Iterated mixed problems for d'Alembertians

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§ 1. Introduction and main results

Let $\boldsymbol{R}_+^{n+1}$ be the open half space $\{(t, x); x=(x', x_n)=(x_1, \cdots, x_{n-1}, x_n), x_n>0\}$ with boundary $x_n=0$. By $(P, B_j; j=1, \cdots, l)$, briefly (P, B_j) we shall mean a mixed, or hyperbolic boundary value problem for a t-strictly hyperbolic operator P and boundary differential operators B_j:

$$P(t, x; D_t, D_x)u(t, x)=f(t, x) \quad \text{in } \boldsymbol{R}_+^{n+1},$$
$$B_j(t, x'; D_t, D_x)u(t, x', 0)=g_j(t, x') \quad (j=1, \cdots, l) \quad \text{on } \boldsymbol{R}^n.$$

Here $D_t=-i\dfrac{\partial}{\partial t}(i=\sqrt{-1})$, $D_k=-i\dfrac{\partial}{\partial x_k}$ and $D_x=(D_1, \cdots, D_n)$. Throughout this paper we assume that all the coefficients of P and B_j are C^∞ and constant outside a compact subset of $\boldsymbol{R}^{n+1}$. Moreover, Q^0 denotes the principal part of a differential operator Q and (τ, σ, λ) denote the dual variables of (t, x', x_n) respectively.

Let P_j^0 $(j=1, \cdots, m)$ be d'Alembertians:

$$P_j^0(t, x; \tau, \sigma, \lambda)=-\tau^2+a_j(t, x)^2\left(\lambda^2+\sum_{k=1}^{n-1}\sigma_k^2\right),$$
$$0<a_m(t, x)<\cdots<a_1(t, x)$$

and let B_j $(j=1, \cdots, m)$ be boundary differential operators of first order:

$$B_j^0(t, x'; \tau, \sigma, \lambda)=\lambda-\sum_{k=1}^{n-1} b_{jk}(t, x')\sigma-c_j(t, x')\tau,$$

where it will be assumed, unless otherwise indicated, that the $b_{jk}(t, x')$, $c_j(t, x')$ are real valued. Then for a permutation $\chi=\begin{pmatrix}1, & \cdots, & m\\ j_1, & \cdots, & j_m\end{pmatrix}$ a mixed problem $(P, {}^\chi B_j)=(P, {}^\chi B_j; j=1, \cdots, m)$ is said to be an iterated mixed, or boundary value problem, if the symbols of P^0 and ${}^\chi B_j^0$ have the following forms:

$$P^0(t, x; \tau, \sigma, \lambda)=\prod_{j=1}^m P_j^0(t, x; \tau, \sigma, \lambda),$$
$${}^\chi B_1^0(t, x'; \tau, \sigma, \lambda)=B_{j_1}^0(t, x'; \tau, \sigma, \lambda),$$
$${}^\chi B_k^0(t, x'; \tau, \sigma, \lambda)=B_{j_k}^0(t, x'; \tau, \sigma, \lambda)\prod_{h=1}^{k-1} P_{j_h}^0(t, x; \tau, \sigma, \lambda), \qquad (k=2, \cdots, m).$$

An iterated mixed problem $(P, {}^{\chi}B_j)$ can be formally written the iterated problems for d'Alembertians:

$$\begin{cases} P_{j_1}u = v_1 \\ B_{j_1}u = g_1 \end{cases}, \quad \begin{cases} P_{j_2}v_1 = v_2 \\ B_{j_2}v_1 = g_2 \end{cases}, \cdots, \quad \begin{cases} P_{j_m}v_{m-1} = f \\ B_{j_m}v_{m-1} = g_m \end{cases}.$$

The purpose of this paper is to study the iterated mixed problems.

In the paper [9] Sakamoto treated the problem of the following type: If (P_j, B_j) are L^2-well posed(1) then is $(P, \hat{B}_j)$ so? Here $\hat{B}^0_j = B^0_j Q^0_j$ and $Q^0_j = \prod_{k \neq j} P^0_k$. However, this does not contain Neumann problem $(P, D_n^{2j-1}; j=1, \cdots, m)$ which is important and critical. Our results will show in particular that Neumann problem occupies a very critical position in our iterated mixed problems.

Now we shall state main results. Let $(P^0, B^0_j)_{(t,x')}$ denotes a constant coefficient problem resulting from freezing the coefficients at a boundary point $(t, x', 0)$. Then we have the following.

THEOREM 1. *Suppose that an iterated mixed problem $(P, {}^{\chi}B_j)$ is L^2-well posed. Then every frozen problem $(P^0_j, B^0_j)_{(t,x')}$ is also L^2-well posed for any $j=1, \cdots, m$, and furthermore it holds for every (t, x') and every pair (j_k, j_{k+1}) $(k=1, \cdots, m-1)$ that if $(P^0_{j_k}, B^0_{j_k})_{(t,x')}$ $((P^0_{j_{k+1}}, B^0_{j_{k+1}})_{(t,x')})$ does not satisfy the uniform Lopatinski condition,(2) $(P^0_{j_{k+1}}, B^0_{j_{k+1}})_{(t,x')}$ $((P^0_{j_k}, B^0_{j_k})_{(t,x')})$ must be Neumann problem, that is, $B^0_{j_{k+1}}(B^0_{j_k}) = D_n$, corresponding to $j_k < j_{k+1}$ $(j_{k+1} < j_k)$.*

In order to reformulate Theorem 1 we shall classify L^2-well posed mixed problems (P, B) of second order with constant coefficients. We say that (P, B) is of type U if it satisfies the uniform Lopatinski condition and, among other L^2-well posed problems, Neumann problem or another problem is of type N or NU respectively. Moreover, we call (P, B) to be of type $\overline{NU}$ if it is of type N or NU and call for convenience every L^2-well posed problem to be of type $\overline{U}$. For example, let $P^0(D_t, D_x) = -D_t^2 + a^2(D_1^2 + \cdots + D_n^2)$ and $B^0(D_t, D_x) = D_n - \sum_{k=1}^{n-1} b_k D_k - cD_t$. Then (P, B) is L^2-well posed (of type $\overline{U}$) if and only if $ac \geqq \sqrt{b_1^2 + \cdots + b_{n-1}^2}$, and it is of type U, NU, N if $ac > \sqrt{b_1^2 + \cdots + b_{n-1}^2}$, $ac = \sqrt{b_1^2 + \cdots + b_{n-1}^2}$ and $c \neq 0$, $c = b_1 = \cdots = b_{n-1} = 0$, respectively. Regarding $(c, b, \cdots, b_{n-1})$ as $(t, x_1, \cdots, x_{n-1}, 0)$, this shows that the set of L^2-well posed mixed problems, i.e. the closed cone with vertex at the origin, coincides with the section of the propagation cone for P by the boundary $x_n = 0$. For these facts see § 4.

An ordered set $((P_1, B_1), \cdots, (P_m, B_m))$ of the constant coefficient problems

(1), (2); For the definitions see § 2.

(P_j, B_j) of second order is said, for instance, to be of type $(U, \cdots, U)$ if all the (P_j, B_j) are of type U. When the types of (P_j, B_j) are mixed, we define in a similar way a type of $((P_1, B_1), \cdots, (P_m, B_m))$. Then we have

THEOREM 2. *Suppose that a permutation* χ *is the unit of the permutation group. Then if an iterated mixed problem* $(P, {}^{\chi}B_j)$ *is* L^2*-well posed, for any fixed* (t, x') *a type of* $((P_1^0, B_1^0)_{(t,x')}, \cdots, (P_m^0, B_m^0)_{(t,x')})$ *then becomes one of the following m-types*:

$$\begin{gathered}(U, \cdots\cdots\cdots\cdots, U, \bar{U}),\\ (U, \cdots\cdots, U, \overline{NU}, N),\\ \vdots\\ (\overline{NU}, N, \cdots\cdots\cdots, N).\end{gathered}$$

Moreover, this condition is sufficient to be L^2*-well posed in the case of constant coefficients and of two space variables.*

The condition in Theorem 2 is not enough to be L^2-well posed for a general permutation χ, for example, $\chi=\begin{pmatrix}1, 2, 3\\ 2, 1, 3\end{pmatrix}(m=3)$. We shall discuss this problem in the succeeding article.

We shall next consider the problem presented in [2]: If every frozen problem $(P^0, B_j^0)_{(t,x')}$ is L^2-well posed then so is a variable coefficient problem (P, B_j)? It seems much difficult to solve in general this problem. A case where we can give an affirmative answer to this problem is as follows:

$$B_j^0(t, x'; D_t, D_x) = B^0(t, x'; D_t, D_x) \qquad \text{for any } j=1, \cdots, m.$$

In this case all the iterated mixed problems $(P, {}^{\chi}B_j)$ are equivalent to the problem $(P, \tilde{B}_j)$ with

$$\tilde{B}_j^0(t, x'; \tau, \sigma, \lambda) = B^0(t, x'; \tau, \sigma, \lambda)\lambda^{2j-2} \qquad (j=1, \cdots, m).$$

Then we have the following

THEOREM 3. *A mixed problem* $(P, \tilde{B}_j)$ *is* L^2*-well posed if and only if every frozen problem* $(P_m^0, B^0)_{(t,x')}$ *is* L^2*-well posed.*

The condition in Theorem 3 can be stated in the other words: Every $(P_j^0, B^0)_{(t,x')}$ is L^2-well posed for any $j=1, \cdots, m$, or every $(P^0, \tilde{B}_j^0)_{(t,x')}$ is so. In this case a type of $((P_1, B_1)_{(t,x')}, \cdots, (P_m, B_m)_{(t,x')})$ is one of only two types:

$$\begin{gathered}(U, \cdots, U, \bar{U})\\ (N, \cdots, N, N).\end{gathered}$$

A partial result analogue to Theorem 3, where the coefficients of B^0 are complex valued, is given in § 9.

Another approach to solve our problem is to assume that a type of $((P_1, B_1)_{(t,x')}, \cdots, (P_m, B_m)_{(t,x')})$ is independent of (t, x'). If $((P_1, B_1)_{(t,x')}, \cdots, (P_m, B_m)_{(t,x')})$ is of type $(N, \cdots, N)$ uniformly in (t, x') then the problem becomes Neumann problem for iterated d'Alemdertians and if it is of type $(U, \cdots, U)$ uniformly in (t, x') then $(P, {}^{\chi}B_j)$ for every χ becomes L^2-well posed problem satisfying the uniform Lopatinski condition. The former was solved in [8] by the semigroup method when the coefficients are independent of t (For a general P see e.g. [13]). The latter is a special case of [5] or [10].

In this direction we have the following

THEOREM 4. *If $((P_1^0, B_1^0)_{(t,x')}, \cdots, (P_m^0, B_m^0)_{(t,x')})$ is of type $(U, \cdots, U, \overline{U})$ uniformly in (t, x'), then an iterated mixed problem $(P, {}^{\chi}B_j)$ for any permutation χ is L^2-well posed.*

The statement in Theorem 4 is also valid, if all the coefficients of B_j^0 $(j=1, \cdots, m)$ are complex valued.

For simplicity we confined ourselves here to problems in the half space, but our results can be extended to the case of general domains with smooth boundary.

The contents of this paper are as follows. An L^2-well posed mixed problem is defined in § 2. A characterization of L^2-well posed mixed problem with constant coefficients, obtained by applying results in [3] to our iterated mixed problem, is given in § 3. In § 4 we give characterizations of L^2-well posed mixed problem for d'Alembertian. They are special cases of results in [1], [2], [4], [12]. Theorems 1, 2, 3 and 4 are proved in § 5, 6, 7 and 8, respectively. An analogue to Theorem 3, where the coefficients of B^0 are complex valued, is given in § 9.

§ 2. Definition of L^2-well posedness

Throughout this paper we use the following function spaces with non-zero parameter γ:

$$H_{k,\gamma}(\boldsymbol{R}_+^{n+1}) = \left\{u(t, x);\ e^{-\gamma t}u(t, x) \in H^k(\boldsymbol{R}_+^{n+1})^{(3)}\right\} \qquad (k \geqq 0\,;\ \text{integer}),$$

$$H_{s,\gamma}(\boldsymbol{R}^n) = \left\{u(t, x');\ e^{-\gamma t}u(t, x') \in H^s(\boldsymbol{R}^n)^{(4)}\right\} \qquad (s\,;\ \text{real})$$

with norms defined by

$$\|u\|_{k,\gamma}^2 = \sum_{j+l+|\alpha|=k} \int_{\boldsymbol{R}_+^{n+1}} |e^{-\gamma t}\gamma^j D_t^l D_x^\alpha u(t, x)|^2 dtdx\,,$$

$$\langle\!\langle u \rangle\!\rangle_{s,\gamma}^2 = \int_{\boldsymbol{R}^n} |e^{-\gamma t} \Lambda^s u(t,x')|^2 dtdx'$$

(3), (4); They are the usual Sobolev spaces.

respectively, where

$$\Lambda^s u(t, x') = (2\pi)^{-n} \int_{R^n} e^{i\tau t + i\sigma x'} \Lambda(\xi, \sigma, \gamma)^s \hat{u}(\tau, \sigma) d\xi d\sigma ,$$

$$\hat{u}(\tau, \sigma) = \int_{R^n} e^{-i\tau t - i\sigma x'} u(t, x') dt dx' ,$$

$$\Lambda(\xi, \sigma, \gamma) = (|\tau|^2 + |\sigma|^2)^{\frac{1}{2}}, \quad |\sigma|^2 = \sigma_1^2 + \cdots + \sigma_{n-1}^2 ,$$

$$\tau = \xi - i\gamma , \quad \sigma x' = \sigma_1 x_1 + \cdots + \sigma_{n-1} x_{n-1} , \quad \sigma \in \boldsymbol{R}^{n-1} .$$

For a non-negative integer s, $\langle\!\langle u \rangle\!\rangle_{s,\gamma}^2$ is equivalent to

$$\sum_{j+l+|\alpha'|=s} \int_{R^n} |e^{-\gamma t} \gamma^j D_t^l D_{x'}^{\alpha'} u(t, x')|^2 dt dx' ,$$

where

$$\alpha = (\alpha', \alpha_n) = (\alpha_1, \cdots, \alpha_n) \ (\alpha_j \geqq 0 ; \text{ integer}),$$

$$D_x^\alpha = D_{x'}^{\alpha'} D_n^{\alpha_n} = D_1^{\alpha_1} \cdots D_n^{\alpha_n} .$$

Moreover, we use the following operator introduced in [7]:

$$\Lambda_{x'}^{\frac{1}{2}} u(t, x') = (2\pi)^{-n} \int_{R^n} e^{i\tau t + i\sigma x'} (\gamma^2 + |\sigma|^2)^{\frac{1}{4}} \hat{u}(\tau, \sigma) d\xi d\sigma .$$

Let the orders of P and B_j be m and $m_j < m$ respectively, let the m_j be mutually distinct and the boundary $x_n = 0$ be non-characteristic for P and B_j. Then we see from the hyperbolicity of P that

$$P^0(t, x ; \tau, \sigma, \lambda) = \prod_{j=1}^{l} \Big(\lambda - \lambda_j^+(t, x ; \tau, \sigma, \lambda)\Big) \prod_{k=1}^{m-l} \Big(\lambda - \lambda_k^-(t, x ; \tau, \sigma)\Big)$$

where $\operatorname{Im} \lambda_j^+ > 0$ ($\operatorname{Im} \lambda_k^- < 0$) if $\operatorname{Im} \tau = -\gamma < 0$ and l is independent of $(t, x ; \tau, \sigma)$ ($\operatorname{Im} \tau < 0$). The number of boundary operators B_j is assumed to be equal to l.

Definition. *A mixed problem $(P, B_j ; j = 1, \cdots, l)$ is said to be L^2-well posed, if there exist positive constants C and γ_0 such that for every $\gamma \geqq \gamma_0$ and $f \in H_{1,\gamma}(\boldsymbol{R}_+^{n+1})$ the problem (P, B_j) with $g_j \equiv 0$ has a unique solution $u \in H_{m,\gamma}(\boldsymbol{R}_+^{n+1})$, which satisfies*

$$\gamma^2 \|u\|_{m-1,\gamma}^2 \leqq C \|f\|_{0,\gamma}^2 . \tag{2.1}$$

This definition is clearly equivalent to the one in [3], where P^0 and B_j^0 are with constant coefficients. Furthermore it is also equivalent to them in [2] and [6]. This fact follows from Proposition 2.2 below.

Let $R(t, x' ; \tau, \sigma)$ denote Lopatinski determinant for $(P^0, B_j^0)_{(t,x')}$:

$$R(t, x'; \tau, \sigma)=B(t, x'; \tau, \sigma)/\Delta(t, x'; \tau, \sigma),$$

$$B(t, x'; \tau, \sigma)=det\left(B_k(t, x'; \tau, \sigma, \lambda_j^+(t, x', 0; \tau, \sigma))_{k\downarrow}^{j\rightarrow}1, \cdots, l\right),$$

$$\Delta(t, x'; \tau, \sigma)=\prod_{j<k}\left(\lambda_j^+(t, x', 0; \tau, \sigma)-\lambda_k^+(t, x', 0; \tau, \sigma)\right),$$

$$(\sigma \in \boldsymbol{R}^{n-1}, \quad \tau=\xi-i\gamma, \quad \gamma \geqq 0, \quad |\tau|^2+|\sigma|^2=1).$$

Here R is continuously extended to $\gamma \geqq 0$. Then we say that (P, B_j) satisfies *the uniform Lopatinski condition* if $R(t, x'; \tau, \sigma) \neq 0$ for any $(t, x'; \tau, \sigma)$ with $\operatorname{Im} \tau=-\gamma \leqq 0$. It is shown in [5] or [10] that a problem (P, B_j) with uniform Lopatinski condition is L^2-well posed. Their proofs give in particular the following.

LEMMA 2.1. *Let $R(t_0, x_0'; \tau_0, \sigma_0) \neq 0$ for some $(t_0, x_0'; \tau_0, \sigma_0)$ and let $\beta'(t, x; \xi, \sigma, \gamma)$ $(\xi^2+\gamma^2+|\sigma|^2=1)$ be C^∞ in all variables and with its support contained in a compact neighbourhood of $(t_0, x_0', 0; \tau_0, \sigma_0)$. Then there exist positive constants C and γ_0 such that for any $u \in H_{m,\gamma}(\boldsymbol{R}_+^{n+1})$*

$$\gamma^2\|\beta u\|_{m-1,\gamma}^2+\gamma \sum_{j=1}^{m-1}\langle\langle D_n^j \beta u\rangle\rangle_{m-1-j,\gamma}^2 \leqq C\left(\|P u\|_{0,\gamma}^2+\gamma \sum_{j=1}^{l}\langle\langle B_j u\rangle\rangle_{m-m_j-1,\gamma}^2+\gamma\|u\|_{m-1,\gamma}^2\right),$$

where

$$\beta u(t, x)=\beta(D_t, D_x, \gamma) u =\frac{1}{(2\pi)^n} \int_{R^n} e^{i\tau t+i\sigma x'} \beta(t, x; \xi, \sigma, \gamma) \hat{u}(\tau, \sigma, x_n) d\xi d\sigma,$$

$$\beta(t, x; \xi, \sigma, \gamma)=\beta'(t, x; \Lambda^{-1}\xi, \Lambda^{-1}\sigma, \Lambda^{-1}\gamma)$$

Making use of results in [6] and Lemma 2.1 we have

PROPOSITION 2.2. *A mixed problem (P, B_j) is L^2-well posed if and only if there exist positive constants C and γ_0 such that for every $\gamma \geqq \gamma_0$, $f \in H_{1,\gamma}(\boldsymbol{R}_+^{n+1})$ and $\Lambda_{x'}^{\frac{1}{2}} g_j \in H_{m-m_j,\gamma}(\boldsymbol{R}^n)$ the problem (P, B_j) has a unique solution $u \in H_{m,\gamma}(\boldsymbol{R}_+^{n+1})$, which satifies*

$$\gamma^2\|u\|_{m-1,\gamma}^2+\gamma^2 \sum_{j=1}^{m-1}\langle\langle\Lambda_{x'}^{-\frac{1}{2}} D_n^j u\rangle\rangle_{m-j-1,\gamma} \leqq C\left(\|f\|_{0,\gamma}^2+\sum_{j=1}^{l}\langle\langle\Lambda_{x'}^{\frac{1}{2}} g_j\rangle\rangle_{m-m_j-1,\gamma}^2\right). \tag{2.2}$$

Moreover, a priori estimate of higher order holds for $\gamma \geqq \gamma_k$ and $u \in H_{m+k,\gamma}(\boldsymbol{R}_+^{n+1})$ $(k \geqq 0$, integer$)$:

$$\gamma^2\|u\|^2_{m-1+k,\gamma}+\gamma^2\sum_{j=0}^{m-1}\langle\!\langle \Lambda_{x'}^{-\frac{1}{2}} D_n^j u\rangle\!\rangle_{m-1-m_j+k,\gamma} \tag{2.3}$$
$$\leqq C_k\Big(\|Pu\|^2_{k,\gamma}+\sum_{j=1}^{l}\langle\!\langle \Lambda_{x'}^{\frac{1}{2}} B_j u\rangle\!\rangle^2_{m-m_j-1+k,\gamma}\Big)$$

where γ_k, C_k are some positive constants.

PROOF. The results in [6] show that the statement of Proposition 2.2 is valid if $\Lambda_{x'}$ is replaced by Λ. The symbols of Λ and $\Lambda_{x'}$ are equivalent in $|\sigma|>\delta\Lambda$ ($\delta>0$) and is not equivalent elsewhere. But we see from [2], Theorem 1 and [12], Theorem 3 that $R(t, x'; \tau, \sigma)\neq 0$ in $|\sigma|<\delta\Lambda$, taking δ small. Hence, using partition of unity and Lemma 2.1, we obtain the estimate (2.2).

The estimate (2.2) was shown in [7] for L^2-well posed problems of second order with real boundary conditions.

We finally remark that the solution u of an L^2-well posed problem (P, B_j) has zero initial data on $t=0$ provided $f=0$ and $g_j=0$ in $t<0$ (for instance see [6]).

§ 3. A characterization of an L^2-well posed problem with constant coefficients

In this section and the succeeding sections we assume that P and B_j ($j=1, \cdots, m$) have the form in question:

$$P^0(t, x; \tau, \sigma, \lambda)=\prod_{j=1}^{m} P_j^0(t, x, \tau, \sigma, \lambda),$$
$$P_j^0(t, x; \tau, \sigma, \lambda)=-\tau^2+a_j(t, x)^2\Big(\lambda^2+\sum_{k=1}^{n-1}\sigma_k^2\Big),$$
$$\big(0<a_m(t, x)<\cdots<a_1(t, x)\big),$$
$$B_j^0(t, x'; \tau, \sigma, \lambda)=\lambda^2-\sum_{k=1}^{n-1} b_{jk}(t, x')\sigma_k-c(t, x')\tau .$$

Furthermore we shall drop the variables (t, x) for the sake of simplicity, since we consider frozen problems.

We recall Lopatinski determinant and reflection coefficients for a mixed problem. Let ${}^{\chi}R$ and R_j denote Lopatinski determinants for an iterated mixed problem (P^0, ${}^{\chi}B_j^0$) and second order problems (P_j^0, B_j^0) respectively:

$$R_j(\tau, \sigma)=B_j^0\big(\tau, \sigma, \lambda_j^+(\tau, \sigma)\big)$$
$$=\lambda_j^+(\tau, \sigma)-\sum_{k=1}^{n-1} b_{jk}\sigma_k-c_j\tau , \tag{3.1}$$
$${}^{\chi}R(\tau, \sigma)={}^{\chi}B(\tau, \sigma)/\Delta(\tau, \sigma),$$
$$(\tau=\xi-i\gamma ,\ \gamma\geqq 0 ,\ \sigma\in \boldsymbol{R}^n ,\ |\tau|^2+|\sigma|^2=1)$$

where

$$P_j^0(\tau, \sigma, \lambda) = a_j^2\left(\lambda - \lambda_j^+(\tau, \sigma)\right)\left(\lambda - \lambda_j^-(\tau, \sigma)\right),$$

$$^{\chi}B(\tau, \sigma) = \det\left(^{\chi}B_k^0(\tau, \sigma, \lambda_j^+(\tau, \sigma))_{k\downarrow}^{j\rightarrow} 1, \cdots, m\right),$$

$$\Delta(\tau, \sigma) = \prod_{j<k}\left(\lambda_j^+(\tau, \sigma) - \lambda_k^+(\tau, \sigma)\right).$$

Since for $\chi = \begin{pmatrix} 1, \cdots, m \\ j_1, \cdots, j_m \end{pmatrix}$

$$^{\chi}B_k^0 = B_{j_k}^0 \prod_{h=1}^{k-1} P_{j_h}^0,$$

we see easy that

$$^{\chi}B(\tau, \sigma) = \prod_{j=1}^{m} R_j(\tau, \sigma) \prod_{j>k} P_j^0\left(\tau, \sigma, \lambda_k^+(\tau, \sigma)\right),$$

which implies

$$^{\chi}R(\tau, \sigma) = \prod_{j=1}^{m} R_j(\tau, \sigma) \prod_{j>k}\left(\lambda_j^+(\tau, \sigma) + \lambda_k^+(\tau, \sigma)\right). \tag{3. 2}$$

Here we use the fact that $\lambda_j^+ + \lambda_j^- = 0$ and $P_j^0(\lambda_j^+) = 0$.

Since the $\lambda_j^{\pm}(\tau, \sigma)$ are mutually distinct for $\operatorname{Im} \tau = -\gamma < 0$, the reflection coefficients $C_j(\tau, \sigma)$ and $^{\chi}C_{jk}(\tau, \sigma)$ for (P_j^0, B_j^0) and $(P, {}^{\chi}B_j^0)$ are well defined respectively and they can be written by the form (see [3], §5):

$$C_j(\tau, \sigma) = B_j^0\left(\tau, \sigma, \lambda^-(\tau, \sigma)\right) / R_j(\tau, \sigma), \tag{3. 3}$$

$$^{\chi}C_{jk}(\tau, \sigma) = {}^{\chi}B_{jk}(\tau, \sigma) / {}^{\chi}B(\tau, \sigma), \qquad (j, k = 1, \cdots, m) \tag{3. 4}$$

where $^{\chi}B_{jk}(\tau, \sigma)$ is the determinant arising from replacing $\lambda_j^+(\tau, \sigma)$ by $\lambda_k^-(\tau, \sigma)$ in $^{\chi}B(\tau, \sigma)$.

Let S be the set $\{(\tau, \sigma); |\tau|^2 + |\sigma|^2 = 1, \gamma \geqq 0\}$. Then we have from [3], Theorem 5. 1 the following.

LEMMA 3. 1. *An iterated mixed problem $(P, {}^{\chi}B_j)$ is L^2-well posed if and only if the followings are filfulled:*

(i) $^{\chi}R(\tau, \sigma) \neq 0$ *for* $\operatorname{Im} \tau = -\gamma < 0$,

(ii) *for every real* $(\xi_0, \sigma_0) \in S$ *with* $\xi_0 \neq 0$ *there exist a constant* $C(\xi_0, \sigma_0)$ *and a neighbourhood* $U(\xi_0, \sigma_0)$ *in* S *such that for any* $(\tau, \sigma) \in U(\xi_0, \sigma_0) \cap \{\gamma > 0\}$

$$|C_{jk}(\tau, \sigma)| \leqq C(\xi_0, \sigma_0) \left| \operatorname{Im} \lambda_j^+(\tau, \sigma) \operatorname{Im} \lambda_k^-(\tau, \sigma) \right|^{\frac{1}{2}} \left| P_\lambda^0\left(\tau, \sigma, \lambda_k^-(\tau, \sigma)\right) \right| \gamma^{-1} \tag{3. 5}$$

$$(j, k = 1, \cdots, m)$$

where $P_\lambda^0 = \partial P^0 / \partial \lambda$.

PROOF. Applying [12], Theorem 3 or [4], Theorem 3.1 to our case, we see that if $(P, {}^{\gamma}B_j)$ is L^2-well posed then (i) is valid. By the definitions of $\lambda_j^{\pm}$, $\lambda_j^{\pm}(0, \sigma_0)=\pm i|\sigma_0|$ for all $j=1, \cdots, m$, respectively, but ${}^{\gamma}R(0, \sigma_0)\neq 0$ because of the assumption on coefficients of B_j^0. Note that the $\lambda_j^-(\tau, \sigma)$ are simple for $\gamma>0$. Therefore, using residue formula, the lemma follows from [3], Theorem 5.1. Here we use a similar technique in the proof of the theorem.

Finally we remark that, by setting $m=1$, Lemma 3.1 contains a characterization of L^2-well posed problem (P_j, B_j) of second order.

§ 4. Characterizations of an L^2-well posed mixed problem of second order

Let (P, B) denote any one of (P_j, B_j) $(j=1, \cdots, m)$ and

$$P^0(t, x; \tau, \sigma, \lambda) = -\tau^2+a(t, x)^2\left(\lambda^2+\sum_{k=1}^{n-1}\sigma_k^2\right),$$

$$B^0(t, x'; \tau, \sigma, \lambda) = \lambda-\sum_{k=1}^{n-1}b_k(t, x')\sigma_k-c(t, x')\tau .$$

Moreover, let $R(t, x'; \tau, \sigma)$ be Lopatinski determinant for a frozen problem $(P^0, B^0)_{(t,x')}$. Then we have the following

LEMMA 4.1. *The following statements are equivalent*:

(i) (P, B) *is L^2-well posed.*

(ii) *Every frozen problem* $(P^0, B^0)_{(t,x')}$ *is L-well posed.*

(iii) *For every* (t, x') $R(t, x'; \tau, \sigma)\neq 0$ *if either* $\operatorname{Im}\tau<0$ *or* $\operatorname{Im}\tau=0$ *and* $\xi^2>a(t, x', 0)^2|\sigma|^2$.

(iv) $a(t, x', 0)c(t, x')|\sigma|\geqq\left|\sum_{k=1}^{n-1}b_k(t, x')\sigma_k\right|$ *for any* $(t, x'; \sigma)$.

(v) $a(t, x', 0)c(t, x')\geqq\left(b_1(t, x')^2+\cdots+b_{n-1}(t, x')^2\right)^{\frac{1}{2}}$ *for any* (t, x').

REMARK. Let (t, x') be fixed and $(c, b, \cdots, b_{n-1})$ vary. Then (v) shows that the set of all the L^2-well posed mixed problems, i.e. the closed cone with vertex at the origin, coincides with the section of the propagation cone by the boundary hyperplane. This fact is also valid for a general P of second order (see [1], the conditions $((C_1), (C_2))$.

PROOF. The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(i) are special cases of [2], Theorem 1, [12], Theorem 3, [4], Remark of Theorem 4.2, [1], Theorem (also see [2], [7]), respectively. However, we give here the proofs of the implications (iii)$\Rightarrow$(iv)$\Rightarrow$(i) together with the equivalence (iv)$\Leftrightarrow$(v) for the sake of completeness.

The implication (iii)$\Rightarrow$(iv). This is originally owe to [1], § 2, but the proof in [7] which will be stated here is somewhat simple. We may drop parameters (t, x') for simplicity. We can shows from (iii) that $c \geqq 0$. Let

$$P(\tau, \sigma, \lambda) = a^2\left(\lambda - \lambda^+(\tau, \sigma)\right)\left(\lambda - \lambda^-(\tau, \sigma)\right) \qquad (\tau = \xi - i\gamma,\ \gamma \geqq 0).$$

Then, by the definition of $\lambda^+(\tau, \sigma)$, we have

$$\lambda^+(\xi, \sigma) = -sgn\, \xi \sqrt{\xi^2/a^2 - |\sigma|^2} \quad \text{in } \xi^2 > a^2|\sigma|^2,$$

where $\sqrt{1} = 1$ and $sgn\, \xi = \begin{cases} 1 & (\xi > 0) \\ -1 & (\xi < 0) \end{cases}$. Hence

$$R(\xi, \sigma) = -sgn\, \xi \sqrt{\xi^2/a^2 - |\sigma|^2} - \sum_{k=1}^{n-1} b_k \sigma_k - c\xi \quad (\xi^2 > a^2|\sigma|^2).$$

Note that for any fixed ξ the surface $\lambda = \lambda^+(\xi, \sigma)$ in $(\lambda, \sigma) \in \boldsymbol{R}^n$ is the open $(n-1)$-hemisphere. Therefore if (iv) is not valid for some σ then the hyperplane $\lambda = \sum_{k=1}^{n-1} b_k \sigma_k + c\xi$ in $(\lambda, \sigma) \in \boldsymbol{R}^n$ intersects the open hemisphere. This contradicts to the fact that $R(\xi, \sigma) \neq 0$ in $\xi^2 > a^2|\sigma|^2$.

The implication (iv)→(i). Since $b_k(t, x')$ and $c(t, x')$ are real valued, it sufficies to derive a priori estimate for real $u \in C_0^\infty(\overline{\boldsymbol{R}}_+^{n+1})$. Put

$$\begin{aligned} &Q(t, x; D_t, D_x) \\ &= -\frac{1}{2}\frac{\partial P^0}{\partial \tau}(t, x; D_t, D_x) - a(t, x)^2 c(t, x) B^0(t, x; D_t, D_x)^{(5)}, \end{aligned}$$

where $c(t, x) = c(t, x')$ and $b_k(t, x) = b_k(t, x')$. Then, using integration by parts, we obtain

$$\begin{aligned} &2(Pu, iQu)_{0,\gamma} \\ &= 2\gamma\Bigg\{\left((a^2c^2+1)\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right)_{0,\gamma} + 2\sum_{k=1}^{n-1}\left(a^2 b_k c\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_k}\right)_{0,\gamma} \\ &\qquad - 2\left(a^2 c\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_n}\right)_{0,\gamma} + \sum_{k=1}^{n}\left((a^2c^2+1)a^2\frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_k}\right)_{0,\gamma}\Bigg\} \\ &\qquad + \Bigg\{\left\langle a^2c(a^2c^2+1)\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right\rangle_{0,\gamma} + 2\sum_{k=1}^{n-1}\left\langle a^2 b_k(a^2c^2+1)\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_k}\right\rangle_{0,\gamma} \\ &\qquad\qquad + \sum_{k,l=1}^{n-1}\left\langle a^4 c(b_k b_l + \delta_{kl})\frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_l}\right\rangle_{0,\gamma}\Bigg\} \\ &\qquad\qquad + \langle Bu, Su\rangle_{0,\gamma} + R(u, u) - \langle a^4 cBu, Bu\rangle_{0,\gamma}, \end{aligned}$$

where $(\cdot, \cdot)_{0,\gamma}$ and $\langle\cdot, \cdot\rangle_{0,\gamma}$ are the innerproducts in $H_{0,\gamma}(\boldsymbol{R}_+^{n+1})$ and $H_{0,\gamma}(\boldsymbol{R}^n)$ respectively, S is a first order differential operator in $D_t, D_{x'}$, δ_{jk} is Kro-

(5); This is the same one as in [1].

necker's symbol and $|R(u, u)| \leqq C\|u\|_{1,\gamma}^2$. Hence we see from (iv) that the integrand of (boundary) volume integral is positive (semi) definite respectively. Using Schwarz inequality we have, for a large $\gamma \geqq \gamma_0$,

$$\gamma^2 \|u\|_{1,\gamma}^2 \leqq C(\|Pu\|_{0,\gamma}^2 + \langle\!\langle \Lambda^{\frac{1}{2}} Bu \rangle\!\rangle_{0,\gamma}^2).$$

The symbols of $\Lambda^{\frac{1}{2}}$ and $\Lambda_{x'}^{\frac{1}{2}}$ are equivalent in $|\sigma| \geqq \delta\Lambda$ ($\delta > 0$) and not equivalent elsewhere. But it follows from the fact $c(t, x') \geqq 0$ that $R_j(t, x'; \tau, 0) \neq 0$. Therefore the same argument as in the proof of Proposition 2.2 gives that, with some constants γ_0, C,

$$\gamma^2 \|u\|_{1,\gamma}^2 \leqq C(\|Pu\|_{0,\gamma}^2 + \langle \Lambda_{x'}^{\frac{1}{2}} Bu \rangle_{0,\gamma}^2) \quad (\gamma \geqq \gamma_0).$$

To show the existence of a solution we use the dual problem.

The equivalence (iv)$\Longleftrightarrow$(v). Put

$$(A\sigma, \sigma) = a^2 c^2 |\sigma|^2 - \left(\sum_{k=1}^{n-1} b_k \sigma_k\right)^2$$

where $a = a(t, x', 0)$, $c = c(t, x')$ and $b_k = b_k(t, x')$. Then it is proved by the mathematical induction on n that the eigenvalues of symmetric matrix A are

$$\underbrace{a^2c^2, \cdots, a^2c^2}_{n-2}, \quad a^2c^2 - (b_1^2 + \cdots + b_{n-1}^2).$$

In fact,

$$|\lambda I - A| = \begin{vmatrix} s+b_1^2, & b_1 b_2, & \cdots, & b_1 b_n \\ b_2 b_1, & s+b_2^2, & \cdots, & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ b_n b_1, & b_n b_2, & \cdots, & s+b_n^2 \end{vmatrix}$$

where $s = \lambda - a^2c^2$. Expand this determinant with respect to the first row. Then we obtain by the assumption of the induction and a simple calculation that

$$\begin{aligned} |\lambda I - A| &= (s + b_1^2) s^{n-2} (s + b_2^2 + \cdots + b_n^2) - s^{n-2} b_1^2 (b_2^2 + \cdots + b_n^2) \\ &= s^{n-1}(s + b_1^2 + \cdots + b_n^2). \end{aligned}$$

Therefore there exists an orthogonal matrix T such that

$$(A\sigma, \sigma) = \left\{a^2c^2 - (b_1^2 + \cdots + b_{n-1}^2)\right\} \hat{\sigma}_1^2 + a^2c^2 \sum_{k=2}^{n-1} \hat{\sigma}_k^2$$

where $\hat{\sigma} = T\sigma$. This implies immediately the equivalence (iv)$\Longleftrightarrow$(v).

The proof of the implication (iii)$\Rightarrow$(iv) shows in particular the followings which are used in §5 and 6.

COROLLARY 4. 2. *If a constant coefficient problem* (P^0, B^0) *is* L^2*-well posed and* $c>0$, *then* $c\xi+\sum_{k=1}^{n-1} b_k\sigma_k\neq 0$ *in* $\xi^2>a^2|\sigma|^2$.

COROLLARY 4. 3. *If a constant coefficient problem* (P^0, B^0) *is of type* $\overline{NU}$, *that is,* $ca=(b_1^2+\cdots+b_{n-1}^2)^{\frac{1}{2}}$, *then there exists a posnt* (ξ_0, σ_0) *on the sheet* $\xi^2=a^2|\sigma|^2$ *such that* $c\xi_0+\sum_{k=1}^{n-1} b_k(\sigma_k)_0=0$ *and* $ac|\sigma_0|=\left|\sum_{k=1}^{n-1} b_k(\sigma_k)_0\right|$.

Finally, from Lemma 4. 1, (v) and the fact $0<a_m<\cdots<a_1$, we obtain the following

COROLLARY 4. 4. *If* (P_m, B) *is* L^2*-well posed, then* (P_j, B) *is also* L^2*-well posed for any* $j=1,\cdots,m$.

§5. Proof of Theorem 1

Applying [2], Theorem 1 to our case we see that if an iterated mixed problem $(P, {}^{\chi}B_j)$ is L^2-well posed then every frozen problem $(P^0, {}^{\chi}B_j^0)_{(t,x')}$ is also L^2-well posed. Then we shall drop parameters (t, x') in this section and the following one.

We recall the definition of reflection coefficients ${}^{\chi}C_{jk}(\tau, \sigma)$ for $(P^0, {}^{\chi}B_j^0)$ (see §3, (3. 4)) and write explicitly them for some pair (j, k) needed for the proof of Theorem 1. By the definition we have

$$ {}^{\chi}C_{jk}(\tau, \sigma)={}^{\chi}B_{jk}(\tau, \sigma)/{}^{\chi}B(\tau, \sigma) $$

where the ${}^{\chi}B_{jk}(\tau, \sigma)$ are the determinants resulting from replacing $\lambda_j^+(\tau, \sigma)$ by $\lambda_k^-(\tau, \sigma)$ in ${}^{\chi}B(\tau, \sigma)$. Let $\chi=\begin{pmatrix}1, & \cdots, & m\\ j_1, & \cdots, & j_m\end{pmatrix}$ and $sgn\ \chi=\begin{cases}1 & \text{(even permutation } \chi)\\ -1 & \text{(odd permutation } \chi)\end{cases}$.
Then we have

$$ \begin{aligned} {}^{\chi}B(\tau, \sigma) &= \det\left({}^{\chi}B_k^0(\tau, \sigma\,;\, \lambda_j^+(\tau, \sigma))_{k\downarrow}^{j\rightarrow}\ 1, \cdots, m\right) \\ &= sgn\ \chi \begin{vmatrix} B_{j_1}^0(\lambda_{j_1}^+), & B_{j_1}^0(\lambda_{j_2}^+) & \cdots & B_{j_1}^0(\lambda_{j_m}^+) \\ & B_{j_2}^0P_{j_1}^0(\lambda_{j_2}^+) & \cdots & B_{j_2}^0P_{j_1}^0(\lambda_{j_m}) \\ & & \ddots & \vdots \\ \Large 0 & & & B_{j_m}^0P_{j_1}^0\cdots P_{j_{m-1}}^0(\lambda_{j_m}^+) \end{vmatrix}. \end{aligned} $$

Here we denote simply $B_j^0(\tau, \sigma, \lambda_k^+(\tau, \sigma))$, $P_j^0(\tau, \sigma, \lambda_k^+(\tau, \sigma))$ by $B_j^0(\lambda_k^+)$, $P_j^0(\lambda_k^+)$ respectively. Note that $\lambda_j^++\lambda_j^-=0$ and $P_j^0(\lambda_k^+)=P_j^0(\lambda_k^-)$ for $j, k=1, \cdots, m$. Then we have

$$ {}^{\chi}C_{jj}(\tau, \sigma)=\frac{B_j^0(\lambda_j^-)}{B_j^0(\lambda_j^+)}=C_j(\tau, \sigma) \qquad (j=1, \cdots, m) \tag{5. 1} $$

$$(5.2)\qquad {}^{\chi}C_{j_kj_{k+1}}(\tau,\sigma)=\frac{\begin{vmatrix} B^0_{j_k}(\lambda^-_{j_{k+1}}), & B^0_{j_k}(\lambda^+_{j_{k+1}}) \\ B^0_{j_{k+1}}(\lambda^-_{j_{k+1}}), & B^0_{j_{k+1}}(\lambda^+_{j_{k+1}}) \end{vmatrix}}{B^0_{j_k}(\lambda^+_{j_k})B^0_{j_{k+1}}(\lambda^+_{j_{k+1}})}\frac{P^0_{j_1}\cdots P^0_{j_{k-1}}(\lambda^+_{j_{k+1}})}{P^0_{j_1}\cdots P^0_{j_{k-1}}(\lambda^+_{j_k})}$$
$$=\frac{2(\alpha_{j_{k+1}}-\alpha_{j_k})\lambda^+_{j_{k+1}}P^0_{j_1}\cdots P^0_{j_{k-1}}(\lambda^+_{j_{k+1}})}{R_{j_k}(\tau,\sigma)R_{j_{k+1}}(\tau,\sigma)P^0_{j_1}\cdots P^0_{j_{k-1}}(\lambda^+_{j_k})}$$

where $\alpha_j(\tau,\sigma)=\sum_{k=1}^{n-1}b_{jk}\sigma_k+c_j\tau$ $(j=1,\cdots,m)$ and $R_j(\tau,\sigma)$ is Lopatinski determinant for (P^0_j, B^0_j) (see (3.1)). The equalities (5.1) show that ${}^{\chi}C_{jj}$ is equal to the reflection coefficient C_j for (P^0_j, B^0_j) (see (3.3)).

We first prove the first part of Theorem 1. Let ${}^{\chi}R(\tau,\sigma)$ be Lopatinski determinant for $(P^0, {}^{\chi}B^0_j)$. Then we have from (3.2)

$${}^{\chi}R(\tau,\sigma)=\prod_{j=1}^{m}R_j(\tau,\sigma)\prod_{j<k}\left(\lambda^+_j(\tau,\sigma)+\lambda^+_k(\tau,\sigma)\right).$$

Hence it follows from Lemma 3.1, (i) that $R_j(\tau,\sigma)\neq 0$ in $\operatorname{Im}\tau=-\gamma<0$ for any $j=1,\cdots,m$, because $\lambda^+_j(\tau,\sigma)+\lambda^+_k(\tau,\sigma)\neq 0$ in $\gamma>0$. Furthermore, we obtain from (5.1) and Lemma 3.1, (ii) that for every $(\xi_0,\sigma_0)\in S=\{(\tau,\sigma);|\tau|^2+|\sigma|^2=1, \gamma\geqq 0\}$ with $\xi_0\neq 0$ it holds in $U(\xi_0,\sigma_0)\cap\{\gamma>0\}$

$$\begin{aligned}|C_j(\tau,\sigma)|&\leqq C(\xi_0,\sigma_0)|\operatorname{Im}\lambda^+_j(\tau,\sigma)\operatorname{Im}\lambda^-_j(\tau,\sigma)|^{\frac12}\\&\quad\times|P^0_\lambda(\tau,\sigma,\ \lambda^-_j(\tau,\sigma)|\gamma^{-1}\\&\leqq C(\xi_0,\sigma_0)|\operatorname{Im}\lambda^+_j(\tau,\sigma)\operatorname{Im}\lambda^-_j(\tau,\sigma)|^{\frac12}\\&\quad\times P^0_{j\lambda}(\tau,\sigma,\ \lambda^-_j(\tau,\sigma)|\gamma^{-1}\qquad\left(P^0_{j\lambda}=\frac{\partial P_j}{\partial\lambda}\right).\end{aligned}$$

Here we use that fact that $\lambda^+_j(\xi_0,\sigma_0)-\lambda^-_k(\xi_0,\sigma_0)\neq 0$ for $j\neq k$ and $\xi_0\neq 0$. We denote hereafter various constants depending only on (ξ_0,σ_0) by $C(\xi_0,\sigma_0)$. Note that $R_j(0,\sigma)=i|\sigma|-\sum_{k=1}^{n-1}b_k\sigma_k\neq 0$ because the b_k are real. Therefore the first part of Theorem 1 follows from Lemma 3.1 for $m=1$, that is, for second order problems.

Next we prove the second part of Theorem 1. We may assume $j_k<j_{k+1}$, since the same argument as below is applicable for $j_k>j_{k+1}$. Let $(\xi_0,\sigma_0)\in S$ with $\xi_0^2=a^2_{j_k}|\sigma_0|^2$ be arbitrary but fixed. Then it follows from Lemma 3.1, (ii) that in particular for small $\gamma>0$

$$\begin{aligned}&|{}^{\chi}C_{j_kj_{k+1}}(\xi_0-i\gamma,\sigma_0)|\\(5.3)\qquad&\leqq C(\xi_0,\sigma_0)|\operatorname{Im}\lambda^+_{j_k}(\xi_0-i\gamma,\sigma_0)\operatorname{Im}\lambda^-_{j_{k+1}}(\xi_0-i\gamma,\sigma_0)|^{\frac12}\\&\quad\times|P_\lambda(\xi_0-i\gamma,\sigma_0,\lambda^-_{j_{k+1}}(\xi_0-i\gamma,\sigma_0)|\gamma^{-1}\end{aligned}$$

Since $(P^0_{j_{k+1}}, B^0_{j_{k+1}})$ is L^2-well posed, $R_{j_{k+1}}(\xi_0,\sigma_0)\neq 0$ from Lemma 4.1, (iii).

Here we use $a_{j_k}>a_{j_{k+1}}$. Hence we have from (5.2)

$$\begin{aligned} &|{}^{\gamma}C_{j_k j_{k+1}}(\xi_0-i\gamma,\sigma_0)| \\ (5.4)\qquad &\geqq C(\xi_0,\sigma_0)\frac{|\alpha_{j_{k+1}}(\xi_0-i\gamma,\sigma_0)-\alpha_{j_k}(\xi_0-i\gamma,\sigma_0)|}{R_{j_k}(\xi_0-i\gamma,\sigma_0)} \end{aligned}$$

On the other hand, we see easy from the hyperbolicity of P_j that

$$\begin{aligned} &|\mathrm{Im}\,\lambda_j^{\pm}(\xi_0-i\gamma,\sigma_0)| = 0(\gamma) \quad (\xi_0^2>a_j^2|\sigma_0|^2), \\ (5.5)\qquad &|\mathrm{Im}\,\lambda_j^{\pm}(\xi_0-i\gamma,\sigma_0)| = 0(\gamma^{\frac{1}{2}}) \quad (\xi_0^2=a_j^2|\sigma_0|^2). \end{aligned}$$

Hence we have from (5.3), (5.4) and (5.5)

$$\begin{aligned} &|\alpha_{j_{k+1}}(\xi_0-i\gamma,\sigma_0)-\alpha_{j_k}(\xi_0-i\gamma,\sigma_0)| \\ (5.6)\qquad &\leqq C(\xi_0,\sigma_0)|R_{j_k}(\xi_0-i\gamma,\sigma_0)|\gamma^{-\frac{1}{4}}, \end{aligned}$$

where $\gamma>0$ is small.

Now assume that $c_{j_{k+1}}>0$ and $(P_{j_k}^0, B_{j_k}^0)$ be of type $\overline{NU}$, that is, does not satisfy the uniformly Lopatinski condition. Then it follows from Corollaries 4.2 and 4.3 that there exists a real point $(\xi_0,\sigma_0)\in S$ such that $\xi_0^2=a_{j_k}^2|\sigma_0|^2$, $\alpha_{j_k}(\xi_0,\sigma_0)=0$ and $\alpha_{j_{k+1}}(\xi_0,\sigma_0)\neq 0$. Also it follows from Lemma 4.1, (v) that

$$(b_{j_k 1},\cdots,b_{j_k n-1},c_{j_k})\neq(b_{j_{k+1}1},\cdots,b_{j_{k+1}n-1},c_{j_{k+1}}),$$

because the contrary leads that $(P_{j_k}^0, B_{j_k}^0)$ must be of type U. Hence we have

$$(5.7)\qquad \alpha_{j_{k+1}}(\xi_0,\sigma_0)-\alpha_{j_k}(\xi_0,\sigma_0)\neq 0.$$

On the other hand, we obtain from (5.5) that

$$\begin{aligned} &|R_{j_k}(\xi_0-i\gamma,\sigma_0)| = |\lambda_{j_k}^{+}(\xi_0-i\gamma,\sigma_0)+ic_{j_k}\gamma| \\ (5.8)\qquad &\leqq C(\xi_0,\sigma_0)\gamma^{\frac{1}{2}}. \end{aligned}$$

Therefore it follows from (5.6), (5.7) and (5.8) that for small $\gamma>0$

$$1\leqq C(\xi_0,\sigma_0)\gamma^{\frac{1}{4}},$$

which implies the contradiction. Hence we conclude that either $(P_{j_k}^0, B_{j_k}^0)$ is of type U or if it is of type $\overline{NU}$ then $c_{j_{k+1}}=0$, that is, $B_{j_{k+1}}=D_n$, because of Lemma 4.1, (v).

§6. Proof of Theorem 2

Let χ be the unit in the permutation group. Then the first part of

Theorem 2 follows immediately from applying results of Theorem 1 to this case. Hence it is enough to prove the sufficiency in the case of constant coefficients. We shall omit the upper suffix χ in this section.

Since all the problem (P_j, B_j) $(j=1, \cdots, m)$ are L^2-well posed for any one of m-types:

$$\begin{gathered}(U, \cdots\cdots\cdots\cdots, U, \overline{U}),\\ (U, \cdots\cdots, U, \overline{NU}, N),\\ \vdots\\ (\overline{NU}, N, \cdots\cdots\cdots, N),\end{gathered}$$

we see that Lemma 4. 1, (iii) and the assumption on coefficients of B_j^0 that $R_j(\tau, \sigma)\neq 0$ except the sheets $\xi^2=a_j^2|\sigma|^2$ $(j=1, \cdots, m)$ $(\tau=\xi-i\gamma)$, which implies from (3. 2) that $R(\tau, \sigma)\neq 0$ except the above sheets. Therefore the assertion (i) of Lemma 3. 1 is valid. Moreover, for every real $(\xi_0, \sigma_0)\in S$ not on the sheets the assertion (ii) of Lemma 3. 1 is also valid, because the right hand side of (3. 5) is estimated below by $C(\xi_0, \sigma_0)$ (see [3], Lemma 6. 1).

By the above consideration it suffices to show that the conditions (3. 5) are satisfied for each point on the sheets $\xi_0^2=a_j^2|\sigma_0|^2$. To do this we write explicitly the reflection coefficients C_{jk}:

$$\begin{aligned}&C_{jk}(\tau, \sigma)=0 \quad (k<j),\\ &C_{jj}(\tau, \sigma)=\frac{B_j^0(\lambda_j^-)}{R_j(\tau, \sigma)}=C_j(\tau, \sigma),\\ &C_{jk}(\tau, \sigma)=\frac{T_{jk}(\tau, \sigma)}{\prod_{h=j}^{k} R_h(\tau, \sigma)\prod_{h=j}^{k-1} S_h(\lambda_h^+)} \quad (k>j)\end{aligned} \tag{6. 1}$$

where

$$T_{jk}(\tau, \sigma)=\begin{vmatrix} B_j^0S_j(\lambda_k^-), & B_j^0S_j(\lambda_{j+1}^+), & \cdots\cdots, & B_j^0S_j(\lambda_k^+)\\ B_{j+1}^0S_{j+1}(\lambda_k^-), & B_{j+1}^0S_{j+1}(\lambda_{j+1}^+), & & \vdots\\ \vdots & 0 & \ddots & \vdots\\ B_{k-1}^0S_{k-1}(\lambda_k^-) & \vdots & \ddots & B_{k-1}^0S_{k-1}\lambda_k^+)\\ B_k^0(\lambda_k^-), & 0, \cdots\cdots\cdots, & 0, & B_k^0(\lambda_k^+)\end{vmatrix}$$

$$S_j(\lambda_k^{\pm})=\prod_{h=1}^{j-1} P_h^0(\lambda_k^{\pm}).$$

Here we use the same abbreviation as § 5. Hence it follows from L^2-well posedness of (P_j, B_j) that the assertion (ii) of Lemma 3. 1 is valid for $C_{jk}(k<j)$ and C_{jj} (refer to § 5 for the latter).

Let a type of $((P_1, B_1), \cdots, (P_{l-1}, B_{l-1}), (P_l, B_l), (P_{l+1}, B_{l+1}), \cdots, (P_m, B_m))$ assume $(U, \cdots, U, \overline{NU}, N, \cdots, N)$. Then

$$R_h(\tau, \sigma) \neq 0 \quad (h<l) \quad \text{in } S \tag{6.2}$$

and

$$B_h^0(\lambda_k^\pm) = \lambda_k^\pm \qquad (h>l,\ k=1, \cdots, m). \tag{6.3}$$

We shall prove that the conditions (3.5) are satisfied for C_{jk} $(j<k)$ in each of the following cases.

(i) The case $k<l$. From [3], Lemma 6.1 we have

$$D_{jk}(\tau, \sigma) \geqq C(\xi_0, \sigma_0) \quad \text{in } U_p(\xi_0, \sigma_0) \qquad (p=1, \cdots, m). \tag{6.4}$$

Here

$$D_{jk}(\tau, \sigma) = |\mathrm{Im}\, \lambda_j^+(\tau, \sigma)\, \mathrm{Im}\, \lambda_k^-(\tau, \sigma)|^{\frac{1}{2}} \ |P_\lambda(\tau, \sigma, \lambda_k^-(\tau, \sigma)|\, \gamma^{-1}$$

and we denote simply a neighbourhood in S of (ξ_0, σ_0) satisfying $\xi_0^2 = a_p^2|\sigma_0|^2$ by $U_p(\xi_0, \sigma_0)$. On the other hand, since no $R_h(h \geqq l)$ appear in the denominator of (6.1), it follows from (6.1) and (6.2) that

$$|C_{jk}(\tau, \sigma) \leqq C(\xi_0, \sigma_0) \quad \text{in } U_p(\xi_0, \sigma_0) \qquad (p=1, \cdots, m). \tag{6.5}$$

Hence we see from (6.4) and (6.5) that (3.5) holds for this case.

(ii) The case $k=l$. From [3], Lemma 6.1 we obtain

$$D_{jl}(\tau, \sigma) \geqq C(\xi_0, \sigma_0) \quad \text{in } U_p(\xi_0, \sigma_0)\ (p \neq l).$$

Note that $R(\tau, \sigma) \neq 0$ except the sheet $\xi^2 = a_l^2|\sigma|^2$. The same argument as in (i) shows that (3.5) holds in $U_p(\xi_0, \sigma_0)$ $(p \neq l)$. Expand T_{jk} with respect to the last row. Then we have from (6.1) and (6.2)

$$|C_{jl}(\tau, \sigma)| \leqq C(\xi_0, \sigma_0)\left(1 + \left|\frac{B_l^0(\lambda_l^-)}{R_l(\tau, \sigma)}\right|\right) \quad \text{in } U_l(\xi_0, \sigma_0). \tag{6.6}$$

Since (P_l, B_l) is L^2-well posed, we see that

$$\begin{aligned} \frac{B_l^0(\lambda^-)}{R_l(\tau, \sigma)} &\leqq C(\xi_0, \sigma_0)|\mathrm{Im}\, \lambda_l^+\ \mathrm{Im}\, \lambda_l^-|^{\frac{1}{2}}\, |P_{l\lambda}^0(\lambda_l^-)|\, \gamma^{-1} \\ &\leqq C(\xi_0, \sigma_0)|\mathrm{Im}\, \lambda_l^-|^{\frac{1}{2}}\, |P_\lambda^0(\lambda_l^-)|\, \gamma^{-1} \quad \text{in } U_l(\xi_0, \sigma_0). \end{aligned} \tag{6.7}$$

On the other hand, we see from [3], Lemma 6.1 that

$$D_{jl}(\tau, \sigma) \geqq C(\xi_0, \sigma_0)|\mathrm{Im}\, \lambda_l^-|^{\frac{1}{2}}\, |P_\lambda(\lambda_l^-)|\gamma^{-1} \geqq C(\xi_0, \sigma_0)\,. \tag{6.8}$$

Hence it follows from (6.6), (6.7) and (6.8) that

$$|C_{jl}(\tau, \sigma)| \leqq C(\xi_0, \sigma_0) D_{jl}(\tau, \sigma) \quad \text{in } U_l(\xi_0, \sigma_0),$$

which is nothing but (3. 5).

When $k>l$ we shall rewrite, using (6. 3), the reflection coefficients C_{jk} $(j<k)$ in a simple form. By adding the last cloumn to the first one in T_{jk} we obtain from (6. 1)

$$C_{jk}(\tau,\sigma)=\begin{cases}\dfrac{\tilde{T}_{jk}(\tau,\sigma)}{\prod_{h=j}^{l}R_h(\tau,\sigma)S_h(\lambda_h^+)} & (k>l>j)\\ 0 & (k>j>l)\end{cases} \tag{6.9}$$

where

$$T_{jk}(\tau,\sigma)=\begin{vmatrix} -2\alpha_j S_j(\lambda_k^-), & B_j^0 S_j(\lambda_{j+1}^+), & \cdots\cdots, & B_j^0 S_j(\lambda_l^+)\\ -2\alpha_{j+1}S_{j+1}(\lambda_k^-), & B_{j+1}^0 S_{j+1}(\lambda_{j+1}^+), & \cdots, & \\ \vdots & 0, & \ddots & \ddots\\ \vdots & \vdots & \ddots & \\ -2\alpha_l S_l(\lambda_k^-) & 0, & \cdots\cdots, 0, & B_l^0 S_l(\lambda_l^+)\end{vmatrix}$$

and

$$B_j(\tau,\sigma,\lambda)=\lambda-\alpha_j(\tau,\sigma)=\lambda-\sum_{k=1}^{n-1}b_{jk}\sigma_k-c_j\tau\,.$$

Since the conditions (3. 5) are clearly satisfied for C_{jk} $(k>j>l)$, it suffices to consider the following case.

(iii) The case $k>l\geqq j$. From [3], Lemma 6. 1 we have

$$D_{jk}(\tau,\sigma)\geqq C(\xi_0,\sigma_0)\quad\text{in } U_p(\xi_0,\sigma_0)\quad(p=1,\cdots,m)$$

Since the denominator of (6. 9) does not vanish from (6. 2) outside the sheet $\xi^2=a_l^2|\sigma|^2$, we see in the same way as (i) that the conditions (3. 5) are satisfied in $U_p(\xi_0,\sigma_0)$ $(p\neq l)$. From (6. 2) and (6. 9) we have

$$|C_{jk}(\tau,\sigma)|\leqq C(\xi_0,\sigma_0)\left(1+\left|\frac{\alpha_l(\tau,\sigma)}{R_l(\tau,\sigma)}\right|\right)\quad\text{in } U_l(\xi_0,\sigma_0).$$

Therefore the proof finishes if it is proved that

$$\left|\frac{\alpha_l(\tau,\sigma)}{R_l(\tau,\sigma)}\right|\leqq C(\xi_0,\sigma_0)\quad\text{in } U_l(\xi_0,\sigma_0). \tag{6.10}$$

Let $n=2$ and put $b_l=b_{l1}$ for simplicity. Then we assume without loss of generality from Corollary 4. 3 that $\xi_0=a_l\sigma_0(\sigma_0>0)$ and

$$a_lc_l+b_l=0\,. \tag{6.11}$$

Since

$$\tau-\xi_0=a_l\sqrt{\lambda^2+\sigma^2}-a_l\sigma_0$$
$$=a_l(\sigma-\sigma_0)+a_l\sigma^{-1}\lambda^2+0(\lambda^3),$$

we obtain the peiseux expansion of $\lambda_l^+(\tau,\sigma)$ in $U_l(\xi_0,\sigma_0)$:

$$\lambda_l^+(\tau,\sigma)=Z^{\frac{1}{2}}+0(Z), \tag{6.12}$$

where $Z=\sigma(\tau/a_l-\sigma)$ and $\operatorname{Im} Z^{\frac{1}{2}}>0$ if $\operatorname{Im}\tau<0$. Hence we have from (6.11)

$$\alpha_l(\tau,\sigma)=\sigma^{-1}a_lc_lZ.$$

Therefore it follows from this and (6.12) that

$$\left|\frac{\alpha_l(\tau,\sigma)}{R_l(\tau,\sigma)}\right|\leqq C(\xi_0,\sigma_0)|Z|^{\frac{1}{2}}\leqq C(\xi_0,\sigma_0)\quad\text{in } U_l(\xi_0,\sigma_0).$$

§7. Proof of Theorem 3

In virture of Theorem 1 it suffices to prove that $(P,\tilde{B}_j)$ is L^2-well posed if every frozen problem $(P_m^0,B^0)_{(t,x')}$ is L^2-well posed.

The following lemma is due to [9], [11].

LEMMA 7.1. *Let V be an open set of $S=\{(\tau,\sigma);|\tau|^2+|\sigma|^2=1,\gamma\geqq 0\}$ such that its closure does not contain $(0,\sigma)$. Moreover, let $\beta'(\xi,\sigma,\gamma)$ be a C^∞-function in S with its support contained in V and put $\beta(\xi,\sigma,\gamma)=\beta'(\xi\Lambda^{-1},\sigma\Lambda^{-1},\gamma\Lambda^{-1})$. Then, for every non-negative integer s, there exist positive constants C_s, γ_s such that it holds for $\gamma\geqq\gamma_s$ and $u\in H_{2m-2+s,\gamma}(\boldsymbol{R}_+^{n+1})$*

$$\|u\|_{2m-3+s,\gamma}^2+\sum_{j=1}^m\|Q_j\beta u\|_{s,\gamma}^2\geqq C_s\|\beta u\|_{2m-2+s,\gamma}^2.$$

Here $Q_j=\prod_{k\neq j}P_k$ and

$$\beta u(t,x)=\beta(D_t,D_{x'},\gamma)u(t,x)$$
$$=(2\pi)^{-n}\int_{R^n}e^{i\tau t+i\sigma x'}\beta(\xi,\sigma,\gamma)\hat{u}(\tau,\sigma,x_n)d\xi d\sigma.$$

PROOF. Remark that $\lambda_j^+(t,x;\tau,\sigma)=\lambda_k^+(t,x;\tau,\sigma)$ $(j\neq k)$ if and only if $\tau=0$. Then it follows from the proof of [9], Theorem 2.1 that for $u\in H_{2+s,\gamma}(\boldsymbol{R}_+^{n+1})$ $(\gamma\geqq\gamma_s')$

$$\|u\|_{1+s,\gamma}^2+\|P_j\beta u\|_{s,\gamma}^2+\|P_k\beta u\|_{s,\gamma}^2\geqq C_s'\|\beta u\|_{2+s,\gamma}^2,$$
$$(j\neq k,j,k=1,\cdots,m).$$

Therefore the lemma follows from this formula and the mathematical induction on m.

We shall first derive a priori estimate for (P, B_j).

PROPOSITION 7.2. *Suppose that* (P_m, B) *is* L^2*-well posed. Then, for every real* s *and integer* $k \geqq 0$, *there exist positive constant* $\gamma_{s,k}$, $C_{s,k}$ *such that it holds for* $\gamma \geqq \gamma_{s,k}$ *and* $\Lambda^s u \in H_{2m+k,\gamma}(\boldsymbol{R}_+^{n+1})$

$$\gamma^2 \|\Lambda^s u\|^2_{2m-1+k,\gamma} \leqq C_{s,k}\left(\|\Lambda^s Pu\|^2_{k,\gamma} + \sum_{j=1}^{m} \langle\!\langle \Lambda_{x'}^{\frac{1}{2}} B_j u \rangle\!\rangle^2_{2m-2j+s+k,\gamma} \right). \tag{7.1}$$

PROOF. It suffices to prove the proposition for $k=s=0$, because $x_n=0$ is non-characteristic for P.

Let U be a neighbourhood of the set $(0, \sigma)$ in $S=\{(\tau, \sigma);\ |\tau|^2+|\sigma|^2=1, \gamma \geqq 0\}$ and let $\alpha'(\xi, \sigma, \gamma)$ be a C^∞-function in S with its support contained in U. Moreover, put $\alpha(\xi, \sigma, \gamma)=\alpha'(\xi\Lambda^{-1}, \sigma\Lambda^{-1}, \gamma\Lambda^{-1})$ and $\beta(\xi, \sigma, \gamma)=1-\alpha(\xi, \sigma, \gamma)$.

From Proposition 2.2 and Corollary 4.4, there exist positive constants C, γ_0 such that

$$\gamma^2 \|Q_j \beta u\|^2_{1,\gamma} \leqq C(\|P_j Q_j \beta u\|^2_{0,\gamma} + \langle\!\langle \Lambda_{x'}^{\frac{1}{2}} B Q_j \beta u \rangle\!\rangle^2_{0,\gamma}),$$
$$\left(\gamma \geqq \gamma_0,\ u \in H_{2m,\gamma}(\boldsymbol{R}_+^{n+1}),\quad j=1, \cdots, m\right),$$

where $Q_j = \prod_{k \neq j} P_k$. Summing up them with respect to j, then it follows from Lemma 7.1 that, with some new constants C, γ_0,

$$\gamma^2 \|\beta u\|^2_{2m-1,\gamma} \leqq C\left(\|P\beta u\|^2_{0,\gamma} + \sum_{j=1}^{m} \langle\!\langle \Lambda_{x'}^{\frac{1}{2}} B Q_j \beta u \rangle\!\rangle^2_{0,\gamma} + \gamma^2 \|u\|^2_{2m-2,\gamma} \right) \quad (\gamma \geqq \gamma_0). \tag{7.2}$$

By the definitions of Q_j and B_j we have

$$BQ_j = \sum_{k=1}^{m} C_{jk} B_k$$

where $C_{jk}=C_{jk}(t, x; D_t, D_{x'})$ is of homogeneous order $2m-2k$. Note that β is a pseudo-differential operator of order zero. Therefore it follows from (7.2) that, with some new constants C, γ_0,

$$\gamma^2 \|\beta u\|^2_{2m-1,\gamma} \leqq C\left(\|Pu\|^2_{0,\gamma} + \sum_{j=1}^{m} \langle\!\langle \Lambda_{x'}^{\frac{1}{2}} B_j u \rangle\!\rangle^2_{2m-2j,\gamma} + \|u\|^2_{2m-1,\gamma} \right) \quad (\gamma \geqq \gamma_0). \tag{7.3}$$

Since the b_k are real, we see from (3.2) that $R(t, x'; \tau, \sigma) \neq 0$ on the support of α. Hence it follows from Lemma 2.1 that, with some positive constants C, γ_0,

$$\gamma^2 \|\alpha u\|^2_{2m-1,\gamma} \leqq C\left(\|Pu\|^2_{0,\gamma} + \gamma \sum_{j=1}^{m} \langle\!\langle B_j u \rangle\!\rangle^2_{2m-2j,\gamma} + \gamma \|u\|^2_{2m-1,\gamma} \right) \quad (\gamma \geqq \gamma_0). \tag{7.4}$$

Therefore we see from (7.3) and (7.4) that (7.1) holds for s, $k=0$.

We shall next consider a dual problem (P^*, B_j^*) of (P, B_j), where P^* is the formal adjoint of P.

Denote the inner products in $L^2(\boldsymbol{R}_+^{n+1})$ and $L^2(\boldsymbol{R}^n)$ by $(\cdot,\ \cdot)$ and $\langle\cdot,\ \cdot\rangle$ respectively. Then, using the integration by parts, we obtain

$$(P_j u, v)-(u, P_j^* v)=i(\langle Bu, C_j v\rangle+\langle C_j^* u, B_j^* v\rangle),$$
$$\left(u, v\in C_0^\infty(\boldsymbol{R}_+^{n+1})\right),$$

where $B_j^{*0}=D_x+\sum_{k=1}^{n-1} b_k(t, x')D_k+c(t, x')D_t$ and C_j, C_j^* are of order zero. By repeating this formulas and noting the form of P_j, there exist Dirichlet sets (B_j, H_j^*) and (H_j, B_j^*) of boundary differential operators such that

$$(Pu, v)-(u, P^* v)=i\left(\sum_{j=1}^{m} \langle B_j u, H_j v\rangle+\langle H_j^* u, B_j^* v\rangle\right),$$
$$\left(u, v\in C_0^\infty(\overline{\boldsymbol{R}}_+^{n+1})\right).$$

Here

$$B^{*0}=D_x+\sum_{k=1}^{n-1} b_k(t, x')D_x+c(t, x')D_t,$$
$$B_j^{*0}=B^{*0}D_x^{2j-2} \qquad (j\geqq 2).$$

For the existence of solutions of (P, B_j) it suffices to prove the following ([6]).

PROPOSITION 7.3. *Suppose that (P_m, B) is L^2-well posed. Then, for every real s and integer $k\geqq 0$, there exist positive constants $C_{s,k}^*$, $\gamma_{s,k}^*$ such that it holds for $\gamma\geqq\gamma_{s,k}^*$ and $\Lambda^s u\in H_{2m+k,\gamma}(\boldsymbol{R}_+^{n+s})$*

$$\gamma^2\|\Lambda^s u\|^2_{2m-1+k,-\gamma}\leqq C_s\left(\|\Lambda^s P^* u\|^2_{k,-\gamma}+\sum_{j=1}^{m}\langle\!\langle \Lambda_{x'}^{\frac{1}{2}} B_j^* u\rangle\!\rangle_{2m-2j+s+k,-\gamma}\right).$$

PROOF. Put

$$P'(t, x; D_t, D_x)=P^*(-t, x; -D_t, D_x),$$
$$B_j'(t, x'; D_t, D_x)=B_j^*(-t, x'; -D_t, D_x).$$

Then $B'^0=D_x+\sum_{k=1}^{n-1} b_k(-t, x')D_k-c(-t, x')D_t$. Hence we see from Lemma 4.1, (iv) that if (P_m, B) is L^2-well posed then (P_m^*, B^*) is also L^2-well posed.

§ 8. Proof of Theorem 4

In this section the coefficients of B_j^0 $(j=1, \cdots, m)$ will be complex valued.

Since $((P_1^0, B_1^0)_{(t,x')}, \cdots, (P_m^0, B_m^0)_{(t,x')})$ is of type $(U, \cdots, U, \overline{U})$ uniformly in (t, x'), we see from (3.2) and Lemma 4.1, (iii) that for any permutation χ

(8.1) $$\quad {}^\chi R(\mathrm{t}, x'; \tau, \sigma)\neq 0 \quad \text{in } \xi^2>a_m(t, x', 0)^2|\sigma|^2.$$

Here we use that Lemma 4.1, (iii) is also valid for the case of complex valued coefficients. Let a point $(t_0, x_0', \xi_0, \sigma_0)$ satisfying $\xi_0^2 \leqq a_m(t_0, x_0', 0)|\sigma_0|^2$ be arbitrary but fixed and let $\beta'(t, x; \xi, \sigma, \gamma)$ $(\xi^2+\gamma^2+|\sigma|^2=1)$ be C^∞ and with its support contained in a compact neighbourhood of $(t_0, x_0', 0; \xi_0, \sigma_0)$. Then making use of partition of unity we can obtain, together with (8.1) and Lemma 2.1, a priori estimate for $(P, {}^\chi B_j)$ if we show that

$$\gamma^2\|\beta u\|^2_{2m-1,\gamma} \leqq C\Big(\|Pu\|^2_{0,\gamma} + \sum_{j=1}^{m} \langle\!\langle \Lambda_{x'}^{\frac{1}{2}}\, {}^\chi B_j u\rangle\!\rangle^2_{2m-2j,\gamma} + \|u\|^2_{2m-1,\gamma}\Big) \tag{8.2}$$
$$\Big(\gamma \geqq \gamma_0,\ u \in H_{m,\gamma}(\boldsymbol{R}_+^{n+1})\Big)$$

where γ_0, C are positive constants and

$$\beta u(t, x) = \beta(D_t, D_x, \gamma)u(t, x)$$
$$= \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n} e^{i\tau t + i\sigma x'} \beta(t, x; \xi, \sigma, \gamma)\hat{u}(\tau, \sigma, x_n)d\xi d\sigma\,,$$
$$\beta(t, x; \xi, \sigma, \gamma) = \beta'(t, x; \xi\Lambda^{-1}, \sigma\Lambda^{-1}, \gamma\Lambda^{-1})\,.$$

Now we shall transform an iterate mixed problem $(P, {}^\chi B_j)$ to the iteration for d'Alembertians:

$$\begin{cases} P_{j_1}u = v_1 \\ B_{j_1}u = g_1 \end{cases}, \quad \begin{cases} P_{j_2}v_1 = v_2 \\ B_{j_2}v_1 = g_2 \end{cases}, \dots, \quad \begin{cases} P_{j_m}v_{m-1} = f \\ B_{j_m}v_{m-1} = g_m \end{cases}. \tag{8.3}$$

We assume $j_{k+1}=m$. Since $P_{j_h}(t_0, x_0', 0; \xi_0, \sigma_0) \neq 0$ and $R_{j_h}(t_0, x_0'; \xi_0, \sigma_0) \neq 0$ for $h \neq k+1$, the following coecive estimates for elliptic boundary value problems hole from (8.3) (for instance see [11]): For $h=0, \cdots, k-1, k+1, \cdots, m-1$ there exist positive constants C_h, γ_h such that for any $\gamma \geqq \gamma_h$ and $u \in H_{2m,\gamma}(\boldsymbol{R}_+^{n+1})$

$$\|\beta v_h\|^2_{2m-1-2h,\gamma} \leqq C_h(\|\beta v_{h+1} + [P_{j_{h+1}}, \beta]v_h\|^2_{2m-3-2h,\gamma}$$
$$+ \langle\!\langle \Lambda^{\frac{1}{2}} g_{h+1}\rangle\!\rangle^2_{2m-3-2h,\gamma} + \|v_h\|^2_{2m-3-2h,\gamma})\,, \tag{8.4}$$
$$(h=0, \cdots, k-1)\,,$$

$$\|\beta v_h\|^2_{2m-2h,\gamma} \leqq C_h(\|\beta v_{h+1} + [P_{j_{h+1}}, \beta]v_h\|^2_{2m-2-2h,\gamma}$$
$$+ \langle\!\langle \Lambda^{\frac{1}{2}} g_{h+1}\rangle\!\rangle^2_{2m-2-2h,\gamma} + \|v_h\|^2_{2m-2-2h,\gamma})\,, \tag{8.5}$$
$$(h=k+1, \cdots, m-1)\,.$$

where $v_0=u$, $v_m=f$ and $[P_j, \beta]$ denotes the commutator of P and β. Note that the symbols of Λ and $\Lambda_{x'}$ are equivalent in a starshaped neighbourhood of (ξ_0, σ_0) and $\|[P_j, \beta]v_h\|^2_{0,\gamma} \leqq C_h\|v_h\|^2_{1,\gamma}$,

$$\|v_h\|^2_{0,\gamma} \leqq C_h \|u\|^2_{2h,\gamma} \qquad (h=0, \cdots, m-1).$$

Then, multiplying (8.4) by γ^2, we obtain

$$(8.6)_h \qquad \begin{aligned} \gamma^2 \|\beta v_h\|^2_{2m-1-2h,\gamma} &\leqq C_h(\gamma^2 \|\beta v_{h+1}\|^2_{2m-3-2h,\gamma} \\ &\quad + \langle\!\langle \Lambda_{x'}^{\frac{1}{2}} g_{h+1} \rangle\!\rangle^2_{2m-2-2h,\gamma} + \|u\|^2_{2m-1,\gamma}), \\ &\quad (h=0, \cdots, k-1), \\ \|\beta v_h\|^2_{2m-2h,\gamma} &\leqq C_h(\|\beta v_{h+1}\|^2_{2m-2-2h,\gamma} \\ &\quad + \langle\!\langle \Lambda_{x'}^{\frac{1}{2}} g_{h+1} \rangle\!\rangle^2_{2m-2-2h,\gamma} + \|u\|^2_{2m-1,\gamma}), \\ &\quad (h=k+1, \cdots, m-1). \end{aligned}$$

On the other hand, since $(P_{j_k}, B_{j_k})=(P_m, B_m)$ is L^2-well posed, it follows from Proposition 2.2 and (8.3) that

$$(8.6)_k \qquad \begin{aligned} \gamma^2 \|\beta v_k\|^2_{2m-1-2k,\gamma} &\leqq C_k(\|\beta v_{k+1}\|^2_{2m-2-2k,\gamma} \\ &\quad + \langle\!\langle \Lambda_{x'}^{\frac{1}{2}} g_{k+1} \rangle\!\rangle_{2m-2-2k,\gamma} + \|u\|^2_{2m-1,\gamma}), \\ &\quad \left(\gamma \geqq \gamma_k,\ u \in H_{2m,\gamma}(\boldsymbol{R}^{n+1}_+)\right) \end{aligned}$$

where C_k, γ_k are positive constants. Therefore, substituting $(8.6)_h$ to $(8.6)_{h+1}$ successively, we obtain a priori estimate (8.2). The existence of a solution of $(P, {}^{\chi}B_j)$ is proved in a similar way to § 7.

§ 9. Remarks

In this section we shall consider an analogue to Theorem 3, where the coefficients of B^0 are complex valued, that is,

$$B(t, y; D_y, D_x) = D_x - ib(t, y)D_y, \quad b(t, y) \text{ is real valued,}$$

where $(t, y, x)=(t, x_1, x_2)$ (n=2).

LEMMA 9.1. *For every $j=1, \cdots, m$, $(P_j, B)_{(t,y)}$ is L^2-well posed if and only if $|b(t, y)|<1$.*

PROOF. Since $(P_j, B)_{(t,y)}$ is a constant coefficient problem, we drop a parameter (t, y) for simplicity. Moreover, we restrict the variables (τ, σ) to the set $S=\{(\tau, \sigma); |\tau|^2+\sigma^2=1, \operatorname{Im}\tau=-\gamma\leqq 0\}$. From Lemma 3.1 and Lemma 4.1, (iii) we see that (P_j, B) is L^2-well posed if and only if

i) $R_j(\tau, \sigma)\neq 0$ if either $\operatorname{Im}\tau<0$ or $\operatorname{Im}\tau=0$ and $\xi^2>a_j^2\sigma^2$,

ii) for every (ξ_0, σ_0) with $\xi_0^2=a_j^2\sigma_0^2$ there exist a positive constant $C(\xi_0, \sigma_0)$ and a neighbourhood $U(\xi_0, \sigma_0)$ in S such that

$$(9.1)\qquad \left|\frac{B^0(\tau, \sigma, \lambda_j^-(\tau, \sigma))}{R_j(\tau, \sigma)}\right|$$
$$\leqq C(\xi_0, \sigma_0)|\operatorname{Im}\lambda_j^+(\tau, \sigma)\operatorname{Im}\lambda_j^-(\tau, \sigma)|^{\frac{1}{2}}|\lambda_j^+(\tau, \sigma)-\lambda_j^-(\tau, \sigma)|\gamma^{-1}$$

in $U(\xi_0, \sigma_0)\cap\{\gamma>0\}$.

iii) for every (ξ_0, σ_0) with $\xi_0^2<a_j^2\sigma_0^2$ there exist a positive constant $C(\xi_0, \sigma_0)$ and a neighbourhood $U(\xi_0, \sigma_0)$ in S such that

$$(9.2)\qquad |R_j(\tau, \sigma)|\geqq C(\xi_0, \sigma_0)\gamma \text{ in } U(\xi_0, \sigma_0)\cap\{\gamma>0\}.$$

From the definitions of $R_j(\tau, \sigma)$ and $\lambda_j^+(\tau, \sigma)$ we have $R_j(\tau, \sigma)=\lambda_j^+(\tau, \sigma)-ib\sigma$,

$$\lambda_j^+(\tau, \sigma)=\begin{cases} i\sqrt{\sigma^2-\tau^2/a_j^2} & \text{if } \operatorname{Im}\tau<0 \\ -sgn\,\xi\sqrt{\xi^2/a_j^2-\sigma^2} & \text{if } \operatorname{Im}\tau=0 \text{ and } \xi^2\geqq a_j^2\sigma^2 \\ i\sqrt{\sigma^2-\xi^2/a_j^2} & \text{if } \operatorname{Im}\tau=0 \text{ and } \xi^2<a_j^2\sigma^2 \end{cases}$$

where $\sqrt{1}=1$. Hence we can easy verify that

$$(9.3)\qquad R(\tau, \sigma)\neq 0 \text{ in } \operatorname{Im}\tau<0 \text{ if and only if } |b|\leqq 1,$$

$$(9.4)\qquad R(\xi_0, \sigma_0)=0\ (\xi_0^2<a_j^2\sigma_0^2) \text{ if and only if } a_j^2(1-b^2)\sigma_0^2=\xi_0^2,\ b\sigma_0>0 \text{ and } 0<|b|\leqq 1$$

and, in a neighbourhood of (ξ_0, σ_0),

$$(9.5)\qquad R(\tau, \sigma)=i(\sqrt{\sigma^2-\xi^2/a_j^2}-b\sigma)+\gamma\frac{\xi}{a_j^2\sqrt{\sigma^2-\xi^2/a_j^2}}+0(\gamma^2).$$

If $|b|>1$ then it follows from (9.3) that the condition (i) is not valid. Let $|b|=1$. Then by (9.4) $R(0, \sigma_0)=0$ for $b\sigma_0>0$. Hence it follows from this and (9.5) that

$$|R_j(i\gamma, \sigma_0)|\leqq C(0, \sigma_0)\gamma^2,$$

which contradict to (9.2) for small $\gamma>0$. Therefore (P_j, B) is not L^2-well posed if $|b|\geqq 1$.

Next we show that (P_j, B) is L^2-well posed if $|b|<1$. We may assume that $b\neq 0$, because (P_j, D_x) is Neumann problem. It is obvious that $R_j(\xi, 0)\neq 0$. Since $\operatorname{Im} R_j(\xi, \sigma)=b\sigma$ in $\xi^2\geqq a_j^2\sigma^2$, we see that $R_j(\xi, \sigma)\neq 0$ there. Then it follows from this and (9.3) that (i) is valid. Note that from [3], Lemma 6.1 the right hand side of (9.1) is estimated below by a constant. Then (ii) is valid, because $R_j(\xi, \sigma)\neq 0$ in $\xi^2=a_j^2\sigma^2$. It is obvious that if $R_j(\xi_0, \sigma_0)\neq 0$ in $\xi_0^2<a_j^2\sigma_0^2$ then (9.2) holds. From (9.4) we see that if

$R_j(\xi_0, \sigma_0)=0$ then $\xi_0 \neq 0$. Therefore it follows from (9. 5) that

$$|R_j(\tau, \sigma)| \geqq |\mathrm{Re}\, R_j(\tau, \sigma)| \geqq C(\xi_0, \sigma_0)\gamma\ .$$

PROPOSITION 9. 2. *Suppose that* $0<b(t, y)<1$ *or* $-1<b(t, y)<0$. *Then* $(P, \tilde{B}_j)$ *is* L^2*-well posed. Here* $\tilde{B}_j = BD_x^{2j-2}$.

PROOF. Remark that the condition in Lemma 9. 1 is independent of $j=1, \cdots, m$. Then it follows from [2], §5 that (P_j, B) is L^2-well posed for every $j=1, \cdots, m$. To show a priori estimate for $(P, \tilde{B}_j)$, we only verify that $R_j(t, y\,; 0, \sigma) \neq 0$ for every $j=1, \cdots, m$ (see Lemma 7. 1). However, this follows from the fact $|b(t, y)| \neq 1$.

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