Submanifolds of codimension greater than 1 with certain properties

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§ 1. Introduction

H. Liebmann [5] has proved that the only ovaloid with constant mean curvature in a 3-dimensional Euclidean space is a sphere. Various generalizations of this theorem have been obtained recently. Y. Katsurada [1], [2] and K. Yano [8] have generalized this theorem to a hypersurface of a Riemannian manifold admitting an infinitesimal conformal or homothetic transformation.

On the other hand Y. Katsurada [3], [4], H. Kōjyo [3], T. Nagai [4], and K. Yano [9] studied this problem when the enveloping manifold admits an infinitesimal conformal transformation, and proved that under some conditions the submanifold in consideration is umbilical only with respect to the mean curvature normal.

Furthermore, M. Okumura [7] studied the same problem as that in [3], [4], [9] and proved that under certain conditions the submanifold in consideration is not only umbilical with respect to the mean curvature normal but also is totally umbilical.

The purpose of the present paper is to obtain a theorem which generalizes the result that M. Okumura has proved in [7]. The author wishes to express to Prof. Y. Katsurada and Dr. T. Nagai his very sincere thanks for their kind guidance.

§ 2. Submanifolds in a Riemannian manifold

Let M^n be an n-dimensional orientable differentiable manifold, and ι be an immersion of M^n into an m-dimensional Riemannian manifold $\widetilde{M}^m$. Then the Riemannian metric $\tilde{g}$ of $\widetilde{M}^m$ induces naturally a Riemannian metric g on M^n by the immersion ι in such a way that

$$g(X, Y)=\tilde{g}\big(d\iota(X),\ d\iota(Y)\big),$$

where we denote by $d\iota$, the differential map of ι, and by X, Y vector fields in M^n. In order to simplify the presentation we identify, for each point $p\in M^n$, the tangent space $T_p(M)$ with $d\iota(T_p(M))\subset T_{\iota(p)}(M)$ by means of the immersion.

Since M^n is orientable, if we assume that $\widetilde{M}^m$ is also orientable, in a certain neighborhood U of $p \in M^n$ we can choose two fields of mutually orhtogonal unit normal vectors C_v of M^n at each point of U in such a way that, if $(B_1, \cdots, B_n)_p$ is a positively oriented frame of tangent vectors at p then the frame $(d\iota(B_1), \cdots, d\iota(B_n), C_v)_{\iota(p)}$ is also positively oriented, where here and in the sequel the indices u, v run over the range $\{n+1, \cdots, m\}$.

Let X, Y be tangent to M^n. Then the covariant derivative of $d\iota(X)$ in the direction of $d\iota(Y)$ is expressed as

$$\tilde{\nabla}_{d\iota(Y)}\, d\iota(X) = \nabla_Y X + \sum_v h_v(X, Y)\, C_v\,. \tag{2.1}$$

Although $\nabla_Y X$ denotes the tangential components of $\tilde{\nabla}_{d\iota(Y)}\, d\iota(X)$, it is easily verified that $\nabla_Y X$ is identical with the covariant derivative of X in the direction of Y with respect to the induced Riemannian metric g.

The tensor h_v of type $(0, 2)$ over M^n is called the second fundamental tensor of M^n in $\widetilde{M}^m$ with respect to the normal vector C_v. The normal vector C_v is unit normal vector, and we can put

$$\tilde{\nabla}_X C_v = -A_v(X) + \sum_u l_{vu}(X)\, C_u\,, \tag{2.2}$$

where $A(X)$ denote the tangential component of $\tilde{\nabla}_X C_v$ on M^n, and l is the third fundamental form of M^nin $\widetilde{M}^m$.

Let X, $Y \in T_p(M)$. Then we have the equation of Weingarten:

$$\tilde{g}(\tilde{\nabla}_X C_v, Y) = -h_v(X, Y)\,. \tag{2.3}$$

Let $\{x^a\}$, $a=1, 2, \cdots, n$ be local coordinate in an open neighborhood U of $p \in M^n$.

Using the local coordinate, (2. 2) is expressed as

$$\tilde{\nabla}_{B_c} C_v = -H_{vc}^a B_a + \sum_u L_{vuc} C_u\,, \tag{2.4}$$

where $L_{vuc} = l_{vu}(B_c)$.

By virture of (2. 3)

$$\begin{aligned} H_{vba} &\overset{\text{def}}{=} h_v\left(\frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^a}\right) \\ &= -\tilde{g}(\tilde{\nabla}_{B_b} C_v, B_a) \\ &= H_{vb}^c \tilde{g}(B_c, B_a) \\ &= H_{vb}^c g_{ca}\,, \end{aligned} \tag{2.5}$$

where $g_{ca} = \tilde{g}(B_c, B_a)$, and we use Einstein's summation convention for simplicity. Let $\tilde{R}$ and R be curvature tensors of $\widetilde{M}^m$ and M^n respectively.

Then the equation of Mainardi-Codazzi is given by

$$(2.6)\qquad \tilde{g}\left(\tilde{R}(B_c, B_b)B_a, C_v\right) = \nabla_c H_{vba} - \nabla_b H_{vca} + \sum_u (L_{vub} H_{uca} - L_{vuc} H_{uba}),$$

where $R_{abcd} = g\left(R\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right)\frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^d}\right)$, and ∇_a denotes the operation of covariant differentiation in classical tensor calculus.

§ 3. Submanifolds and some vector fields

Let g^{ab} be the inverse matrix of g_{ab}, and put

$$H_{va}^{a} = g^{ba} H_{vab}, \qquad H_v^{ab} = g^{ac} H_{vc}^{b}.$$

Then the vector H defined by

$$(3.1)\qquad H = \frac{1}{n} \sum_v H_{va}^{a} C_v$$

is dependent of the choice of mutually orthogonal unit normal vectors of M^n, and so defines a vector field along M^n. We call this vector field the mean curvature vector field along M^n with respect to $\tilde{M}^m$.

When at each point of M^n there exist function h'_v such that $h_v(X, Y) = h'_v g(X, Y)$ or equivalently

$$(3.2)\qquad H_{vba} = h'_v g_{ba},$$

we call M^n a totally umbilical submanifold. From this definition, if M^n is totally umbilical we have

$$(3.3)\qquad h'_v = \frac{1}{n} H_{va}^{a}.$$

PROPOSITION 3.1. *A necessary and sufficient condition for a submanifold to be umbilical is that the following equation is satisfied*:

$$(3.4)\qquad H_{vba} H_v^{ba} = \frac{1}{n}(H_{va}^{a})^2.$$

PROOF. This follows from the identity

$$\left(H_{vba} - \frac{H_{vc}^{c}}{n} g_{ba}\right)\left(H_v^{ba} - \frac{H_{vc}^{c}}{n} g^{ba}\right) = H_{vba} H_v^{ba} - \frac{1}{n}(H_{vc}^{c})^2,$$

and the positive definiteness of Riemannian metric g_{ba}.

Next we consider the normal bundle $N(M^n)$ of M^n. For $X \in T(M^n)$, $N \in N(M^n)$, a connection $'\nabla$ on $N(M^n)$ is defined by

$$'\nabla_X N = (\tilde{\nabla}_X N)^N, \tag{3.5}$$

where $(\tilde{\nabla}_X N)^N$ denotes the normal part of $\tilde{\nabla}_X N$. When $\tilde{\nabla}_X N$ vanishes identically along M^n we say that N is parallel with respect to the connection of the normal bundle $N(M^n)$.

PROPOSITION 3.2. *The mean curvature vector field H is parallel with respect to the connection of the normal bundle if and only if the following equation is valid,*

$$\nabla_c H_{ua}^{\ \ a} = -\sum_v H_{va}^{\ \ a} L_{vuc}. \tag{3.6}$$

PROOF.

$$\begin{aligned}\tilde{\nabla}_{B_c} H &= \frac{1}{n}\sum_v \tilde{\nabla}_{B_c}(H_{va}^{\ \ a} C_v)\\ &= \frac{1}{n}\sum_v \nabla_c H_{va}^{\ \ a} C_v + \frac{1}{n}\sum_v H_{va}^{\ \ a}(-H_{vc}^{\ \ b} B_b + \sum_u L_{vuc} C_u)\\ &= \frac{1}{n}\sum_v H_{va}^{\ \ a} H_{vc}^{\ \ b} B_b + \frac{1}{n}\sum_a (\nabla_c H_{ua}^{\ \ a} + \sum_u H_{av}^{\ \ v} L_{vuc}) C_u,\end{aligned}$$

from which we have Proposition 3.2.

PROPOSITION 3.3. *If the mean curvature vector field H is parallel with respect to the induced connection of the normal bundle, then $\sum_v (H_{va}^{\ \ a})^2$ is constant.*

PROOF. From Proposition 3.2. this is easily verified.

PROPOSITION 3.4. *Let M^n be a totally umbilical submanifold on $\tilde{M}^m$ such that at each point of M^n the tangent space is invariant under the curvature transformation of $\tilde{M}^m$. Then the mean curvature vector field H is parallel with respect to the induced connection of the normal bundle.*

PROOF. Since at each point of M^n the tangent space is invariant under the curvature transformation of $\tilde{M}^m$, equation (2.6) reduces to

$$\nabla_c H_{va}^{\ \ a} - \nabla_b H_{vc}^{\ \ b} = \sum_u (L_{vuc} H_{ua}^{\ \ a} - L_{vub} H_{uc}^{\ \ b}). \tag{3.7}$$

Assuming M^n to be totally umbilical submanifold of $\tilde{M}^m$, we have

$$\begin{aligned} H_{va}^{\ \ a} &= nh'_v \\ H_{vc}^{\ \ b} &= H_{vca} g^{ab} = h'_v g_{ca} g^{ab} = h'_v \delta_c^b, \end{aligned} \tag{3.8}$$

and consequently, inserting the relation (3.8) into (3.7), we obtain

$$\nabla_c h'_v = \sum_u h'_u L_{vuc} = -\sum_u h'_u L_{uvc}$$

$$\frac{1}{n}\nabla_c H_{va}^{a} = -\frac{1}{n}\sum_u H_{ua}^{a} L_{uvc}\,,$$

from which, together with Proposition 3. 2, we have Proposition 3. 4.

§ 4. Integral formulas

Let M^n be a compact, orientable submanifold of $\tilde{M}^m$ in which there exists an infinitesimal conformal transformation $\tilde{X}$, that is, in which $\tilde{X}$ is a field of $\tilde{M}^m$ and satisfies for any vector fields $\tilde{Y}$, $\tilde{Z} \in T(\tilde{M}^m)$,

$$(4.\,1) \qquad \left(\mathscr{L}(\tilde{X})\tilde{g}\right)(\tilde{Y}, \tilde{Z}) = \tilde{g}(\tilde{\nabla}_{\tilde{Y}}\tilde{X}, \tilde{Z}) + \tilde{g}(\tilde{Y}, \tilde{\nabla}_{\tilde{Z}}\tilde{X}) = 2\rho\tilde{g}(\tilde{Y}, \tilde{Z}),$$

where $\mathscr{L}(\tilde{X})$ is the operator of Lie derivative with respect to $\tilde{X}$ and ρ is a function on $\tilde{M}^m$. $\tilde{X}$ is represented as a linear combination of B_a, C_v. Hence we put

$$(4.\,2) \qquad \tilde{X} = X + \sum_v \alpha_v C_v,\ X = v^a B_a\,.$$

Since (2. 1) and (4. 2) yield that

$$\begin{aligned} g(\nabla_Y X, Z) &= \tilde{g}\left(\tilde{\nabla}_Y X - \sum_v h_v(X, Y)\, C_v, Z\right) \\ &= \tilde{g}(\tilde{\nabla}_Y X, Z) \\ &= \tilde{g}\left\{\tilde{\nabla}_Y(\tilde{X} - \sum_v \alpha_v C_v),\ Z\right\} \\ &= \tilde{g}(\tilde{\nabla}_Y \tilde{X}, Z) - \sum_v \alpha_v \tilde{g}(\tilde{\nabla}_Y C_v, Z) \\ &= \tilde{g}(\tilde{\nabla}_Y \tilde{X}, Z) + \sum_v \alpha_v \tilde{g}(H_{vc}^{a} B_a, Z)\,. \end{aligned}$$

Similarly we have

$$g(Y, \nabla_Z X) = \tilde{g}(Y, \tilde{\nabla}_Z \tilde{X}) + \sum_v \alpha_v \tilde{g}(Y, H_{vb}^{a} B_a)\,.$$

Substituting $v^a\partial/\partial x^a$, $\partial/\partial x^a$ and $\partial/\partial x^b$ for X, Y and Z respectivety, we get

$$(4.\,3) \qquad \nabla_c v_b + \nabla_b v_c = 2(\rho g_{bc} + \sum_v \alpha_v H_{vbc})$$

because of (4. 1), which implies that

$$(4.\,4) \qquad \operatorname{div} X = \nabla_a v^a = n\rho + \sum_v \alpha_v H_{va}^{a} = n\left\{\rho + \tilde{g}(H, \tilde{X})\right\}.$$

Since M^n is compact, orientable we have

$$(4.\,5) \qquad \int_{M^n} \rho\, dM = -\int_{M^n} \tilde{g}(H, \tilde{X})\, dM\,.$$

Now we put $F = \sum_v A_v H_{vd}^{d}$. Then it is easily verified that F is independent of the choice of mutually orthogonal normal vector C_v and consequently that F defines a linear transformation on $T(M^n)$. Let $Y=FX$, that is,

$$\begin{aligned} Y &= \sum_v H_{vd}^{d} A_v(X) \\ &= \sum_v H_{vd}^{d} A_v\left(v^b \frac{\partial}{\partial x^b}\right) \\ &= \sum_v H_{vd}^{d} H_{vb}^{a} \frac{\partial}{\partial x^a} v^b . \end{aligned}$$

Putting $Y=u^c\partial/\partial x^c$, we have

$$u_e = g(Y, \partial/\partial x^e) = \sum_v H_{vd}^{d} H_{vbe} v^b, \tag{4.6}$$

from which

$$\begin{aligned} \nabla_a u_e = \sum_v (\nabla_a H_{vd}^{d})\, H_{vbe} v^b + \sum_v H_{vd}^{d} (\nabla_a H_{vbe})\, v^b \\ + \sum_v H_{vd}^{d} H_{vbe} (\nabla_a v^b) . \end{aligned}$$

Thus we get

$$\begin{aligned} \operatorname{div} Y = \nabla_a u^a = \sum_v (\nabla_a H_{vd}^{d})\, H_{vb}^{a} v^b + \sum_v H_{vd}^{d} (\nabla^e H_{vbe})\, v^b \\ + \sum_v H_{vd}^{d} H_{vb}^{a} (\nabla_a v^b) . \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_v H_{vd}^{d} H_{vb}^{a} \nabla_a v^b &= \sum_v H_{vd}^{d} H_v^{ac} g_{cb} \nabla_a v^b \\ &= \sum_v H_{vd}^{d} H_v^{ac} \nabla_a v_c \\ &= \frac{1}{2} \sum_v (H_{vd}^{d} H_v^{ac} \nabla_a v_c + H_{vd}^{d} H_v^{ca} \nabla_c v_a) \\ &= \frac{1}{2} \sum_v H_{vd}^{d} H_v^{ac} (\nabla_a v_c + \nabla_c v_a) \\ &= \sum_v H_{vd}^{d} H_v^{ac} (\rho\, g_{ac} + \sum_u \alpha_u H_{uac}) \\ &= \rho \sum_{u,v} (H_{vd}^{d})^2 + \sum \alpha_u H_{uac} H_{vd}^{d} H_v^{ac} . \end{aligned} \tag{4.7}$$

LEMMA. $W = \sum_{u,v} (H_{ud}^{d} H_{vac} H_u^{ac} - H_{vd}^{d} H_{uac} H_u^{ac})\, C_v$.

It can be easily veriefied that W is also independent of the choice of mutually orthogonal unit normal vectors of M^n, and so defines a vector field along M^n. The vector field W vanishes identically if the submanifold is totally umbilical.

PROOF. $W = \sum_{u,v} (nh_u' h_v' g_{ac} h_u' g^{ac} - nh_v' h_u' g_{ac} h_u' g^{ac})\, C_v = 0$

Using the vector field W defined in the Lemma,

$$\tilde{g}(W, \tilde{X}) = \sum_{u,v} \alpha_v (H_{ud}^{d} H_{vac} H_u^{ac} - H_{vd}^{d} H_{uac} H_u^{ac}), \tag{4.8}$$

moreover,

$$n\tilde{g}(H, \tilde{X})(\sum_u H_{uac} H_u^{ac}) = \sum_{u,v} \alpha_v H_{vd}^{d} H_{uac} H_u^{ac}. \tag{4.9}$$

Adding (4. 8) and (4. 9), we have

$$\tilde{g}(W, \tilde{X}) + n\tilde{g}(H, \tilde{X})(\sum_u H_{uac} H_u^{ac}) = \sum_{u,v} \alpha_v H_{ud}^{d} H_{vac} H_u^{ac}. \tag{4.10}$$

Thus we get

$$\begin{aligned} \operatorname{div} Y = {} & n\tilde{g}(H, \tilde{X})(\sum_u H_{uac} H_u^{ac}) + \rho \sum_v (H_{vd}^{d})^2 \\ & + \tilde{g}(W, \tilde{X}) + \sum_v H_{vd}^{d} (\nabla^a H_{vba}) v^b \\ & + \sum_v (\nabla_a H_{vd}^{d}) H_{vb}^{a} v^b. \end{aligned}$$

Since M^n is compact, orientable we have

$$\begin{aligned} -\int_{M^n} \rho \sum_v (H_{vd}^{d})^2 dM = \int_{M^n} \Big\{ & n\tilde{g}(H, \tilde{X})(\sum_u H_{uac} H_u^{ac}) \\ & + \tilde{g}(W, \tilde{X}) + \sum_v H_{vd}^{d} (\nabla^a H_{vba}) v^b \\ & + \sum_v (\nabla_a H_{vd}^{d}) H_{vb}^{a} v^b \Big\} dM. \end{aligned} \tag{4.11}$$

§ 5. Compact submanifolds with certain Properties

In this section we assume that M^n is a compact, orientable submanifold of $\tilde{M}^m$ in which there exists an infinitesimal conformal transformation $\tilde{X}$ of $\tilde{M}^m$ and that M^n satisfies the following condition:

1) The tangent space at each point of M^n is invariant under the curvature transformation of $\tilde{M}^m$.

2) The mean curvature vector of M^n in $\tilde{M}^m$ is parallel with respect to the connection of the normal bundle and is non-vanishing at almost everywhere.

Then the condition 1), together with (2. 6), implies that

$$\nabla_c H_{vba} - \nabla_b H_{vca} = \sum_u (L_{vuc} H_{uba} - L_{vvb} H_{uca}). \tag{5.1}$$

Furthermore, from condition 2), Proposition 3. 2. and (5. 1) it follows that

$$\nabla_b H_{vc}^{b} = \sum_u L_{vub} H_{uc}^{b}. \tag{5.2}$$

Using (5. 1) and 5. 2),

$$\sum_{v} H_{vd}^{d}(\nabla^{a} H_{vba})\, v^{b} + \sum_{v} (\nabla_{a} H_{vd}^{d})\, H_{vb}^{a} v^{b}$$
$$= \sum_{u,v} H_{vd}^{d} L_{vuc} H_{ub}^{c} v^{b} + \sum_{u,v} H_{ud}^{d} L_{vua} H_{vb}^{a} v^{b}$$
$$= 0 .$$

Thus we get

$$-\sum_{v} (H_{vd}^{d})^{2} \int_{M^{n}} \rho\, dM = \int_{M^{n}} \{ n\tilde{g}(H, \tilde{X}) (\sum_{u} H_{uac} H_{u}{}^{ac}) + \tilde{g}(W, \tilde{X}) \}\, dM \tag{5.3}$$

because of (4. 11) and Proposition 3. 3. Substituting (4. 5) into (5. 3), we have

$$\int_{M^{n}} \Big[n\tilde{g}(H, \tilde{X}) \Big\{ \sum_{v} H_{vab} H_{v}{}^{ab} - \frac{1}{n} \sum_{v} (H_{va}^{a})^{2} \Big\} + \tilde{g}(W, \tilde{Y}) \Big]\, dM = 0 . \tag{5.4}$$

Thus, if the vector field W is on the same side as H with respect to the normal part of $\tilde{X}$ in the normal bundle, and $\tilde{g}(H, \tilde{X})$ has fixed sign on M^{n}, then tne integrand of (5. 4) has a definite sign. In this case we have

$$\sum_{v} \Big(H_{vab} H_{v}{}^{ab} - \frac{1}{n} (H_{va}^{a})^{2} \Big) = 0 .$$

Consequently we have, from Proposition 3. 1,

THEOREM 5. 1. *Let M^{n} be a compact, orientable submanifold of $\tilde{M}^{m}$ whose tangent space at each point is invariant under the curvature transformation of $\tilde{M}^{m}$. Suppose that $\tilde{M}^{m}$ admits an infinitesimal conformal transformation $\tilde{X}$ and the mean curvature vector field of M^{n} in $\tilde{M}^{m}$ is parallel with respect to the connection of the normal bundle and $\tilde{g}(H, \tilde{X})$ is nonvanishing at almost everywhere on M^{n}. If, with respect to the normal part of $\tilde{X}$, the vector field W defined by Lemma is on the same side as the mean curvature vector field in the normal bundle and $\tilde{g}(H, \tilde{X})$ has fixed sign, then M^{n} is a totally umbilical submanifold of $\tilde{M}^{m}$.*

Theorem 5. 1 is due to M. Okumura ([7] p. 464) for the case where $m = n + 2$.

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