The endomorphism ring of an indecomposable module with an Artinian projective cover

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As is well known, each endomorphism of an Artinian, Noetherian and indecomposable module is either nilpotent or an automorphism. C. I. Vinsonhaler has proved that each endomorphism of an indecomposable module with a Noetherian injective hull is either nilpotent or an automorphism (see [6], [7] and [5; p. 75]). In this note we shall verify that each endomorphism of an indecomposable module with an Artinian projective cover is either nilpotent or an automorphism.

We will assume throughout that M denotes a nonzero unital left R-module, where R is a nonzero ring with identity.

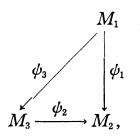
THEOREM. Let M be an indecomposable left R-module and let P be an Artinian left R-module which is a projective cover of M with a minimal epimorphism $\varphi: P \rightarrow M$. Then each endomorphism α of M is either nilpotent or an automorphism.

REMARK 1. M is called a semiperfect module iff every factor module of M has a projective cover (cf. Mares [4]).

Let N be a submodule of M. A cocomplement N^c of N in M is a minimal submodule of M such that $N+N^c=M$. M is called a cocomplemented module iff every submodule of M has a cocomplement in M.

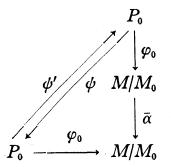
As is easily proven, semiperfect modules are cocomplemented. Conversely, projective cocomplemented modules are semiperfect (see Kasch-Mares [3]).

REMARK 2. In the following commutative diagram with left *R*-modules M_i and homomorphisms ϕ_i (i=1, 2, 3):



if ψ_1 is an epimorphism and if ψ_2 is a minimal epimorphism, then ψ_3 is necessarily an epimorphism.

PROOF of THEOREM. Since also M is Artinian, there exists a positive integer r such that $M\alpha^r = M\alpha^{r+1} = \cdots$. Evidently, every Artinian module is cocomplemented, so that P is semiperfect (Remark 1). Then, so is Msince every factor module of a semiperfect module is semiperfect. Put M_0 =Ker α^r and let P_0 be a projective cover of M/M_0 with a minimal epimorphism $\varphi_0: P_0 \longrightarrow M/M_0$. If $\pi: M \longrightarrow M/M_0$ is the natural epimorphism, then there exists a homomorphism $\psi_0: P \longrightarrow P_0$ such that $\psi_0\varphi_0 = \varphi\pi$, by the projectivity of P. But ψ_0 is an epimorphism (Remark 2), whence P_0 is Artinian. Since $M_0\alpha \subset M_0$, α induces the endomorphism $\bar{\alpha}$ of M/M_0 by $(a+M_0)\bar{\alpha} = a\alpha + M_0$ $(a \in M)$. Moreover, $M\alpha^r = M\alpha^{r+1}$ asserts that $\bar{\alpha}$ is an epimorphism. Then we have, by the projectivity of P_0 , an endomorphism ψ of P_0 such that $\psi\varphi_0 = \varphi_0\bar{\alpha}$.



But ψ is an epimorphism (again by Remark 2), so that the projective P_0 has a splitting endomorphism $\psi'; \psi'\psi$ is the identity mapping of P_0 . Since P_0 is Artinian, there exists a positive integer s such that $P_0\psi'^s=P_0\psi'^{s+1}$. Since ψ' is monomorphic, we obtain $P_0=P_0\psi'$, so that $(\psi' \text{ or}) \psi$ is an isomorphism.

Now, put $K = \text{Ker } \varphi_0 \bar{\alpha}$. Considering that $K\psi$ is included in K which is also Artinian, we know the existence of a positive integer t satisfying $K\psi^t = K\psi^{t+1}$. The monomorphism ψ deduces $K = K\psi$. Then, this yields that $\bar{\alpha}$ is a monomorphism. Next, let a be an arbitrary element of Ker α^{2r} . Then, $a\alpha^r$ is in M_0 and hence $(a + M_0)\bar{\alpha}^r = 0$. Since $\bar{\alpha}$ is monomorphic, a is contained in M_0 , showing that Ker $\alpha^r = \text{Ker } \alpha^{2r}$. This implies, together with $M\alpha^r = M\alpha^{2r}$, that M is the direct sum of $M\alpha^r$ and Ker α^r . It follows from the assumption of M to be indecomposable that $M\alpha^r = 0$ or that $M\alpha^r = M$ and Ker $\alpha^r = 0$. Consequently, α is nilpotent or α^r is, and hence, α is an automorphism, as required.

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