On valuations of polynomial rings of many variables

Part Three

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W is a given discrete valuation of a ring $K[x, y]$, where x and y are algebraically independent over a field K and W induces a valuation V_{00} in K. Then owing to Theorem 8. 2, a function V_{11} which is defined as follows for a polynomial $f(x, y)$ is a valuation of the ring $K[x, y]$, consequently it is also a valuation of its quotient field $K(x, y)$;

$$f(x, y) = \sum_{i,j} f_{ij} x^i y^j$$

$$V_{11} f(x, y) = \underset{i,j}{Min}\, [V_{00} f_{ij} + i\mu_1 + j\nu_1]$$

, *where* $f_{ij} \in K$ *and* $\mu_1 = Wx$ and $\nu_1 = Wy$.

We begin with V_{11} and repeating x-simply augmented valuations, we arrive at a valuation V_{s1} and next repeating y-simply augmented valuations, we arrive at a valuation V_{st} which is decided uniquely by W and is called the last simply augmented valuation of W. Next, furthermore repeating xy-doubly augmented valuations, we arrive at W at last. The method was explained in Part One of this paper.

$$V_{11} \overset{xs}{<} V_{21} \overset{xs}{<} \cdots \overset{xs}{<} V_{s1} \overset{ys}{<} V_{s2} \overset{ys}{<} \cdots \overset{ys}{<} V_{st} \overset{xy}{<} \cdots \overset{xy}{<} W.$$

V_{pq} is an arbitrary valuation in the series. In this paper, I want to study about these valuations, namely about value groups Γ_{pq} of V_{pq} in $K[x, y]$, structures of the residue-class ring Δ_{pq} of V_{pq} in $K[x, y]$, factorizations of polynomials $f(x, y)$ into equivalence-irreducible factors in V_{pq} and about structures of the key polynomials which produce these valuations.

In this paper, I will often quote the Theorems, Corollaries and Definitions which are printed as References at the end of this paper, with the following notations;

Theorem 16. 8. or § 18. These denote Theorem 16. 8, or § 18. in "On valuations of polynomial rings of many variables" by Hiroshi Inoue.

M. Theorem 12. 1. This denotes Theorem 12. 1. in "Construction for absolute values in polynomial rings" by Saunders MacLane.

§ 28. Conditions given in this paper.

In the beginning, we give the two following conditions to W in this paper.

CONDITION 28. 1. *F_{11} is the residue-class field of V_{00} in K and Δ_W is the residue-class ring of W in $K[x, y]$. Every residue-class in Δ_W that is algebraic over F_{11} is separable over F_{11}.*

Next, V_{00} is a so-called P-adic valuation, so there is such a prime number P that $V_{00}\ P>0$.

CONDITION 28. 2. *Γ_{00} is the value group of V_{00} in K and Γ_W is the value group of W in $K[x, y]$. $[\Gamma_W: \Gamma_{00}]$ is not a multiple of P.*

I shall state the case when we do not give these two conditions to W, in § 42.

Let W induce in $K[x]$ such a series of x-augmented inductive valuations that

$$V_{s0}=[V_{00},\ V_{10}x=\mu_1,\ V_{20}\phi_2=\mu_2, \cdots, V_{s0}\phi_s=\mu_s]$$

, where $\phi_i=\phi_i(x)$ is the x-key polynomial which produces an x-augmented valuation V_{i0} of $V_{i-1,0}$ in $K[x]$ and we write it simply as follows;

$$V_{i0}=[V_{i-1,0},\ V_{i0}\phi_i=\mu_i] \qquad \text{for } i=1, 2, \cdots, s.$$

And W induces in $K[y]$ such a series of y-augmented inductive valuations that

$$V_{0t}=[V_{00},\ V_{01}y=\nu_1,\ V_{02}\zeta_2=\nu_2, \cdots, V_{0t}\zeta_t=\nu_t]$$

, where $\zeta_j=\zeta_j(y)$ is the y-key polynomial which produces a y-augmented valuation V_{0j} of $V_{0,j-1}$ in $K[y]$.

$$V_{0j}=[V_{0,j-1},\ V_{0j}\zeta_j=\nu_j] \qquad \text{for } j=1, 2, \cdots, t.$$

§ 29. Residue-class ring of V_{11}

Many matters which hold in the case of V_{11} also hold in the same way in the case of V_{pq}, in spite of the fact that their calculations become complex.

V_{11} induces a valuation V_{01} in $K[y]$, namely for a polynomial $f(y)=\sum_j f_j y^j$

$$V_{01}f(y)=\operatorname*{Min}_j\,[V_{00}f_j+j\nu_1]$$

, where $f_j \in K$ and $\nu_1=Wy=V_{11}y$.

σ_1 is the smallest natural number that $\sigma_1\nu_1=V_{01}(y^{\sigma_1})$ belongs to Γ_{00} and d_1 is such a number in K that $V_{01}(d_1y^{\sigma_1})=0$. Then, by M. Theorem 10. 2, Δ_{01}, the residue-class ring of V_{01} in $K[y]$ is isomorphic to a ring $F_{11}[Y_{01}]$, where $Y_{01}=H_{01}(d_1y^{\sigma_1})$ is transcendental over F_{11} and H_{01} is the natural homomorphism from $K[y]^+$ to Δ_{01} and $K[y]^+$ is a set of all such polynomials $f(y)$ in $K[y]$ that $V_{01}f(y)\geqq 0$.

Γ_{11}, the value group of V_{11} in $K[x, y]$, consists of such real numbers of the form $e+m\mu_1+n\nu_1$, where $e\in\Gamma_{00}$ and m and n are integers.

Let τ_1 be the smallest natural number that $\tau_1\mu_1=V_{11}(x^{\tau_1})$ belongs to Γ_{01}, the value group of V_{01} in $K[y]$, then in K there exist such a number a_1 and such an integer λ_1 that $V_{11}(a_1x^{\tau_1}y^{\lambda_1})=0$, where $0\leqq\lambda_1\leqq\sigma_1-1$. Let be $X_{11}=H_{11}(a_1x^{\tau_1}y^{\lambda_1})$, where H_{11} is the natural homomorphism from $K[x, y]^+$ to Δ_{11}.

Here we have the two following cases;

(1) $\lambda_1=0$,

(2) $\lambda_1>0$.

In the case (1) when $\lambda_1=0$, $V_{11}(a_1x^{\tau_1})=0$ and $X_{11}=H_{11}(ax^{\tau_1})$ and Δ_{11} is very simple.

If $V_{11}(gx^{\alpha}y^{\beta})=0$, where $g\in K$, then α is a multiple $\alpha_1\tau_1$ of τ_1, so

$$\begin{aligned} 0 &= V_{11}(gx^{\alpha_1\tau_1}y^{\beta}) \\ &= V_{11}(a_1x^{\tau_1})^{\alpha_1}+V_{11}(ga_1^{-\alpha_1}y^{\beta}) \\ &= V\ \ (ga_1^{-\alpha_1}y^{\beta}). \end{aligned}$$

Therefore, β is a multiple $\beta_1\sigma_1$ of σ_1.

$$0=V_{11}(ga_1^{-\alpha_1}y^{\beta_1\sigma_1})=V_{11}(d_1y^{\sigma_1})^{\beta_1}+V_{11}(ga_1^{-\alpha_1}d_1^{-\beta_1})$$

$$0=V_{11}(ga_1^{-\alpha_1}d^{-\beta_1})$$

$$\therefore\ H_{11}(ga_1^{-\alpha_1}d_1^{-\beta_1})=\bar{k}\in F_{11} \qquad \because\ ga_1^{-\alpha_1}d_1^{-\beta_1}\in K.$$

Thus, when $V_{11}(gx^{\alpha_1\tau_1}y^{\beta_1\sigma_1})=0$,

$$H_{11}(gx^{\alpha_1\tau_1}y^{\beta_1\sigma_1})=\bar{k}X_{11}^{\alpha_1}Y_{11}^{\beta_1}, \quad \text{where} \quad Y_{11}=Y_{01}.$$

So, when $V_{11}f(x, y)=V_{11}(\sum_{i,j}f_{ij}x^iy^j)=0$ and $V_{11}(f_{ij}x^iy^j)=0$ for every term of $f(x, y)$,

$$H_{11}f(x, y)=\sum_{i_1,j_1}\bar{f}_{i_1j_1}X_{11}^{i_1}Y_{11}^{j_1}$$

, where $i=i_1\tau_1$ and $j=j_1\sigma_1$ and $\bar{f}_{i_1j_1}\in F_{11}$.

Therefore, in the case when $\lambda_1=0$,

$$\Delta_{11}\cong F_{11}[X_{11}, Y_{11}].$$

By M. Theorem 10. 2, $Y_{11}=Y_{01}=H_{01}(d_1y^{\sigma_1})$ is transcendental over F_{11}. And again by M. Theorem 10. 2, $X_{11}=H_{11}(a_1x^{\tau_1})$ is transcendental over the field $F_{11}(Y_{01})$. Consequently X_{11} and Y_{11} are algebraically independent over F_{11}.

Generally when $\lambda_1 \geqq 0$, $V_{11}(a_1x^{\tau_1}y^{\lambda_1})=0$ and $H_{11}(a_1x^{\tau_1}y^{\lambda_1})=X_{11}$.
If $V_{11}(gx^{\alpha}y^{\beta})=0$, then α is a multiple $\alpha_1\tau_1$ of τ_1,

$$0 = V_{11}(gx^{\alpha_1\tau_1}y^{\beta}) = V_{11}(a_1x^{\tau_1}y^{\lambda_1})^{\alpha_1} + V_{11}(ga_1^{-\alpha_1}y^{\beta-\lambda_1\alpha_1})$$
$$0 = V_{11}(ga_1^{-\alpha_1}y^{\beta-\lambda_1\alpha_1}).$$

According to the definition of σ_1, $\beta-\lambda_1\alpha_1$ is a multiple $\beta_1\sigma_1$ of σ_1, but in this case β_1 is only an integer, namely β_1 may be positive, or negative, or zero.

$$0 = V_{11}(ga_1^{-\alpha_1}y^{\beta_1\sigma_1}) = V_{11}(d_1y^{\sigma_1})^{\beta_1} + V_{11}(ga_1^{-\alpha_1}d_1^{-\beta_1})$$
$$0 = V_{11}(ga_1^{-\alpha_1}d_1^{-\beta_1}).$$

Thus, when $V_{11}(gx^{\alpha}y^{\beta})=0$,

$$H_{11}(gx^{\alpha}y^{\beta}) = \bar{k}X_{11}^{\alpha_1}Y_{11}^{\beta_1}, \quad \text{where } \bar{k} \in F_{11}$$

and α_1 is a non-negative integer and β_1 is an integer.

However, here $\beta-\alpha_1\lambda_1=\beta_1\sigma_1$

$$\frac{\lambda_1}{\sigma_1}\alpha_1+\beta_1 = \frac{\beta}{\sigma_1} \geqq 0.$$

$\dfrac{\lambda_1}{\sigma_1} = R_1$ is a fixed non-negative number which is independent of both α and β.

Therefore, when $H_{11}(gx^{\alpha}y^{\beta})=\bar{k}X_{11}^{\alpha_1}Y_{11}^{\beta_1}$, $\alpha_1 \geqq 0$ and $R_1\alpha_1+\beta_1 \geqq 0$.

So, if $V_{11}f(x, y) = V_{11}(\sum_{i,j} f_{ij}x^iy^j)=0$ and $V_{11}(f_{ij}x^iy^j)=0$ for every term of $f(x, y)$, $H_{11}f(x, y)= \sum_{i_1,j_1} \bar{f}_{i_1j_1}X_{11}^{i_1}Y_{11}^{j_1}$, where $\bar{f}_{i_1j_1} \in F_{11}$ and $R_1i_1+j_1 \geqq 0$ and i_1 are non-negative integers and R_1 is a fixed non-negative number for every term of $H_{11}f(x, y)$.

Therefore, here we give a definition to such polynomials.

DEFINITION 29.1. *Such a polynomial $f(X, Y)=\sum_{i,j} f_{ij}X^iY^j$ is called a R Y-quotient polynomial of X, where i are non-negative integers and j are integers and for every term of $f(X, Y)$ there exists such a fixed non-negative number R that $Ri+j \geqq 0$.*

THEOREM 29.2. *A set S_R of all R Y-quotient polynomials of X is a subring of the ring $K_Y[X]$, where $K_Y=K(Y)$ is the coefficient field of the ring $K_Y[X]$.*

PROOF. This proof can be done very easily by the definition of R Y-quotient polynomials of X.

Therefore, Δ_{11} is isomorphic to a ring S_{R_1} of all R_1 Y_{11}-quotient polynomials of X_{11}.

THEOREM 29. 3. *Λ_{11}, the residue-class field of V_{11} in the field $K(x, y)$ is isomorphic to $F_{11}(X, Y)$, where F_{11} is the residue-class field of V_{00} in K and X and Y are algebraically independent over F_{11}.*

PROOF. Λ_{11} is a quotient field of the ring Δ_{11}, so this theorem holds, both in the case when $\lambda_1=0$ and in the case when $\lambda_1>0$.

§ 30. Factorizations in $F[X, Y]$

Now we must consider factorizations of polynomials of two variables X and Y whose coefficients are in a field F.

We define that a term $X^{m_1}Y^{n_1}$ is the term of higher order than another term $X^{m_2}Y^{n_2}$ when $m_1>m_2$ or $m_1=m_2$ and $n_1>n_2$. In this paper we prescribe that the leading coefficient, namely the coefficient of the term of the highest order, of every irreducible polynomial $P(X, Y)$ is always 1. Then there is a famous theorem which is called as a theorem of uniqueness of factorization.

Every polynomial $f(X, Y)$ can be resolved uniquely into irreducible factors in $F[X, Y]$;

$$f(X, Y)=k\prod_i p_i(X, Y)$$

, where $k\in F$.

THEOREM 30. 1. *When a polynomial $f(X)$ is irreducible in a ring $F[X]$, $f(X)$ is also irreducible in a ring $F_Y[X]$ whose coefficient field F_Y is $F(Y)$ and X and Y are algebraically independent over F.*

PROOF. Assumed that $f(X)$ is reducible in $F_Y[X]$,

$$f(X)=\prod_b b_Y(X)$$

, then multiply both sides by the least common multiple $g(Y)$ of denominators of all irreducible factors $b_Y(X)$, then

$$f(X)g(Y)=\prod_b b(X, Y).$$

All polynomials $b(Y, X)$ include X. It contradicts the theorem of uniqueness of factorization.

COROLLARY 30. 2. *In the ring S_R of all R Y-quotient polynomials of X with coefficients in a field F, every polynomial is uniquely resolved into irreducible R Y-quotient polynomials of X in S_R.*

PROOF. Assumed that a R Y-quotient polynomial $f(Y, X)$ of X has two factorizations f_1 and f_2 into irreducible R Y-quotient polynomials of X in S_R, multiply f_1 and f_2 by the least common multiple Y^n of denominators of f_1 and f_2. Then, by the theorem of uniqueness of factorization, we find that these two factorizations coincide with each other.

COROLLARY 30. 3. *If a polynomial $f(X)$ is irreducible in $F[X]$ and another polynomial $g(Y)$ is irreducible in $F[Y]$, then both $f(X)$ and $g(Y)$ are irreducible in the ring S_R of all R Y-quotient polynomials of X with coefficients in F.*

PROOF. This is self-evident by Theorem 30. 1.

§ 31. Factorizations in V_{11}

THEOREM 31. 1. *Every residue-class in Δ_{11} is uniquely resolved into irreducible factors in Δ_{11}.*

PROOF. In the case when $\lambda_1=0$, $\Delta_{11}\cong F_{11}[X_{11}, Y_{11}]$. So, in this case this theorem is evident by the theorem of uniqueness of factorization. Generally when $\lambda_1\geqq 0$, Δ_{11} is isomorphic to the ring S_{R_1} of all R_1 Y_{11}-quotient polynomials of X_{11}, so in this case, this theorem is evident by Corollary 30. 2.

Now we select only one polynomial respectively as a representative of all such polynomials that are equivalent to each other in V_{11} and we define a canon class in Δ_{11}, a canon polynomial in V_{11} and its key part in the same way as in § 16.

DEFINITION 31. 2.

If $$\bar{f}(X_{11}, Y_{11})=X_{11}^m Y_{11}^n+\cdots$$

is a polynomial in Δ_{11} and $m^2+n^2\geqq 1$ and the leading coefficient is 1, *then $\bar{f}(X_{11}, Y_{11})$ is called a canon class in Δ_{11}. And when $H_{11}(f(x, y))$ is a canon class in Δ_{11}, $f(x, y)$ is called a canon polynomial in V_{11}. And let k be the leading coefficient of $f(x, y)$, then $\frac{1}{k}f(x, y)$ is called a key part of $f(x, y)$.*

H_{11} is the natural homomorphism from $K[x, y]^+$ to Δ_{11}, therefore we have the following theorem.

THEOREM 31. 3. *A necessary and sufficient condition that a canon class $H_{11}f(x, y)$ is irreducible in Δ_{11} is that $f_{11}(x, y)$ is equivalence-irreducible in V_{11} in $K[x, y]$.*

COROLLARY 31. 4. *Every polynomial $f(x, y)$, for which $V_{11}f(x, y)=0$, is uniquely resolved into equivalence-irreducible factors in V_{11} in $K[x, y]$.*

This corollary is self-evident by Theorem 31. 3.

I explained about Δ_{11} pretty minutely, because a residue-class ring of a valuation in a ring is very important in theory of valuation. But, after we prove Corollary 31. 4, we have a more simple method to carry it out. The method like this will often be used below in this paper. Now I will explain the method.

Let $\overline{\Gamma}_{11}$ be an intersection of Γ_{01} and Γ_{10}, where Γ_{10} is the value group of V_{10} in $K[x]$ and Γ_{01} is the value group of V_{01} in $K[y]$. a_1 is the smallest natural number that $a_1\mu_1=V_{11}(x^{a_1})$ belongs to $\overline{\Gamma}_{11}$ and b_1 is the smallest natural number that $b_1\nu_1=V_{11}(y^{b_1})$ belongs to $\overline{\Gamma}_{11}$.

If $V_{11}(gx^iy^j)=0$, then $V_{00}g+i\mu_1=-j\nu_1$, both $i\mu_1$ and $j\nu_1$ must belong to $\overline{\Gamma}_{11}$ and i and j are respectively multiples i_1a_1 and j_1b_1 of a_1 and b_1.

$$0=V_{11}(gx^iy^j)=V_{11}(gx^{i_1a_1}y^{j_1b_1}).$$

Therefore, every equivalence-irreducible factor in Corollary 31. 4. is a polynomial of a ring $K[x^{a_1}, y^{b_1}]$.

§ 32. The valuation V_{s1} in $K[x, y]$

$V_{i0}=[V_{i-1,0}, V_{i0}\phi_i=\mu_i]$ is a valuation in $K[x]$. The field $K_x=K(x)$ has valuations V_{i0} $(i=1, 2, \cdots, s)$, so, by M. Theorem 3. 1, the function V_{i1} which are defined as follows for a polynomial $f(x, y)=\sum_j f_j(x)y^j$ are valuations of the ring $K_x[y]$ for $i=1, 2, \cdots, s$;

$$V_{i1}f(x, y)=\operatorname*{Min}_j\left[V_{i0}f_j(x)+j\nu_1\right]$$

, where $V_{i1}\phi_i(x)=V_{i1}\phi_i>V_{i-1,0}\phi_i=V_{i-1,1}\phi_i$.

Therefore V_{i1} is an x-simply augmented valuation of $V_{i-1,1}$,

$$V_{11}\overset{xs}{<}V_{21}\overset{xs}{<}\cdots\overset{xs}{<}V_{i-1,1}\overset{xs}{<}V_{i1}\overset{xs}{<}\cdots\overset{xs}{<}V_{s1}.$$

All that I state hence in § 32 hold for every valuation V_{i1} between V_{11} and V_{s1}, however here I will state only V_{s1} that is the most important, because I fear that this paper becomes too long.

Let be $f_j(x)=\sum_j f_{ji}(x)\phi_s^i$ an expansion of $f_j(x)$ by $\phi_s(x)$, namely $0\leqq \deg_x f_{ji}(x)<\deg_x\phi_s$ for each term,

then
$$V_{s0}=[V_{s-1,0}, V_{s0}\phi_s=\mu_s]$$
$$V_{s1}f(x, y)=V_{s1}\Big(\sum_{i,j}f_{ji}(x)\phi_s^iy^j\Big)$$
$$=\operatorname*{Min}_{j,i}\left[V_{s-1,0}f_{ji}(x)+i\mu_s+j\nu_1\right].$$

We want to study about Γ_{s1}, the value group of V_{s1} in $K[x, y]$ and about Δ_{s1}, the residue-class ring of V_{s1} in $K[x, y]$. We can do it in the same way when we did in V_{11}, after we define units in V_{s1} in $K[x, y]$.

DEFINITION 32. 1. *V_Q is a valuation of $K[x, y]$ and if $a(x, y)b(x, y) \sim 1$ in V_Q, then the polynomials $a(x, y)$ and $b(x, y)$ are called equivalent-units in V_Q, or shortly units in V_Q.*

This definition is an extension of the definition of a unit in V_s in $K[x]$ which I gave in § 15. A set U_Q of all units in V_Q in $K[x, y]$ is a group with respect to multiplication.

THEOREM 32. 2. *Every unit in V in $K[x, y]$ is also a unit in an augmented valuation D of V in $K[x, y]$.*

PROOF. $a(x, y)b(x, y) \sim 1$ in V, then

$$D(a(x, y)b(x, y)-1) \geqq V(a(x, y)b(x, y)-1) > V(1) = D(1) = 0.$$

This inverse statement is not true generally. In § 15, I used a notion "effective degree" and in the same way as in § 15, we can easily prove that every unit in V_{s1} in $K[x, y]$ is equivalent to a polynomial which does not include y and whose degree with respect to x is less than that of $\phi_s(x)$. Namely every unit in V_{s1} in $K[x, y]$ is equivalent to a unit in V_{s0} in $K[x]$. Therefore we can obtain the following Corollary 32. 4. immediately after we have Theorem 32. 3.

THEOREM 32. 3. *In Δ_{s0}, the residue-class ring of V_{s0} in $K[x]$, a set N of all such classes that include units in V_{s0} in $K[x]$ is a field F_{s1} in Δ_{s0} and $\Delta_{s0} \cong F_{s1}[X]$, where F_{s1} is a finite-dimensional extension of F_{11} and X is transcendental over F_{s1}.*

PROOF. This theorem is a part of M. Theorem 12. 1.

COROLLARY 32. 4. *In Δ_{s1}, the residue-class ring of V_{s1} in $K[x, y]$, a set of all such classes that include units in V_{s1} in $K[x, y]$ is isomorphic to N in Theorem 32. 3.*

After we obtain these theorems, we can establish Δ_{s1} in the same way as we establish Δ_{11}, using U_{s1}, a set of all units in V_{s1} in $K[x, y]$ which is correspondent to K in the case of V_{11} in $K[x, y]$.

Every unit in V_{s0} in $K[x]$ is equivalent to a polynomial whose degree with respect to x is less than that of $\phi_s(x)$. Therefore the value group of V_{s1} in U_{s1} is equal to the value group $\Gamma_{s-1,0}$ of $V_{s-1,0}$ in $K[x]$.

Γ_{s1}, the value group of V_{s1} in $K[x, y]$ consists of such real numbers of the form $k + m\mu_s + n\nu_1$, where $k \in \Gamma_{s-1,0}$ and m and n are integers. σ_s is the smallest natural number that $\sigma_s\nu_1 = V_{s1}(y^{\sigma_s})$ belongs to $\Gamma_{s-1,0}$ which is equal to the value group of V_{s1} in U_{s1}. Then there exists such a unit d_s in V_{s1} that $V_{s1}(d_s y^{\sigma_s}) = 0$. Let be $H_{s1}(d_s y^{\sigma_s}) = Y_{s1}$, where H_{s1} is the natural homomorphism of $K[x, y]^+$ to Δ_{s1}.

A set L_{s1} of all such real numbers of the form $k+n\nu_1$, where $k\in\Gamma_{s-1,0}$ and n are integers, is a subgroup of a cyclic group Γ_{s1}. τ_s is the smallest natural number that $\tau_s\mu_s=V_{s1}(\phi_s^{\tau_s})$ belongs to L_{s1} and there exist such a unit a_s in V_{s1} and such an integer λ_s that $V_{s1}(a_s\phi_s^{\tau_s}y^{\lambda_s})=0$, where $0\leqq\lambda_s\leqq\sigma_s-1$. Here we have the two following cases;

(1) $\lambda_s=0$ and $V_{s1}(a_s\phi_s^{\tau_s})=0$

(2) $\lambda_s>0$ and $V_{s1}(a_s\phi_s^{\tau_s}y^{\lambda_s})=0$.

In the case (1) when $\lambda_s=0$, $H_{s1}(a_s\phi_s^{\tau_s})=X_{s1}$ and Δ_{s1} is isomorphic to $F_{s1}[Y_{s1}, X_{s1}]$, where X_{s1} and Y_{s1} are algebraically independent over F_{s1} which is a coefficient field of the residue-class ring of V_{s0} in $K[x]$ and is a finite-dimensional extension of F_{11}, owing to M. Theorem 12. 1.

Because, if $\qquad V_{s1}(g\phi_s^{\alpha}y^{\beta})=0, \qquad$ where $g\in U_{s1}$,

α is a multiple $\alpha_s\tau_s$ of τ_s and

$$0=V_{s1}(g\phi_s^{\alpha_s\tau_s}y^{\beta})=V_{s1}(a_s\phi_s^{\tau_s})^{\alpha_s}+V_{s1}(ga_s^{-\alpha_s}y^{\beta})$$
$$0=V_{s1}(ga_s^{-\alpha_s}y^{\beta}).$$

So, β is a multiple $\beta_s\sigma_s$ of σ_s and

$$0=V_{s1}(ga_s^{-\alpha_s}y^{\beta_s\sigma_s})=V_{s1}(d_sy^{\sigma_s})^{\beta_s}+V_{s1}(ga_s^{-\alpha_s}d_s^{-\beta_s})$$
$$0=V_{s1}(ga_s^{-\alpha_s}d_s^{-\beta_s}).$$

$ga_s^{-\alpha_s}d_s^{-\beta_s}$ is a unit in V_{s1}, so $H_{s1}(ga_s^{-\alpha_s}d_s^{-\beta_s})=\bar{k}_s$ belongs to F_{s1}. Namely, when $V_{s1}(g\phi_s^{\alpha}y^{\beta})=0$ and $g\in U_{s1}$, $H_{s1}(g\phi_s^{\alpha}y^{\beta})=\bar{k}_sX_{s1}^{\alpha_s}Y_{s1}^{\beta_s}$ and $\bar{k}_s\in F_{s1}$.

So, by Theorem 32. 4. in this case $\Delta_{s1}\cong F_{s1}[Y_{s1}, X_{s1}]$. Generally when $\lambda_s\geqq0$, $V_{s1}(a_s\phi_s^{\tau_s}y^{\lambda_s})=0$ and Δ_{s1} is isomorphic to a ring S_{R_s} of R_s Y_{s1}-quotient polynomials of X_{s1} with coefficients in F_{s1}, where R_s is a fixed non-negative number and $X_{s1}=H_{s1}(a_s\phi_s^{\tau_s}y^{\lambda_s})$ and $Y_{s1}=H_{s1}(d_sy^{\sigma_s})$.

Because, if $V_{s1}(g\phi_s^{\alpha}y^{\beta})=0$ and $g\in U_{s1}$, then α is a multiple $\alpha_s\tau_s$ of τ_s and

$$0=V_{s1}(g\phi_s^{\alpha_s\tau_s}y^{\beta})=V_{s1}(a_s\phi_s^{\tau_s}y^{\lambda_s})^{\alpha_s}+V_{s1}(ga_s^{-\alpha_s}y^{\beta-\lambda_s\alpha_s})$$
$$0=V_{s1}(ga_s^{-\alpha_s}y^{\beta-\lambda_s\alpha_s}).$$

So, $\beta-\lambda_s\alpha_s$ must be a multiple $\beta_s\sigma_s$ of σ_s and

$$0=V_{s1}(ga_s^{-\alpha_s}y^{\beta_s\sigma_s})=V_{s1}(d_sy^{\sigma_s})^{\beta_s}+V_{s1}(ga_s^{-\alpha_s}d_s^{-\beta_s})$$
$$0=V_{s1}(ga_s^{-\alpha_s}d_s^{-\beta_s}). \qquad (32.\ 1)$$

And $ga_s^{-\alpha_s}d_s^{-\beta_s}\in U_{s1}$.

However, here $\beta-\lambda_s\alpha_s=\beta_s\sigma_s$

$$\frac{\lambda_s}{\sigma_s}\alpha_s+\beta_s=\frac{\beta}{\sigma_s}\geqq 0$$

$\frac{\lambda_s}{\sigma_s}=R_s$ is a fixed non-negative number which is independent of α and β.

We can prove that X_{s1} and Y_{s1} are algebraically independent over F_{s1} in the same way as in M. Theorem 12. 1. Namely assumed that X_{s1} and Y_{s1} are algebraically dependent with respect to F_{s1}, then such an equation holds,

$$\sum_{i,j}\bar{k}_{ij}Y_{s1}^jX_{s1}^i=0\,,\qquad \text{where}\quad \bar{k}_{ij}\in F_{s1}\,.$$

There are such polynomials $k_{ij}(x, y)$ in $K[x, y]$ that $H_{s1}(k_{ij}(x, y))=\bar{k}_{ij}X_{s1}^iY_{s1}^j$ for every term of the above-mentioned equation. $V_{s1}(k_{ij}(x, y))$ are all zero, so according to the definition of V_{s1},

$$V_{s1}\Big(\sum_{i,j}k_{ij}(x, y)\Big)=\operatorname*{Min}_{i,j}\Big[V_{s1}(k_{ij}(x, y))\Big]=0\,.$$

While $\sum_{i,j}\bar{k}_{ij}X_{s1}^iY_{s1}^j=0$, so the residue-class $H_{s1}(\sum_{i,j}k_{ij}(x, y))=0$ in Δ_{s1}, namely $V_{s1}(\sum_{i,j}k_{ij}(x, y))$ must be positive. Thus a contradiction takes place, therefore X_{s1} and Y_{s1} are algebraically independent over F_{s1}.

THEOREM 32. 5. *The residue-class field Λ_{s1} of V_{s1} in the field $K(x, y)$ is isomorphic to $F_{s1}(X, Y)$, where F_{s1} is a finite-dimensional extension of F_{11} and* X and Y *are algebraically independent of F_{s1}.*

THEOREM 32. 6. *Every polynomial $f(x, y)$ for which $V_{s1}\,f(x, y)=0$ can be uniquely resolved into equivalence-irreducible factors in V_{s1} in $K[x, y]$.*

Theorem 32. 5 and 32. 6 can be verified completely in the same way as we did in V_{11}

§ 33. Construction of V_{s2} in $K[x, y]$

After we find a series of x-augmented inductive valuations which W induces in $K[x]$, the method to make a series of x-simply augmented inductive valuations of $K[x, y]$ which begins with V_{11} and ends at V_{s1} is not so complex, as I made them in this paper already. But, after we find the following series of y-augmented inductive valuations which W induces in $K[y]$;

$$[V_{00},\ V_{01}y=\nu_1,\ V_{02}\zeta_2=\nu_2,\cdots,\ V_{0t}\zeta_t=\nu_t]$$

, it is pretty complex and troublesome to make a series of y-simply augmented inductive valuations in $K[x, y]$ that

$$V_{s1}\overset{ys}{<}V_{s2}\overset{ys}{<}\cdots\overset{ys}{<}V_{st}<\cdots$$

, where $V_{sj}\zeta_j = W\zeta_j = V_{0j}\zeta_j$ for $j=1, 2, \cdots t$. $\zeta_j = \zeta_j(y)$ is the y-key polynomial which produces the y-augmented valuation V_{0j} of $V_{0,j-1}$ in $K[y]$.

The reason is as follows,

$$V_{i0} = [V_{i-1,0}, \quad V_{i0}\phi_i = \mu_i]$$
$$V_{i1} = [V_{i-1,1}, \quad V_{i1}\phi_i = \mu_i].$$

Namely when we make an x-simply augmented valuation V_{i1} of $V_{i-1,1}$ in $K[x, y]$, we can adopt $\phi_i(x)$ itself as the x-key polynomial.

$$V_{02} = \left[V_{01}, \; V_{01}\zeta_2(y) = \nu_2\right].$$

But when we try to make such a y-simply augmented valuation V_{s2} of V_{s1} in $K[x, y]$ that $V_{s2}\zeta_2 = V_{02}\zeta_2$, it happens often that we can not adopt $\zeta_2(y)$ as the y-key polynomial which produces V_{s2}. For, $\zeta_2(y)$ is equivalence-irreducible in V_{01} in $K[y]$, but $\zeta_2(y)$ is not always equivalence-irreducible in V_{s1} in $K[x, y]$, but it happens often that ζ_2 is equivalence-reducible in V_{s1} in $K[x, y]$. Consequently in such a case, we must find a factorization of ζ_2 into equivalence-irreducible factors in V_{s1} in $K[x, y]$ and we must adopt one of the factors as the y-key polynomoal which produces a y-simply augmented valuation V_{s2} of V_{s1} in $K[x, y]$.

Let be $\zeta_2(y) = y^{n\sigma_1} + \cdots + b_i y^{i\sigma_1} + \cdots + b_0$, where $b_i \in K$ and $V_{01}(y^{n\sigma_1}) = V_{01}b_0$ and a value of every term of $\zeta_2(y)$ in V_{01} equals each other.

As I explained in § 29, Δ_{01}, the residue-class ring of V_{01} in $K[y]$ is isomorphic to $F_{11}[Y_{01}]$, where $H_{01}(d_1 y^{\sigma_1}) = Y_{01}$ and $V_{01}(d_1 y^{\sigma_1}) = 0$.

$\sum_i f_i(y)\zeta_2^i$ is an expansion of a polynomial $f(y)$ in $K[y]$ by $\zeta_2(y)$ and V_{02} is defined as follows;

$$V_{02}\big(f(y)\big) = \operatorname*{Min}_i \left[V_{01}f_i(y) + i\nu_2\right]$$

, where $\nu_2 = V_{02}\zeta_2 > V_{01}\zeta_2 = V_{s1}\zeta_2$.

So, Γ_{02}, the value group of V_{02} in $K[y]$, includes Γ_{01} and ν_2.

$$V_{01}(d_1^n\zeta_2) = \operatorname*{Min}_i \left[V_{01}(d_1 y^{\sigma_1})^n, \cdots\right] = 0$$

and $g(Y_{01}) = H_{01}(d_1^n\zeta_2) = Y_{01}^n + \cdots + \bar{b}_i Y_{01}^i + \cdots + \bar{b}_0$, where $\bar{b}_i \in F_{11}$.

$g(Y_{01})$ is irreducible in $\Delta_{01} \cong F_{11}[Y_{01}]$, because ζ_2 is equivalence-irreducible in V_{01} in $K[y]$, for ζ_2 is the key polynomial which produces a y-augmented valuation V_{02} of V_{01} in $K[y]$.

Let ρ_2 be the smallest natural number that $\rho_2\nu_2 = V_{02}(\zeta_2^{\rho_2})$ belongs to Γ_{01} and C_2 be such a unit in V_{02} in $K[y]$ that $V_{02}(c_2\zeta_2^{\rho_2}) = 0$. Then, by M. Theorem 12. 1, Δ_{02}, the residue-class ring of V_{02} in $K[y]$, is isomorphic to

$F_{02}[Y_{02}]$, where $F_{02}=F_{11}(\theta)$ and θ is a root of $g(Y_{01})=0$ and $Y_{02}=H_{02}(c_2\zeta_2^{\rho_2})$ is transcendental over F_{02} and H_{02} is the natural homomorphism from $K[y]^+$ to Δ_{02}.

σ_s is the smallest natural number that $\sigma_s\nu_1=V_{01}(y^{\sigma_s})=V_{s1}(y^{\sigma_s})$ belongs to $\Gamma_{s-1,0}$ which equals the value group of V_{s1} in U_{s1}.

$$\sigma_s\nu_1\in\Gamma_{s-1,0} \quad \text{and} \quad \sigma_1\nu_1\in\Gamma_{00}$$

and Γ_{00} is a subgroup of a cyclic group $\Gamma_{s-1,0}$. Therefore, σ_1 is a multiple $\delta_1\sigma_s$ of σ_s.

$[\Gamma_{01}:\Gamma_{00}]=\sigma_1$, because Γ_{01} is the value group of V_{01} in $K[y]$ and σ_1 is the smallest natural number that $\sigma_1\nu_1$ belongs to Γ_{00}. Γ_{01} is a subgroup of Γ_w, so $[\Gamma_w:\Gamma_{00}]=[\Gamma_w:\Gamma_{01}][\Gamma_{01}:\Gamma_{00}]$ and according to Condition 28. 2, $[\Gamma_w:\Gamma_{00}]$ is not a multiple of P, so σ_1 is not a multiple of P and δ_1 is also not a multiple of P.

$$\begin{aligned}
&0=V_{01}(d_1y^{\sigma_1})=V_{s1}(d_1y^{\sigma_1})=V_{s1}(d_1y^{\delta_1\sigma_s})\\
&0=V_{s1}(d_sy^{\sigma_s})^{\delta_1}+V_{s1}(d_1d_s^{-\delta_1})=V_{s1}(d_1d_s^{-\delta_1})\\
\because\quad &V_{s1}(d_sy^{\sigma_s})=0 \quad \text{and} \quad H_{s1}(d_sy^{\sigma_s})=Y_{s1}\\
&d_1\in K \quad \textit{and} \quad d_s\in U_{s1} \qquad \therefore\quad d_1d_s^{-\delta_1}\in U_{s1}
\end{aligned}$$

, *so* $H_{s1}(d_1d_s^{-\delta_1})=\bar{l}_s\in F_{s1}$.

$$\begin{aligned}
&Y_{01}=H_{01}(d_1y^{\sigma_1})=H_{s1}(d_1y^{\delta_1\sigma_s})=\left[H_{s1}(d_sy^{\sigma_s})\right]^{\delta_1}\cdot H_{s1}(d_1d_s^{-\delta_1})\\
&Y_{01}=\bar{l}_sY_{s1}^{\delta_1}\\
\therefore\quad &H_{s1}(d_1^n\zeta_2)=H_{01}(d_1^n\zeta_2)=g(Y_{01})\\
&g(Y_{01})=Y_{01}^n+\cdots+\bar{b}_iY_{01}^i+\cdots+\bar{b}_0\\
&\qquad=(\bar{l}_sY_{s1}^{\delta_1})^n+\cdots+\bar{b}_i(\bar{l}_sY_{s1}^{\delta_1})^i+\cdots+\bar{b}_0\\
&\qquad=\bar{l}_s^n[Y_{s1}^{n\delta_1}+\cdots+\bar{C}_iY_{s1}^{i\delta_1}+\cdots+\bar{C}_0]\\
&H_{s1}(d_1^n\zeta_2)=\bar{l}_s^n\cdot L(Y_{s1})\cdots \qquad (33)
\end{aligned}$$

, *where* $\bar{C}_i\in F_{s1}$ *and* $\bar{b}_i\in F_{11}$.

According to Condition 28. 1, F_w is separable over F_{11}, so F_{s1}, a subfield of F_w, is also separable over F_{11}. $g(Y_{01})$ is in $F_{11}[Y_{01}]$ and $g(Y_{01})=0$ is a separable equation over F_{11} and $L(Y_{s1})$ is in $F_{s1}[Y_{s1}]$ and $L(Y_{s1})=0$ is a separable equation over F_{s1}, for δ_1 is not a multiple of P.

We resolve $L(Y_{s1})$ uniquely into irreducible factors in Δ_{s1}.

$$L(Y_{s1})=G_1(Y_{s1})\cdots G_q(Y_{s1}) \qquad \text{in } \Delta_{s1}.$$

$L(Y_{s1})$ does not include X_{s1}, so these $G_i(Y_{s1})$ are polynomials of Y_{s1} with coefficients in F_{s1}. $L(Y_{s1})$ is a separable polynomial over F_{s1}, so $G_1(Y_{s1}), \cdots, G_q(Y_{s1})$ are prime each other. The leading coefficient of $L(Y_{s1})$ is 1, so these $G_i(Y_{s1})$ are all canon classes in Δ_{s1}.

$$Y_{s1} = H_{s1}(d_s y^{\sigma_s}), \qquad \text{where } d_s \in U_{s1}.$$

Let $g_i(x, y^{\sigma_s})$ be the canon polynomial of $G_i(Y_{s1})$ for $i=1, 2, \cdots, q$, namely $H_{s1}(g_i(x, y^{\sigma_s}))=G_i(Y_{s1})$. From (33) $d_1^n\zeta_2 \sim h_0 g_1(x, y^{\sigma_s})\cdots g_q(x, y^{\sigma_s})$ in V_{s1}, where $h_0 \in U_{s1}$.

Let be $g_i(x, y^{\sigma_s})=h_i g_i^*(x, y^{\sigma_s})$, where h_i is the leading coefficient of $g_i(x, y^{\sigma_s})$ and $g_i^*(x, y^{\sigma_s})$ is a key part of $g_i(x, y^{\sigma_s})$ for $i=1, 2, \cdots, q$.

All $G_i(Y_{s1})$ are irreducible in Δ_{s1}, so all $g_i(x, y^{\sigma_s})$ are equivalence-irreducible in V_{s1} in $K[x, y]$ and all $g_i^*(x, y^{\sigma_s})$ are also equivalence-irreducible in V_{s1} in $K[x, y]$. Let be $k=d_1^n(h_0 h_1 \cdots h_q)^{-1}$, then

$$k\zeta_2 \sim g_1^*(x, y^{\sigma_s})\cdots g_q^*(x, y^{\sigma_s}) \qquad \text{in } V_{s1}.$$

Therefore, in the same way as in Corollary 18. 2, we know that there exist such polynomials $l_1(x, y^{\sigma_s}), \cdots, l_q(x, y^{\sigma_s})$ that

$$k\zeta_2 \overset{\omega}{\approx} l_1(x, y^{\sigma_s})\cdots l_q(x, y^{\sigma_s}) \qquad \text{in } V_{s1}$$

, where

$$g_i^*(x, y^{\sigma_s}) \sim l_i(x, y^{\sigma_s}) \qquad \text{in } V_{s1}$$

and

$$deg_y g_i^*(x, y^{\sigma_s}) = deg_y l_i(x, y^{\sigma_s}) \qquad \text{for } i=1, 2, \cdots, q$$

and ω is an arbitrary given positive number.

$$V_{s1}\Big(k\zeta_2 - l_1(x, y^{\sigma_s})\cdots l_q(x, y^{\sigma_s})\Big) - V_{s1}(k\zeta_2) > \omega.$$

$g_i^*(x, y^{\sigma_s})$ is equivalence-irreducible in V_{s1} in $K[x, y]$, then $l_i(x, y^{\sigma_s})$ is also equivalence-irreducible in V_{s1} in $K[x, y]$ and the leading coefficient of $l_i(x, y^{\sigma_s})$ is 1. Therefore every $l_i(x, y^{\sigma_s})$ satisfies a sufficient condition that $l_i(x, y^{\sigma_s})$ becomes a y-key polynomial which produces a y-simply augmented valuation of V_{s1} in $K[x, y]$.

Here let be $$\omega = V_{02}\zeta_2 - V_{01}\zeta_2 > 0.$$

k is a unit in V_{s1} in $K[x, y]$, so $V_{s2}k = V_{s1}k$.

$$\therefore \quad V_{s2}(k\zeta_2) - V_{s1}(k\zeta_2) = V_{02}\zeta_2 - V_{01}\zeta_2 = \omega.$$

Therefore, one and only one out of $l_1(x, y^{\sigma_s}), \cdots, l_q(x, y^{\sigma_s})$ must increase its value when we make a y-simply augmented valuation V_{s2} of V_{s1} in $K[x, y]$. Let it be $l_1(x, y^{\sigma_s})=l(x, y^{\sigma_s})$.

Namely $$V_{s2}\left(l(x, y^{\sigma_s})\right) = V_{s1}\left(l(x_1 y^{\sigma_s})\right) + \omega = \eta_2 \cdots \qquad (33.2)$$

, then $$V_{s2}\left(l_i(x, y^{\sigma_s})\right) = V_{s1}\left(l_i(x, y^{\sigma_s})\right) \qquad \text{for } i=2,3,\cdots,q.$$

So, by M. Lemma 9.1,

$$\prod_{i=2}^{q} l_i(x, y^{\sigma_s}) = \varepsilon \text{ is a unit in } V_{s2} \text{ in } K[x, y].$$

$$k\zeta_2 \overset{\omega}{\approx} l(x, y^{\sigma_s})\varepsilon \qquad \text{in } V_{s1}$$

, namely $$V_{s1}\left(k\zeta_2 - l(x, y^{\sigma_s})\varepsilon\right) > \omega + V_{s1}\left(l(x, y^{\sigma_s})\varepsilon\right)$$

$$V_{s2}\left(k\zeta_2 - l(x, y^{\sigma_s})\varepsilon\right) \geqq V_{s1}\left(k\zeta_2 - l(x, y^{\sigma_s})\varepsilon\right).$$

And from (33.2)

$$V_{s2}\left(k\zeta_2 - l(x, y^{\sigma_s})\varepsilon\right) > V_{s2}\left(l(x, y^{\sigma_s})\right).$$

$$k\zeta_2 \sim l(x, y^{\sigma_s})\varepsilon \qquad in \ V_{s2},$$

$$\therefore \quad \zeta_2 \sim l(x, y^{\sigma_s})\varepsilon_2 \qquad in \ V_{s2} \qquad (33.3)$$

, where $\varepsilon_2 \doteqdot \varepsilon k^{-1}$ is a unit in V_{s2} in $K[x, y]$.

$$V_{s2} = \left[V_{s1},\ V_{s2}\, l(x, y^{\sigma_s}) = \eta_2\right].$$

V_{s2} is a y-simply augmented valuation of V_{s1} in the ring $K_x[y]$ whose coefficient field $K_x = K(x)$ has valuation V_{s0}.

§ 34. Units in V_{s2} in $K[x, y]$

Now we investigate the structure of $l(x, y^{\sigma_s}) = l$ and the structure of V_{s2}. By. M. Theorem 9.4.

$$l = y^{m\sigma_s} + \cdots + b_i(x, y)y^{i\sigma_s} + \cdots + b_0(x, y)$$

, where $b_i(x, y) \in U_{s1}$ and

$$V_{s1} l = V_{s1}(y^{m\sigma_s}) = V_{s1}\left(b_0(x, y)\right) = V_{s1}\left(b_i(x, y)y^{i\sigma_s}\right)$$

for every term of l.

Let be $\sum_j f_j(x, y)l^j$ an expansion of a polynomial $f(x, y)$ by l

, where $$deg_y f_j(x, y) < deg_y l = m\sigma_s.$$

$$f_j(x, y) = \sum_i f_{ji}(x) y^i = \sum_i \left(\sum_h f_{jih}(x)\phi_s^h\right) y^i.$$

, where $deg_x f_{jih}(x) < deg_x \phi_s$.

$$f(x, y) = \sum_{j,h} \left(\sum_i f_{jih}(x) y^i\right) \phi_s^h l^j$$

, where $i < deg_y l = m\sigma_s$.

So, consequently

$$V_{s2} f(x, y) = \operatorname*{Min}_{j,h} \left[V_{s1}(\sum_i f_{jih}(x) y^i) + h\mu_s + j\eta_2 \right]$$

, where $\mu_s = V_{s0}\phi_s = V_{s2}\phi_s$ and $\eta_2 = V_{s2} l$.

Now I will prove that such a polynomial $\sum_i f_i(x) y^i$

, where $\quad deg f_i(x) < deg \phi_s(x) \quad$ and $\quad deg_y\left(\sum_i f_i(x) y^i\right) < deg_y l$

, is a unit in V_{s2} in $K[x, y]$.

THEOREM 34. 1. *Such a polynomial $\sum_i f_i(x) y^i$, where $f_i(x) \in U_{s1}$ and $deg_y(\sum_i f_i(x) y^i) < deg_y l$, is a unit in V_{s2} in $K[x, y]$.*

PROOF. All $f_i(x)$ are units in V_{s1} in $K[x, y]$, so

, $\quad k\left(\sum_i f_i(x) y^i\right) \sim y^n \left(\sum_j f_j^*(x) y^{j\sigma_s}\right) \quad$ in V_{s1}, also in V_{s2}

, where $k \in U_{s1}$ and $0 \leqq n \leqq \sigma_s - 1$ and $f_j^*(x) \in U_{s1}$

$$V_{s1}\left(\sum_j f_j^*(x) y^{j\sigma_s}\right) = 0 .$$

This is evident from the definition of σ_s. Here we have the two following cases

(1) $n = 0$

(2) $n > 0$.

when $n = 0$, let be $Q(Y_{s1}) = H_{s1}(\sum_j f_j^*(x) y^{j\sigma_s})$

$$G_1(Y_{s1}) = H_{s1}\left(g_1(x, y^{\sigma_s})\right) = H_{s1}(h_1 l).$$

Both $Q(Y_{s1})$ and $G_1(Y_{s1})$ are polynomials of the ring $F_{s1}[Y_{s1}]$ and $G_1(Y_{s1})$ is irreducible in $F_{s1}[Y_{s1}]$ and

$$deg_Y Q(Y_{s1}) < deg_Y G_1(Y_{s1}).$$

Then there exist such two polynomials $A(Y_{s1})$ and $B(Y_{s1})$ that

$$Q(Y_{s1}) A(Y_{s1}) + G_1(Y_{s1}) B(Y_{s1}) = 1$$

, where $\quad deg_Y A(Y_{s1}) < deg_Y G_1(Y_{s1})$.

We adopt the polynomials which are correspondent to these classes, then

$$\left(\sum_j f_j^*(x) y^{j\sigma_s}\right) \cdot a(x, y) + h_1 l \cdot b(x, y) \sim 1 \qquad \text{in } V_{s1}$$

$$V_{s2}\big(h_1 l \cdot b(x, y)\big) > V_{s1}\big(h_1 l \cdot b(x, y)\big)$$

$$V_{s2} 1 = 0 = V_{s1} 1$$

and $$V_{s2}\Big(\big(\sum_j f_j^*(x) y^{j\sigma_s}\big) a(x, y)\Big) = V_{s1}\Big(\big(\sum_j f_j^*(x) y^{j\sigma_s}\big) a(x, y)\Big)$$

, because $$deg_y\, a(x, y) < deg_y\, l\,.$$

Therefore

$$\left(\sum_j f_j^*(x) y^{j\sigma_s}\right) a(x, y) \sim 1 \qquad \text{in } V_{s2}\,.$$

$$\left(\sum_j f_i(x) y^i\right)\big(k \cdot a(x, y)\big) \sim 1 \qquad \text{in } V_{s2}\,.$$

Thus $\sum_i f_i(x) y^i$ is a unit in V_{s2} in $K(x, y)$.

When $n > 0$,

$$k'\left(\sum_i f_i(x) y^i\right) \sim \frac{1}{y^{\sigma_s - n}} \left(\sum_j f_j^*(x) y^{j\sigma_s}\right)(d_s y^{\sigma_s}) \qquad \text{in } V_{s1}, \text{ also in } V_{s2}$$

, where $k' \in U_{s1}$.

Let be $$Y_{s1} \cdot Q(Y_{s1}) = H_{s1}\Big[(d_s y^{\sigma_s})\big(\sum_j f_j^*(x) y^{j\sigma_s}\big)\Big]$$

, then $G_1(Y_{s1})$ and $Y_{si} \cdot Q(Y_{s1})$ are prime to each other.

And $$Y_{s1} Q(Y_{s1}) A(Y_{s1}) + G_1(Y_{s1}) B(Y_{s1}) = 1$$

and in the same way, we know that

$$d_s y^{\sigma_s}\left(\sum_j f_j^*(x) y^{j\sigma_s}\right) a(x, y) \sim 1 \qquad \text{in } V_{s2}$$

$$k'\left(\sum_i f_i(x) y^i\right) a(x, y) \sim \frac{1}{y^{\sigma_s - n}} \qquad \text{in } V_{s2}$$

$$\left(\sum_i f_i(x) y^i\right)\big(k' y^{\sigma_s - n} a(x, y)\big) \sim 1 \qquad \text{in } V_{s2}\,.$$

Thus, when $n > 0$, $\sum_i f_i(x) y^i$ is also a unit in V_{s2} in $K[x, y]$.

§ 35. Residue-class field of V_{s2} in $K[x, y]$

Next, we make Δ_{s2}, the residue-class ring of V_{s2} in $K[x, y]$ in the same way as we did in V_{s1}. Let σ be the smallest natural number that $\sigma\eta_2 = V_{s2}(l^\sigma)$

belongs to the value group Γ_U of V_{s2} in U_{s2} which is a set of all units in V_{s2} in $K[x, y]$. Then d is such a unit in U_{s2} that $V_{s2}(dl^{\sigma})=0$.

Let Γ_{Ul} be the value group which consists of such real numbers of the form $k+n\eta_2$, where $k\in\Gamma_U$ and n are integers. τ is the smallest natural number that $\tau\mu_s=V_{s0}(\phi_s^{\tau})$ belongs to Γ_{Ul}. Then there exist such a unit a in U_{s2} and such an integer λ that $V_{s2}(a\phi_s^{\tau}l^{\lambda})=0$. Here we have the two following cases;

(1) $\lambda=0$ and $V_{s2}(a\phi_s^{\tau})=0$

(2) $\lambda>0$ and $V_{s2}(a\phi_s^{\tau}l^{\lambda})=0$.

When $\lambda=0$, $\Delta_{s2}\cong F_{s2}[X_{s2}, Y_{s2}]$, where F_{s2} is a finite-dimensional algebraic extension of F_{s1} and $X_{s2}=H_{s2}(a\phi_s^{\tau})$ and $H_{s2}(dl^{\sigma})=Y_{s2}$ and X_{s2} and Y_{s2} are algebraically independent over F_{s2} and H_{s2} is the natural homomorphism from $K[x, y]^+$ to Δ_{s2}.

These matters are calculated in the same way as we did them in V_{s1}, except that F_{s2} is a finite-dimensional algebraic extension of F_{s1}. It is to be proved at the end of § 35. Generally when $\lambda\geqq 0$, Δ_{s2} is isomorphic to a ring of R Y_{s2}-quotient of polynomials as X_{s2} with coefficients in F_{s2}, where $X_{s2}=H_{s2}(a\phi_s^{\tau}l^{\lambda})$.

THEOREM 35. 1. *Δ_{s2}, the residue-class field of V_{s2} in $K(x, y)$ is isomorphic to $F_{s2}(X, Y)$, where X and Y are algebraically independent over F_{s2}.*

Every class in Δ_{s2} can be resolved uniquely into irreducible factors in Δ_{s2} and every polynomial $f(x, y)$ for which $V_{s2}(x, y)=0$ can be resolved uniquely into equivalence-irreducible factors in V_{s2} in $K[x, y]$. These matters can be proved in the same way as in V_{s1}.

Now I must prove that F_{s2}, which is a constant field of Δ_{s2}, is isomorphic to an extension of F_{s1}.

If $$V_{s2}(g\phi_s^{\alpha}\eta^{\beta})=0$$

, where $g\in U_{s2}$, then in the same way when we obtain (32. 1),

$$V_{s2}(ga^{-\alpha_1}d^{-\beta_1})=0$$

, where g, a and d are such polynomials in Theorem 34. 1.

And again by Theorem 34. 1, $a^{-\alpha_1}$ and $d^{-\beta_1}$ are also equivalent to such polynomials in Theorem 34. 1.

Therefore $ga^{-\alpha_1}d^{-\beta_1}$ is equivalent to a polynomial of y with coefficients in U_{s1}, but its degree with respect to y is not bounded. So $H_{s1}(ga^{-\alpha_1}b^{-\beta_1})$ is a polynomial of Y_{s1} with coefficients in F_{s1}. Let be

$$H_{s1}(ga^{-\alpha_1}b^{-\beta_1})=Q_1(Y_{s1})G_1(Y_{s1})+R(Y_{s1})$$

, where $deg_Y R(Y_{s1}) < deg_Y G_1(Y_{s1})$. We adopt the polynomials which are correspondent to these classes and $deg_y r(x, y) < deg_y l$

$$ga^{-\alpha_1}b^{-\beta_1} \sim q(x, y)h_1 l + r(x, y) \quad \text{in } V_{s1}$$

and
$$V_{s1}(ga^{-\alpha_1}b^{-\beta_1}) = V_{s1}\big(q(x, y)h_1 l\big) = V_{s1}\big(r(x, y)\big) = 0$$

, but only $V_{s2}(q(x, y)h_1 l) > 0$.

So,
$$ga^{-\alpha_1}b^{-\beta_1} \sim r(x, y) \quad \text{in } V_{s2}.$$

Namely
$$H_{s2}(ga^{-\alpha_1}b^{-\beta_1}) = H_{s2}\big(r(x, y)\big)$$

, where H_{s2} is the natural homomorphism from $K[x, y]^+$ to Δ_{s2}.

$$H_{s1}(ga^{-\alpha_1}b^{-\beta_1}) = H_{s2}(ga^{-\alpha_1}b^{-\beta_1}) = R(Y_{s1}).$$

Therefore every class in Δ_{s2}, which includes units in V_{s2} in $K[x, y]$, includes such a polynomial in Theorem 34. 1. And a set of all such classes in Δ_{s2}, which include units in V_{s2} in $K[x, y]$, is equal to a set of all such polynomials $R(Y_{s1})$ in $F_{s1}[Y_{s1}]$ whose degree with respect to Y_{s1} is less than that of $G_1(Y_{s1})$.

Therefore, the constant field, namely, a set of all such classes in Δ_{s2}, which include units in V_{s2} in $K[x, y]$, is isomorphic to the field $F_{s2} = F_{s1}(\theta)$, where θ is a root of $G_1(Y_{s1}) = 0$.

§ 36. The last simply augmenten valuation V_{st} of W

Thus we can make such a series of y-simply augmented inductive valuations

$$V_{s1} \overset{ys}{<} V_{s2} \overset{ys}{<} \cdots \overset{ys}{<} V_{st}$$

, that
$$V_{sj}\big(\zeta_j(y)\big) = V_{0j}\big(\zeta_j(y)\big) = \nu_j \qquad \text{for } j = 1, 2, \cdots, t.$$

Now we must study the last valuation V_{st}, because otherwise, we can not make an augmented valuations of V_{st}.

$$V_{st} = [V_{s,t-1},\ V_{st} l_t = \eta_t].$$

$$l_t = l_{t-1}^{n\sigma_{t-1}} + \cdots + b_i(x, y) l_{t-1}^{i\sigma_{t-1}} + \cdots + b_0(x, y)$$

, where every $b_i(x, y) \in U_{s,t-1}$ and

$$V_{s,t-1} l_t = V_{s,t-1}(l_{t-1}^{n\sigma_{t-1}}) = V_{s,t-1}\big(b_0(x, y)\big) = V_{s,t-1}\big(b_i(x, y) l_{t-1}^{i\sigma_{t-1}}\big)$$

for every term of l_t.

And l_{t-1} is the y-key polynomial which produces the y-simply augmented valuation $V_{s,t-1}$ of $V_{s,t-2}$ in $K[x, y]$ and σ_{t-1} is the smallest natural number that $\sigma_{t-1}\eta_{t-1}=V_{s,t-1}(l_{t-1}^{\sigma_{t-1}})$ belongs to the value group of $V_{s,t-1}$ of $U_{s,t-1}$ which is a set of all units in $V_{s,t-1}$ in $K[x, y]$.

An arbitrary polynomial $f(x, y)$ can become an expansion of $\phi_s(x)$ and l_t with coefficients in $U_{s,t}$ which is a set of all units in $V_{s,t}$ in $K[x, y]$.

$$f(x, y) = \sum_{i,j} f_{ij}\phi_s^i l_t^j$$

, where $\qquad deg_y f_{ij} < deg_y l_t$ and $deg_x f_{ij} < deg_x \phi_s$.

$$V_{st} f(x, y) = \operatorname*{Min}_{i,j} [V_{s,t-1} f_{ij} + i\mu_s + j\eta_t]$$

, where $\qquad \mu_s = V_{s0}\phi_s = W\phi_s$ and $\eta_t = V_{st} l_t = W l_t$.

Let Γ_{t-1} be the value group of V_{st} in $U_{s,t}$ and σ_t the smallest natural number that $\sigma_t\eta_t = V_{st}(l_t^{\sigma_t})$ belongs to Γ_{t-1}, then in $U_{s,t}$ there exists such a unit d_t that $V_{st}(d_t l_t^{\sigma_t})=0$.

Γ_t is the value group which consists of such real numbers of the form $k+n\eta_t$, where $k\in\Gamma_{t-1}$ and n are integers and τ_t is the smallest natural number that $\tau_t u_s = V_{st}(\phi_s^{\tau_t})$ belongs to Γ_t, then there exist such a unit a_t in V_{st} and such an integer λ_t that $V_{st}(a_t\phi_s^{\tau_t} l_t^{\lambda_t})=0$, where $0\leqq\lambda_t\leqq\sigma_t-1$.

C_t is such a unit in $V_{s,t-1}$ in $K[x, y]$ that $V_{s,t-1}(c_t l_t)=0$ and $G(Y)= H_{s,t-1}(c_t l_t)$ is an irreducible class in $\Delta_{s,t-1}$, where $H_{s,t-1}$ is the natural homomorphism from $K[x, y]^+$ to $\Delta_{s,t-1}$, the residue-class ring of $V_{s,t-1}$ in $K[x, y]$. And θ_t is one root of $G(Y)=0$.

Then, in the same way as we did above, we can prove that Δ_{st}, the residue-class ring of V_{st} in $K[x, y]$ is isomorphic to a ring of R_t Y_t-quotient polynomials of X_t with coefficients in a field F_{st}, where $F_{st}=F_{s,t-1}(\theta_t)$ and $R_t=\dfrac{\lambda_t}{\sigma_t}$ is a non-negative fixed number and X_t and Y_t are algebraically independent over F_{st}.

Λ_{st}, the residue-class field of V_{st} in $K(x, y)$ is isomorphic to $F_{st}(X_t, Y_t)$.

Every residue-class of Δ_{st} can be uniquely resolved into irreducible factors in Δ_{st} and every polynomial $f(x, y)$ for which $V_{st}f(x, y)=0$, can be uniquely resolved into equivalence-irreducible factors in V_{st} in $K[x, y]$.

§ 37. Factorization in V_{st}

Here we try to carry out real factorization of a polynomial $f(x, y)$ for which $V_{st}f(x, y)=0$, into equivalence-irreducible factors in V_{st} in $K[x, y]$, because this will be used below in this paper.

Γ_ϕ is a value group which consists of such real numbers of the form $k+n\mu_s$, where $k\in\Gamma_{t-1}$ and n are integers. Let Γ^* be an intersection of Γ_t and Γ_ϕ. Then a is the smallest natural number that $a\mu_s=V_{st}(\phi_s^a)$ belongs to Γ^* and b is the smallest natural number that $b\eta_t=V_{st}(l_t^b)$ belongs to Γ^*.

When $$V_{st}(f\phi_s^i l_t^j)=0$$

, where $f\in U_{s,t}$,

i is a multiple i_1a of a and j is a multiple j_1b of b, namely

$$f\phi_s^i l_t^j=f\phi_s^{i_1a}l_t^{j_1b}\,.$$

Therefore, when $V_{st}f(x,y)=0$

$$f(x,y)\sim h\prod_i P_i(\phi_s^a,\ l_t^b) \qquad \text{in } V_{st}$$

, where $h\in U_{s,t}$ and every $P_i(\phi_s^a,\ l_t^b)$ is an equivalence-irreducible polynomial of ϕ_s^a and l_t^b with coefficients in U_{st}.

Next we resolve specially a polynomial $c(x)$ in $K[x]$ and another polynomial $d(y)$ in $K[y]$ into equivalence-irreducible factors in V_{st} in $K[x,y]$.

$\phi_s(x)$ is a polynomial in $K[x]$, so $c(x)$ can be expressed as an expansion by ϕ_s,

$$c(x)=\sum_i c_i(x)\phi_s^i$$

, where $deg_x c_i(x)<deg_x\phi_s$, so $c_i(x)\in U_{s,t}$.

$$V_{st}c(x)=V_{s0}c(x)=\operatorname*{Min}_i\left[V_{t0}c_s(x)\phi_s^i\right].$$

We adopt only the homogeneous part of $c(x)$, namely we abandon all such terms that $V_{s0}(c_i(x)\phi_s^i)>V_{st}c(x)$

, then $$c(x)\sim\sum_l c_l(x)\phi_s^l \qquad \text{in } V_{st}\cdots \qquad (37.1)$$

, where $c_l(x)\in U_{s,t}$.

Next we must prove that

$$d(y)\sim\sum_j d_j(x,y)l_t^j \qquad \text{in } V_{st}$$

, where $d_j(x,y)\in U_{st}$.

For the sake of it, we must make another series of augmented inductive valuations as follows.

§ 38. $\overline{V}_{st}$

W induces in $K[x]$ the following series;

$$V_{10} < V_{20} < \cdots < V_{s0}$$

, where $V_{i0} = [V_{i-1,0},\ V_{i0}\phi_i(x) = \mu_i]$ for $i=1, 2, \cdots, s$.

And W induces in $K[y]$ another series as follows;

$$V_{01} < V_{02} < \cdots < V_{0t}$$

, where $V_{0j} = [V_{0,j-1},\ V_{0j}\zeta_j(y) = \nu_j]$ for $j=1, 2, \cdots, \mathrm{t}$.

At first, we defined the valuation V_{s1} of the ring $K_x[y]$ whose coefficient field $K_x = K(x)$ has the valuation V_{s0}. And next we made a series of y-simply augmented inductive valuations and we obtained the valuation V_{st} as the last valuation of this series.

So, here we consider, exchanging x and y. We define a valuation $\overline{V}_{1t}$ of the ring $K_y[x]$ whose coefficient field $K_y = K(y)$ has the valuation V_{0t}, as follows;

$$f(x, y) = \sum_i f_i(y)x^i$$

$$\overline{V}_{1t}\big(f(x, y)\big) = \operatorname*{Min}_i\left[V_{0t}f_i(y) + i\mu_1\right]$$

, where $$\mu_1 = Wx = V_{10}x\,.$$

Next, we make such a series of x-simply augmented inductive valuations that

$$\overline{V}_{1t} \overset{xs}{<} \overline{V}_{2t} \overset{xs}{<} \cdots \overset{xs}{<} \overline{V}_{st}$$

, where $$\overline{V}_{it} = [\overline{V}_{i-1,t}\,,\ \overline{V}_{it}\phi_i = \pi_i]$$

and $\overline{V}_{it}\phi_i(x) = \pi_i = W\phi_i(x)$ for $i=1, 2, \cdots, s$.

$\phi_i(x)$ is an x-key polynomial which produces an x-simply augmented valuation $\overline{V}_{it}$ of $\overline{V}_{i-1,t}$ of the ring $K_y[x]$.

Then $$V_{st} = \overline{V}_{st}\,.$$

Therefore, if $$f(x, y) \sim g(x, y) \quad \text{in } V_{st}$$

, then $$f(x, y) \sim g(x, y) \quad \text{in } \overline{V}_{st}$$

and vice versa.

And U_{st}, a set of all units in V_{st} in $K[x, y]$, coincides completely with that in $\overline{V}_{st}$ in $K[x, y]$.

$$\overline{V}_{st} = [\overline{V}_{s-1,t}\,,\ \overline{V}_{st}\phi_s(x) = \pi_s]\,.$$

In the same way as (33. 3), there exist such units ε_t and ε_s in U_{st} that

$$\zeta_t(y) \sim l_t\varepsilon_t \qquad \text{in } V_{st} \tag{38. 1}$$

and $$\phi_s(x) \sim \psi_s \varepsilon_s \quad \text{in } \overline{V}_{st} \tag{38.2}$$

Γ_ψ is a value group which consists of such real numbers of the form $k+n\pi_s$, where $k \in \Gamma_{t-1}$ and n are integers.

$$\mu_s = \overline{V}_{st}\phi_s = \overline{V}_{st}\psi_s + \overline{V}_{st}\varepsilon_s \qquad \text{from (38.2)}$$

$$\mu_s = \pi_s + \overline{V}_{st}\varepsilon_s$$

, where $$\overline{V}_{st}\varepsilon_s \in \Gamma_{t-1}.$$

Therefore $\Gamma_\psi = \Gamma_\phi$.

Γ_ζ is a value group which consists of such real numbers of the form $k+n\nu_t$, where $k \in \Gamma_{t-1}$ and n are integers. Then, in the same way, from (38.1)

$$\Gamma_\zeta = \Gamma_l.$$

Let a' be the smallest natural number that $a'\pi_s = \overline{V}_{st}(\psi_s^{a'})$ belongs to Γ^* and b' be the smallest natural number that $b'\nu_t = V_{st}(\zeta_t^{b'})$ belongs to Γ^*, then $a'=a$ and $b'=b$, where a and b are defined in § 37.

Therefore every equivalence relation in V_{st} also holds, when we substitute $l_t\varepsilon_t$ for $\zeta_t(y)$ and $\psi_s\varepsilon_s$ for $\phi_s(x)$ in the equivalence relation in V_{st}.

For example; when $$V_{st}f(x, y) = 0$$

$$f(x, y) \sim h \prod_i p_i(\phi_s^a, l_t^b) \qquad \text{in } V_{st}$$

, where $h \in U_{st}$ and every $p_i(\phi_s^a, l_t^b)$ is an equivalence-irreducible polynomial of ϕ_s^a and l_t^b with coefficients in U_{st},

$$f(x, y) \sim h' \prod_i p_i'(\psi_s^a, \zeta_t^b) \qquad \text{in } V_{st}$$

, where $h' \in U_{st}$ and every $p_i'(\psi_s^a, \zeta_t^b)$ is an equivalence-irreducible polynomial of ψ_s^a and ζ_t^b with coefficients in U_{st}.

Now we can prove easily that $d(y) \sim \sum_j d_j(x, y) l_t^j$ in V_{st}. In the same way as in (37.1)

$$d(y) \sim \sum_l d_l(y) \zeta_t^l \qquad \text{in } V_{st}$$

, where $d_l(y) \in U_{st}$ and $deg_y d_l(y) < deg_y \zeta_t(y)$.

From (38.1) $$d(y) \sim \sum_l \left(d_l(y)\varepsilon_t^l\right) l_t(y)^l \qquad \text{in } V_{st}$$

, then $d_l(y)\varepsilon_t^l \in U_{st}$, because U_{st} is a group with respect to multiplication.

§ 39. Structures of key polynomials

In the beginning of Part Three of this paper I assumed that V_{st} is the last simply augmented valuation of W. But hence in this paper I abandon this assumption, namely now we prescribe that we can freely make an x-simply augmented valuation of V_{st}, or a y-simply augmented valuation of V_{st}, or an xy-doubly augmented valuation of V_{st}.

And we want to study structures of these key polynomials.

According to M. Theorem 9. 4, a polynomial $L=L(x, y)$ can be a y-key polynomial which produces a y-augmented valuation of V_{st} in the ring $K_x[y]$ whose coefficient field K_x has the valuation V_{s0}, if and only if the following conditions hold:

$$L=l_t^{m\varphi}+\cdots+q_i(x, y)l_t^{i\varphi}+\cdots+q_0(x, y)\cdots(39.1)$$

, where $V_{st}L=V_{st}(l_t^{m\varphi})=V_{st}(q_0(x, y))=V_{st}(q_il_t^{i\varphi})$ for every term of L and L is equivalence-irreducible in V_{st} in $K_x[y]$. Here $q_i(x, y)$ is a polynomial of y with coefficients in $K(x)$ and $deg_y q_i(x, y)<deg_y l_t$ for every term of L and φ is the smallest natural number that $\varphi\eta_t=V_{st}(l_t^{\varphi})$ belongs to Γ_ϕ.

Let be $$q_i(x, y)=\frac{n_i(x, y)}{d_i(x)}$$

, where $n_i(x, y)$ is a polynomial of x and y and $deg_y n_i(x, y)<deg_y l_t$ and $d_i(x)$ is a polynomial of x.

$$n_i(x, y)\sim\sum_j n_j\phi_s^j \quad \text{in } V_{st}$$

and

$$d_i(x)\sim\sum_j d_j\phi_s^j \quad \text{in } V_{st}$$

, where $n_j\in U_{st}$ and $d_j\in U_{st}$.

Then I will prove the following theorem;

THEOREM 39. 1. *A necessary and sufficient condition that L, the above-mentioned polynomial, is a key polynomial that produces a y-simply augmented valuation of V_{st} in $K_x[y]$, is that none of $n_i(x, y)$ and $d_i(x)$ of L include $\phi_s(x)$ in their homogeneous parts.*

This theorem is equivalent to the following theorem, because if a y-augmented valuation of V_{st} in $K_x[y]$ is not an xy-doubly augmented valuation of V_{st} in $K_x[y]$, then it is a y-simply augmented valuation of V_{st} in $K_x[y]$ and vice versa.

THEOREM 39. 2. *A necessary and sufficient condition that L, the above-mentioned polynomial, is a key polynomial that produces an xy-doubly augmented valuation of V_{st} in $K_x[y]$, is that $\phi_s(x)$ appears in some terms of*

$n_i(x, y)$ or $d_i(x)$ of L in their homogeneous parts.

$$L \sim l_t^{m\varphi} + \cdots + \frac{n_i(x, y)}{d_i(x)} l_t^{i\varphi} + \cdots + \frac{n_0(x, y)}{d_0(x)} \quad \text{in } V_{st}.$$

At first I prove that, if $\phi_s(x)$ appears in some terms of $n_i(x, y)$ or $d_i(x)$ of L and L is a key polynomial which produces a y-simply augmented valuation of V_{st} in $K_x[y]$, then a contradiction takes place. As L produces a y-simply augmented valuation V_y of V_{st} in $K_x[y]$, there exists such a polynomial $f(y)$ in $K[y]$ that $V_y f(y) > V_{st} f(y)$.

So, by M. Theorem 5. 1

$$L \,|\, f(y) \quad \text{in } V_{st} \quad \text{in } K_x[y].$$

Namely $f(y)$ is equivalence-divisible by L in V_{st} in $K_x[y]$.

$$f(y) \sim L \cdot g(x, y) \quad \text{in } V_{st} \quad \text{in } K_x[y]$$

, where $g(x, y)$ is a polynomial of y with coefficients in $K(x)$.

We multiple both sides by the least common multiple $h'(x)$ of $L \cdot g(x, y)$, then

$$f(y) h'(x) \sim L' \cdot g'(x, y) \quad \text{in } V_{st} \quad \text{in } K[x, y].$$

L is equivalence-irreducible in V_{st} in $K_x[y]$, so L' is equivalence-irreducible in V_{st} in $K[x, y]$ and in L', both ϕ_s^a and l_t^b appear. But in $f(y)$, only l_t^b appears and ϕ_s^a does not appear, and in $h'(x)$, only ϕ_s^a appears and l_t^b does not appear as I explained in §37. $f(y)h'(x)$ must be resolved uniquely into equivalence-irreducible factors in V_{st} in $K[x, y]$. Thus a contradiction takes place. Therefore, if $\phi_s(x)$ appears in L, then L is a y-key polynomial which produces an xy-doubly augmented valuation of V_{st} in $K[x, y]$.

Next I will prove that if $\phi_s(x)$ does not appear in L, then L is not a key polynomial which produces an x-augmented valuation of V_{st} in $K[x, y]$. If I can prove it, then L is a key polynomial which produces a y-simply augmented valuation of V_{st} in $K[x, y]$, because L produces a y-augmented valuation, but L can not produce an x-augmented valuation of V_{st} in $K[x, y]$, so owing to the definition of a y-simply augmented valuation, L is a key polynomial which produces a y-simply augmented valuation of V_{st} in $K[x, y]$.

$L = \dfrac{L'(x, y)}{h(x)}$, where $L'(x, y)$ does not include $\phi_s(x)$.

$$V_{st} = \overline{V}_{st} = \left[\overline{V}_{s-1,t},\ \overline{V}\phi_s(x) = \pi_s\right]$$

$$\zeta_t(y) \sim l_t \varepsilon_t \quad \text{and} \quad \phi_s(x) \sim \varphi_s \varepsilon_s \quad \text{in } V_{st}.$$

According to M. Theorem 9. 4., a polynomial $L^*(x, y)=L^*$ can be an x-key polynomial which produces an x-augmented valuation of $\overline{V}_{st}$ in $K_y[x]$ whose coefficient field $K_y=K(y)$ has the valuation V_{0t}, if and only if the following conditions hold:

$$L^* = \psi_s^n + \cdots .$$

$$L^* = \frac{L''(x, y)}{h(x, y)} .$$

Namely the numerator of L^* must include ψ_s in its homogeneous part.

$$\zeta_t(y) \sim l_t \varepsilon_t \quad \text{and} \quad \phi_s(x) \sim \psi_s \varepsilon_s \qquad \text{in} \quad V_{st} .$$

So, the numerator $L'(x, y)$ is equivalent only to such polynomials which include only l_t or $\zeta_t(y)$ and do not include ψ_s. Therefore L can not be such a key polynomial that produces an x-augmented valuation of $\overline{V}_{st}$ in $K_y[x]$.

Thus, Theorem 39. 1. and 39. 2. are proved completely.

§ 40. xy-doubly augmented valuation $V_{s+1,t}$

When L in (39. 1) is a y-key polynomial which produces an xy-doubly augmented valuation of V_{st} in $K_x[y]$, we define a y-augmented valuation $V_{s,t+1}$ of V_{st} in $K_x[y]$ in the same way as we difined a y-augmented valuation V_{st} of $V_{s,t-1}$ in $K_x[y]$.

$$V_{s,t+1} = [V_{st}, V_{s,t+1}L = \eta_{t+1}]$$

, where $$\eta_{t+1} - V_{st}L = V_{s,t+1}L - V_{st}L = \omega > 0 \tag{40. 1}$$

Then $V_{s,t+1}$ is a y-augmented valuation of V_{st}, but it is an xy-doubly augmented valuation of V_{st} in $K_x[y]$.

Next multiply both sides of (39. 1) by the least common multiple $d(x)$ of denominators of L, then

$$d(x)L = g\left(\phi_s^a(x),\, l_t^b\right)$$

, where coefficients $\in U_{st}$.

We substitute $\varepsilon_t^{-1}\zeta_t(y)$ for l_t and $\psi_s(x)\varepsilon_s$ for $\phi_s(x)$ in $g(\phi_s^a, l_t^b)$

and $$d(x)L \sim g' = \sum_j m_j(\zeta_t^b)\psi_s^{ja} \qquad \text{in} \quad V_{st}$$

, where coefficients $\in U_{st}$.

Let $m_p(\zeta_t^b)$ be a coefficient of ψ_s^{pa}, the highest term of g'.

$$d(x)L \sim m_p(\zeta_t^b)g''\left(\psi_s^a(x),\, \zeta_t^b\right) \qquad \text{in} \quad V_{st} .$$

g'' is equivalence-irreducible in $V_{st}=\overline{V}_{st}$, so there exist such polynomials m''_p and g^* in $K_y[x]$ that

$$d(x)L \overset{\omega}{\approx} m''_p(\zeta^b_t)g^*\left(\phi^a_s(x),\ \zeta^b_t\right) \qquad \text{in } \overline{V}_{st}. \qquad (40.2)$$

The leading coefficient of g^* is 1 and $g^*(\phi^a_s,\ \zeta^b_t)$ satisfies the necessary and sufficient conditions that g^* is a x-key polynomial which produces an x-augmented valuation of $\overline{V}_{st}$ in $K_y[x]$.

So, we define an x-augmented valuation $\overline{V}_{s+1,t}$ of $\overline{V}_{st}$ in $K_y[x]$, in the same way as we defined an x-augmented valuation $\overline{V}_{st}$ of $\overline{V}_{s-1,t}$ in $K_y[x]$;

$$\overline{V}_{s+1,t}=[\overline{V}_{st},\ \overline{V}_{s+1,t}g^*=\omega+\overline{V}_{st}g^*].$$

$\overline{V}_{s+1,t}$ is an xy-doubly augmented valuation of $\overline{V}_{st}$, so values of $d(x)$ and m''_p do not increase when we make $\overline{V}_{s+1,t}$ of $\overline{V}_{st}$.

Let be $Q=d(x)L-m''_pg^*$, then by (40.2)

$$\begin{aligned}\overline{V}_{s+1,t}Q \geqq \overline{V}_{st}Q &> \omega+\overline{V}_{st}(m''_pg^*)=\overline{V}_{st}m''_p+(\omega+\overline{V}_{st}g^*)\\ &=\overline{V}_{s+1,t}m''_p+\overline{V}_{s+1,t}(g^*)=\overline{V}_{s+1,t}(m''_pg^*).\\ \therefore\quad \overline{V}_{s+1,t}(m''_pg^*)&=\overline{V}_{s+1,t}\left(d(x)L\right).\end{aligned}$$

In the same way as above

$$V_{s,t+1}(m''_pg^*)=V_{s,t+1}\left(d(x)L\right)$$

and

$$V_{st}=\overline{V}_{st}.$$

So,

$$\overline{V}_{s+1,t}=V_{s,t+1}.$$

§ 41. Residue-class fields of doubly augmented valuations

We make a y-augmented valuation $V_{s,t+1}$ of V_{st} in $K_x[y]$ as follows;

$$V_{s,t+1}=[V_{st},\ V_{s,t+1}L=\eta_{t+1}]$$

, where L is the polynomial in (39.1) and η_{t+1} is the value in (40.1).

$V_{s,t+1}$ is an xy-doubly augmented valuation of V_{st}, so $\Lambda_{s,t+1}$, the residue-class field of $V_{s,t+1}$ in $K(x, y)$, is pretty different from Λ_{st}, the residue-class field of V_{st} in $K(x, y)$.

Here we change representation of Δ_{st} a little as follows, because this method is necessary for us to study the structure of $\Lambda_{s,t+1}$.

Γ_{t-1} is the value group of V_{st} in U_{st} and Γ_ϕ is the value group which is defined in the beginning of § 37 and α is the smallest natural number that $\alpha\mu_s=V_{st}(\phi^a_s)$ belongs to Γ_{t-1}. φ is the natural number which is defined in (39.1).

$V_{st}(\phi_s^{\alpha})$ belongs to Γ_{t-1}, then there exists such a unit h_1 that $V_{st}(h_1\phi_s^{\alpha})=0$. $V_{st}(l_t^{\varphi})$ belongs to Γ_{ϕ}, so there exist such a unit h_2 and such an integer π that $V_{st}(h_2\phi_s^{\pi}l_t^{\varphi})=0$, where $0\leqq\pi\leqq\alpha-1$.

Let be $H_{st}(h_1\phi_s^{\alpha})=\bar{X}$ and $H_{st}(h_2\phi_s^{\pi}l_t^{\varphi})=\bar{Y}$, then $\bar{X}$ and $\bar{Y}$ are algebraically independent over F_{st} and $\Lambda_{st}\cong F_{st}(\bar{X},\bar{Y})$.

Moreover Δ'_{st}, the residue-class ring of V_{st} in the ring $K_x[y]$ whose coefficient field $K_x=K(x)$ has the valuation V_{s0}, is isomorphic to $F_{st\bar{x}}[\bar{Y}]$, where $F_{st\bar{x}}=F_{st}(\bar{X})$. Because, if $f(x,y)\in K_x[y]$ and $V_{st}f(x,y)=0$, then $f(x,y)=\dfrac{n(x,y)}{d(x)}$, where $n(x,y)$ are a polynomial of x and y.

$$H_{st}f(x,y)=\frac{H_{st}n(x,y)}{H_{st}d(x)}=\frac{\sum\limits_{i,j}n_{ij}\bar{X}^i\bar{Y}^j}{d'(\bar{X})}$$

, where $n_{ij}\in F_{st}$.

$$H_{st}f(x,y)=\sum_j\left(\frac{\sum n_{ij}\bar{X}^i}{d'(\bar{X})}\right)\bar{Y}^j.$$

Now we can make $\Lambda_{s,t+1}$ by M. Theorem 12. 1. M. Theorem 12. 1. states as follows:
V_{t+1} is a given discrete valuation of $k(y)$ and V_{t+1} induces a valuation V_0 in k and F_1 is the residue-class field of V_0 in k. V_1 is defined for a polynomial $f(y)=\sum\limits_i f_iy^i$ in $k(y)$ as

$$V_1f(y)=\operatorname*{Min}_i\,[V_0f_i+iV_{t+1}y].$$

Thus we make such a series of y-augmented inductive valuations in $k[y]$ that

$$V_1<V_2<\cdots<V_t<V_{t+1}$$

, where $\quad V_i=[V_{i-1},\ V_il_i=\eta_i] \quad$ for $i=1,2,\cdots,t$.

When $a(y)b(y)\sim1$ in V_i, $a(y)$ is called a unit in V_i. And U_i is a set of all units in V_i.

$$V_t=[V_{t-1},\ V_tl_t=\eta_t].$$

Let ξ_t be the smallest natural number that $\xi_t\eta_t=V_t(l_t^{\xi_t})$ belongs to the value group of V_t in U_t, then there exists such a unit d_t in V_t that $V_t(d_tl_t^{\xi_t})=0$.

Δ_t, the residue-class ring of V_t in $k[y]$, is isomorphic to $F_t[Y]$, where F_t is a set of all such classes in Δ_t that include units in V_t and F_t is a finite-dimensional extension of F_1 and $Y=H_t(d_tl_t^{\xi_t})$ is transcendental with respect to F_t and H_t is the natural homomorphism of $k[y]^+$ to Δ_t.

$$V_{t+1}=[V_t,\ V_{t+1}l_{t+1}=\eta_{t+1}].$$

There exists such a unit k in V_t that

$$V_t(kl_{t+1})=0$$

and $H_t(kl_{t+1})=g(Y)$ is a polynomial of Y with coefficients in F_t. θ is a root of $g(Y)=0$.

Then Δ_{t+1}, the residue-class ring of V_{t+1} in $k[y]$ is isomorphic to $F_{t+1}[Y_{t+1}]$, where $F_{t+1}=F_t(\theta)$ and Y_{t+1} is transcendental with respect to F_{t+1}.

M. Theorem 12. 1. states so. Comparing his theorey with ours, we have the following complete correspondences;

$$\begin{array}{ccccccccccccc} k & y & V_0 & V_1 & V_t & l_t & \eta_t & U_t & H_t & F_t & Y & V_{t+1} & l_{t+1} \\ | & | & | & | & | & | & | & | & | & | & | & | & | \\ K(x) & y & V_{s0} & V_{s1} & V_{st} & l_t & \eta_t & U_{st} & H_{st} & F_{st} & \overline{Y} & V_{s,t+1} & L \end{array}$$

Now we will make $\Delta'_{s,t+1}$, the residue-class ring of $V_{s,t+1}$ in the ring $K_x[y]$. From (39. 1)

$$V_{st}L=V_{st}q_0(x,y), \qquad \text{so}\quad V_{st}(q_0^{-1}L)=0$$

, then $H_{st}(q_0^{-1}L)=G(\overline{Y})$ is a polynomial of $\overline{Y}$ with coefficients in $F_{st}(\overline{X})$ and its degree with respect to $\overline{Y}$ is m. And $\phi_s(x)$ appears in some coefficients of L in their homogeneous parts, so some coefficients of $G(\overline{Y})$ include $\overline{X}$. Let be $G(\theta)=0$, then θ is algebraic with respect to $F_{st}(\overline{X})$, but θ is not algebraic with respect to F_{st}. And by M. Theorem 12. 1,

$$\Delta'_{s,t+1}\cong F_1^*[Y_{t+1}]$$

, where $F_1^*=F_{st\bar{x}}(\theta)$ and $F_{st\bar{x}}=F_{st}(\overline{X})$ and Y_{t+1} is transcendental with respect to F_1^*.

Therefore $$\Lambda_{s,t+1}\cong F_1^*(Y_{t+1}).$$

I summarize the relations between valuations V_{ij} and their residue-class field Λ_{ij} in the following table;

$$\begin{array}{l} V_{11}\overset{xs}{<}V_{21}\overset{xs}{<}\cdots\overset{xs}{<}V_{s1}\overset{ys}{<}V_{s2}\overset{ys}{<}\cdots\overset{xy}{<}V_{st}\overset{xy}{<}V_{s,t+1}\overset{xy}{<}\cdots\overset{xy}{<}V_{s,t+q} \\ F_{11}\subset F_{21}\subset\cdots\subset F_{s1}\subset F_{s2}\subset\cdots\subset F_{st} \\ F_{st}(X)\subset F_1^*\subset\cdots\subset F_q^* \end{array}$$

, where F or F^* is respectively a finite-dimensional algebraic extension of its preceding F or F^* on the same line.

(α) $\Lambda_{ij}\cong F_{ij}(X,Y)$, while simply augmented valuations continue.

(β) $\Lambda_{s,t+j}\cong F_j^*(Y)$, while xy-doubly augmented valuations continue.

Of course (β) always holds, even if when simply augmented valuations take place. But, I, Inoue, struggled very hard to prove that (α) holds while simply augmented valuations continue.

§ 42. The case without Conditions 28.1 and 28.2.

When we do not give Conditions 28.1 and 28.2, after V_{s2} only (β) holds. Namely Λ_{s1}, the residue-class field of V_{s1} in $K(x, y)$ is isomorphic to $F_{s1}(X, Y)$, but Λ_{s2}, the residue-class field of V_{s2} in $K(x, y)$ is isomorphic to $F_1^*(Y)$, where F_1^* is a finite-dimensional algebraic extension of $F_{s1}(X)$ and X is transcendental with respect to F_{s1} and after Λ_{s2}, every $\Lambda_{s,1+j} \cong F_j^*(Y)$, where F_j^* is a finite-dimensional algebraic extension of F_{j-1}^* for $j=2, 3, 4, \cdots$.

Corrections in Part Two of this paper.

Part Two of this paper was pretty long and I am really sorry that I could not find completely some careless misses in it. I think that you can easily find them, but here I correct them as follows;

1. To the assumptions of Corollary 13.2. I add such a condition that $V_k' G(X) > UG(X)$ *when* U *is a descended valuation of* V_k'.
2. To the assumptions of Lemma 21.3. I add such a condition that $f(x, y) \sim g(x, y)$ *in* V_{1q}.
3. In Page 272, "n" is "θ", of course.

In near future I want to establish valuations of polynomial rings of three or more variables, namely, in Part Four of this paper.

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APPENDIX

In front of Corollary 30. 2, I add the four following definitions;

DEFINITION 30. A. *If $f=ay^n g$, where $a \in K$ and $f \in S_R$ and $g \in S_R$ and n is an integer, then we say that f and g are associately equal to each other in S_R.*

DEFINITION 30. B. *If a polynomial $p(X, Y)$ which is not equal to Y is irreducible in $K[X, Y]$, then $p(X, Y)$ is said to be irreducible in S_R. And if $f(X, Y)=ay^n p(X, Y)$, where $a \in K$ and n is an integer, then $f(X, Y)$ is said to be associately irreducible in S_R.*

DEFINITION 30. C. *If $l(x, y) \sim ay^n g(x, y)$ in V_{11}, where $a \in K$ and n is an integer, then we say that these two polynomials $l(x, y)$ and $g(x, y)$ are associately equivalent to each other in V_{11}.*

DEFINITION 30. D. *$p(x, y) \in K[x, y]$ and $p[x, y] \neq y$ and $p(x, y)$ is not in K. And if $p(x, y) \sim a(x, y)b(x, y)$ in V_{11} in $K[x, y]$, then always one of $a(x, y)$ and $b(x, y)$ is in K. In this case we say that $p(x, y)$ is equivalence-irreducible in V_{11} in $K[x, y]$.*

Consequently I rewrite the word "*uniquely*" into "*associately uniquely*" in Corollary 30. 2, in Theorem 31. 1, and in Corollary 31. 4, and "*irreducible in Δ_{11}*" into "*associately irreducible in Δ_{11}*" and "*equivalence-irreducible*" into "*associately equivalence-irreducible*" in Theorem 31. 3. And I rewrite in the same way in Δ_{s1} and in V_{s1} and so on.

Immediately after my first proofreading of this paper, I visited Professor Malcolm Griffin of Queen's University in Canada who gave me many useful advices by which I determined to write supplementary explanations about Part I, II and III of this papers in Part IV of this papers in future. And after my long research in Zentral Blatt recently at last I could find Van der Put's opinions about this papers to which I want to answer in Part IV of this papers, if possible. And I want to write about polynomial rings of n variables after them.

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