Some remarks on nonlinear differential equations in Banach spaces

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§ 1. Introduction and results.

Let E be a Banach space with the dual space E^*. The norms in E and E^* are denoted by $\|\|$. We denote by $S(u, r)$ the closed sphere of center u with radius r.

In this paper we are concerned with nonlinear abstract Cauchy problems of the forms

$$(D_1) \qquad \frac{d}{dt} u(t) = f(t, u(t)), \qquad u(0) = u_0 \in E,$$

and

$$(D_2) \qquad \frac{d}{dt} u(t) = Au(t) + f(t, u(t)), \qquad u(0) = u_0 \in D(A).$$

Here A is a nonlinear operator with domain $D(A)$ and range $R(A)$ in E, and f is a E-valued mapping defined on $[0, T] \times S(u_0, r)$ or on $[0, \infty) \times E$.

It is well known that in the case of $E = R^n$, the n-dimensional Euclidean space, the continuity of f in a neighbourhood of $(0, u_0)$ alone implies the existence of a local solution of (D_1). This is the classical Peano's theorem. However, this theorem cannot be generalized to the infinite-dimensional case (see [3], [16]).

It is our object in this paper to give sufficient conditions for the existence of the unique solutions to the Cauchy problems of the forms (D_1) and (D_2).

Let the functionals $\langle , \rangle_1$ and $\langle , \rangle_2$ be defined as follows (cf. M. Hasegawa [6]):

$$\langle u, v \rangle_1 = \lim_{h \downarrow 0} \frac{1}{h} (\|u + hv\| - \|u\|),$$

and

$$\langle u, v \rangle_2 = \frac{1}{2} (\langle u, v \rangle_1 - \langle u, -v \rangle_1)$$

for u, v in E.

In order to prove the existence of the unique solution of the equation

(D_1) we consider the following scalar equation

(1.1) $$w'(t) = g(t, w(t)),$$

where $g(t, \tau)$ is a scalar-valued function defined on $(0, a] \times [0, b]$ which is measurable in t for fixed τ, and continuous nondecreasing in τ for fixed t.

We say w is a solution of (1.1) on an interval I contained in $[0, a]$ if w is absolutely continuous on I and if

$$w'(t) = g(t, w(t)) \qquad \text{for a.e. } t \in I^0,$$

where I^0 is the set of all interior points of I.

We assume furthermore that g satisfies the following conditions: $(\mathrm{i_a})$ There exists a function m defined on $(0, a)$ such that

$$|g(t, \tau)| \leqq m(t) \qquad \text{for } (t, \tau) \in (0, a] \times [0, b]$$

and for which m is Lebesgue integrable on (ε, a) for every $\varepsilon > 0$. $(\mathrm{ii_a})$ For each $t_0 \in (0, a]$, $w \equiv 0$ is the only solution of the equation (1.1) on $[0, t_0]$ satisfying the conditions that $w(0) = (D^+ w)(0) = 0$, where $D^+ w$ denotes the right-sided derivative of w.

First, we can state the following result.

THEOREM 1. *Let f be a strongly continuous mapping of $[0, T] \times S(u_0, r)$ into E such that*

(1.2) $$\langle u-v, f(t, u)-f(t, v)\rangle_2 \leqq g(t, \|u-v\|)$$

for all (t, u), $(t, v) \in (0, T] \times S(u_0, r)$, where g satisfies $(\mathrm{i_a})$, $(\mathrm{ii_a})$ with $a = T$ and $b = 2r$.

Then (D_1) has a unique strongly continuously differentiable solution u defined on some interval $[0, T_0]$.

We next consider a global analogue of Theorem 1, and we assume that $g(t, \tau)$ is a scalar-valued function defined on $(0, \infty) \times [0, \infty)$ which is measurable in t for fixed τ, and continuous nondecreasing in τ for fixed t. We assume furthermore that g satisfies the following conditions: $(\mathrm{i_b})$ $g(t, 0) = 0$ for all $t \in (0, \infty)$, and for every bounded subset B of $(0, \infty) \times [0, \infty)$ let there exist a locally Lebesgue integrable function m_B defined on $(0, \infty)$ such that

$$|g(t, \tau)| \leqq m_B(t) \qquad \text{for } (t, \tau) \in B.$$

$(\mathrm{ii_b})$ There exists a strictly increasing continuous function α defined on $[0, \infty)$ satisfying $\alpha(0) = 0$ and

$$|g(t, \tau) - g(t, \tilde{\tau})| \leqq m_B(t)\, \alpha(|\tau - \tilde{\tau}|)$$

for (t, τ), $(t, \tilde{\tau}) \in B$.

(iii$_b$) For every $\delta > 0$, $\int_0^\delta d\tau/\alpha(\tau) = \infty$.

Under these conditions we can prove the following

THEOREM 2. *Let f be a strongly continuous mapping of* $[0, \infty) \times E$ *into E, carring bounded sets in* $[0, \infty) \times E$ *into bounded sets in E. Suppose furthermore that*

$$\langle u-v, f(t, u)-f(t, v)\rangle_2 \leqq g(t, \|u-v\|) \tag{1.3}$$

for (t, u), $(t, v) \in (0, \infty) \times E$.
Then (D_1) *has a unique strongly continuously differentiable solution u defined on* $[0, \infty)$.

Finally, we consider the equation (D_2) in a Banach space E whose dual space E^* is uniformly convex.

We say u is a solution of (D_2) on $[0, \infty)$ with $u(0) = u_0$ if u is strongly absolutely continuous on any finite interval of $[0, \infty)$ and if

$$u(t) \in D(A), \qquad \frac{d}{dt} u(t) = Au(t) + f(t, u(t))$$

for a. e. $t \in [0, \infty)$.

We assume that A satisfies

$$\langle u-v, Au-Av\rangle_2 \leqq 0 \qquad \text{for } u, v \in D(A), \tag{1.4}$$

and $R(I-\lambda_0 A) = E$ for some $\lambda_0 > 0$.
If the strongly continuous mapping f of $[0, \infty) \times E$ into E has the strongly continuous derivative f_t with respect to t and if both f and f_t carry bounded sets in $[0, \infty) \times E$ into bounded sets in E, then we have

THEOREM 3. *Let A, f and f_t satisfy the assumptions mentioned above. Furthermore, if f satisfies*

$$\langle u-v, f(t, u)-f(t, v)\rangle_1 \leqq \beta(t)\|u-v\| \tag{1.5}$$

for (t, u), $(t, v) \in (0, \infty) \times E$, *where β is a locally Lebesgue integrable function defined on* $(0, \infty)$.
Then (D_2) *has a unique solution u on* $[0, \infty)$ *for each* $u_0 \in D(A)$.

In the paper [1] F. E. Browder proved the global existence in a Hilbert space of the unique solution of (D_1) under the monotonicity condition.

Recently T. M. Flett ([4], [5]) has given the sufficient conditions for both local and global existence in Banach and Hilbert spaces of the unique solution of (D_1).

The contents of this paper are as follows: Some lemmas concerning the scalar differential equation (1.1) are given in § 2. Theorems 1, 2 and 3 are proved in § 3, 4 and 5, respectively. In § 6 we shall give a simple example and some remarks about the relations between our results and those of F. E. Browder and T. M. Flett.

§ 2. Some lemmas.

In the following Lemmas 2.1, 2.2 and 2.3 we assume that g satisfies the assumptions $(\mathrm{i_a})$ and $(\mathrm{ii_a})$ stated in § 1.

LEMMA 2.1. *Let $\{w_n\}$ be a sequence of functions from $[0, a]$ into $[0, b]$ converging uniformly on $[0, a]$ to a function w_0. Let $M>0$ such that*

$$|w_n(t)-w_n(s)| \leqq M|t-s| \qquad \text{for } s, t\in [0, a] \text{ and } n\geqq 1.$$

Suppose furthermore that for each $n\geqq 1$

$$w_n'(t)\leqq g(t, w_n(t)) \qquad \text{for } t\in (0, a) \text{ such that } w_n'(t) \text{ exists}.$$

Then

$$w_0'(t)\leqq g(t, w_0(t)) \qquad \text{for a. e. } t\in (0, a).$$

PROOF. Since $|w_0(t)-w_0(s)|\leqq M|t-s|$ for $s, t\in [0, a]$, $w_0'(t)$ exists for a. e. $t\in [0, a]$.
Let $A_n=\{t\in [0, a]\,;\ w_n'(t) \text{ does not exist}\}$ and let $A=\bigcup_{n=0}^{\infty} A_n$, then mes $(A)=0$.
Set

$$B=\{t\in (0, a]\,;\ \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} g(s, w_0(s))\,ds=g(t, w_0(t))\}.$$

Then, by $(\mathrm{i_a})$, we have mes $([0, a]-B)=0$.
For each $t\in \{[0, a]-A\}\cap B$, $n\geqq 1$ and for sufficiently small $h>0$

$$w_n(t+h)-w_n(t)\leqq \int_t^{t+h} g(s, w_n(s))\,ds.$$

By the Lebesgue's dominated convergence theorem, we have

$$w_0(t+h)-w_0(t)\leqq \int_t^{t+h} g(s, w_0(s))\,ds.$$

Dividing both sides by $h>0$ and letting $h\to 0$, we have $w_0'(t)\leqq g(t, w_0(t))$. Thus we have the inequality

$$w_0'(t)\leqq g(t, w_0(t)) \qquad \text{for a. e. } t\in (0, a).$$

LEMMA 2.2. *Let $M>0$ and Φ be a set of functions from $[0, a]$ into $[0, b]$ with the property that for all $s, t\in[0, a]$ and $w\in\Phi$*

$$|w(t)-w(s)|\leqq M|t-s|. \tag{2.1}$$

Let $z=\sup\{w; w\in\Phi\}$, and suppose that for each $w\in\Phi$

$$w'(t)\leqq g(t, w(t)) \quad \text{for } t\in(0, a) \text{ such that } w'(t) \text{ exists.} \tag{2.2}$$

Then

$$z'(t)\leqq g(t, z(t)) \quad \text{for a. e. } t\in(0, a).$$

PROOF. We follow an argument essentially given in T. M. Flett [4]. By the definition of z and (2.1), z satisfies

$$|z(t)-z(s)|\leqq M|t-s|$$

and

$$0\leqq z(t)-w(t)\leqq z(s)-w(s)+2M|t-s|$$

for all $s, t\in[0, a]$ and all $w\in\Phi$. From this it follows that for each positive integer n we can find a positive integer k, a partition of $[0, a]$ into k subintervals of equal length, and k functions $w_1, \cdots, w_k\in\Phi$ such that in the jth subinterval

$$0\leqq z(t)-w_j(t)\leqq 1/n.$$

We put $w^{(n)}=\mathrm{Max}\,\{w_1, \cdots, w_k\}$. Then $w^{(n)}$ satisfies (2.1) and (2.2). Since

$$0\leqq z(t)-w^{(n)}(t)\leqq 1/\mathrm{n}$$

for all $t\in[0, a]$, the sequence $\{w^{(n)}\}$ converges uniformly to z on $[0, a]$, and the required result follows from Lemma 2.1.

LEMMA 2.3. *Let w be an absolutely continuous function from $[0, a]$ into $[0, b]$ such that $w(0)=(D^+w)(0)=0$ and*

$$w'(t)\leqq g(t, w(t)) \quad \text{for a. e. } t(0, a).$$

Then $w\equiv 0$ on $[0, a]$.

PROOF. The method of the following proof is essentially due to that of Theorem 2.2 in [2].

Suppose that there exists a σ, $0<\sigma\leqq a$ such that $w(\sigma)>0$. Then there exists a solution z of (1.1) with $z(\sigma)=w(\sigma)$ on some interval to the left of σ. As far to the left of σ as z exists, it satisfies the inequality $z(t)\leqq w(t)$, for if this were not the case there would exist a positive σ_1 to the left of σ where $z(\sigma_1)=w(\sigma_1)$, and $z(t)>w(t)$ for $t<\sigma_1$, and sufficiently near σ.

By the assumptions on w we have for sufficiently small $h>0$

$$w(\sigma_1)-w(\sigma_1-h)\leqq\int_{\sigma_1-h}^{\sigma_1} g(t, w(t))\,dt.$$

On the other hand, from the definition of z we have, since $z(\sigma_1)=w(\sigma_1)$,

$$w(\sigma_1)-z(\sigma_1-h) = \int_{\sigma_1-h}^{\sigma_1} g(t, z(t))\,dt,$$

where h is assumed so small that z exists on $[\sigma_1-h, \sigma_1]$.
Thus

$$z(\sigma_1-h)-w(\sigma_1-h)\leqq\int_{\sigma_1-h}^{\sigma_1} [g(t, w(t))-g(t, z(t))]\,dt.$$

Since g is nondecreasing in τ and $z(t)>w(t)$ on $[\sigma_1-h, \sigma_1)$ we have the contradiction $z(\sigma_1-h)\leqq w(\sigma_1-h)$.

We shall next show that $z(t)>0$ on $0<t\leqq\sigma$, as far as it exists. Otherwise $z(t_0)=0$ for some t_0, $0<t_0<\sigma$, and the function $\tilde{z}$ defined by

$$\tilde{z}(t) = \begin{cases} 0 & (0\leqq t\leqq t_0) \\ z(t) & (t_0\leqq t\leqq \sigma) \end{cases}$$

would be a function on $[0, \sigma]$ not identically zero, which satisfies

$$\tilde{z}'(t) = g(t, \tilde{z}(t)), \qquad \tilde{z}(0) = (D^+\tilde{z})(0) = 0.$$

This contradicts the assumption (ii_a). Therefore

$$0<z(t)\leqq w(t)$$

as far to the left of σ as z exists.
It therefore follows that z can be continued as a solution, call it z again, on the whole interval $0<t\leqq\sigma$. Since $\lim_{t\downarrow 0} z(t)=0$, we define $z(0)=0$. Since

$$0<z(t)/t\leqq w(t)/t \qquad \text{for } 0<t\leqq\sigma$$

and $(D^+w)(0)=0$, we have $(D^+z)(0)=0$.
From (ii_a) it follows $z\equiv 0$ on $[0, \sigma]$, but this contradicts the fact $z(\sigma)=w(\sigma)>0$.

LEMMA 2.4. *If g satisfies the assumptions* (i_b), (ii_b) *and* (iii_b) *stated in* § 1, *then for each $T>0$ and $d\geqq 0$ there exists a unique solution w of* (1.1) *on* $[0, T]$ *with the initial condition* $w(0)=d$.

PROOF. Suppose that there are two solutions w_1 and w_2 of (1.1) on $[0, T]$ satisfying $w_1(0)=w_2(0)=d$. Let z be the function defined by

$$z(t) = |w_1(t)-w_2(t)| \qquad \text{for } t\in[0, T].$$

Then there exist $\sigma\in(0, T]$ and $\sigma_0\in[0, \sigma)$ such that $z(\sigma_0)=0$ and $z(t)>0$ for $t\in(\sigma_0, \sigma]$.
Since z is absolutely continuous, $z'(t)$ exists for a. e. $t\in[\sigma_0, \sigma]$ and, by (ii$_b$), we have

$$\begin{aligned}z'(t)&\leqq|w_1'(t)-w_2'(t)| = |g(t, w_1(t))-g(t, w_2(t))|\\&\leqq m_B(t)\,\alpha(z(t)),\end{aligned}$$

where $B=\{(t, w_1(t)), (t, w_2(t));\ t\in[\sigma_0, \sigma]\}$.
Since α is continuous and z is absolutely continuous, we have for sufficiently small $\varepsilon>0$

$$\int_{\sigma_0+\varepsilon}^{\sigma} z'(t)/\alpha\Big(z(t)\Big)dt = \int_{z(\sigma_0+\varepsilon)}^{z(\sigma)} d\tau/\alpha(\tau)\leqq\int_{\sigma_0+\varepsilon}^{\sigma} m_B(\tau)\,d\tau$$

(see [13], p. 211).

By (iii$_b$) and by letting $\varepsilon\downarrow 0$, we have a contradiction.

§ 3. Proof of Theorem 1.

Let the functionals $\langle,\rangle_1$ and $\langle,\rangle_2$ be as in § 1.
We shall give the following two lemmas which are used throughout this paper.

LEMMA 3. 1. (*cf. M. Hasegawa* [6]). *For u, v and w in E,*

(1) $\quad |\langle u ,v\rangle_1|\leqq\|v\|,$

(2) $\quad \langle u, v+w\rangle_1\leqq\langle u, v\rangle_1+\langle u, w\rangle_1$

(3) $\quad \langle u, du+v\rangle_2=d\|u\|+\langle u, v\rangle_2$ *for real number d,*

(4) $\quad \langle u, v\rangle_2\leqq\langle u, v\rangle_1,$

(5) $\quad \langle u, v+w\rangle_2\leqq\langle u, v\rangle_2+\langle u, w\rangle_1,$

(6) $\quad \langle u, v\rangle_2\leqq\langle u, v-w\rangle_2+\|w\|.$

PROOF. (1) and (2) are easy consequences of the definition. For any real number d we have

$$\begin{aligned}\langle u, du+v\rangle_2 &= \frac{1}{2}\lim_{h\downarrow 0}\frac{1}{h}\Big(\|u+h(du+v)\|-\|u-h(du+v)\|\Big)\\&=\frac{1}{2}\Big\{\lim_{h\downarrow 0}\frac{1+dh}{h}\Big(\|u+\frac{1}{1+dh}v\|-\|u\|\Big)\\&\qquad-\lim_{h\downarrow 0}\frac{1-dh}{h}\Big(\|u-\frac{h}{1-dh}v\|-\|u\|\Big)\Big\}+d\|u\|\\&=d\|u\|+\frac{1}{2}\Big(\langle u,v\rangle_1-\langle u, -v\rangle_1\Big)=d\|u\|+\langle u, v\rangle_2,\end{aligned}$$

which proves (3).

(4) follows readily from (2). By the definitions and (2) we have

$$
\begin{aligned}
&\langle u, v\rangle_2+\langle u, w\rangle_1-\langle u, v+w\rangle_2 \\
&\geqq \frac{1}{2}\Big(\langle u, w\rangle_1+\langle u, -(v+w)\rangle_1-\langle u, -v\rangle_1\Big) \\
&\geqq \frac{1}{2}\Big(\langle u, -v\rangle_1-\langle u, -v\rangle_1\Big)=0,
\end{aligned}
$$

which implies (5).

To prove (6) we note that

$$
\begin{aligned}
\langle u, v\rangle_2 &= \frac{1}{2}\lim_{h\downarrow 0}\frac{1}{h}\Big(\|u+hv\|-\|u-hv\|\Big) \\
&= \frac{1}{2}\lim_{h\downarrow 0}\frac{1}{h}\Big(\|u+h(v-w)+hw\|-\|u-h(v-w)-hw\|\Big) \\
&\leqq \frac{1}{2}\lim_{h\downarrow 0}\frac{1}{h}\Big(\|u+h(v-w)\|-\|u-h(v-w\|+2h\|w\|\Big) \\
&= \langle u, v-w\rangle_2+\|w\|.
\end{aligned}
$$

LEMMA 3.2. *Let $u(t)$ be a E-valued function defined on a real interval I such that $u'(t)$ and $\frac{d}{dt}\|u(t)\|$ exist for a. e. $t\in I$. Then*

$$
\frac{d}{dt}\|u(t)\|=\langle u(t),\ u'(t)\rangle_2 \qquad \textit{for a. e. } t\in I.
$$

PROOF. If we denote $D^+u(t)$ and $D^-u(t)$ respectively the right and left derivatives of $u(t)$. Then

$$
\begin{aligned}
&\Big|\frac{1}{h}\Big(\|u(t+h)\|-\|u(t-h)\|\Big)-\frac{1}{h}\Big(\|u(t)+hD^+u(t)\|-\|u(t)\|\Big) \\
&\qquad\qquad +\frac{1}{h}\Big(\|u(t)-hD^-u(t)\|-\|u(t)\|\Big)\Big| \\
&=\frac{1}{h}\Big|\|u(t+h)\|-\|u(t-h)\|-\|u(t)+hD^+u(t)\|+\|u(t)-hD^-u(t)\|\Big| \\
&\leqq \Big\|\frac{1}{h}\Big(u(t+h)-u(t)\Big)-D^+u(t)\Big\|+\Big\|\frac{1}{h}\Big(u(t-h)-u(t)\Big)+D^-u(t)\Big\| \\
&\qquad\qquad \to 0 \text{ as } h\downarrow 0 \qquad \text{for a. e. } t\in I.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&D^+\|u(t)\|+D^-\|u(t)\|=\langle u(t),\ \ D^+u(t)\rangle_1-\langle u(t),\ \ -D^-u(t)\rangle_1 \\
&\qquad \text{for a. e. } t\in I.
\end{aligned}
$$

It follows from the assumptions that

$$\frac{d}{dt}\|u(t)\| = \langle u(t),\ u'(t)\rangle_2 \qquad \text{for a. e. } t\in I.$$

PROOF of THEOREM 1. Since f is strongly continuous on $[0, T]\times S(u_0, r)$ there exist constants $0<r_0\leqq r$, $0<T_1\leqq T$ and $M>0$ such that

$$\|f(t, u)\|\leqq M \text{ for } (t, u)\in[0, T_1]\times S(u_0, r_0).$$

Let $T_0=\text{Min }\{r_0/M, T_1\}$ and let n be a positive integer.
We set $t_0^n=0$, and $u_n(t_0^n)=u_0$. Inductively for each positive integer i, define δ_i^n, t_i^n, $u_n(t_{i-1}^n)$ as follows (cf. G. Webb [14]):

(3. 1) $\qquad \delta_i^n\geqq 0,\qquad t_{i-1}^n+\delta_i^n\leqq T_0;$

(3. 2) $\qquad$ If $\|v-u_n(t_{i-1}^n)\|\leqq M\delta_i^n$ and $t_{i-1}^n\leqq t\leqq t_{i-1}^n+\delta_i^n$, then $\|f(t, v)-f(t_{i-1}^n, u_n(t_{i-1}^n))\|\leqq 1/n;$

(3. 3) $\qquad \|u_n(t_{i-1}^n)-u_0\|\leqq r_0,$

and δ_i^n is the largest number such that (3. 1), (3. 2) and (3. 3) hold.
Let $t_i^n=t_{i-1}^n+\delta_i^n$. We set

$$u_n(t) = u_n(t_{i-1}^n)+\int_{t_{i-1}^n}^{t} f\big(s, u_n(t_{i-1}^n)\big)\,ds \qquad \text{for each } t\in[t_{i-1}^n, t_i^n].$$

Then for each $t\in[t_{k-1}^n, t_k^n]$

$$\begin{aligned}u_n(t) &= u_n(t_{k-1}^n)+\int_{t_{k-1}^n}^{t} f\big(s, u_n(t_{k-1}^n)\big)\,ds\\ &= u_n(t_{k-1}^n)+\int_{t_{k-2}^n}^{t_{k-1}^n} f\big(s, u_n(t_{k-2}^n)\big)\,ds+\int_{t_{k-1}^n}^{t} f\big(s, u_n(t_{k-1}^n)\big)\,ds\\ &=\cdots= u_0+\sum_{j=1}^{k-1}\int_{t_{j-1}^n}^{t_j^n} f\big(s, u_n(t_{j-1}^n)\big)\,ds+\int_{t_{k-1}^n}^{t} f\big(s, u_n(t_{k-1}^n)\big)\,ds.\end{aligned}$$

For each t, s (say $t>s$) in $[0, T_0]$ there exist i, k such that $t\in[t_{i-1}^n, t_i^n]$ and $s\in[t_{k-1}^n, t_k^n]$. Then

$$\begin{aligned}\|u_n(t)-u_n(s)\| \leqq &\int_{s}^{t_k^n}\|f\big(s, u_n(t_{k-1}^n)\big)\|\,ds+\sum_{j=k+1}^{i-1}\int_{t_{j-1}^n}^{t_j^n}\|f\big(s, u_n(t_{j-1}^n)\big)\|\,ds\\ &+\int_{t_{i-1}^n}^{t}\|f\big(s, u_n(t_{i-1}^n)\big)\|\,ds\end{aligned}$$

$$\leqq M(t_k^n - s) + \sum_{j=k+1}^{i-1} M(t_j^n - t_{j-1}^n) + M(t - t_{i-1}^n)$$
$$= M(t-s).$$

On the other hand

$$\|u_n(t) - u_0\| \leqq \sum_{j=1}^{i-1} \int_{t_{j-1}^n}^{t_j^n} \|f(s, u_n(t_{j-1}^n))\| ds + \int_{t_{i-1}^n}^{t} \|f(s, u_n(t_{i-1}^n))\| ds$$
$$\leqq Mt \leqq r_0.$$

We shall show that there exists some positive integer $N = N(n)$ such that $t_N^n = T_0$. Suppose, on the contrary, that this were not true. Then, since $\{t_i^n\}$ is a nondecreasing sequence bounded from above, there is a t_0 in $(0, T_0]$ such that $\lim_{i\to\infty} t_i^n = t_0$.

Since $\|u_n(t_i^n) - u_n(t_k^n)\| \leqq M|t_i^n - t_k^n| \to 0$ as $i, k \to \infty$, $\lim_{i\to\infty} u_n(t_i^n) = v_0$ exists. Let $\sigma_1 > 0$ such that

$$\|f(t, v) - f(t_0, v_0)\| \leqq 1/2n \tag{3.5}$$

whenever $\|v - v_0\| \leqq 2\sigma_1$ and $|t - t_0| \leqq 2\sigma_1$.

Since $\lim_{k\to\infty} f(t_k^n, u_n(t_k^n)) = f(t_0, v_0)$ there exist $\sigma_2 > 0$ and sufficiently large positive integer i such that

$$\|f(t_0, v_0) - f(t_{i-1}^n, u_n(t_{i-1}^n))\| \leqq 1/2n \tag{3.6}$$

whenever $t_0 - t_{i-1}^n \leqq \sigma_2$ and $\|v_0 - u_n(t_{i-1}^n))\| \leqq \sigma_2$.

Set $\sigma = \mathrm{Min}\ \{\sigma_1, \sigma_2\}$. Then there exists a positive integer j such that

$$\delta_j^n < \mathrm{Min}\ \{\sigma/2M, \sigma\}. \tag{3.7}$$

Thus (3. 5), (3. 6) and (3. 7) hold for σ and $k = \mathrm{Max}\ \{i, j\}$.

Consequently, if $\|v - u_n(t_{k-1}^n)\| \leqq M(\delta_k^n + \sigma/4M)$ and $t_{k-1}^n \leqq t \leqq t_{k-1}^n + \sigma$, then

$$\|v - v_0\| \leqq \|v - u_n(t_{k-1}^n)\| + \|u_n(t_{k-1}^n) - v_0\| \leqq 3\sigma/4 + \sigma < 2\sigma,$$

and

$$|t - t_0| \leqq |t - t_{k-1}^n| + |t_0 - t_{k-1}^n| \leqq 2\sigma.$$

It therefore follows that

$$\|f(t, v) - f(t_{k-1}^n, u_n(t_{k-1}^n))\| \leqq \|f(t, v) - f(t_0, v_0)\|$$
$$+ \|f(t_0, v_0) - f(t_{k-1}^n, u_n(t_{k-1}^n))\|$$
$$\leqq 1/2n + 1/2n = 1/n.$$

This is a contradiction to the choice of δ_k^n.

We next show that the sequence of continuous functions $\{u_n(t)\}$ converges uniformly to a E-valued function $u(t)$ on $[0, T_0]$.
For this we set $w_{mn}(t)=\|u_m(t)-u_n(t)\|$ for $m>n\geqq 1$ and $t\in[0, T_0]$, and remark first that, since

$$(3.8)\qquad |w_{mn}(t)-w_{mn}(s)|\leqq 2M|t-s| \qquad \text{for } s, t\in[0, T_0],$$

$w'_{mn}(t)$ exists for a. e. $t\in[0, T_0]$.

For each $t\in(0, T_0)$ such that $w'_{mn}(t)$ exists there exist positive integers i and j such that $t\in(t^n_{i-1}, t^n_i)$ and $t\in(t^m_{j-1}, t^m_j)$.
By Lemma 3.1 (1), (6) and Lemma 3.2 we have

$$(3.9)\qquad \begin{aligned} w'_{mn}(t) &= \langle u_m(t)-u_n(t), f(t, u_m(t^m_{j-1}))-f(t, u_n(t^n_{i-1}))\rangle_2 \\ &\leqq g(t, w_{mn}(t))+\|f(t, u_m(t))-f(t, u_m(t^m_{j-1}))\| \\ &\qquad +\|f(t, u_n(t))-f(t, u_n(t^n_{i-1}))\|. \end{aligned}$$

On the other hand

$$\|u_m(t)-u_m(t^m_{j-1})\|\leqq M|t-t^m_{j-1}|\leqq M\delta^m_j \text{ and } \|u_n(t)-u_n(t^n_{i-1})\|\leqq M\delta^n_i.$$

Thus we have by (3.2)

$$(3.10)\qquad w'_{mn}(t)\leqq g\big(t, w_{mn}(t)\big)+1/m+1/n\leqq g\big(t, w_{mn}(t)\big)+2/n$$

for a. e. $t\in(0, T_0)$. Let $w_n(t)=\sup_{m>n}\{w_{mn}(t)\}$ for $t\in[0, T_0]$.
Then $w_n(0)=(0)$ for all n. It thus follows from (3.8), (3.10) and Lemma 2.2 that

$$(3.11)\qquad |w_n(t)-w_n(s)|\leqq 2M|t-s| \qquad \text{for } s, t\in[0, T_0],$$

and

$$(3.12)\qquad w'_n(t)\leqq g(t, w_n(t))+2/n \qquad \text{for a. e. } t\in(0, T_0).$$

Since

$$0\leqq w_n(t)\leqq w_n(0)+2Mt\leqq 2MT_0 \qquad \text{for } n\geqq 1 \text{ and } t\in[0, T_0]$$

the sequence $\{w_n\}$ is equicontinuous and uniformly bounded, and hence it has a subsequence converging uniformly on $[0, T_0]$ to a function w, and obviously $w(0)=0$. From (3.12) and the proof of Lemma 2.1 we have

$$w'(t)\leqq g\big(t, w(t)\big) \qquad \text{for a. e. } t\in(0, T_0).$$

We show next that $(D^+w)(0)=0$. Since f is continuous at $(0, u_0)$, given $\varepsilon>0$ we can find $\delta>0$ such that $\|f(t, u)-f(t, u_0)\|<\varepsilon$ whenever $0\leqq t\leqq\delta$ and $\|u-u_0\|\leqq\delta$. Let $\delta_0=\text{Min}\,\{\delta, \delta/M\}$. Since $\|u_n(t)-u_0\|\leqq Mt\leqq\delta$, $\|f(t, u_m(t))-f(t, u_n(t))\|<2\varepsilon$ whenever $m>n\geqq 1$ and $t\in[0, \delta_0]$. By Lemma 3.1 (1) and (3.9) we have

$$\begin{aligned} w'_{mn}(t) &= \langle u_m(t)-u_n(t),\ f(t, u_m(t^m_{j-1}))-f(t, u_n(t^n_{i-1}))\rangle_2 \\ &\leqq \|f(t, u_m(t^m_{j-1}))-f(t, u_m(t^n_{i-1}))\| \\ &\leqq \|f(t, u_m(t))-f(t, u_n(t))\| + 2/n \leqq 2(\varepsilon+1/n) \end{aligned}$$

for a. e $t\in(0, \delta_0)$, and hence, by integrating the above inequality,

$$0 \leqq w_{mn}(t) \leqq 2(\varepsilon+1/n)\, t,$$

whence $(D^+w)\ (0)=0$.

From Lemma 2. 3 we deduce now that $w\equiv 0$, and this implies that the sequence $\{u_n\}$ is uniformly convergent on $[0, T_0]$. The limit u of this sequence satisfies

$$u(t) = u_0 + \int_0^t f\big(s, u(s)\big)ds \qquad \text{for } t\in[0, T_0].$$

To show this, note that

$$\int_0^t f\big(s, u(s)\big)ds = \sum_{j=1}^{k-1}\int_{t^n_{j-1}}^{t^n_j} f\big(s, u(s)\big)ds + \int_{t^n_{k-1}}^{t} f\big(s, u(s)\big)ds$$

for $t\in[t^n_{k-1}, t^n_k]$. Then we have by (3. 4)

$$\begin{aligned} &\Big\| u_n(t) - \Big(u_0 + \int_0^t f\big(s, u(s)\big)ds\Big)\Big\| \\ &\leqq \sum_{j=1}^{k-1}\int_{t^n_{j-1}}^{t^n_j} \Big\| f\big(s, u_n(t^n_{j-1})\big) - f\big(s, u(s)\big)\Big\| ds \\ &\qquad + \int_{t^n_{k-1}}^{t} \Big\| f\big(s, u_n(t^n_{k-1})\big) - f\big(s, u(s)\big)\Big\| ds \\ &\leqq \Big[1/n + \operatorname*{Max}_{0\leqq s\leqq T_0} \Big\| f\big(s, u_n(s)\big) - f\big(s, u(s)\big)\Big\|\Big] T. \end{aligned}$$

Because of the uniform convergence of $\{u_n\}$ to u on $[0, T_0]$, $C=\{u_n(t), u(t)\,;\ 0\leqq t\leqq T_0,\ n=1, 2, \cdots\}$ is a compact set in E. Since $f(t, u)$ is uniformly continuous on $[0, T_0]\times C$ we have

$$\operatorname*{Max}_{0\leqq s\leqq T_0} \Big\| f\big(s, u_n(s)\big) - f\big(s, u(s)\big)\Big\| \to 0 \ \text{ as } n\to\infty,$$

and hence the required result follows.

Thus u is a strongly continuously differentiable solution of (D_1) on $[0, T_0]$.

Let v be another strongly continuously differentiable solution of (D_1) on $[0, T_0]$ and let $z(t)=\|u(t)-v(t)\|$. Then $z(0)=0$, and

$$z'(t) = \langle u(t)-v(t),\ f(t, u(t)) - f(t, v(t))\rangle_2 \leqq g(t, z(t))$$

for a. e. $t\in(0, T_0)$. The fact $(D^+z)(0)=0$ follows from

$$0\leqq z(t)/t = \|(u(t)-v(t))/t\|\to 0 \text{ as } t\to 0.$$

It therefore follows from Lemma 2.3 that $z\equiv 0$. The proof is complete.

§ 4. Proof of Theorem 2.

PROOF of THEOREM 2. It follows from Lemma 2.4 and Theorem 1 that there exists a unique local solution u of (D_1) on some interval $[0, T_0^*)$. We assume that $[0, T_0^*)$ is a maximal interval of existence of u. We have only to show that $T_0^*<\infty$ leads to a contradiction.

Let $w(t)=\|u(t)-u_0\|$ for $t\in[0, T_0^*)$. Then, by Lemma 3.1 (6), we have

$$\begin{aligned} w'(t) &= \langle u(t)-u_0,\ f(t, u(t))\rangle_2 \\ &\leqq \langle u(t)-u_0,\ f(t, u(t))-f(t, u_0)\rangle_2+\|f(t, u_0)\| \\ &\leqq g(t, w(t))+L \end{aligned} \tag{4.1}$$

for a. e. $t\in(0, T_0^*)$, where $L=\operatorname*{Max}_{0\leqq t\leqq T_0^*}\|f(t, u_0)\|$.

In virtue of (i_b), (ii_b) and (iii_b) the differential equation

$$z'(t) = g(t, z(t))+L \tag{4.2}$$

has a unique solution z on $[0, T_0^*]$ with the initial condition $z(0)=0$.
It therefore follows from (4.1) that

$$w(t)\leqq z(t) \qquad \text{for all } t\in[0, T_0^*). \tag{4.3}$$

In fact, if we assume that there exists a $\sigma\in(0, T_0^*)$ such that $w(\sigma)>z(\sigma)$. Then there exists a $\sigma_0\in[0, \sigma)$ such that $w(\sigma_0)=z(\sigma_0)$ and $w(t)>z(t)$ for $t\in(\sigma_0, \sigma]$.
Let $\theta(t)=w(t)-z(t)$. Then, by (4.1), (4.2) and (ii_b), we have

$$\theta'(t)=w'(t)-z'(t)\leqq g(t, w(t))-g(t, z(t))\leqq m_B(t)\alpha(\theta(t))$$

for a. e. $t\in[\sigma_0, \sigma]$, where $B=\{(t, w(t)), (t, z(t))\,;\ \sigma_0\leqq t\leqq\sigma\}$.
Since α is continuous and θ is absolutely continuous, we have for sufficiently small $\varepsilon>0$

$$\int_{\sigma_0+\varepsilon}^{\sigma}\theta'(t)/\alpha\big(\theta(t)\big)dt = \int_{\theta(\sigma_0+\varepsilon)}^{\theta(\sigma)}d\tau/\alpha(\tau)\leqq\int_{\sigma_0+\varepsilon}^{\sigma}m_B(t)\,dt.$$

By (iii_b) and by letting $\varepsilon\downarrow 0$, we have a contradiction.

(4.3) implies that

$$\|u(t)\|\leqq\|u_0\|+\operatorname*{Max}_{0\leqq t\leqq T_0^*}\{z(t)\} \text{ for } t\in[0, T_0^*).$$

Since $\{f(t, u(t))\,;\ t\in[0, T_0^*)\}$ is a bounded set in E, we have

$$\|u(t)-u(s)\| \leqq \left|\int_s^t \left\|f\left(\tau, u(\tau)\right)\right\| d\tau\right| \to 0 \text{ as } s, t \uparrow T_0^*.$$

Let $v_0=\lim_{t\uparrow T_{0*}} u(t)$, then we can apply Theorem 1 once more with the initial condition $u(T_0^*)=v_0$, and obtain a unique continuation of the solution u beyond T_0^*, which contradicts the assumption on T_0^*.

§ 5. Proof of Theorem 3.

Throughout this section we assume that the dual space E^* is uniformly convex.

We say that F is a duality mapping of E into E^* if to each u in E it assigns (in general a set) $F(u)$ in E^* determined by

$$F(u) = \left\{x^* ;\ x^* \in E^* \text{ such that } (u, x^*) = \|u\|^2 = \|x^*\|^2\right\},$$

where (u, x^*) denotes the value of x^* at u.

Since E^* is uniformly convex F is single-valued and uniformly continuous on any bounded subset of E (see [9]).

LEMMA 5.1. *For each $u \neq 0$ and v in E*

$$\langle u, v\rangle_2 = Re(v, F(u))/\|u\|.$$

PROOF. Since $\langle u, v\rangle_1=\mathrm{Re}(v, F(u))/\|u\|$ for each $u\neq 0$ and v in E (see the proof of Proposition 2.5 in [11]),

$$\langle u, v\rangle_2 = \frac{1}{2}\mathrm{Re}\left(v, F(u)\right)-\mathrm{Re}\left(-v, F(u)\right) = \mathrm{Re}\left(v, F(u)\right).$$

We recall that A satisfies

$$\langle u-v,\ Au-Av\rangle_2 \leqq 0 \qquad \text{for } u, v \in D(A), \tag{5.1}$$

and $\mathrm{R}(I-\lambda_0 A)=E$ for some $\lambda_0>0$.
For such an operator A we have

LEMMA 5.2. *$(I-\lambda A)^{-1}$ exists for any $\lambda>0$.*
Set $J_n = (I-\frac{1}{n} A)^{-1}$ *and* $A_n=A\ J_n=n(J_n-I)$ *for* $n=1, 2, \cdots,$.
Then

(1) $\|J_n u-J_n v\| \leqq \|u-v\|$ *for* $u, v\in E$,

(2) $\|A_n u\| \leqq Au\|$ *for* $u\in D(A)$,

(3) $\langle u-v,\ A_n u-A_n v\rangle_2 \leqq 0$ *for* $u, v\in E$,

and

(4) *A is demiclosed, that is, if $u_n \in D A)$, $n=1, 2, \cdots$, $u_n \to u$ (strongly in E) and $Au_n \to v$ (weakly in E), then $u \in D(A)$ and $v=Au$.*

PROOF. In virtue of Lemma 5.1, $-A$ is m-monotonic in the sense of T. Kato [9], and hence, the existence of $(I-\lambda A)^{-1}$ and (1), (2) and (4) follows from Lemma 2.5 in [9]. To prove (3) note that

$$\begin{aligned}\langle u-v,\ A_n u - A_n v\rangle_2 &= n\langle u-v,\ J_n u - J_n v - (u-v)\rangle_2 \\ &= n(\langle u-v,\ J_n u - J_n v\rangle_2 - \|u-v\|) \\ &\leqq n(\|J_n u - J_n v\| - \|u-v\|) \leqq 0,\end{aligned}$$

where we used (1) and Lemma 3.1 (1), (4).

In Theorem 2, if $g(t, \tau) = \beta(t)\tau$, where β is a locally Lebesgue integrable function defined on $(0, \infty)$, then the conclusion of Theorem 2 remains valid. In fact, it is obvious that this function $\beta(t)\tau$ satisfies the conditions ($\mathrm{i_b}$), ($\mathrm{ii_b}$) and ($\mathrm{iii_b}$) except that $\beta(t)\tau$ need not be nondecreasing in τ for fixed t. However, the nondecreasing nature of g in τ was used in establishing Lemma 2.3 which is valid for this $\beta(t)\tau$.

LEMMA 5.3. *Under the hypothesis of Theorem 3 the differential equation*

$$\frac{d}{dt} u_n(t) = A_n u_n(t) + f\Big(t, u_n(t)\Big), \qquad u_n(0) = u_0 \in E,$$

has a unique strongly continuously differentiable solution u_n defined on $[0, \infty)$.

PROOF. Since $\|A_n u - A_n v\| \leqq 2n\|u-v\|$ for u, v in E, $A_n u + f(t, u)$ carries bounded sets in $[0, \infty)\times E$ into bounded sets in E. By Lemma 3.1 (5) and Lemma 5.2 (3) we have

$$\begin{aligned}&\langle u-v,\ A_n u + f(t, u) - (A_n v + f(t, v))\rangle_2 \\ &\leqq \langle u-v,\ A_n u - A_n v\rangle_2 + \langle u-v,\ f(t, u) - f(t, v)\rangle_1 \\ &\leqq \beta(t)\|u-v\|\end{aligned}$$

for (t, u), $(t, v) \in [0, \infty)\times E$.

Hence the assertion follows directly from Theorem 2 and the above mentioned remark.

We shall now deduce some estimates for $u_n(t)$.

LEMMA 5.4. *Let $u_0 \in D(A)$. Then $\{u_n(t)\}$ and $\{u_n' t)\}$ are bounded on any finite interval of $[0, \infty)$.*

PROOF. By Lemma 3.1 (3) and Lemma 5.2 (2), (3)

$$\begin{aligned}
&\frac{d}{dt}\|u_n(t)-u_0\| = \langle u_n(t)-u_0,\ A_nu_n(t)+f(t,u_n(t))\rangle_2 \\
&\leqq \langle u_n(t)-u_0,\ A_nu_n(t)\rangle_2+\langle u_n(t)-u_0,\ f(t,u_n(t))\rangle_1 \\
&\leqq \langle u_n(t)-u_0,\ f(t,u_n(t))-f(t,u_0)\rangle_1+\|f(t,u_0)\|+\|A_nu_0\| \\
&\leqq \beta(t)\|u_n(t)-u_0\|+\|f(t,u_0)\|+\|A\,u_0\|.
\end{aligned}$$

Thus we have

$$\|u_n(t)-u_0\| \leqq \int_0^t \exp\left(\int_s^t \beta(\tau)\,d\tau\right)\left(\|f(s,u_0)\|+\|A\,u_0\|\right)ds$$

for $n=1, 2, \cdots$. This implies

$$\|u_n(t)\| \leqq \|u_0\|+\int_0^t \exp\left(\int_s^t \beta(\tau)\,d\tau\right)\left(\|f(s,u_0)\|+\|A\,u_0\|\right)ds \tag{5.2}$$

for $t\in[0,\infty)$ and $n=1, 2, \cdots$.

For each fixed $h>0$ we have, by Lemma 3.1 (5) and Lemma 5.2 (3),

$$\begin{aligned}
&\frac{d}{dt}\|u_n(t+h)-u_n(t)\| = \langle u_n(t+h)-u_n(t),\ A_nu_n(t+h)-A_nu_n(t) \\
&\qquad\qquad +f(t+h,\ u_n(t+h))-f(t,u_n(t))\rangle_2 \\
&\leqq \langle u_n(t+h)-u_n(t),\ f(t+h,u_n(t+h))-f(t,u_n(t))\rangle_1 \\
&\leqq \langle u_n(t+h)-u_n(t),\ f(t+h,u_n(t+h))-f(t,u_n(t))\rangle_1 \\
&\qquad\qquad +\|f(t+h,u_n(t))-f(t,u_n(t))\| \\
&\leqq \beta(t+h)\|u_n(t+h)-u_n(t)\|+\|f(t+h,u_n(t))-f(t,u_n(t))\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|u_n(t+h)-u_n(t)\| \leqq \|u_n(h)-u_n(0)\| \\
&\qquad +\int_0^t \exp\left(\int_s^t \beta(\tau+h)\,d\tau\right)\left\|f\left(s+h,u_n(s)\right)-f\left(s,u_n(s)\right)\right\|ds
\end{aligned}$$

By dividing the above inquality by h and letting $h\downarrow 0$, we have

$$\|u_n'(t)\| \leqq \|u_n'(0)\|+\int_0^t \exp\left(\int_s^t \beta(\tau)\,d\tau\right)\left\|f_s\left(s,u_n(s)\right)\right\|ds \tag{5.3}$$

for $n=1, 2, \cdots$. This completes the proof.

We shall now give the proof of Theorem 3.

PROOF of THEOREM 3. By (5.2) and (5.3) there exists constant $M_T>0$ for each $T>0$ such that

$$\|u_n'(t)\|+\|f(t,u_n(t))\| \leqq M_T \qquad \text{for } t\in[0,T] \text{ and } n\geqq 1. \tag{5.4}$$

By Lemma 3.1 (5) and Lemma 5.1, for each $t\in[0,T]$ such that

$$\frac{d}{dt}\|u_n(t)-u_m(t)\| \text{ exists and } u_n(t)-u_m(t)\neq 0,$$

$$\frac{d}{dt}\|u_n(t)-u_m(t)\| = \langle u_n(t)-u_m(t),\ A_nu_n(t)-A_mu_m(t) + f(t, u_n(t))-f(t, u_m(t))\rangle_2$$

$$\leqq \beta(t)\|u_n(t)-u_m(t)\| + 2M_T\|F(u_n(t)-u_m(t))-F(J_nu_n(t)-J_mu_m(t))\|/\|u_n(t)-u_m t)\|.$$

It follows that

$$\frac{d}{dt}\|u_n(t)-u_m(t)\|^2 \leqq 2\beta(t)\|u_n(t)-u_m(t)\|^2 + 4M_T\|F(u_n(t)-u_m(t))-F(J_nu_n(t)-J_mu_m(t))\|.$$

On the other hand, for each $t\in[0, T]$ such that $\frac{d}{dt}\|u_n(t)-u_m(t)\|$ exists and $u_n(t)-u_m(t)=0$,

$$\frac{d}{dt}\|u_n(t)-u_m(t)\| = \langle 0,\ A_nu_n(t)-A_mu_m(t)\rangle_2 = 0.$$

Thus we have

$$\frac{d}{dt}\|u_n(t)-u_m(t)\|^2 \leqq 2\beta(t)\|u_n(t)-u_m(t)\|^2 + 4M_T\|F(u_n(t)-u_m(t))-F(J_nu_n(t)-J_mu_m(t))\|$$

for a. e. $t\in[0, T]$ and $n, m\geqq 1$.
Consequently

$$\|u_n(t)-u_m(t)\|^2 \leqq 4M_T\int_0^t \exp\left(\int_s^t 2\beta(\tau)\,d\tau\right)\|F(u_n(s)-u_m(s))-F(J_nu_n(s) -J_mu_m(s))\|\,ds$$

for $t\in[0, T]$ and $n, m\geqq 1$.
In virtue of (5. 4) and the definition of A_n

$$\|u_n(s)-u_m(s)-(J_nu_n(s)-J_mu_m(s))\| \leqq \frac{1}{n}\|A_nu_n(s)\| + \frac{1}{m}\|A_mu_m(s)\|$$

$$\leqq M_T(1/n+1/m)\to 0 \text{ as } n, m\to\infty.$$

Since $F(u)$ is uniformly continuous on any bounded set in E, $\{u_n(t)\}$ converges uniformly to a continuous function $u(t)$ on $[0, T]$ for each $T>0$. The absolute continuity of $u(t)$ on $[0, T]$ follows from the inequality

$$\|u_n(t)-u_n(s)\| \leqq \left|\int_s^t \|u_n'(\tau)\|\,d\tau\right| \leqq M_T|t-s| \qquad \text{for } t, s\in[0, T].$$

We show next that $u(t)$ is a solution of (D_1).
By (5. 4) we have

(5. 5) $$\|A_n u_n(t)\| \leqq \|u'_n(t)\| + \|f(t, u_n(t))\| \leqq M_T$$

for $t \in [0, T]$ and $n \geqq 1$.

This implies that $\{A_n u_n(t)\}$ is a bounded set in $L^2_E [0, T]$ for each $T>0$, where $L^2_E [0, T]$ denotes the set of all square integrable E-valued strongly measurable functions on $[0, T]$.

Thus some subsequence of $\{A_n u_n(t)\}$ converges to an element z weakly in $L^2_E [0, T]$. For notational convenience we assume that $\{A_n u_n(t)\}$ itself converges to z weakly in $L^2_E [0, T]$.

Let $C[t]$ be the set of all weak limit in E of a subsequence of $\{A_n u_n(t)\}$ for each fixed $t \in [0, T]$.

We will show that $u(t) \in D(A)$ for all $t \in [0, T]$ and $z(t) = A\ u(t)$ for a. e. $t \in [0, T]$ (cf. T. Kato [10]).
To show this we note that for each $v \in C[t]$ there exists a subsequence $\{A_{nm} u_{nm}(t)\}$ such that $w\text{-}\lim_{m \downarrow \infty} A_{nm} u_{nm}(t) = v$, where w-lim denotes weak limit in E. Since $J_{nm} u_{nm}(t) \to u(t)$, $J_{nm} u_{nm}(t) \in D(A)$ and $A_{nm} u_{nm}(t) = A\ J_{nm} u_{nm}(t)$, it follows from the demiclosedness of A that

$$u(t) \in D(A) \text{ and } v = A\ u(t).$$

Hence $C[t]$ consists of only one element for each $t \in [0, T]$. Since any subsequence of $\{A_n u_n(t)\}$ has a subsequence converging weakly to the same element $v = v(t)$, $\{A_n u_n(t)\}$ itself converges weakly to $v(t)$ for each $t \in [0, T]$. Since $\{A_n u_n(t)\}$ converges to z weakly in $L^2_E [0, T]$, z is the strong limit of the type $\sum_i a_i A_{n+i} u_{n+i}$. Here $\{a_i\}$ is a finite set of nonnegative numbers such that $\sum_i a_i = 1$.
Thus we can find a subsequence of the above sequence converging to $z(t)$ strongly in E for a. e. $t \in [0, T]$.

Let U be any open convex neighbourhood of 0 in the weak topology of E. Then there exists an open convex neighbourhood V of 0 in the same topology of E such that $V + V \subset U$.
Since $v(t) + V$ is open convex in the weak topolopy of E, there is a n_0 such that

$$A_n u_u(t) \in v(t) + V \qquad \text{for } n \geqq n_0.$$

Thus the convex combination of the type $\sum_i a_i A_{n+i} u_{n+i}(t)$ belongs to $v(t) + V$ for $n \geqq n_0$. Hence $z(t) \in (v(t) + V)^{-w}$, where $(v(t) + V)^{-w}$ denotes the closure of $v(t) + V$ with respect to the weak topology of E. Since

$$(v(t)+V)^{-w}\subset(v(t)+V)+V\subset v(t)+U,$$

it follows that $z(t)-v(t)\in U$. This implies that

$$z(t)=v(t) \qquad \text{for a. e. } t\in[0,T].$$

Since $\|A_n u_n(t)\|\leqq M_T$ the norm of a convex combination of $A_n u_n(t)$'s is also $\leqq M_T$. It follows that $\|z(t)\|\leqq M_T$ for a. e. $t\in[0,T]$ and that $z(t)$ is Bochner integrable on $[0,T]$. Since $L^2_E[0,T]^*=L^2_{E^*}[0,T]$ and since

$$(u_n(t),x^*)=(u_0,x^*)+\int_0^t\Big(A_n u_n(s)+f\big(s,u_n(s)\big),\ x^*\Big)ds$$

for each $x^*\in E^*$ and $t\in[0,T]$, we have by going to $n\to\infty$

$$(u(t),x^*)=(u_0,x^*)+\int_0^t\Big(z(s)+f\big(s,u(s)\big),x^*\Big)ds.$$

Thus we obtain that $\frac{d}{dt}u(t)$ exists for a. e. $t\in[0,T]$ and

$$\frac{d}{dt}u(t)=z(t)+f(t,u(t)=A\ u(t)+f(t,u(t)) \qquad \text{for a. e. } t\in[0,T].$$

Since T is arbitrary, the proof is complete.

§ 6. Remarks and an example.

In this section we give some remarks about the relations between our results and those of F. E. Browder and T. M. Flett. We give also a simple example to which our Theorem 2 applies.

REMARK 1. In the papers [4] and [5] T. M. Flett has given sufficient conditions for the existence in Banach and Hilbert spaces of the unique local solution of (D_1) on some interval $[0,T_0]$ under the following conditions:
(A) E is a Banach space and f is a continuous mapping of $[0,T]\times S(u_0,r)$ into E such that for all (t,u), $(t,v)\in(0,T]\times S(u_0,r)$

(6.1) $$\|f(t,u)-f(t,v)\|\leqq g(t,\|u-v\|);$$

(B) E is a Hilbert space with inner product (,) and f is a continuous mapping of $[0,T]\times S(u_0,r)$ into E such that for all (t,u), $(t,v)\in(0,T]\times S(u_0,r)$

(6.2) $$\operatorname{Re}(f(t,u)-f(t,v),\ u-v)\leqq\|u-v\|g(t,\|u-v\|),$$

where g is a continuous function defined on $(0,T]\times[0,2r]$ satisfying the condition ($\mathrm{ii_a}$) in § 1 in this paper.

In Theorem 1 if we assume that $g(t,\tau)$ is continuous on $(0,T]\times[0,2r]$,

then we can drop the assumption that $g(t, \tau)$ is nondecreasing in τ for fixed t (cf. [2]).
In virtue of this fact and Lemma 3.1 (1), (4) our result is an extension of (A). If E is a Hilbert space with inner product (,), then we can easily see that

$$\langle u, v\rangle_2 = \mathrm{Re}(v, u)/\|u\| \qquad \text{for } u\neq 0 \text{ and } v \text{ in } E,$$

and hence, our condition of Theorem 1 becomes

$$\mathrm{Re}(f(t, u)-f(t, v),\ u-v) \leqq \|u-v\| g(t, \|u-v\|)$$

for all (t, u) $(t, v) \in (0, T] \times S(u_0, r)$. Thus our result is also an extension of (B).

Let $F(u)$ be the duality mapping of E into E^* defined in § 5. Then for each $u\neq 0$ and v in E

$$\langle u, v\rangle_2 \leqq \mathrm{Re}(v, x^*)/\|u\| \qquad \text{for some } x^* \in F(u)$$

(see the proof of Proposition 2.5 in [11]).

Thus we can replace the condition of Theorem 1 by the following one.

$$\mathrm{Re}(f(t, u)-f(t, v), x^*) \leqq \|u-v\| g(t, \|u-v\|)$$

for (t, u), $(t, v) \in (0, T] \times S(u_0, r)$ and for all $x^* \in F(u-v)$.

Hence our result is a generalization of (B) into a general Banach space.

Remark 2. In [1] F. E. Browder proved the existence and uniqueness of a strongly continuously differentiable solution of (D_1) on $[0, \infty)$ under the following conditions:
(I) E is a Hilbert space with inner product (,) and f is a continuous mapping of $[0, \infty)\times E$ into E, carring bounded sets in $[0, \infty)\times E$ into bounded sets in E.
(II) There exists a real-valued continuous function $c(t)$ defined on $[0, \infty)$ such that

$$\mathrm{Re}\,(f(t, u)-f(t, v),\ u-v) \leqq c(t)\|u-v\|^2 \tag{6.3}$$

for all (t, u), $(t, v) \in [0, \infty)\times E$.

By the same argument as in Remark 1 we see that Theorem 2 is a generalization into a general Banach space of the above result of F. E. Browder.

The following example shows that the conditions of Theorem 2 are more general than those of F. E. Browder.

EXAMPLE. Let $E=R^1$ and let $a(t)$ be the function defined by

$$a(t)=\begin{cases} t\,(0\leqq t\leqq\varepsilon) \\ \varepsilon\,(t>\varepsilon)\end{cases}$$

where ε is a positive constant. We consider the differential equation

$$\frac{d}{dt}u=f(t,u)=\begin{cases} 1+\dfrac{1}{1+\sqrt{u}} & (t\geqq 0,\ u>a(t)) \\ 1+\dfrac{1}{1+\sqrt{a(t)}} & (t\geqq 0,\ u\leqq a(t)).\end{cases}$$

Obviously, the function $f(t,u)$ is continuous from $[0,\infty)\times R^1$ into R. However the function $f(t,u)$ does not satisfy the monotonicity condition (6.3) but does satisfy all our conditions of Theorem 2.

In fact, for $u\neq v$ and $t>0$

$$\begin{aligned}&\langle u-v, f(t,u)-f(t,v)\rangle_2=(f(t,u)-f(t,v))\,(u-v)/|u-v|\\ &=\pm(f(t,u)-f(t,v))\\ &\leqq\begin{cases}(1/2\sqrt{a(t)}\,)|u-v| & (u,v>a(t),t>0)\\ (1/2\sqrt{a(t)}\,)|u-v| & (u>a(t),\ 0\leqq v\leqq a(t),\ t>0)\\ (1/2\sqrt{a(t)}\,)|u-v| & (u>a(t),\ v<0,t>0)\\ 0 & (u,v\leqq a(t,)\ t>0).\end{cases}\end{aligned}$$

Thus we have

$$\langle u-v,\ f(t,u)-f(t,v\rangle_2\leqq\ (1/2\sqrt{a(t)}\,)|u-v|$$

for all (t,u), $(t,v)\in(0,\infty)\times R^1$.

Set $g(t,\tau)=(1/2\sqrt{a(t)}\,)\,\tau$ and $\alpha(t)=t$, then it follows easily that g and α satisfy all our conditions of Theorem 2.

On the other hand we have

$$(f(t,u)-f(t,v),\ u-v)\leqq(1/2\sqrt{a(t)}\,)|u-v|^2$$

for all (t,u), $(t,v)\in(0,\infty)\times R^1$.

Since $1/2\sqrt{a(t)}$ is discontinuous at 0, the condition (6.3) does not hold.

REMARK 3. In Theorem 3 if A is linear and $D(A)$ is dense in E, then A is the infinitesimal generator of a strongly continuous contraction semi-group $\{T(t);\ t\geqq 0\}$ (see M. Hasegawa [6]).
In this case the integral equation

$$v(t) = u_0 + \int_0^t T(t-s)\, f\Big(s, v(s)\Big) ds$$

has a unique solution for each $u_0 \in D(A)$ by the same argument as G. Webb [15]. We don't know whether the solution of the above integral equation is a solution of (D_2).

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