# Discrimination of the space-time $V$, II.

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This paper is a continuation of [6][1]. There, we have developed a theory by which we can discriminate the space-times belonging to $V_{\mathrm{I}}$, $V_{\mathrm{II}}$ and $V_{\mathrm{III}}$, which are not $V_0$. Now, in the present paper, we first deal with the same problem assuming that the given $V$ is $V_{\mathrm{IV}}$ or $V_{\mathrm{V}}$, which is not $V_0$, and then proceed to the problem of $V_0$. The same notations, terminologies, numbers of sections, of equations and of references as those in [6] are used.

## § 14. Discrimination of $V_{\mathrm{IV}}$.

Now we deal with the problem of discriminating $V_{\mathrm{IV}}$, assuming that the given $U$ is a $U_{\mathrm{IV}}$. Just as in the case of $V_{\mathrm{III}}$, we further assume that the $U_{\mathrm{IV}}$ is not $U_0$ and that $\overset{1}{u}_i$ is known. $V_{\mathrm{IV}}$'s are classified into two subclasses $V_{\mathrm{IV}a}$ and $V_{\mathrm{IV}b}$, each of which is defined by $\{\nu_1=\nu_2\neq\nu_3=\nu_4\}$ (or $\{\nu_1=\nu_3\neq\nu_2=\nu_4\}$) or $\{\nu_1=\nu_4\neq\nu_2=\nu_3\}$ respectively. Corresponding to this, we classify all $U_{\mathrm{IV}}$'s into two subclasses. A $U_{\mathrm{IV}}$ is $U_{\mathrm{IV}a}$ or $U_{\mathrm{IV}b}$ according as the two-dimensional eigenspace $E_s$ composed of only space-like eigenvectors of $K_i^{\cdot j}$ contains $\overset{1}{u}_i$ or not respectively. It goes without saying that the other two-dimensional eigenspace $E_t$ contains space-like, null and time-like eigenvectors.

First we consider $U_{\mathrm{IV}a}$'s. Determine the unit vector which is contained in $E_s$ and orthogonal to the $\overset{1}{u}_i$, and denote it by $\overset{2}{u}_i$. Let $v_i$ and $w_i$ be an arbitrary pair of mutually orthogonal space-like and time-like unit vectors belonging to $E_t$. Then by the c.v. test in which $(\overset{\alpha}{u}_i)=(\overset{1}{u}_i, \overset{2}{u}_i, v^*_i, w^*_i)$, where $v^*_i$ and $w^*_i$ are given by (5.2), we can determine whether the given $U_{\mathrm{IV}a}$ is a $V_{\mathrm{IV}a}$ or not. In the latter case, the $U_{\mathrm{IV}b}$ is not $V$.

Next we consider $U_{\mathrm{IV}b}$'s. In this case, $\overset{1}{u}_i$ must belong to $E_t$. Since $\overset{4}{u}_i$ must be orthogonal to $\overset{1}{u}_i$, we can easily determine it. Then, if $v_i$ and $w_i$ are any pair of mutually orthogonal unit space-like vectors belonging to $E_s$, the c.v. test in which $(\overset{\alpha}{u}_i)=(\overset{1}{u}_i, v^*_i, w^*_i, \overset{4}{u}_i)$, where $v^*_i$ and $w^*_i$ are given by (5.1), is sufficient for the discrimination.

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1) Numbers in brackets refer to the references at the end of the paper.

## § 15. Discrimination of $V_{\mathrm{V}}$.

Lastly, we deal with the discrimination of $V_{\mathrm{V}}$, assuming that the given $U$ is a $U_{\mathrm{V}}$, i.e. a $U$ whose four principal values are the same. In other words, the $U$ is a space-time which is an Einstein space in the sense of the differential geometry. As in the former sections, we further assume that the $U_{\mathrm{V}}$ is not $U_0$ and that $\overset{1}{u}_i$ is known.

Such $V_{\mathrm{V}}$'s are studied in detail in §§ 8, 9 and 10 of [2], and the properties of their c.s. are made clear in § 10 of [3]. The most important results are as follows: Such a $V_{\mathrm{V}}$ is neither $S(A)$ nor $S(B)$. The scalar curvature $K$ is a constant. These $V_{\mathrm{V}}$'s are classified into the following four classes:

($A_1$) $V_{\mathrm{V}}$'s satisfying $K=0$ and appearing in Proposition 9.2 of [2].
($A_2$) " " $K=0$ and " in ($ii_{\rho b}$) of Proposition 9.4 of [2].
($B$) " " $K<0$ and " in Proposition 10.1 of [2].
($C$) " " $K>0$ and " in ($C$) of § 10 of [2].

$\{\lambda\}$'s of these space-times are of the form $\{\rho_2, \rho_2, \rho_3, \rho_3, \rho_4, \rho_4\}$, where $\rho_\alpha$'s are non-constant (and accordingly, non-vanishing) functions which do not satisfy $\rho_2=\rho_3=\rho_4$. In other words, $\{\lambda\}$ is of type {three double eigenvalues} or {one quadruple and one double eigenvalues}, or if we use the notations in § 13 of [6],

$$(15.1) \qquad (5) \quad \{a_1, a_1, a_2, a_2, a_3, a_3\}, \quad \text{or} \quad (9) \quad \{a_1, a_1, a_1, a_1, a_2, a_2\},$$

where $a_\rho \neq a_\sigma$ when $\rho \neq \sigma$.

Now we shall show that, in the case of type (5), all $\overset{a}{u}_i$'s are determined by the use of $\overset{1}{u}_i$ and $u_{\rho|A}$'s, and that, in the case of type (9), one of $\overset{a}{u}_i$'s is determined and the remaining two are determined to within a transformation of the type (5.1) or (5.2), although in both cases the numberings of the vectors are not determined uniquely.

As is seen in § 8 of [2], the condition that a $V$ be $V_{\mathrm{V}}$ is given, in terms of $\lambda_{\alpha\beta}$'s, by

$$(15.2) \qquad \lambda_{12}=\lambda_{34}, \quad \lambda_{13}=\lambda_{24}, \quad \lambda_{14}=\lambda_{23}.$$

Hence, when ($\lambda_{12}, \lambda_{13}, \lambda_{14} \neq$), the six-dimensional eigenspace is composed of three two-dimensional eigenspaces, each of which is of signature type $(+-)$. The eigenvectors corresponding to, for example, $\lambda_{12}$ $(=\lambda_{34})$, are given by

$$(15.3) \qquad u_A = a u_{12|A} + b u_{34|A}, \qquad (u_A u^A = a^2 - b^2),$$

or, in terms of four-dimensional expressions,

$$u_{ij} = a\overset{1}{u}_{[i}\overset{2}{u}_{j]} + b\overset{3}{u}_{[i}\overset{4}{u}_{j]}, \tag{15.4}$$

where $a$ and $b$ are arbitrary scalars. Thus we have

$$2u_{ij}\overset{1}{u}^{j} = a\overset{2}{u}_{i}, \tag{15.5}$$

which shows that we can determine $\overset{2}{u}_i$ from $\overset{1}{u}_i$ and $u_A$. Similarly, we can determine $\overset{3}{u}_i$ and $\overset{4}{u}_i$ by using the eigenvectors corresponding to $\lambda_{13}$ and $\lambda_{14}$ respectively. We can distinguish $\overset{4}{u}_i$ from the other two by the condition that it must be time-like. If we consider the fact that we cannot distinguish beforehand $\lambda_{12}$, $\lambda_{13}$ and $\lambda_{14}$ from $\{\lambda\}$ only, we can conclude that when a $V_{\text{V}}$ of type (5) is given, we can determine $\overset{a}{u}_i$'s to within the interchange of $\overset{2}{u}_i$ and $\overset{3}{u}_i$ by using $\overset{1}{u}_i$ and the six-dimensional eigenvectors corresponding to $a_1$, $a_2$ and $a_3$.

Next we consider the case of (9), i.e. the case in which two of $\lambda_{12}$, $\lambda_{13}$ and $\lambda_{14}$ are equal. In this case by dealing with the two-dimensional eigenspace in the same way as the above, we can determine one of $\overset{a}{u}_i$'s. Then the remaining two, which are orthogonal to both $\overset{1}{u}_i$ and the determined $\overset{a}{u}_i$, are determined in the form of (5.1) or (5.2) according as the determined $\overset{a}{u}_i$ is $\overset{4}{u}_i$ or $\overset{2}{u}_i$ (or $\overset{3}{u}_i$) respectively.

Now we shall show that $\lambda_a$'s are determined from $\rho_a$'s. As the result of § **10** of [3], we can find that $(\lambda_2, \lambda_3, \lambda_4)$ is a permutation of $(\lambda_2^*, \lambda_3^*, \lambda_4^*)$, where $\lambda_a^*$'s are determined from $\rho_a$'s by

$$(\lambda_2^*\lambda_3^*\lambda_4^*)^2 = \rho_2\rho_3\rho_4\,; \quad (\lambda_2^*)^2 = \rho_3\rho_4/\rho_2, \cdots; \quad \lambda_3^*/\lambda_2^* = \lambda_2/\lambda_3, \cdots. \tag{15.6}$$

Thus when a $U_{\text{V}}$ which is not $U_0$ is given, we can determine to a great extent the quantities which must be identical with $\overset{a}{u}_i$'s and $\lambda_a$'s when the $U_{\text{V}}$ is a $V_{\text{V}}$. Therefore it will not be difficult to execute the c.v. test. We shall omit the discrimination theorem, which is evident, for brevity's sake.

We have completed the discriminations of $V_{\text{I}}$ and $V_{\text{II}}$, and those of $V_{\text{III}}$, $V_{\text{IV}}$ and $V_{\text{V}}$ which are not $V_0$.

## § 16. Discrimination of $V_0$, 1.

As is frequently stated, when a $V$ is not $V_0$, we can determine $\overset{1}{u}_i$ as the unit space-like vector proportional to the gradient of any non-constant $\lambda_{\alpha\beta}$. But if we consider a $V_0$, this method cannot be applied. This is the reason why we deal with the problem of discriminating $V_0$ separately. In the present section, we shall make some preparatory investigations.

It is shown in Proposition 3.1 of [2] that $V_0$'s are classified into the four types (I), (II), (III) and (IV). The types (II) and (III) are further classified into subtypes $\{(\mathrm{II}_{\rho a}), (\mathrm{II}_{\rho b}), (\mathrm{II}_{\rho b'})\}$ and $\{(\mathrm{III}_{\rho a}), (\mathrm{III}_{\rho b}), (\mathrm{III}_{\rho b'})\}$, $(\rho=2, 3, 4)$, respectively. The actual forms of $(\beta, \gamma, \delta)$, $(B, C, D)$, $\lambda_a$'s and $\lambda_{\alpha\beta}$'s in the standard coordinate system for the c.s. are given in detail in [2]. Of these $V_0$'s, those of type $(\mathrm{III}_{\rho a})$ or (IV) are nothing but the flat space-time $S(B)$, which is characterized by $K_{ij}{}^{mn}=0$, or, in terms of c.s., $\lambda_{\alpha\beta}=0$. Hence we shall assume hereafter that the $V_0$ is non-flat. In other words, the $V_0$'s dealt with in the following are restricted to those belonging to (I), $(\mathrm{II}_{\rho a})$, $(\mathrm{II}_{\rho b})$, $(\mathrm{II}_{\rho b'})$, $(\mathrm{III}_{\rho b})$ and $(\mathrm{III}_{\rho b'})$.

It is also shown in the same Proposition that $\lambda_{\alpha\beta}$'s of these space-times are given respectively by

(I) $\{p_2^2, p_3^2, p_4^2, p_3p_4, p_4p_2, p_2p_3\}$, $(p_2p_3p_4\neq 0)$.

$(\mathrm{II}_{4a})$ $\{p_2^2, p_3^2, 0, 0, 0, p_2p_3\}$, $(p_2p_3\neq 0)$.

$(\mathrm{II}_{4b})$ or $(\mathrm{II}_{4b'})$ $\{P, P, 0, 0, 0, P\}$, $(P=\pm p^2\neq 0)$.

$(\mathrm{III}_{4b})$ or $(\mathrm{III}_{4b'})$ $\{0, 0, P, 0, 0, 0\}$, $(P=\pm p^2\neq 0)$.

Those for $\{(\mathrm{II}_{2a}), (\mathrm{II}_{2b}), (\mathrm{II}_{2b'}), (\mathrm{II}_{3a}), (\mathrm{II}_{3b}), (\mathrm{II}_{3b'})\}$ and $\{(\mathrm{III}_{2b}), (\mathrm{III}_{2b'}), (\mathrm{III}_{3b}), (\mathrm{III}_{3b'})\}$ are given by the expressions similar to those for $\{(\mathrm{II}_{4a}), (\mathrm{II}_{4b}), (\mathrm{II}_{4b'})\}$ and $\{(\mathrm{III}_{4b}), (\mathrm{III}_{4b'})\}$ respectively. The $p_a$'s and $P$ are constants satisfying the conditions in brackets respectively.

The $\lambda_{\alpha\beta}$'s in the above are written in the order $\{\lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{34}, \lambda_{42}, \lambda_{23}\}$ respectively. If we change the orders suitably, the types of the $\{\lambda\}$'s are given by the following four:

(a) $\{e_1, e_2, e_3, e_4, e_5, e_6\}$, $(e_\rho\neq 0,\ \rho=1, 2, \cdots, 6;\ e_1, e_2, e_3>0)$.

(b) $\{0, 0, 0, e_1, e_2, e_3\}$, $(e_\rho\neq 0,\ \rho=1, 2, 3;\ e_1, e_2>0)$.

(c) $\{0, 0, 0, e, e, e\}$, $(e\neq 0)$.

(d) $\{0, 0, 0, 0, 0, e\}$, $(e\neq 0)$.

Here $e$'s are arbitrary constants satisfying the conditions in the brackets respectively.

Now we consider the problem of determining $p_a$'s from $\{\lambda\}$, or, in other words, from $e$'s. If we consider the properties of $\{\lambda\}_s$ at the same time, the results will become more useful in the discrimination theory. But we start only with the values of $\lambda_{\alpha\beta}$'s for brevity's sake. (See the next section.)

First we deal with the case of type (a). Then the $V_0$ is of type (I) and its line element in the standard coordinate system stated above is given by

$$ds^2=-dx^2-c_2e^{2p_2x}dy^2-c_3e^{2p_3x}dz^2+c_4e^{2p_4x}dt^2, \qquad (p_2p_3p_4\neq 0), \tag{16.1}$$

and we have $\lambda_a = -p_a$. Our present problem is to solve

$$e_1 = q_2^2, \quad e_2 = q_3^2, \quad e_3 = q_4^2, \quad e_4 = q_3 q_4, \quad e_5 = q_4 q_2, \quad e_6 = q_2 q_3, \tag{16.2}$$

and to determine $q_a$'s and then $p_a$'s. Here it should be noted that we use the notations $q_a$'s in place of $p_a$'s, since, as will be seen in due course, $q_a$'s do not necessarily coincide with $p_a$'s. As a matter of course, $q_a$'s are assumed to be real. As is easily seen, we have $q_2 q_3 q_4 \neq 0$. When $(q_2, q_3, q_4)$ are solutions of (16.2), $(-q_2, -q_3, -q_4)$ are also solutions. (This corresponds to the fact that the transformation $x' = -x$, by which the $V_0$ is kept invariant, changes the signs of $p_a$'s in the line element (16.1).) Therefore we can assume without any loss of generality that we have either $(q_2, q_3, q_4 > 0)$ or (two of $q_a$'s are positive and the remaining one negative).

First we consider the case in which $(e_4, e_5, e_6 > 0)$ holds. Then we have $(q_2, q_3, q_4 > 0)$. By eliminating $q_a$'s from (16.2), we have

$$e_4 = \sqrt{e_2 e_3}, \quad e_5 = \sqrt{e_3 e_1}, \quad e_6 = \sqrt{e_1 e_2}. \tag{16.3}$$

Conversely, when (16.3) is satisfied,

$$q_2 = \sqrt{e_1}, \quad q_3 = \sqrt{e_2}, \quad q_4 = \sqrt{e_3}, \tag{16.4}$$

satisfy (16.2). Hence we have

PROPOSITION 16.1. *A necessary and sufficient condition that* $\{e_\rho\}$ *of type* (a), *in which* $e_\rho > 0$ *for all* $\rho$, *admit* $(q_a)$, *which satisfies* $q_a > 0$ *for all* $a$, *is given by* (16.3).

Next we consider the problem of the freedom of $(q_a)$, i.e. the problem of determining, when $\{e_\rho\}$ satisfying $e_\rho > 0$ and (16.3) is given, whether we have any solution other than that given by (16.4). If we consider the fact that the numberings of $e_\rho$'s are arbitrary, we can restate this problem in more precise form: Let a set of six positive constants $\{e_\rho\}$ be given, and $(1', 2', \cdots, 6')$ be any permutation of $(1, 2, \cdots, 6)$. Then the problem is to determine all sets of positive constants $(q_2, q_3, q_4)$ satisfying

$$e_{1'} = q_2^2, \quad e_{2'} = q_3^2, \cdots, \quad e_{6'} = q_2 q_3. \tag{16.2'}$$

To solve this problem, we must make clear whether or not we have a solution of (16.2) of the form, for example, $(q_2 = \sqrt{e_4}, q_3 = \sqrt{e_5}, q_4 = \sqrt{e_6})$. After examining all possible cases, we obtain

PROPOSITION 16.2. *The set* $(q_a)$ *stated in the above problem is determined uniquely to within the freedom of the numberings of* $q_a$*'s. If we remove the condition "*$q_a > 0$ *for all* $a$*", the set obtained by changing the signs of all* $q_a$*'s also gives a solution.*

If we elucidate the first part of the Proposition in more detail, the circumstances are as follows: If we have some relations among $e_\rho$'s (for example, $e_1=e_2=\cdots=e_6$), we may have some other solutions $(q_a)$'s (for example, $q_2=\sqrt{e_1}$, $q_3=\sqrt{e_4}$, $q_4=\sqrt{e_6}$) formally different from $(q_a)$ given by (16.4). But all solutions are the same if we disregard the numberings of $q_a$'s and regard each $(q_a)$ as a set of three posistive constants.

(Note that if we consider $\{\lambda\}_s$ together with $\{\lambda\}$, or, in terms of $e_\rho$'s, $\{e_\rho\}_s$ together with $\{e_\rho\}$, the six $e_\rho$'s are classified into two classes, each of which is composed of three $e_\rho$'s, according as the corresponding eigenvectors are minus or plus respectively, and the results take more complicated but useful forms. We omit such considerations, however, as is stated in the above.)

Next we consider the case in which two of $(e_4, e_5, e_6)$ are negative and the remaining one is positive. In the same way as before, we can prove

PROPOSITION 16.3. *Let* $\{e_\rho\}$, $(e_1, e_2, e_3, e_6>0;\ e_4, e_5<0)$, *be given. A necessary and sufficient condition that there exist* $(q_a)$ *satisfying* (16.2) *and* $(q_2, q_3>0, q_4<0)$ *is given by*

$$e_4=-\sqrt{e_2e_3}\,,\quad e_5=-\sqrt{e_1e_3}\,,\quad e_6=\sqrt{e_1e_2}\,. \tag{16.5}$$

*When* (16.5) *holds, a set of solutions* $(q_a)$ *is given by* $(\sqrt{e_1}\,,\ \sqrt{e_2}\,,\ -\sqrt{e_3}\,)$.

PROPOSITION 16.4. *When* (16.5) *holds, the set* $(q_a)$ *satisfying* $(q_2, q_3>0, q_4<0)$ *and* (16.2′) *with* $(e_{1'}, e_{2'}, e_{3'}, e_{6'}>0;\ e_{4'}, e_{5'}<0)$, *is determined uniquely to within the interchange of* $q_2$ *and* $q_3$.

For example, if we consider the case in which $e_1e_2=e_3^2$ holds, $(\sqrt{e_1}\,, \sqrt{e_2}\,, -\sqrt{e_6}\,)$ gives a set of solutions, but this is identical with the above $(\sqrt{e_1}\,, \sqrt{e_2}\,, -\sqrt{e_3}\,)$. If we consider the fact that (16.2) and (16.2′) are symmetric with respect to $q_a$'s, Propositions 16.2 and 16.4 can be rewritten in the following form:

PROPOSITION 16.5. *When* (16.3) *or* (16.5) *holds, the solution of* (16.2) *are determined uniquely to within the freedom of numberings of* $q_a$*'s and that of the change of signs of all* $q_a$*'s.*

Now we come back to the problem of the discrimination of $V_0$. From the above we can conclude that, when a $V_0$ of type (a) is given, its $\{\lambda\}$ must be one of the following two types:

$$(\mathrm{a_i})\quad e_1, e_2, \cdots, e_6>0\,, \qquad (\mathrm{a_{ii}})\quad e_1, e_2, e_3, e_6>0\,;\quad e_4, e_5<0\,,$$

by changing the numberings of $e_\rho$'s when necessary. Then (16.3) or (16.5) must hold respectively. Thus we have

PROPOSITION 16. 6. *When a* $U_0$ *is given, calculate its* $\{e_\rho\}$. *If it satisfies the conditions stated above, it has a possibility of being a candidate for* $V_0$ *of type* (a), *and if it does not satisfy, it is not* $V_0$ *and accordingly is not* $V$.

The actual method of the discrimination for $U_0$ satisfying the conditions in the Proposition will be considered in the following sections.

Lastly, we touch on a special kind of $V_0$ of type (a). We consider a $V_0$ satisfying

$$(16.6) \qquad e_1 = e_2 = \cdots = e_6 = p^2 (>0), \quad \text{i.e.} \quad \{\lambda\} = \{p^2, \cdots, p^2\}.$$

In this case, the $V_0$ satisfies

$$(16.7) \qquad K_{ij}^{\cdot\cdot mn} = p^2 (\delta_i^m \delta_j^n - \delta_j^m \delta_i^n), \qquad (K = -12p^2 < 0),$$

and the $V_0$ is nothing but the $S(A)$ whose scalar curvature is negative. (Compare this result with (8. 7) of [2]. The $p$ in the (8. 7) is $-2p$ in terms of the present $p$.)

## § 17. Discrimination of $V_0$, 2.

Now we consider $V_0$ whose $\{\lambda\}$ is of type (b). In this case, the $V_0$ belongs to ($\mathrm{II}_{\rho a}$), and the line element is given by (16. 1) with $p_\rho = 0$. The equation corresponding to (16. 2) is

$$(17.1) \qquad e_1 = q_2^2, \quad e_2 = q_3^2, \quad e_3 = q_2 q_3, \qquad (e_4 = e_5 = e_6 = 0; \quad q_4 = 0).$$

Similarly to Propositions 16. 1 and 16. 2, we have

PROPOSITION 17. 1. *A necessary and sufficient condition that* $\{e_\rho\}$ *of type* (b) *admit a solution of* (17. 1) *is given by*

$$(17.2) \qquad e_3^2 = e_1 e_2.$$

*When this condition is satisfied, the set* $(q_a)$ *is determined uniquely to within the freedom of the numberings and the change of all signs of* $q_a$*'s.*

Hence, when a $U_0$ of type (b) satisfies the condition stated above, it has a possibility of being a candidate for $V_0$ of type ($\mathrm{II}_{\rho a}$). The actual method of the discrimination will be considered in the following sections.

In the above investigations, if we consider the case in which $e_1 = e_2 = e_3$ holds, $\{e_\rho\}$ is of type (c) with $e > 0$, and the $V_0$ belongs to type ($\mathrm{II}_{\rho b}$) at the same time. The $V_0$ is nothing but the $S(C)$ or $S(\bar{C})$. This is also seen from the fact that the line elements of ($\mathrm{II}_{\rho b}$) given in § 3 of [2] can contain those of ($\mathrm{II}_{\rho a}$) if we consider the special case in which $a = 0$ or $b = 0$ holds. In connection with these circumstances, we give the following Proposition, written for $\rho = 4$:

PROPOSITION 17. 2. *A necessary and sufficient condition that a $V_0$ of type* ($\mathrm{II}_{4a}$) *given by* $(p_2, p_3, 0)$ *be of type* ($\mathrm{II}_{4b}$) (*i.e.* $S(C)$) *at the same time is given by* $p_2=p_3$.

The proof is easy if we use the facts: (1) $\delta_4^i$ is the unique parallel vector field, and (ii) a necessary and sufficient condition that the three-dimensional space orthogonal to this vector be of constant curvature is given by $p_2^2=p_3^2=p_2p_3$, together with Proposition 19. 1 below.

As a matter of course, we have similar Propositions for $\rho=3$ and 2, in which cases $S(C)$ should be replaced by $S(\bar{C})$.

Thus we have completed the investigations of the problem of determining $p_a$'s from $\{\lambda\}$ assuming that the $V_0$ belongs to (I) or ($\mathrm{II}_{\rho a}$). The discussions have been made only from the values of $\lambda_{\alpha\beta}$'s. As is stated in the last section, however, the consideration of $\{\lambda\}_s$ together with $\{\lambda\}$ will be of use in determining the numberings of $p_a$'s and, as a result, in the discrimination process itself. If the given $U_0$ is a $V_0$, the six-dimensional eigenvectors corresponding to three $e_\rho$'s must be plus and those corresponding to the remaining three must be minus vectors. Therefore, when $\{\lambda\}_s$ does not satisfy this condition, the $U_0$ is not $V_0$. Next, as an example, we consider a $U_0$ whose $\{\lambda\}$ is of type (a). When, for example, $e_1$ is a simple eigenvalue and the eigenvector $u_{1|A}$ is plus, the $e_1$ should be identified with one of $p_2^2$, $p_3^2$ and $p_2p_3$. On the contrary, if $u_{1|A}$ is minus, the $e_1$ should be one of $p_4^2$, $p_2p_4$ and $p_3p_4$. It will be easily understood that such considerations are of use in determining the numberings of $p_a$'s. Similar circumstances also hold for all cases belonging to (a) or (b). But we stop here for brevity's sake.

## § 18. Discrimination of $V_0$, 3.

Let a $V_0$ belong to type (I) or ($\mathrm{II}_{\rho a}$). From the formulas in § 2, we find that such a $V_0$ has a c.s. satisfying

$$\lambda_2=-p_2, \quad \lambda_3=-p_3, \quad \lambda_4=-p_4; \tag{18.1}$$

$$\begin{aligned} \nu_1 &= -(p_2^2+p_3^2+p_4^2), & \nu_2 &= -p_2(p_2+p_3+p_4), \\ \nu_3 &= -p_3(p_2+p_3+p_4), & \nu_4 &= -p_4(p_2+p_3+p_4). \end{aligned} \tag{18.2}$$

As a matter of course, these $\nu_a$'s are constants, and we can determine them as the eigenvalues of the Ricci tensor $K_i^{\cdot j}$. Further, the eigenvectors of $K_i^{\cdot j}$ corresponding to $\nu_1$ are time-like, and those to $\nu_a$'s are space-like. On the other hand, $p_a$'s are almost determined from $\lambda_{\alpha\beta}$'s by the methods stated in the preceding sections. If we consider these circumstances, it is evident that (18.2) is of use in determining $p_a$'s more precisely.

From Proposition 3.2 of **[2]**, we have the following table concerning the relation between the type (I) or ($\mathrm{II}_{pa}$) and the classification ($V_{\mathrm{I}}$, $V_{\mathrm{II}}$, $\cdots$, $V_{\mathrm{V}}$).

| | | |
|---|---|---|
| (I), ($\mathrm{II}_{pa}$) | | $V_{\mathrm{I}}$ |
| (I) | $V_{\mathrm{II}\,a}$, $V_{\mathrm{II}\,b}$, $V_{\mathrm{II}\,c}$, $V_{\mathrm{II}\,d}$ | $V_{\mathrm{II}}$ |
| (I) | $V_{\mathrm{III}\,c}$ | $V_{\mathrm{III}}$ |
| ($\mathrm{II}_{pa}$) | $V_{\mathrm{III}\,a}$, $V_{\mathrm{III}\,b}$, $V_{\mathrm{III}\,c}$ | $V_{\mathrm{III}}$ |
| (I) | $S(A)$ | $V_{\mathrm{V}}$ |

From this table, we find, for example, that $V_0$ belonging to $V_{\mathrm{I}}$ is of type (I) or ($\mathrm{II}_{pa}$), and that the only $V_0$ belonging to $V_{\mathrm{V}}$ is $S(A)$, which is of type (I).

Now we come back to the problem of the discrimination. When the given $U_0$ is $U_{\mathrm{I}}$ or $U_{\mathrm{II}}$, the discrimination is easy by using the theory developed in §**4** and §**5**. In this case, the execution of the c.v. test will become much simpler if we use the values of $\lambda_a$'s obtained from (18.1) by using $p_a$'s almost determined by the methods studied in the preceding sections. On the other hand, $S(A)$ is characterized by (16.7), or, in terms of $\{\lambda\}$, by (16.6). Thus the method of the discrimination is evident. Hence, if it is known that the given $U_0$ is of type (I) or ($\mathrm{II}_{pa}$), the only remaining case to be studied is that in which the $U_0$ belongs to $U_{\mathrm{III}}$.

By virtue of the circumstances stated above, we consider in the following the $V_0$'s, which are of type (I) or ($\mathrm{II}_{pa}$) and belong to $V_{\mathrm{III}}$. The line element of such a $V_0$ in the coordinate system of (1.1) is given by (16.1) when the $V_0$ is of type (I), and by the same expression in which one of $p_a$'s is 0 and the remaining two are non-zero when it is of type ($\mathrm{II}_{pa}$).

First we consider the case of $V_{\mathrm{III}\,a}$. From the definition of $V_{\mathrm{III}\,a}$, we have $\nu_1=\nu_2=\nu_3\neq\nu_4$, or, in terms of $p_a$'s,

$$p_2^2+p_3^2+p_4^2=p_2(p_2+p_3+p_4)=p_3(\quad\text{''}\quad)\neq p_4(\quad\text{''}\quad), \tag{18.3}$$

from which we can obtain

$$p_2=p_3(\equiv p),\quad p_4=0;\quad \nu_1=\nu_2=\nu_3=-2p^2\neq\nu_4=0, \tag{18.4}$$

$$\{\lambda\}=\{0,0,0,P,P,P\},\quad \{\lambda\}_s=\{-\,-\,-,\ +\,+\,+\},\qquad (P\equiv p^2>0). \tag{18.5}$$

Therefore the $V_0$ is nothing but the $S(C)$ studied in detail in §**5** of **[2]**, and the method of its discrimination is given by Proposition 19.1 below.

Next we consider the case of $V_{\mathrm{III}\,b}$. In the same way as in the above, we can obtain from $\nu_1=\nu_3=\nu_4\neq\nu_2$ (or $\nu_1=\nu_2=\nu_4\neq\nu_3$),

$$p_3 = p_4(\equiv p), \quad p_2 = 0\,; \quad \nu_1 = \nu_3 = \nu_4 = -2p^2 \neq \nu_2 = 0\,, \tag{18.6}$$

and

$$\{\lambda\} = \{0, 0, 0, P, P, P\}\,, \quad \{\lambda\}_s = \{+\,+\,-,\ +\,-\,-\}\,, \quad (P \equiv p^2 > 0). \tag{18.7}$$

The equation for the case $(\nu_1 = \nu_2 = \nu_4 \neq \nu_3)$ corresponding to (18.6) is evident. The $V_0$ is nothing but the $S(\bar{C})$, whose discrimination theorem is given by the same Proposition 19.1 below.

Lastly we deal with $V_0$ belonging to $V_{\mathrm{III}\,c}$. In this case, we have $\nu_1 \neq \nu_2 = \nu_3 = \nu_4$, or, in terms of $p_a$'s,

$$p_2^2 + p_3^2 + p_4^2 \neq p_2(p_2 + p_3 + p_4) = p_3(\quad \text{"} \quad) = p_4(\quad \text{"} \quad)\,, \tag{18.8}$$

from which we have

$$p_2 + p_3 + p_4 = 0\,. \tag{18.9}$$

Considering the case $p_2 p_3 p_4 \neq 0$, we can easily prove from these equations

PROPOSITION 18.1. *When a $V_0$ of type* (I) *belongs to $V_{\mathrm{III}\,c}$, its $\{\lambda\}$ is of type stated in Propositions* 16.3 *and* 16.4. *Further, a necessary and sufficient condition that $\{e_\rho\}$ have a solution of* (16.2) *and* (18.9) *is given by that in Proposition* 16.3 *and*

$$e_3 + e_4 + e_5 = 0\,. \tag{18.10}$$

*When this condition is satisfied, $p_a$'s are determined from $\lambda_{\alpha\beta}$'s uniquely to within the interchange of $p_2$ and $p_3$ and the change of all signs, if we take the $\{\lambda\}_s$ into consideration.*

For example, ($p_2$ and $p_3$) or ($p_3$ and $p_4$) (or $p_2$ and $p_4$) are of the same sign according as the eigenvectors corresponding to $e_6$, which is positive, are plus or minus vectors respectively.

Next we proceed to the case in which one of $p_a$'s, say $p_4$, is 0. From (18.9), we have

$$p_2 = -p_3(\equiv p) \neq 0\,; \quad \nu_2 = \nu_3 = \nu_4 = 0 \neq \nu_1 = 2P\,, \qquad (P \equiv p^2), \tag{18.11}$$

$$\{\lambda\} = \{0, 0, 0, P, P, -P\}\,, \quad \{\lambda\}_s = \{+\,+\,-,\ +\,-\,-\}\,. \tag{18.12}$$

Considering similarly the cases $p_3 = 0$ and $p_2 = 0$, we have

PROPOSITION 18.2. *When a $V_0$ of type* ($\mathrm{II}_{\rho a}$) *belongs to $V_{\mathrm{III}\,c}$, its $\{\lambda\}$ is of type given in* (18.12). *When this condition is satisfied, $p_a$'s are determined by $\lambda_{\alpha\beta}$'s uniquely to within the interchange of $p_2$ and $p_3$ and the change of all signs, if we consider $\{\lambda\}_s$.* (It should be noted here that one of $p_a$'s is 0.)

Now we shall show a theorem which is of use in discriminating $V_0$ which belongs to $V_{\mathrm{III}\,c}$ and is of type (I) or ($\mathrm{II}_{\rho a}$). In such a $V_0$, $\overset{1}{u}_i$ is known from $K_i^{\cdot j}$ *by* considering the unit eigenvector corresponding to the simple eigenvalue.

PROPOSITION 18.3. *Let a* $V_0$ *defined by* (16.1) *be given. Then the equations*

$$(18.13)\qquad \nabla_i u_j = -p_2 \overset{1}{u}_i u_j\,, \quad u_i u^i = -1\,, \quad \overset{1}{u}_i u^i = 0\,,$$

*determine* $u_i$ *uniquely to within its sign, provided* $p_2 \neq p_3$ *and* $p_2 \neq p_4$. *Here* $\overset{1}{u}_i$ *and* $u_i$ *are known and unknown quantities respectively. In other words, we have* $u_i = \epsilon \overset{2}{u}_i$ *in this case. When* $p_2 = p_3 = p_4$ *or* $p_2 = p_3 \neq p_4$ *or* $p_2 = p_4 \neq p_3$ *holds, we have* $u_i = a\overset{2}{u}_i + b\overset{3}{u}_i + c\overset{4}{u}_i$ *or* $u_i = a\overset{2}{u}_i + b\overset{3}{u}_i$ *or* $u_i = a\overset{2}{u}_i + b\overset{4}{u}_i$, *where* $a$, $b$ *and* $c$ *are arbitrary constants satisfying* $a^2 + b^2 - c^2 = 1$ *or* $a^2 + b^2 = 1$ *or* $a^2 - b^2 = 1$ *respectively. Similar Propositions hold if we replace* $p_2$ *by* $p_3$ *or* $p_4$ *in* (18.13), *and, in the case of* $p_4$, *the second equation by* $u_i u^i = 1$.

It is evident that the existence of the arbitrariness of $(a, b, c)$ and $(a, b)$ correspond to the freedoms of the generalized $\omega$- and $\omega$-transformations of c.s. respectively. The proof is evident if we calculate the actual expression of (18.13) in the coordinate system of (16.1), and use the relation $\overset{1}{u}_i = \delta_i^1$.

From the above considerations, we can conclude that when a $V_0$ of type (I) or ($\mathrm{II}_{\rho a}$) is given, and it is known that it belongs to $V_{\mathrm{III}\,c}$, we can determine its $p_a$'s, $\overset{1}{u}_i$ and $\overset{a}{u}_i$'s by using $K_i^{\cdot j}$, $\{\lambda\}$ and $\{\lambda\}_s$ and solving (18.13).

On the other hand, both $V_{\mathrm{III}\,b}$ and $V_{\mathrm{III}\,c}$ (generically called $V_{\mathrm{III}\,2}$) are characterized by the fact that the eigenvector $v_i$ of $K_i^{\cdot j}$ corresponding to the simple eigenvalue is space-like. Further, if we use Proposition 2.1 of [2], we can determine whether a given $V_{\mathrm{III}\,2}$ is $V_{\mathrm{III}\,b}$ or $V_{\mathrm{III}\,c}$ as follows:

PROPOSITION 18.4. *A* $V_{\mathrm{III}\,2}$ *belongs to* $V_{\mathrm{III}\,c}$ *when and only when* $v^i \nabla_i v_j = 0$ *and* $\nabla_i v_j \neq 0$. *Thus, when* $v^i \nabla_i v_j \neq 0$ *or* $\nabla_i v_j = 0$ *holds, the* $V_{\mathrm{III}\,2}$ *belongs to* $V_{\mathrm{III}\,b}$.

Now we come back to the problem of the discrimination. Let a $U_0$ belonging to $U_{\mathrm{III}}$ be given. We assume that it is known to belong to $U_{\mathrm{III}\,c}$, by examining the sign of the magnitude of the $v_i$ and trying the test stated in the above Proposition. Further we assume that its $\{\lambda\}$ be of type (a) or (b). (It is evident that the $U_0$ is neither $S(C)$ nor $S(\bar{C})$, since they belong to $V_{\mathrm{III}\,a}$ and $V_{\mathrm{III}\,b}$ respectively, and accordingly that the $\{\lambda\}$ of the given $U_0$ cannot be of type (c).) Then from the discussions in **§16** and the present section, we can conclude that, if the $U_0$ is a $V_0$, it must be of type (I) or

($\mathrm{II}_{\rho a}$), and its $\{\lambda\}$ and $\{\lambda\}_s$ must satisfy the conditions stated in Proposition 18.1 or 18.2. (If this is not the case, the $U_0$ cannot be a $V_0$.) Determine $p_a$'s from $\lambda_{\alpha\beta}$'s as is stated in these Propositions, $\overset{1}{u}_i$ from $K_i^{\cdot j}$ by putting $\overset{1}{u}_i=v_i$, and $\overset{a}{u}_i$'s by the method stated in Proposition 18.3. Then try c.v. test. If the $U_0$ fails anywhere, it is not $V_0$. It should be noted here that in the process of finding $\overset{a}{u}_i$'s, we may have some arbitrariness given by $a$ and $b$ stated in Proposition 18.3. It is evident in this case, however, that any pair of $\overset{2}{u}_i$ and $\overset{3}{u}_i$, for example, will be of use in the c.v. test. Moreover, we cannot have the case in which $p_2=p_3=p_4$ holds, since the relation (18.9) must hold.

Thus we have finished the investigations concerning the discrimination of $U_0$, which belongs to any of $U_{\mathrm{I}}$, $U_{\mathrm{II}}$, $U_{\mathrm{V}}$ and $U_{\mathrm{III}}$, and whose $\{\lambda\}$ is of type (I) or ($\mathrm{II}_{\rho a}$). But we will not restate the results in the from of Proposition for brevity's sake.

## § 19. Discrimination of $V_0$, 4.

In the previous sections, we have completed the discussions for the cases in which $\{\lambda\}$'s are of type (a) or (b). Now we consider the case of type (c). As is stated in § 17, the case of (c), in which $e>0$ holds, is a special case of (b), and gives $S(C)$ or $S(\bar{C})$. Now we shall give a theorem characterizing $S(C)$ and $S(\bar{C})$, which naturally includes the case $e<0$ also.

PROPOSITION 19.1. *An $S(C)$ (or $S(\bar{C})$) is characterized by the conditions that* (i) *it admit one and only one parallel time-like* (*or space-like*) *vector field $v_i$ to within an arbitrary constant multiplier, and that* (ii) *it be conformally flat.*

PROOF. We prove the theorem for $S(C)$. The necessity is evident by the direct calculations. (Cf. Proposition 5.6 of [2].) Conversely, we assume that (i) and (ii) are satisfied. Just as in [3.1] of [1], the line element can be brought into the form

$$ds^2 = -h_{\rho\sigma}dx^\rho dx^\sigma + dt^2\,, \qquad v^i = v_i = \delta_i^4\,, \tag{19.1}$$

where $h_{\rho\sigma}=h_{\rho\sigma}(x^\tau)$ and $\rho$, $\sigma$, $\tau=1, 2, 3$. Then it is easy to prove that the condition that the space-time be conformally flat is equivalent to that the three-dimensional space defined by $h_{\rho\sigma}$ be of constant curvature. Thus the space-time is $S(C)$. Similarly, we can prove the theorem for $S(\bar{C})$.

Thus we can conclude that when a $U_0$, which belongs to $U_{\mathrm{III}a}$ or $U_{\mathrm{III}b}$ and whose $\{\lambda\}$ is of type (c), is given, we can determine whether it is a $V_0$

of type $(\mathrm{II}_{\rho b})$ or $(\mathrm{II}_{\rho b'})$, i.e. $S(C)$ or $S(\bar{C})$. As a matter of course, the special case of type (b) stated in the above is included in the above discussions.

Proposition 19.1 is written in an invariant form, and is of use as a discrimination theorem for $S(C)$ and $S(\bar{C})$. Moreover, it is shown in Proposition 3.1 of **[2]** that a $V_0$ of type (c) is necessarily $S(C)$ or $S(\bar{C})$. Therefore we can conclude that we have completed the discrimination theory for $V_0$ of type (c).

REMARK. The invariant characterization of $S(C)$ was investigated in detail by the present author in **[7]** from the standpoint that it is a special type of the spherically symmetric space-time. Another method is seen in **[8]** by the same author.

Lastly, we consider the case of $V_0$ whose $\{\lambda\}$ is of type (d). Such $V_0$'s are studied in detail in §3 and §7 of **[2]**. The main results are as follows. We have only three kinds of such $V_0$'s, i.e. those satisfying ($\lambda_{14}=P$, other $\lambda_{\alpha\beta}=0$), ($\lambda_{13}=P$, other $\lambda_{\alpha\beta}=0$) and ($\lambda_{12}=P$, other $\lambda_{\alpha\beta}=0$). The first one belongs to $V_{\mathrm{IV}\,b}$ and the remaining two to $V_{\mathrm{IV}\,a}$. Each class is further classified into two subclasses according as the sign of $P$. They are $\{(\mathrm{III}_{4b}), (\mathrm{III}_{4b'})\}$, $\{(\mathrm{III}_{3b}), (\mathrm{III}_{3b'})\}$ and $\{(\mathrm{III}_{2b}), (\mathrm{III}_{2b'})\}$. The indices $b$ and $b'$ correspond to the cases $P>0$ and $P<0$ respectively. The last two classes are (23)-conjugate to each other and their space-times are the same. Hence we can consider that these two classes are the same.

From these considerations, we arrive at the following results. When a $V_0$ of type (d) is given, we can discriminate to which of $V_{\mathrm{IV}\,a}$ or $V_{\mathrm{IV}\,b}$ it belongs by the condition ($\nu=-P$, $\nu'=0$) or ($\nu=0$, $\nu'=-P$) respectively, or by the condition that the six-dimensional eigenvector $\boldsymbol{u}_A$ corresponding to $P$ be plus or minus vector respectively. Here $\nu$ is the double eigenvalue of $K_i^{\cdot j}$ whose eigenspace is composed of only space-like eigenvectors and $\nu'$ is the other double eigenvalue. In other words, when a $U_0$ belonging to $U_{\mathrm{IV}}$ is given (or when $U_0$ of type (d) is given), and if the $U_0$ is a $V_0$, we can determine the subclass to which the $U_0$ belongs by the method stated above.

Now we assume shat a $U_0$ belonging to $U_{\mathrm{IV}}$ is given, and that it is found by the method stated above that it belongs to $V_{\mathrm{IV}\,a}$. If this $U_0$ is a $V_{\mathrm{IV}\,a}$, its $\overset{\alpha}{u}_i$'s are given by ($\overset{2}{u}_i=\overset{2}{u}{}^*_i$, $\overset{3}{u}_i=\overset{3}{u}{}^*_i$) and

$$(19.2)\qquad \overset{1}{u}_i=\overset{1}{u}{}^*_i\cosh\sigma+\overset{4}{u}{}^*_i\sinh\sigma\,,\qquad \overset{4}{u}_i=\overset{1}{u}{}^*_i\sinh\sigma+\overset{4}{u}{}^*_i\cosh\sigma\,,$$

where $\sigma$ is an arbitrary scalar and ($\overset{2}{u}{}^*_i$, $\overset{3}{u}{}^*_i$) and ($\overset{1}{u}{}^*_i$, $\overset{4}{u}{}^*_i$) are any pairs of the unit eigenvectors of $K_i^{\cdot j}$ corresponding to $-P$ and 0 respectively, and satisfying the orthonormal condition $(\mathrm{F}_1)$. Further, as a result of §7 of **[2]**, we must have $\lambda_2=\lambda_3=0$. (This is the reason why we can take any pair of $(\overset{2}{u}{}^*{}_i, \overset{3}{u}{}^*{}_i)$ as $(\overset{2}{u}_i, \overset{3}{u}_i)$.) Thus we can determine whether the given $U_0$ is a $V_{\mathrm{IV}a}$ or not by the c.v. test using these $\overset{\alpha}{u}_i$'s and $\lambda_\alpha$'s. Similar results can be obtained with respect to the case of $V_{\mathrm{IV}b}$, but we omit them for brevity's sake.

Thus we have completed the discrimination theory for $V_0$ of type (d). As an appendix we add a proposition concerning $V_0$'s of type $(\mathrm{III}_{pb})$ or $(\mathrm{III}_{pb'})$ and corresponding to Proposition 19.1.

PROPOSITION 19.2. *The $V_0$ of type* $(\mathrm{III}_{4b})$ *or* $(\mathrm{III}_{4b'})$ *is a direct product of two-dimensional flat space* (*whose signature is* $-2$) *and a two-dimensional space of constant curvature* (*whose signature is* 0). *The two-dimensional flat space is the linear space spanned by the two mutually orthogonal parallel vector fields, which are both space-like. Similar propositions hold for the cases of* $(\mathrm{III}_{2b})$, $(\mathrm{III}_{2b'})$, $(\mathrm{III}_{3b})$ and $(\mathrm{III}_{3b'})$.

The proof is easy if we use the results of §**5** and §**7** of [**2**]. As a matter of course, we can use this Proposition in the discrimination process of the $V_0$ under consideration.

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(Received March 20, 1972)