Some congruence theorems for closed hypersurfaces in Riemann spaces

(The continuation of Part III)

Dedicated to the memory of Professor Dr. Heinz Hopf

By Yoshie KATSURADA

Introduction. This is the continuation of the previous paper ([1])[1] given by H. Hopf and the present author. In [1], considering an $(m+1)$-dimensional orientable Riemann space S^{m+1} with constant curvature of class C^{ν} $(\nu \geqq 3)$ which admits a one-parameter group G of isometric transformations, we proved the following

THEOREM. Let W^m and $\overline{W}^m$ be two orientable closed hypersurfaces in S^{m+1} which do not contain a piece of a hypersurface covered by the orbits of the transformations and $p\ \bar{p}$ be the corresponding points of these hypersurfaces along an orbit, and $H_r(p)$ and $\overline{H}_r(p)$, $r=1, \cdots, m$ be the r-th mean curvatures of these hypersurfaces at p and $\bar{p}$ respectively. Assume that in case $r \geqq 2$, the second fundamental form of $W^m(t) \overset{\text{def.}}{=} (1-t)W^m + t\overline{W}^m$, $0 \leqq t \leqq 1$, is positive definite. If the relation $H_r(p) = \overline{H}_r(p)$ holds for each point $p \in W^m$, then W^m and $\overline{W}^m$ are congruent mod G.

In the present paper, we shall cancel the assumption that the transformations are isometric, in fact, under a group G of essentially arbitrary transformations it is the purpose of the present paper to generalize the above theorem. Especially, in case of $r=m$, that is, the general theorem relating to the Gauss curvature was already proved in the previous paper [2].

§1. A certain integral form for two closed hypersurfaces. We suppose an $(m+1)$-dimensional orientable Riemann space S^{m+1} with constant curvature of class C^{ν} $(\nu \geqq 3)$ which admits an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x)\delta\tau \tag{1.1}$$

(where x^i are local coordinates in S^{m+1} and ξ^i are the components of a contravariant vector ξ). We assume that orbits of the transformations generated by ξ cover S^{m+1} simply and that ξ is everywhere continuous and $\neq 0$. Let us choose a coordinate system such that the orbits of transfor-

1) Numbers in brackets refer to the references at the end of the paper.

mations are new x^1-coordinate curves, that is, a coordinate system in which the vector ξ^i has components $\xi^i=\delta_1^i$, where the symbol δ_j^i denotes Kronecker's delta; then (1.1) becomes as follows

$$\hat{x}^i = x^i + \delta_1^i \delta\tau \tag{1.2}$$

and S^{m+1} admits a one-parameter continuous group G of transformations which are 1–1-mappings of S^{m+1} onto itself and are given by the expression

$$\hat{x}^i = x^i + \delta_1^i \tau$$

in the new special coordinate system.

Now we consider two orientable closed hypersurfaces W^m and $\overline{W}^m$ of class C^ν imbedded in S^{m+1} which are given as follows

$$\begin{cases} W^m: & x^i = x^i(u^\alpha) \qquad i=1,\cdots,m+1 \quad \alpha=1,\cdots,m. \\ \overline{W}^m: & \bar{x}^i = x^i(u^\alpha)+\delta_1^i\tau(u^\alpha) \end{cases} \tag{1.3}$$

where u^α are local coordinates of W^m and τ is a continuous function attached to each point of the hypersurface W^m. We shall henceforth confine ourselves to Latin indices running from 1 to $m+1$ and Greek indices from 1 to m.

Then we can take the family of the hypersurfaces

$$W^m(t)=(1-t)W^m+t\overline{W}^m \qquad 0\leqq t\leqq 1,$$

genrated by W^m and $\overline{W}^m$ whose points correspond along the orbits of the transformations, where W^m and $\overline{W}^m$ mean $W^m(0)$ and $W^m(1)$ respectively. Thus according to (1.3), $W^m(t)$ is given by the expression

$$W^m(t): \quad x^i(u^\alpha, t)=(1-t)x^i(u^\alpha)+t\bar{x}^i(u^\alpha) \qquad 0\leqq t\leqq 1, \tag{1.4}$$

and (1.4) may be rewritten as follows

$$W^m(t): \quad x^i(u^\alpha, t)= x^i(u^\alpha)+\delta_1^i t\tau(u^\alpha) \qquad 0\leqq t\leqq 1.$$

The relation between $\overline{W}^m$ and $W^m(t)$ becomes as follows

$$\bar{x}^i(u^\alpha)= x^i(u^\alpha, t)+\delta_1^i(1-t)\tau(u^\alpha).$$

If we take the hypersurface $W^m(t_0)$ defined by a fixed value t_0 in $0\leqq t\leqq 1$, then we have the transformation $T_{(1-t_0)\tau(p_0)}\in G$ attached to the point on $W^m(t_0)$ corresponding to $p_0\in W^m$, given by

$$T_{(1-t_0)\tau(p_0)}: \quad \hat{x}^i = x^i+\delta_1^i(1-t_0)\tau(u_0^\alpha),$$
$$(1-t_0)\tau(u_0^\alpha) = \text{constant}.$$

Thus we get the additional hypersurface

$$\widetilde{W}^m_{p_0}(t_0) \overset{\text{def.}}{=} T_{(1-t_0)\tau(p_0)} \cdot W^m(t_0)$$

which passes through the corresponding point $\bar{p}_0$ on $\overline{W}^m$, and is given by

$$\widetilde{W}^m_{p_0}(t_0): \quad \tilde{x}^i_{p_0}(u^\alpha, t_0) = x^i(u^\alpha, t_0) + \delta^i_1(1-t_0)\tau(u^\alpha_0),$$
$$(1-t_0)\tau(u^\alpha_0) = \text{const.}.$$

Therefore we have the additional hypersurfaces

$$\widetilde{W}^m_{p_0}(t) = T_{(1-t)\tau(p_0)}\, W^m(t) \qquad 0 \leqq t \leqq 1$$

for all hypersurfaces in the family, which pass through the corresponding point $\bar{p}_0 \in \overline{W}^m$. Thus we can consider $\widetilde{W}^m_p(t) = T_{(1-t)\tau(\ )}\, W^m(t)$ for each point $p \in W^m$, which pass through the corresponding point $\bar{p} \in \overline{W}^m$, and the normal unit vector $\tilde{n}^i_p(t)$ of $\widetilde{W}^m_p(t)$ at $\bar{p}$.

Let us give henceforth the derivative with respect to t by the dash. We shall calculate $\tilde{n}'^i_p(t)$. Then $\bar{g}_{ij}$ being the metric tensor of S^{m+1} at $\bar{p}$ and differentiating the following relations with respect to t,

$$\bar{g}_{ij}\tilde{n}^i_p(t)\frac{\partial \tilde{x}^j_p(u, t)}{\partial u^\alpha} = 0, \quad \bar{g}_{ij}\tilde{n}^i_p(t)\tilde{n}^j_p(t) = 1, \qquad 0 \leqq t \leqq 1$$

since $\bar{g}_{ij}$ is independent with respect to t, we have

$$\bar{g}_{ij}\tilde{n}'^i_p(t)\frac{\partial \tilde{x}^j_p(u, t)}{\partial u^\alpha} + \bar{g}_{ij}\tilde{n}^i_p(t)\frac{d}{dt}\left(\frac{\partial \tilde{x}^j_p(u, t)}{\partial u^\alpha}\right) = 0, \tag{1.6}$$

$$\bar{g}_{ij}\tilde{n}^i_p(t)\tilde{n}'^j_p(t) = 0. \tag{1.7}$$

From (1.6), (1.7) and

$$\frac{d}{dt}\left(\frac{\partial \tilde{x}^i_p(u, t)}{\partial u^\alpha}\right) = \frac{d}{dt}\left(\frac{\partial x^i(u, t)}{\partial u^\alpha}\right) = \delta^i_1 \tau_\alpha,$$

we obtain

$$\tilde{n}'^i_p(t) = -\tilde{g}^{\alpha\beta}_p(t)\tau_\alpha \delta^l_1 \tilde{n}_{pl}(t)\frac{\partial \tilde{x}^i_p(u, t)}{\partial u^\beta} \tag{1.8}$$

where $\tilde{g}^{\alpha\beta}_p(t)$ is the contravariant metric tensor of $\widetilde{W}^m_p(t)$ and τ_α means $\partial\tau/\partial u^\alpha$. Throughout this paper repeated lower case Latin indices call for summation 1 to $m+1$ and repeated lower case Greek indices for smmation 1 to m; but p is not a summation index. And also for the covariant differential of $\tilde{n}'^i_p(t)$ along $\widetilde{W}^m_p(t)$ at $\bar{p}$, we get

$$\delta\tilde{n}'^i_p(t) = d\tilde{n}'^i_p(t) + \bar{\Gamma}^i_{jl}\tilde{n}'^j_p(t)\tilde{x}^l_{p\gamma}du^\gamma,$$

where $\bar{\Gamma}^i_{jk}$ is the christoffel symbol with respect to the metric tensor of

S^{m+1} at $\bar{p}$ and $\tilde{x}^l_{p\gamma}$ means $\dfrac{\partial \tilde{x}^l_p(u,t)}{\partial u^\gamma}$.

Calculating $(\delta \tilde{n}^i_p(t))'$, we have

$$\delta \tilde{n}^i_p t) = d\tilde{n}^i_p(t) + \Gamma^i_{jl}\tilde{n}^j_p(t)\tilde{x}^l_{p\gamma}du^\gamma ,$$

$$\left(\delta \tilde{n}^i_p(t)\right)' = \left(d\tilde{n}^i_p(t)\right)' + \Gamma^i_{jl}\tilde{n}'^j_p(t)\tilde{x}^l_{p\gamma}du^\gamma + \Gamma^i_{jl}\tilde{n}^j_p(t)(\tilde{x}^l_{p\gamma})'\, du^\gamma$$

because of Γ^i_{jk} is independent with respect to t. Consequently we get the following relation between $\delta \tilde{n}'^i_p$ and $(\delta \tilde{n}^i_p)'$

$$(\delta \tilde{n}^i_p)' = \delta \tilde{n}'^i_p + \Gamma^i_{j1}\tilde{n}^j_p \tau_\gamma du^\gamma \tag{1.9}$$

because of $d\tilde{n}'^i_p = (d\tilde{n}^i_p)'$.

We consider the following differential form of degree $m-1$ attached to each point on the hypersurface $\widetilde{W}^m_p(t)$

$$\begin{aligned}
&\Big((\tilde{n}'_p,\ \delta_1\tau,\ \underbrace{\delta\tilde{n}_p, \cdots, \delta\tilde{n}_p}_{r-1},\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big) \\
&\overset{\text{def.}}{=} \sqrt{g}\,(\tilde{n}'_p,\ \delta_1\tau,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p) \\
&= (-1)^{r-1}\sqrt{g}\,(\tilde{n}'_p,\ \delta_1\tau,\ \tilde{x}_{p\alpha_1}, \cdots, \tilde{x}_{p\alpha_{r-1}},\ \tilde{x}_{p\beta_r}, \cdots, \tilde{x}_{p\beta_{m-1}}) \\
&\quad \times \tilde{b}^{\alpha_1}_{p\beta_1}(t)\cdots \tilde{b}^{\alpha_{r-1}}_{p\beta_{r-1}}(t)du^{\beta_1}\Lambda\cdots\Lambda du^{\beta_r}\Lambda\cdots\Lambda du^{\beta_{m-1}}
\end{aligned} \tag{1.10}$$

where g is the determinant of the metric tensor g_{ij} of S^{m+1}, the symbol ( ) means a determinant of order $m+1$ whose columns are the components of respective vectors and $\tilde{b}_{p\alpha\beta}(t)$ is the second fundamental tensor of $\widetilde{W}^m_p(t)$ and $\tilde{b}^\beta_{p\alpha}(t)$ denotes $\tilde{b}_{p\alpha\gamma}(t)\tilde{g}^{\beta\gamma}_p(t)$.

Then the exterior differential of the differential form (1.10) becomes as follows

$$\begin{aligned}
&d\Big((\tilde{n}'_p,\ \delta_1\tau,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big) \\
&= \Big((\delta\tilde{n}'_p,\ \delta_1\tau,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big) \\
&\quad + \Big((\tilde{n}'_p,\ \delta(\delta_1)\tau,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big) \\
&\quad + \Big((\tilde{n}'_p,\ \delta_1 d\tau,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big)
\end{aligned} \tag{1.11}$$

because since S^{m+1} is a space of constant curvature, we have

$$\Big((\tilde{n}'_p,\ \delta_1\tau,\ \delta\delta\tilde{n}_p,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big) = 0 .$$

From that the quantity $\tilde{n}_{pi}(t)\delta^i_1\sqrt{\tilde{g}^*_p(t)}$ is independent with respect to t, where $\tilde{g}^*_p(t)$ is the determinant of $\tilde{g}_{p\alpha\beta}(t)$, we see

$$r\Big((\delta_1\tau,\ (\delta\tilde{n}_p)',\ \delta\tilde{n}_p,\ \cdots,\delta\tilde{n}_p,\ d\tilde{x}_p,\ \cdots,d\tilde{x}_p)\Big) \tag{1.12}$$
$$=(-1)^r m!\,\tilde{H}'_{pr}\tilde{n}_{pi}\delta_1^i\tau d\tilde{A}_p(t)$$

where $\tilde{H}_{pr}(t)$ and $d\tilde{A}_p(t)$ are the r-th mean curvature and the area element of $\tilde{W}_p^m(t)$ respectively, and using (1.8), we have

$$\Big((\tilde{n}'_p,\ \delta_1 d\tau,\ \delta\tilde{n}_p,\ \cdots,\delta\tilde{n}_p,\ d\tilde{x}_p,\ \cdots,d\tilde{x}_p)\Big)$$
$$=(-1)^{r-1}\tilde{g}_p^{\alpha\beta}(t)\tau_\alpha\tilde{n}_{pi}(t)\delta_1^i$$
$$\times\Big((\delta_1\tau_\gamma,\ \tilde{x}_{p\beta},\ \tilde{x}_{p\alpha_1},\ \cdots,\tilde{x}_{p\alpha_{r-1}},\ \tilde{x}_{p\alpha_r},\ \cdots,\tilde{x}_{p\alpha_{m-1}})\Big) \tag{1.13}$$
$$\times\tilde{b}^{\alpha_1}_{p\beta_1}(t)\cdots\tilde{b}^{\alpha_{r-1}}_{p\beta_{r-1}}du^\gamma\wedge du^{\beta_1}\wedge\cdots\wedge du^{\beta_{r-1}}\wedge du^{\alpha_r}\wedge\cdots\wedge du^{\alpha_{m-1}}$$
$$=(-1)^{r-1}\tilde{g}_p^{\alpha\beta}(t)\tilde{\varepsilon}_{p\beta\alpha_1\cdots\alpha_{r-1}\alpha_r\cdots\alpha_{m-1}}\tilde{\varepsilon}_p^{\gamma\beta_1\cdots\beta_{r-1}\alpha_r\cdots\alpha_{m-1}}$$
$$\times\tilde{b}^{\alpha_1}_{p\beta_1}(t)\cdots\tilde{b}^{\alpha_{r-1}}_{p\beta_{r-1}}(t)(\tilde{n}_{pi}(t)\delta_1^i)^2\tau_\alpha\tau_\beta d\tilde{A}_p(t)$$

because since $\varepsilon_{i_1\cdots i_{m+1}}$ and $\tilde{\varepsilon}_{p\alpha_1\cdots\alpha_m}$ are the ε-symbols of S^{m+1} and $\tilde{W}_p^m(t)$,

$$\varepsilon_{i_1\cdots i_{m+1}}\overset{\text{def.}}{=}\sqrt{g}\ e_{i_1\cdots i_{m+1}},\quad \tilde{\varepsilon}_{p\alpha_1\cdots\alpha_m}\overset{\text{def.}}{=}\sqrt{\tilde{g}_p^*(t)}\ e_{\alpha_1\cdots\alpha_m},$$

the symbol $e_{i_1\cdots i_{m+1}}$ meaning plus one or minus one, depending on whether the indices $i_1\cdots i_{m+1}$ denote an even permutation of $1,2,\cdots,m+1$ or odd permutation, and zero when at least any two indices have the same value, and also the symbol $e_{\alpha_1\cdots\alpha_m}$ meaning similarly for the indices $\alpha_1,\cdots,\alpha_m$ running from 1 to m, we have the relation

$$\tilde{n}_{pi}(t)\tilde{\varepsilon}_{p\alpha\alpha_1\cdots\alpha_{m-1}}(t)=\varepsilon_{ii_2\cdots i_{m+1}}(\bar{p})\tilde{x}^{i_2}_{p\alpha}\tilde{x}^{i_3}_{p\alpha_1}\cdots\tilde{x}^{i_{m+1}}_{p\alpha_{m-1}}. \tag{1.14}$$

On the other hand, from (1.9) we have

$$\Big((\delta\tilde{n}'_p,\ \delta_1\tau,\ \delta\tilde{n}_p,\ \cdots,\delta\tilde{n}_p,\ d\tilde{x}_p,\ \cdots,d\tilde{x}_p)\Big)$$
$$=\Big(((\delta\tilde{n}_p)',\ \delta_1\tau,\ \delta\tilde{n}_p,\ \cdots,\delta\tilde{n}_p,\ d\tilde{x}_p,\ \cdots,d\tilde{x}_p)\Big) \tag{1.15}$$
$$-\Big((\Gamma_{j1}\tilde{n}_p^j\tau_\gamma du^\gamma,\ \delta_1\tau,\ \delta\tilde{n}_p,\cdots,\delta\tilde{n}_p,\ d\tilde{x}_p,\cdots,d\tilde{x}_p)\Big).$$

Then putting

$$(m-1)!\,\tilde{C}^{\alpha\beta}_{p(r)}=\tilde{\varepsilon}^{\alpha}_{p\alpha_1\cdots\alpha_{r-1}\alpha_r\cdots\alpha_{m-1}}\tilde{\varepsilon}_p^{\beta\beta_1\cdots\beta_{r-1}\alpha_r\cdots\alpha_{m-1}}\tilde{b}^{\alpha_1}_{p\beta_1}(t)\cdots\tilde{b}^{\alpha_{r-1}}_{p\beta_{r-1}}(t)$$

and using (1.11), (1.12), (1.13), (1.14) and the relation

$$\delta(\delta_1^i)=\Gamma^i_{j1}\tilde{x}^j_{p\gamma}du^\gamma,$$

we have

$$
\begin{aligned}
d\Big(&(\tilde{n}'_p,\ \delta_1\tau,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big) \\
&= \frac{(-1)^{r-1}}{r} m!\, \widetilde{H}'_{pr}\tilde{n}_{pi}\delta^i_1\tau d\tilde{A}_p(t) \\
&\quad + (-1)^{r-1}(m-1)!\, \widetilde{C}^{\alpha\beta}_{p(r)}\tau_\alpha\tau_\beta(\tilde{n}_i(t)\delta^i_1)^2 d\tilde{A}_p(t) \\
&\quad + \Big((\tilde{n}'_p,\ \tau\overline{\Gamma}_{j1}\tilde{x}^j_{p\gamma}du^\gamma,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big) \\
&\quad - \Big((\overline{\Gamma}_{j1}\tilde{n}^j_p(t)\tau_\gamma du^\gamma,\ \delta_1\tau,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big).
\end{aligned}
$$

Furthermore we shall calculate the following quantity

$$
\begin{aligned}
&\Big((\tilde{n}'_p,\ \tau\overline{\Gamma}_{j1}\tilde{x}^j_{p\gamma}du^\gamma,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big) \\
&\quad - \Big((\overline{\Gamma}_{j1}\tilde{n}^j_p(t)\tau_\gamma du^\gamma,\ \delta_1\tau,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ dx_p, \cdots, d\tilde{x}_p)\Big).
\end{aligned} \tag{1.16}
$$

For the first term of (1.16), making use of (1.8) and from that $\widetilde{C}^{\alpha\beta}_{p(r)}$ is the symmetric tensor, we have the following

$$
\begin{aligned}
&\Big((\tilde{n}'_p,\ \tau\overline{\Gamma}_{j1}\tilde{x}^j_{p\gamma}du^\gamma,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big) \\
&= (-1)^{r-1}\tau\tilde{n}_{pl}(t)\delta^l_1\Big((\overline{\Gamma}_{j1}\tilde{x}^j_{p\gamma},\ \tilde{g}^{\alpha\beta}(t)\tau_\beta\tilde{x}_{p\alpha}, \\
&\qquad \tilde{x}_{p\alpha_1}, \cdots, \tilde{x}_{p\alpha_{r-1}},\ \tilde{x}_{p\alpha_r}, \cdots, \tilde{x}_{p\alpha_{m-1}})\Big) \\
&\qquad \times \tilde{b}^{\alpha_1}_{p\beta_1}(t)\cdots\tilde{b}^{\alpha_{r-1}}_{p\beta_{r-1}}(t)du^\gamma\wedge du^{\beta_1}\wedge\cdots\wedge du^{\beta_{r-1}}\wedge du^{\alpha_r}\wedge\cdots\wedge du^{\alpha_{m-1}} \\
&= (-1)^{r-1}\tau\tilde{n}_{pl}(t)\delta^l_1\overline{\Gamma}^i_{j1}\tilde{n}_{pi}\tilde{x}^j_{p\gamma}\tau_\beta\tilde{g}^{\beta\alpha}(t) \\
&\qquad \times \tilde{\varepsilon}_{p\alpha\alpha_1\cdots\alpha_{r-1}\alpha_r\cdots\alpha_{m-1}}\tilde{\varepsilon}_p^{\gamma\beta_1\cdots\beta_{r-1}\alpha_r\cdots\alpha_{m-1}}\tilde{b}^{\alpha_1}_{p\beta_1}(t)\cdots\tilde{b}^{\alpha_{r-1}}_{p\beta_{r-1}}(t)d\tilde{A}_p(t) \\
&= (-1)^{r-1}(m-1)!\,\tau\tilde{n}_{pl}(t)\delta^l_1\overline{\Gamma}_{ji1}\tilde{n}^i_p(t)\tilde{x}^j_{p(\gamma}\tau_{\beta)}\widetilde{C}^{\beta\gamma}_{p(r)}(t)d\tilde{A}_p(t),
\end{aligned} \tag{1.17}
$$

where $\overline{\Gamma}_{ji1}$ is $\tilde{g}_{li}\overline{\Gamma}^l_{j1}$ and the symbol $(\gamma\beta)$ means the symmetric part for indices γ and β.

On the other hand, putting the vector δ^i_1 by the following expession

$$
\delta^i_1 = \Big(\tilde{n}_{pl}(t)\delta^l_1\Big)\tilde{n}^i_p + \tilde{\varphi}^\beta_p(t)\tilde{x}^i_{p\beta}
$$

for the second term of (1.16), we have

$$
\begin{aligned}
&-\Big((\overline{\Gamma}_{j1}\tilde{n}^j_p(t)\tau_\gamma du^\gamma,\ \delta_1\tau,\ \delta\tilde{n}_p, \cdots, \delta\tilde{n}_p,\ d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big) \\
&\quad = -(-1)^{r-1}\tau\Big\{n_{pl}(t)\delta^l_1\Big((\overline{\Gamma}_{j1}\tilde{n}^j_p(t)\tau_\gamma,\ \tilde{n}_p,\ \tilde{x}_{p\alpha_1}, \cdots, \tilde{x}_{p\alpha_{r-1}},\ \tilde{x}_{p\alpha_r}, \cdots, \tilde{x}_{p\alpha_{m-1}})\Big) \\
&\quad + \tilde{\varphi}^\beta_p(t)\Big((\overline{\Gamma}_{j1}\tilde{n}^j_p(t)\tau_\gamma,\ \tilde{x}_{p\beta},\ \tilde{x}_{p\alpha_1}, \cdots, \tilde{x}_{p\alpha_{r-1}},\ \tilde{x}_{p\alpha_r}, \cdots, \tilde{x}_{p\alpha_{m-1}})\Big)\Big\} \\
&\quad \times \tilde{b}^{\alpha_1}_{p\beta_1}(t)\cdots\tilde{b}^{\alpha_{r-1}}_{p\beta_{r-1}}(t)du^\gamma\wedge du^{\beta_1}\wedge\cdots\wedge du^{\beta_{r-1}}\wedge du^{\alpha_r}\wedge\cdots\wedge du^{\alpha_{m-1}}.
\end{aligned}
$$

Let us take the relation (1.14) and

$$\tilde{\varepsilon}_{p\alpha\alpha_1\cdots\alpha_{m-1}}\tilde{g}_p^{\alpha\beta}(t)\tilde{x}^j_{p\beta}\bar{g}_{ij}=(-1)^m\bar{\varepsilon}_{ii_2\cdots i_{m+1}}\tilde{x}^{i_2}_{p\alpha_1}\cdots\tilde{x}^{i_m}_{p\alpha_{m-1}}\tilde{n}_p^{i_{m+1}},$$

where $\bar{\varepsilon}_{i_1i_2\cdots i_{m+1}}$ means the $\varepsilon_{i_1\cdots i_{m+1}}$ at $\bar{p}$. Then after some calculations, we get

$$\begin{aligned}
&-\Big((\bar{\Gamma}_{j1}\tilde{n}^j_p(t)\tau_\gamma du^\gamma,\ \delta_1\tau,\ \delta\tilde{n}_p,\cdots,\delta\tilde{n}_p,\ d\tilde{x}_p,\cdots,\delta\tilde{x}_p)\Big)\\
&\quad=(-1)^{r-1}(m-1)!\,\tau\Big\{\tilde{n}_{pl}(t)\delta^l_1\bar{\Gamma}_{ij1}\tilde{n}^i_p(t)\tilde{x}^j_{p(\beta}\tau_{\gamma)}\\
&\qquad-\bar{\Gamma}_{ji1}\tilde{n}^j_p(t)\tilde{n}^i_p(t)\tilde{\varphi}_{p(\beta}\tau_{\gamma)}\Big\}\tilde{C}^{\beta\gamma}_p(t)d\tilde{A}_p(t),
\end{aligned}\tag{1.18}$$

where $\tilde{\varphi}_{p\beta}$ is $\tilde{g}_{p\beta\alpha}(t)\tilde{\varphi}^\alpha_p$.

Thus from (1.17), (1.18) and $\bar{\Gamma}_{ji1}+\bar{\Gamma}_{ij1}=\left(\dfrac{\partial g_{ij}}{\partial x^1}\right)_{\bar{p}}$, we have

$$\begin{aligned}
&\Big((\tilde{n}'_p,\ \tau\bar{\Gamma}_{j1}\tilde{x}^j_{p\gamma}du^\gamma,\ \delta\tilde{n}_p,\cdots,\delta\tilde{n}_p,\ d\tilde{x}_p,\cdots,d\tilde{x}_p)\Big)\\
&\quad-\Big((\bar{\Gamma}_{j1}\tilde{n}^j_p(t)\tau_\gamma du^\gamma,\ \delta_1\tau,\ \delta\tilde{n}_p,\cdots,\delta\tilde{n}_p,\ d\tilde{x}_p,\cdots,d\tilde{x}_p)\Big)\\
&=(-1)^{r-1}(m-1)!\,\tau\Big\{\tilde{n}_{pl}(t)\delta^l_1(\bar{\Gamma}_{ji1}+\bar{\Gamma}_{ij1})\tilde{n}^i_p(t)\tilde{x}^j_{p(\gamma}\tau_{\beta)}\\
&\quad-\frac{1}{2}(\bar{\Gamma}_{ji1}+\bar{\Gamma}_{ij1})\tilde{n}^j_p(t)\tilde{n}^i_p(t)\tilde{\varphi}_{p(\beta}\tau_{\gamma)}\Big\}\tilde{C}^{\beta\gamma}_{p(r)}(t)d\tilde{A}_p(t)\\
&=(-1)^{r-1}(m-1)!\,\tau\Big\{\tilde{n}_{pl}(t)\delta^l_1\mathcal{L}\bar{g}_{ij}\tilde{n}^i_p(t)\tilde{x}^i_{p(\gamma}\tau_{\beta)}\\
&\quad-\frac{1}{2}\mathcal{L}\bar{g}_{ij}\tilde{n}^i_p(t)\tilde{n}^j_p(t)\tilde{\varphi}_{p(\beta}\tau_{\gamma)}\Big\}\tilde{C}^{\beta\gamma}_{p(r)}(t)d\tilde{A}_p(t),
\end{aligned}\tag{1.19}$$

where the symbol $\mathcal{L}$ means the Lie derivative and $\mathcal{L}\bar{g}_{ij}$ is the Lie derivative of g_{ij} at $\bar{p}$.

Now putting

$$\begin{aligned}
\tilde{S}_{p(r)}(t)=\tau\Big\{&\tilde{n}_{pl}(t)\delta^l_1\mathcal{L}\bar{g}_{ij}\tilde{n}^i_p(t)\tilde{x}^j_{p(\gamma}\tau_{\beta)}-\frac{1}{2}\mathcal{L}\bar{g}_{ij}\tilde{n}^i_p(t)\tilde{n}^j_p(t)\\
&\times\tilde{\varphi}_{p(\beta}\tau_{\gamma)}\Big\}\tilde{C}^{\beta\gamma}_{p(r)}(t)d\tilde{A}_p(t),
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{(-1)^{r-1}}{(m-1)!}d\Big((\tilde{n}'_p,\ \delta_1\tau,\ \delta\tilde{n}_p,\cdots,\delta\tilde{n}_p,\ d\tilde{x}_p,\cdots,d\tilde{x}_p)\Big)\\
&\quad=\frac{m}{r}\tilde{H}'_{pr}\tilde{n}_{pi}\delta^i_1\tau d\tilde{A}_p(t)\\
&\qquad+\frac{1}{\sqrt{\tilde{g}^*_p(t)}}\tilde{C}^{\alpha\beta}_{p(r)}(t)\tau_\alpha\tau_\beta(\tilde{n}_{pi}\delta^i_1)^2\sqrt{\tilde{g}^*_p(t)}\,d\tilde{A}_p(t)+\tilde{S}_{p(r)}(t).
\end{aligned}\tag{1.20}$$

Since $\tilde{n}_{pi}(t)\delta_1^i\sqrt{\tilde{g}_p^*(t)}$ at $\bar{p}$ is independent with respect to t, we can see the following

$$\tilde{n}_{pi}(t)\delta_1^i\sqrt{\tilde{g}_p^*(t)} = \bar{n}_i\delta_1^i\sqrt{\bar{g}^*}\ ,$$

where $\bar{n}_i$ is the normal unit vector of $\overline{W}^m$ at $\bar{p}$ and $\bar{g}^*$ is the determinant of the metric tensor $\bar{g}_{\alpha\beta}$ of $\overline{W}^m$.

Integrating both members of (1.20) at $\bar{p}$ over the interval $0\leqq t\leqq 1$, we obtain the following

$$\begin{aligned}
&\frac{r(-1)^{r-1}}{(m-1)!}\, d\int_0^1 \Big((\tilde{n}'_p, \delta_1\tau, \delta n_p, \cdots, \delta\tilde{n}_p, d\tilde{x}_p, \cdots, d\tilde{x}_p)\Big)dt \\
(1.21)\qquad &= m\Big(\overline{H}_r - \tilde{H}_{pr}(0)\Big)\bar{n}_i\delta_1^i\tau d\bar{A} \\
&\quad + \sqrt{\bar{g}^*}\int_0^1 \tilde{g}_p^*(t)^{-\frac{1}{2}}\tilde{C}^{\alpha\beta}_{p(r)}dt\tau_\alpha\tau_\beta(\bar{n}_i\delta_1^i)^2 d\bar{A} \\
&\quad + \int_0^1 \tilde{S}_{p(r)}(t)dt,
\end{aligned}$$

where $\overline{H}_r$ and $\tilde{H}_{pr}(0)$ are the r-th mean curvature of $\overline{W}_m$ and $\tilde{W}_p^m(0)$ respectively, and $d\bar{A}$ means the area element of $\overline{W}^m$. Thus we can see that the left hand member of (1.21) is the exterior differential of the differential form attached to each point on the hypersurface W^m. Let us denote a set of $\tilde{W}_p^m(t)$ for all $p\in W^m$ by $\tilde{W}^m(t)$ and sets of the quantities of $\tilde{W}_p^m(t)\in\tilde{W}^m(t)$ by $\tilde{H}_r(t)$, $\tilde{C}^{\alpha\beta}_{(r)}(t)$, $\tilde{S}_{(r)}(t)$, etc.. Then integrating both members of (1.21) over W^m and applying Stokes' theorem, since W^m is closed, we have

$$\begin{aligned}
&m\iint_{W^m}\Big(H_r - \tilde{H}_r(0)\Big)\bar{n}_i\delta_1^i\tau d\bar{A} \\
(1.22)\qquad &+\iint_{W^m}\sqrt{\bar{g}^*}\int_0^1 g^*(t)^{-\frac{1}{2}}\tilde{C}^{\alpha\beta}_{(r)}(t)dt\tau_\alpha\tau_\beta(\bar{n}_i\delta_1^i)^2 d\bar{A} \\
&+\iint_{W^m}\int_0^1 \tilde{S}_{(r)}(t)dt = 0\,.
\end{aligned}$$

(1.22) is the integral form relating to the r-th mean curvature defined by two hypersurfaces W^m and $\overline{W}^m$.

§2. **A main theorem.** We shall prove the following congruence theorem concerning the r-th mean curvature for closed hypersurfaces with the aid of the statements of the preceding section. We shall henceforth confine ourselves to two hypersurfaces W^m and $\overline{W}^m$ which do not contain a piece of a hypersurface covered by the orbits of the transformations, which is expressed by $f(x^2, \cdots, x^{m+1}) = 0$.

THEOREM 2.1. *If the hypersurfaces W^m and $\bar{W}^m$ in S^{m+1} are closod orientable, and if there exists the relation*

$$\bar{H}_r = \tilde{H}_r(0) \tag{2.1}$$

at corresponding points along the orbits of the transformations, and if the following conditions are satisfied

$$\iint_{W^m} \int_0^1 \tilde{S}_{(r)} dt \geqq 0 \tag{2.2}$$

and that in case of $r \geqq 2$, the second fundamental form of $\tilde{W}_p^m(t)$ is positive definite for all $p \in W^m$ and all t of the interval $0 \leqq t \leqq 1$, at the corresponding point $\bar{p}$, then W^m and $\bar{W}^m$ are congruent mod G to ecch other.

PROOF. Using the condition that in case of $r \geqq 2$, the second fundamental form of $\tilde{W}_p^m(t)$ is positive definite at $\bar{p}$, for all r, we have that the quantity

$$\sqrt{\bar{g}^*} \int_0^1 \tilde{g}^*(t)^{-\frac{1}{2}} \tilde{C}_{(r)}^{\alpha\beta} dt V_\alpha V_\beta$$

is positive definite. If we put the conditions (2.1) and (2.2) in the integral form (1.22), we get

$$\iint_{W^m} \sqrt{\bar{g}^*} \int_0^1 \tilde{g}^*(t)^{-\frac{1}{2}} \tilde{C}_{(r)}^{\alpha\beta}(t) dt \tau_\alpha \tau_\beta (\bar{n}_i \delta_1^i)^2 d\bar{A} = 0 .$$

From the hypersurfaces W^m and $\bar{W}^m$ do not contain a piece of a hypersurface covered by the orbits of transformations, that is, a point on $\bar{W}^m$ such that $\bar{n}_i \delta_1^i = 0$ must be an isolate point. Moreover since τ is a continuous function of W^m, we conclude

$$\tau = \text{constant}$$

at every point of W^m. This fact shows that W^m and $\bar{W}^m$ are congruent mod G.

REMARK 1. If G is a group of isometric transformations, then we have that $\tilde{S}_{(r)}(t) = 0$ and $\tilde{H}_r(0) = H_r$, etc., and Theorem 2.1 coincide with the theorem given in the introduction.

REMARK 2. In case of $r=1$, that is, the first mean curvature, we can cancel the assumption that our space S^{m+1} is of constant curvature.

Department of Mathematics
Hokkaido University

References

[1] H. HOPF and Y. KATSURADA: Some congruence theorems for closed hypersurfaces in Riemann spaces (Part III: Method based on Voss' proof), Comment. Math. Helv. 46, (1971), 478–486.

[2] Y. KATSURADA: A certatin congruence theorem for closed hypersurfaces in a space of constant curvature, to appear in Ann. di Mat. p. appl.

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