At}{}^{t})^2\right\} dM = 0\,.$$

THEOREM 3.3. *Let M^m be a compact orientable submanifold of codimension p in a sphere S^{m+p} of radius 1. Suppose that the mean curvature vector field H^λ of M^m is parallel with respect to the connection of the normal bundle and that the connection of the normal bundle is trivial. If the function r has fixed sign on M^m, then M^m is totally umbilical.*

PROOF. Since $H_{Aji}H_A{}^{ji} - \dfrac{1}{m}(H_{At}{}^{t})^2$ is non negative, we have

$$H_{Aji}H_A{}^{ji} - \frac{1}{m}(H_{At}{}^{t})^2 = 0$$

from (3.9) and the assumption. Thus we find that M^m is totally umbilical by virtue of Lemma 2.1.

In the case of $p=1$ and $p=2$, we have the following corollaries by means of Remark in §2.

COROLLARY 3.4. *Let M^m be a compact orientable hypersurface in a sphere S^{m+1} of radius 1. Assume that the mean curvature h of M^m is constant. If the function r has fixed sign on M^m, then M^m is umbilical.*

COROLLARY 3.5. *Let M^m be a compact orientable submanifold of codimension 2 in a sphere S^{m+2} of radius 1. Assume that the mean curvature vector field H^λ of M^m is parallel with respect to the connection of the normal bundle. If the function r has fixed sign on M^m, then M^m is totally umbilical.*

When $p=1$, F_λ that is given by (1.13) is a conformal Killing vector field on S^{m+1}. Therefore Corollary 3.4 is included in the theorem of Y. Katsurada [3], [4]. When $p=2$, Corollary 3.5 is considered as the generalization of the theorem of M. Okumura [11].

§ 4. Examples.

When $m=2n+1$ and $p=2$, M. Okumura [11] has given a example of a submanifold that the function r is constant by making use of the normal contact structure on S^{2n+1}. Similarly we give examples of submanifolds in a sphere such that r is constant in the case of $p=2$ and $p=3$.

(i) Case of $p=2$: We take Φ_{AB} on E^{m+3} in the following way

$$(4.1)\qquad (\Phi_{AB}) = \left(\begin{array}{ccc|cc} 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{array}\right)$$

Then we find easily that Φ_{AB} is a skew symmetric and parallel tensor of degree 2 on E^{m+3}. From (1.4) and (1.13), we have

$$(4.2)\qquad (F_{\lambda\mu}) = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & \dfrac{X^1}{X^{m+3}} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \dfrac{X^{m+1}}{X^{m+3}} \\ \hline -\dfrac{X^1}{X^{m+3}}, & \cdots\cdots, & -\dfrac{X^{m+1}}{X^{m+3}}, & 0 \end{array}\right)$$

Now we consider a submanifold M^m of S^{m+2} whose local representation is given by

$$(4.3)\qquad \begin{cases} X^i = u^i, & (i=1,2,\cdots,m) \\ (X^{m+1})^2 = t - \sum_{i=1}^{m} (u^i)^2, & 0<t<1, \end{cases}$$

$$\begin{cases} X^{m+2}=0\,, \\ X^{m+3}=\sqrt{1-t}\,. \end{cases}$$

Then we see that the submanifold M^m is compact and totally umbilical in S^{m+2}.

We put

$$(4.4)\qquad \begin{aligned} &(C^\lambda)=(0,\cdots,1),\\ &(D^\lambda)=(X^1,\cdots,X^{m+1},0)\times\sqrt{\frac{1-t}{t}}\,. \end{aligned}$$

Then C^λ and D^λ are mutually orthogonal unit normal vectors of M^m.

From (4.2), (4.3) and (4.4), we have

$$\begin{aligned} r=F_{\lambda\mu}C^\lambda D^\mu &= -\frac{1}{X^{m+3}}\sqrt{\frac{1-t}{t}}\sum_{i=1}^{m+1}(X^i)^2\\ &= -\frac{1}{\sqrt{1-t}}\sqrt{\frac{1-t}{t}}\times t=-\sqrt{t}\,. \end{aligned}$$

This shows that the submanifold M^m is a desired one.

(ii) Case of $p=3$: We take three vectors $\underset{1}{\Phi}_A$, $\underset{2}{\Phi}_A$ and $\underset{3}{\Phi}_A$ on E^{m+4} in such way that

$$(4.5)\qquad \begin{aligned} &(\underset{1}{\Phi}_A)=(0,0,\cdots,1)\,,\\ &(\underset{2}{\Phi}_A)=(0,0,\cdots,1,0)\,,\\ &(\underset{3}{\Phi}_A)=(0,\cdots,1,0,0)\,. \end{aligned}$$

We put

$$(4.6)\qquad \Phi_{ABC}=\sum\operatorname{sgn}\begin{pmatrix}1,&2,&3\\ a,&b,&c\end{pmatrix}\underset{a}{\Phi}_A\underset{b}{\Phi}_B\underset{c}{\Phi}_C\,,\qquad (a,b,c=1,2,3).$$

Then we find easily that Φ_{ABC} is a skew symmetric and parallel tensor of degree 3 on E^{m+4}.

We put

$$(4.7)\qquad \underset{1}{F}_\lambda=\underset{1}{\Phi}_A B_\lambda{}^A\,,\quad \underset{2}{F}_\lambda=\underset{2}{\Phi}_A B_\lambda{}^A\,,\quad \underset{3}{F}_\lambda=\underset{3}{\Phi}_A B_\lambda{}^A\,.$$

Then we get

$$(4.8)\qquad \begin{aligned} &(\underset{1}{F}_\lambda)=\left(-\frac{X^1}{X^{m+4}},\cdots,-\frac{X^{m+3}}{X^{m+4}}\right)\\ &(\underset{2}{F}_\lambda)=(0,0,\cdots,0,1)\\ &(\underset{3}{F}_\lambda)=(0,\cdots,0,1,0) \end{aligned}$$

by virtue of (1.4) and (4.7). From (1.13), we have

$$F_{\lambda\mu\nu} = \sum \operatorname{sgn}\begin{pmatrix}1, 2, 3\\ a, b, c\end{pmatrix} \underset{a}{F_\lambda} \underset{b}{F_\mu} \underset{c}{F_\nu}, \qquad (a, b, c = 1, 2, 3). \tag{4.9}$$

Now we consider a submanifold M^m of S^{m+3} whose local representation is given by

$$\begin{cases} X^i = u^i, & (i = 1, 2, \cdots, m) \\ (X^{m+1})^2 = t - \sum_{i=1}^{m} (u^i)^2, & 0 < t < 1, \\ X^{m+2} = 0, \quad X^{m+3} = 0, & \\ X^{m+4} = \sqrt{1-t}. & \end{cases} \tag{4.10}$$

We put

$$(N_{m+1}{}^\lambda) = (0, \cdots, 1),$$
$$(N_{m+2}{}^\lambda) = (0, \cdots, 1, 0),$$
$$(N_{m+3}{}^\lambda) = (X^1, \cdots, X^{m+1}, 0, 0) \times \sqrt{\frac{1-t}{t}}.$$

Then $N_{m+1}{}^\lambda$, $N_{m+2}{}^\lambda$ and $N_{m+3}{}^\lambda$ are mutually orthogonal unit normal vectors of M^m.

Making use of (4.8), (4.9) and (4.10), we have

$$r = F_{\lambda\mu\nu} N_{m+1}{}^\lambda N_{m+2}{}^\mu N_{m+3}{}^\nu = \underset{2}{F_\lambda} \underset{3}{F_\mu} \underset{1}{F_\nu} N_{m+1}{}^\lambda N_{m+2}{}^\mu N_{m+3}{}^\nu$$
$$= -\frac{1}{X^{m+4}} \sqrt{\frac{1-t}{t}} \sum_{i=1}^{m+1} (X^i)^2 = -\sqrt{t}.$$

Thus we see that the submanifold M^m is a desired one.

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