

Parametrices for pseudo-differential equations with double characteristics I.

By Kazuhiro YAMAMOTO

(Received June 19, 1975)

0. Introduction.

In this paper we shall construct parametrices and prove solvability and hypoellipticity for some classes of pseudo-differential operators whose characteristic set is a closed manifold of codimension 2 in the cotangent space.

We shall consider a pseudo-differential operator $L(x, D)$ with double characteristics ;

$$L(x, D) = (P \circ Q)(x, D) + R(x, D),$$

where P , Q and R are pseudo-differential operators whose principal part are essentially transformed into the pseudo-differential operator of the following type $(M) D_n - ix_n^k a(x, D')$ (with $a(x, \xi') \neq 0$) by a canonical transformation.

Theory of the local solvability of pseudo-differential operators with simple characteristics was extensively studied in [1] and [14]. The description of their condition was based on the classical Hamilton-Jacobi theory of characteristics and bicharacteristics. However, the case of multiple characteristics is much more complicated. The good example of pseudo-differential operators with double characteristics are pseudo-differential operators whose principal symbols are written by the product of those of the type (M) .

Investigation of the operator whose principal part is the product of abstract first order evolutionary equations was first made in Treves [18] when $k=1$. Furthermore, for general odd integer k , Gilioli-Treves [7] obtained the necessary and sufficient conditions of local solvability for the differential operator R^2 whose principal part is the product of the differential operators of the type (M) with $a(x, D') = aD'$. However, when the base space is a manifold, their conditions was not intrinsic for coordinate systems and in particular, it is not clear how to microlocalize the pseudo-differential operator in their argument. As a matter of fact, in the special case when $k=1$ and the base space is more general manifold, Boutet de

Monvel-Treves [2], [3], Boutet de Monvel [4] and Sjöstrand [15] showed the condition for denoting the principal symbol by the desired form and got the fairly intrinsic condition for existence of parametrices. In order to transform the principal symbol into the product of pseudo-differential operators of the type (M) essentially, they imposed the condition for L ; the principal symbol is equivalent to the square of the distance function to the characteristic set of L . But, their condition and the winding number of L , which they used, are not useful when k is a general integer.

Our approach is different. We seek the necessary and sufficient condition in order to transform the many pseudo-differential operators into those of type (M) by only one canonical transformation at the same time. In this argument our main tools will be the Hamilton-Jacobi theory. For sake of construction of parametrices of L we use the vector valued pseudo-differential operators ([11], [15], [17]), in particular Sjöstrand [15] method is also useful for us. The simplest example of L was studied by [7] and their condition of local solvability inspired us.

A forthcoming paper [19], we show that the condition of Theorem 1.2 is necessary and sufficient condition for existence of parametrices.

1. Formulation of problem and results.

Let X be a paracompact C^∞ manifold of dimension n and let $T^*(X) \setminus 0$ be the cotangential space minus the zero section. Let $P(x, D)$ be a properly supported classical pseudo-differential operator on X , which in every local coordinate system $U \subset X$ has a symbol of the form

$$(1.1) \quad p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi),$$

where p_{m-j} are an element of $C^\infty(R^n \times (R^n \setminus \{0\}))$ and positively homogeneous of degree $m-j$. Then $p_m(x, \xi)$ is independent of a local coordinate system U . Therefore we denote the principal symbol of P by $p_m(x, \xi) \in C^\infty(T^*(X) \setminus 0)$ with same notation. We denote by $S_c^m(X)$ the space of symbols with asymptotic expansion (1.1). $L_c^m(X)$ is the space of pseudo-differential operators defined by the symbols of $S_c^m(X)$.

The characteristic set of P is written by Σ ;

$$\Sigma = \{(x, \xi) \in T^*(X) \setminus 0; p_m(x, \xi) = 0\}.$$

For arbitrary $C^\infty(T^*(X) \setminus 0)$ functions f and g we write the Poisson bracket for f and g by

$$\{f, g\}(x, \xi) = \sum_{j=1}^n \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right) (x, \xi).$$

We also denote the Hamilton field of f by

$$H_f = \sum_{j=1}^n \left(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \right).$$

Let $q^1(x, \xi)$ and $q^2(x, \xi)$ be the real and imaginary part of $p_m(x, \xi)$ respectively. For each sequence $I = (i_1, \dots, i_s)$, where $i_1 = 1$ or 2 , we then define the function

$$C_I(x, \xi) = \{q^{i_1}, \{\dots, \{q^{i_s}, q^1\} \dots\}\}(x, \xi).$$

Let $|I|$ be the length of I ; $|I| = s$. For $(x, \xi) \in \Sigma$ we put $l(x, \xi) = |I_0|$ if $C_I(x, \xi) = 0$ for $|I| < |I_0|$ and $C_{I_0}(x, \xi) \neq 0$.

Using these notations, we shall introduce the following class of pseudo-differential operators.

DEFINITION 1.1. Let Σ be a connected closed conic submanifold of $T^*(X) \setminus 0$ and $\text{codim } \Sigma = 2$. Then we denote the class of properly supported classical pseudo-differential operators which satisfy the following three conditions by $M^{m,k}(\Sigma, X)$

- i) $P(x, D)$ belongs to $L_c^m(X)$.
- ii) The characteristic set of P is equal to Σ .
- iii) The function $l(x, \xi)$ defined by $p_m(x, \xi)$ is equal to k on Σ .

Furthermore $M^{m,0}(\Sigma, X) = L_c^m(X)$ and if k is odd we define $\sigma(P)$ for any $P \in M^{m,k}(\Sigma, X)$ as $\sigma(P) = 1$ if $\sup (C_{I_1}(x, \xi), C_{I_2}(x, \xi)) > 0$ and $\sigma(P) = -1$ if $\sup (C_{I_1}(x, \xi), C_{I_2}(x, \xi)) \leq 0$. Here $I_1 = (1, \dots, 1, 2)$, $I_2 = (2, \dots, 2)$, $|I_j| = k$ and supremum is taken over Σ .

In the above definition we assume the connectness of Σ , but this is not essential. We remark that Σ is non-involutive manifold by iii).

In this paper we consider the following double characteristic pseudo-differential operator,

$$(1.2) \quad L(x, D) = (P \circ Q)(x, D) + R(x, D).$$

Here $P \in M^{m_1,k}(\Sigma, X)$, $Q \in M^{m_2,k}(\Sigma, X)$, $R \in M^{m_1+m_2-1,j}(\Sigma, X)$. In the following theorem we write $A \equiv B$ for operators A, B ; $\mathcal{D}'(X) \rightarrow \mathcal{D}'(X)$ if $A - B$ is an integral operator with a C^∞ -kernel. We also write $\text{diag}(V) = \{(\rho, \rho); \rho \in V\} \subset (T^*(X) \setminus 0) \times (T^*(X) \setminus 0)$ for any conic subset V of $T^*(X) \setminus 0$. Our main theorem is the following

THEOREM 1.2. Let $L(x, D)$ be the double characteristic pseudo-differential operator defined by (1.2). Let k be an odd integer and $j = k - 1$. We

assume that $(H_{p_{m_1}})^l q_{m_2}(x, \xi) = 0$ on Σ for $l=1, \dots, k-1$ and $(H_{p_{m_1}})^l r_{m_1+m_2-2}(x, \xi) = 0$ on Σ for $l=1, \dots, k-2$ when $k > 1$. Furthermore

i) When $\sigma(P)=1$ and $\sigma(Q)=-1$, whatever the positive integer n , the function

$$(1.3) \quad (H_{m_1})^{k-1} \left(r_{m_1+m_2-1} + i\lambda \{p_{m_1}, q_{m_2}\} \right) (x, \xi)$$

does not vanish at any point of Σ , where $\lambda = (1-n(k+1))/k$ or $-n(k+1)/k$ and $H_{p_{m_1}}^0$ is the identity,

ii) When $\sigma(P)=-1$ and $\sigma(Q)=1$, for every positive integer n the function (1.3) does not vanish on Σ , where $\lambda = (1+(n-1)(k+1))/k$ or $(n-1)(k+1)/k$.

Then there exists properly supported linear operator $F; \mathcal{D}'(X) \rightarrow \mathcal{D}'(X)$ which is continuous $H_s^{\text{loc}}(X) \rightarrow H_{s+m_1+m_2-2k/(k+1)}^{\text{loc}}(X)$ for all $s \in \mathbb{R}$ such that

$$F \circ L(x, D) \equiv L(x, D) \circ F \equiv I \quad \text{and} \quad WF'(F) = \text{diag}(T^*(X) \setminus 0).$$

Here I is the identity in $\mathcal{D}'(X)$.

COROLLARY 1.3. Let $L(x, D)$ be a pseudo-differential operator satisfying the conditions of Theorem. Then $L(x, D)$ is locally solvable on X and strictly hypoelliptic, i.e., for every $u \in \mathcal{D}'(X)$ $WF(u) = WF(Lu)$. Moreover, if $u \in \mathcal{D}'(X)$ such that $Lu \in H_s^{\text{loc}}(X)$ then u belongs to $H_{s+m_1+m_2-2k/(k+1)}^{\text{loc}}(X)$. For any compact set K of X the following estimate holds for all $u \in C_0^\infty(K)$

$$\|u\|_{m_1+m_2-2k/(k+1)} \leq C_K (\|Lu\|_0 + \|u\|_{m_1+m_2-2}).$$

REMARK 1.4. When $k=1$, this type of pseudo-differential operator was studied by Boutet de Monvel and Treves [2], [3] and Sjöstrand [15]. In this case our condition i) and ii) are equivalent to that of [2], [15].

REMARK 1.5. i) The condition, $(H_{p_{m_1}})^l q_{m_2}(x, \xi) = 0$ on Σ for $l=1, \dots, k-1$, $(H_{p_{m_1}})^l r_{m_1+m_2-1}(x, \xi) = 0$ on Σ for $l=1, \dots, k-2$ have many equivalent conditions which are denoted in Proposition 2.2.

ii) In (1.3) $H_{p_{m_1}}^{k-1}$ is exchangeable. If $s_{k-1}(x, \xi)$ is equal to one of $p_{m_1}, \bar{p}_{m_1}, q_{m_2}, \bar{q}_{m_2}, r_{m_1+m_2-1}, \bar{r}_{m_1+m_2-1}$. If $s_{k-1}(x, \xi) = \bar{r}_{m_1+m_2-1}(x, \xi)$, then $\lambda = 2(1-n(k+1))/k$ or $-2n(k+1)/k$. The latter case we need use the canonical transformation in [14], by which we may assume $c(x, \xi')$ in (2.14) is real.

$$H_{s_1} \cdots H_{s_{k-1}} \left(r_{m_1+m_2-1} + i\lambda \{p_{m_1}, q_{m_2}\} \right) (x, \xi) \neq 0,$$

where $\lambda = (1-n(k+1))/k$ or $-n(k+1)/k$ and $s_j (j \leq k-2)$ is equal to one of $p_{m_1}, \bar{p}_{m_1}, q_{m_2}, \bar{q}_{m_2}, r_{m_1+m_2-1}, \bar{r}_{m_1+m_2-1}$. If $s_{k-1}(x, \xi) = \bar{r}_{m_1+m_2-1}(x, \xi)$, then $\lambda = 2(1-n(k+1))/k$ or $-2n(k+1)/k$. The latter case we need use the canonical transformation in [14], by which we may assume $c(x, \xi')$ in (2.14) is real.

iii) Let $A(x, D), B(x, D)$ be elliptic pseudo-differential operators with

the principal symbol $a(x, \xi)$ and $b(x, \xi)$ respectively. If $L(x, D)$ satisfies the condition i) then by the proof of Proposition 2.2 it follows that $(ALB)(x, D)$ satisfies the same condition i) for $(ap_{m_1})(x, \xi)$, $(q_{m_2}b)(x, \xi)$ and $(ar_{m_1+m_2-1}b)(x, \xi)$ when $(x, \xi) \in \Sigma$.

iv) If $L(x, D)$ satisfies the condition i) then so does the adjoint operator $L^*(x, D)$.

REMARK 1.6. Assume that $L(x, D) = (P' \circ Q') + R'$, where $P'(x, D) \in M^{m_1, k}(\Sigma, X)$, $Q'(x, D) \in M^{m_2, k}(\Sigma, X)$, $R'(x, D) \in M^{m_1+m_2-1}(\Sigma, X)$ ($m_1 + m_2 = m'_1 + m'_2$), $\sigma(P') = 1$ and $\sigma(Q') = -1$. Furthermore $(H_{p_{m_1}})^l p'_{m_1}(x, \xi) = (H_{p_{m_1}})^l q'_{m_2}(x, \xi) = 0$ and $(H_{p_{m_1}})^n r'_{m_1+m_2-1}(x, \xi) = 0$, where $(x, \xi) \in \Sigma$, $l = 1, \dots, k-1$ and $n = 1, \dots, k-2$. Then the condition i) in Theorem 1.2 is invariant, i.e., when $(x, \xi) \in \Sigma$ (1.3) is equal to

$$(H_{p_{m_1}})^{k-1} (r'_{m_1+m_2-1} + i\lambda \{p'_{m_1}, q'_{m_2}\}).$$

When k is odd and $\sigma(P)\sigma(Q) = 1$, we can also construct a parametrix-like object and obtain certain informations on local solvability and hypoellipticity. Under different conditions we can construct parametrices of $L(x, D)$ when k is even (Theorem 4.1 and 4.3). These results are stated in Section 4.

2. Reduction to a micro-local situation.

In this section, we transform $L(x, D)$ into a pseudo-differential operator whose principal symbol is a product of those of type (M) .

The following proposition, proved in [19], is the starting point of our discussion.

PROPOSITION 2.1. Let P be an element of $M^{m, k}(\Sigma, X)$. Then there exists a homogeneous canonical transformation χ from an open conic neighbourhood of $\rho \in \Sigma$ to an open conic neighbourhood of $\chi(\rho) \in T^*(R^n) \setminus 0$ such that

$$(p_m \circ \chi^{-1})(x, \xi) = (\xi_n - ix_n^k \xi_{n-1}) \tilde{Q}(x, \xi).$$

Here $\xi_{n-1} \neq 0$ and $\tilde{Q}(x, \xi)$ is a non-vanishing positively homogeneous function of degree $m-1$. Moreover if k is odd, then the sign of ξ_{n-1} is equal to the sign of $C_{I_1}(\rho)$ or $C_{I_2}(\rho)$. Here either $C_{I_1}(\rho)$ or $C_{I_2}(\rho)$ is non-zero and these have the same sign if $C_{I_1}(\rho) \neq 0$, $C_{I_2}(\rho) \neq 0$ and k is odd.

For an odd integer k if $C_{I_1}(x, \xi) > 0$ or $C_{I_2}(x, \xi) > 0$ for some $(x, \xi) \in \Sigma$ then $\sup(C_{I_1}(x, \xi), C_{I_2}(x, \xi)) > 0$. Here supremum is taken over Σ . It implies that the sign of ξ_{n-1} is equal to that of $\sigma(P)$ if k is odd.

If $k=1$, then for any pair $P, Q \in M^{m, 1}(\Sigma, X)$ we can find one and same

canonical transformation χ such that $p_m \circ \chi^{-1}$ and $q_m \circ \chi^{-1}$ are written by the product of non-vanishing function and the polynomial of degree 1 with respect to ξ_n . However, for general k this is not true. The necessary and sufficient condition is given by the following

PROPOSITION 2.2. *Let $P_j(x, D)$ be an element of $M^{m_j, k_j}(\Sigma, X)$, $j=1, \dots, N$. We assume $k_1 \geq \dots \geq k_N$. Then the following statements are equivalent.*

A) *For any j with $k_j > 1$ and $l(1 \leq l \leq k_j - 1)$ there exists a sequence $J_l = (j(1), \dots, j(l))$ such that for any $(x, \xi) \in \Sigma$*

$$(2.1) \quad (H_{p_{j(1)}} \cdots H_{p_{j(l)}}) p_{j, m_j}(x, \xi) = 0,$$

where $p_{j(i)} (i=1, \dots, l-1)$ is an element in $\{p_{n, m_n}(x, \xi); k_n \geq k_j\}$. Furthermore if $k_j = k_1$, then $p_{j(l)} = p_{1, m_1}(x, \xi)$ and if $k_j < k_1$, then $p_{j(l)} = p_{i, m_i}(x, \xi)$ with $k_i > k_j$.

B) *For all j with $k_j > 1$ and $l(1 \leq l \leq k_j - 1)$ we have*

$$\left((H_{p_{1, m_1}})^l p_{j, m_j} \right) (x, \xi) = 0, \quad (x, \xi) \in \Sigma.$$

C) *For any j with $k_j > 1$, $l(1 \leq l \leq k_j - 1)$ and any $(x, \xi) \in \Sigma$, we have*

$$(2.2) \quad \left((H_{p_1} \cdots H_{p_l}) p_{j, m_j} \right) (x, \xi) = 0.$$

Here $p_m(x, \xi) (m=1, \dots, l)$ is an arbitrary element of $\{p_{n, m_n}; k_n \geq k_j\}$.

D) *There exists a homogeneous canonical transformation from an open conic neighbourhood of $\rho \in \Sigma$ to an open conic neighbourhood of $\chi(\rho) \in T^*(R^n) \setminus 0$ such that for all $j=1, \dots, N$*

$$(p_{j, m_j} \circ \chi^{-1})(x, \xi) = (\xi_n - i x_n^{k_j} a_j(x, \xi')) Q_j(x, \xi).$$

Here $a_j(x, \xi')$ and $Q_j(x, \xi)$ are positively homogeneous of degree 1 and $m_j - 1$ respectively and $\xi' = (\xi_1, \dots, \xi_{n-1})$. Moreover $\text{Re} a_j(x, \xi')$ and $Q_j(x, \xi)$ are non-vanishing.

In the above statement D) if k_j is odd, then the sign of $\text{Re} a_j(x, \xi')$ is the same of that of $\sigma(P_j)$.

PROOF. First we shall show that the statement D) implies the statement C). We write $p_m(x, \xi) = (\xi_n - i x_n^{k_m} b_m(x, \xi')) R_m(x, \xi)$ and $q_m(x, \xi) = (\xi_n - i x_n^{k_m} b_m(x, \xi'))$. Then by the induction with respect to l we see that

$$(2.3) \quad \begin{aligned} (H_{p_1} \cdots H_{p_{l-1}} p_l)(x, \xi) &= (R_1 \cdots R_l) (H_{q_1} \cdots H_{q_{l-1}} q_l)(x, \xi) \\ &+ \sum_{|I| < l} A_I(x, \xi) (H_{q_{i(1)}} \cdots H_{q_{i(k-1)}} q_{i(k)})(x, \xi) \\ &+ \sum_{m=1}^l A_m(x, \xi) q_m(x, \xi). \end{aligned}$$

Here $I=(i(1), \dots, i(k))$, $1 \leq i(m) \leq l$ and $A_I(x, \xi)$, $A_m(x, \xi)$ are some positively homogeneous functions. Similary if $k \leq \min(l_{i(k)}, l_{i(k-1)})+1$, then by the induction with respect to k it follows that

$$\begin{aligned}
 (2.4) \quad & (H_{q_{i(1)}} \cdots H_{q_{i(k-1)}} q_{i(k)})(x, \xi) \\
 &= i l_{i(k-1)} \cdots (l_{i(k-1)} - k + 2) x_n^{l_{i(k-1)} - k + 1} b_{i(k-1)}(x, \xi') \\
 &\quad - i l_{i(k)} \cdots (l_{i(k)} - k + 2) x_n^{l_{i(k)} - k + 1} b_{i(k)}(x, \xi') \\
 &\quad + A(x, \xi) x_n^{l_{i(1)} - k + 2} + B(x, \xi) x_n^{l_{i(2)} - k + 2}.
 \end{aligned}$$

Here $A(x, \xi)$, $B(x, \xi)$ are positively homogeneous of degree 1 respectively. By (2.3) and (2.4) it implies (2.2). The implications of C) to B) and B) to A) being clear, we prove that A) implies D).

Assume (2.1). By Proposition 2.1, there exists a homogeneous canonical transformation χ_j from an open conic neighbourhood of $\rho \in \Sigma$ to an open conic neighbourhood of $\chi_j(\rho) \in T^*(R^n) \setminus 0$ such that

$$(2.5) \quad (p_{j, m_j} \circ \chi_j^{-1})(y, \eta) = (\eta_n - i y_n^{k_j} \eta_{n-1}) \tilde{Q}_j(y, \eta).$$

Here $\eta_{n-1} \neq 0$ and $\tilde{Q}_j(y, \eta)$ is a non-zero positively homogeneous function of degree $m_j - 1$. We show that the required canonical transformation χ is χ_1 . We denote the canonical transformation $\chi_j \circ \chi_1^{-1}$ from a conic neighbourhood of $\chi_1(\rho)$ to a conic neighbourhood of $\chi_j(\rho)$ by

$$(y_1(x, \xi), \dots, y_n(x, \xi), \eta_1(x, \xi), \dots, \eta_n(x, \xi)).$$

Here $y_i(x, \xi)$ and $\eta_i(x, \xi)$ are positively homogeneous of degree 0 and 1 respectively, $i=1, \dots, n$. Then $(p_{j, m_j} \circ \chi_1^{-1})(x, \xi)$ is equal to

$$(2.6) \quad (\eta_n(x, \xi) - i(y_n(x, \xi))^{k_j} \eta_{n-1}(x, \xi)) \tilde{Q}_j(x, \xi).$$

Since $\chi_j(\Sigma) = \{(y, \eta) \in T^*(R^n) \setminus 0; y_n = \eta_n = 0\}$ and $\chi_1(\Sigma) = \{(x, \xi) \in T^*(R^n) \setminus 0; x_n = \xi_n = 0\}$, we see that $y_n(x', 0, \xi', 0) = \eta_n(x', 0, \xi', 0) = 0$. Therefore by the definition of a canonical transformation we obtain

$$(2.7) \quad \{y_n, \eta_n\}(x', 0, \xi', 0) = \left(\frac{\partial y_n}{\partial \xi_n} \frac{\partial \eta_n}{\partial x_n} - \frac{\partial y_n}{\partial x_n} \frac{\partial \eta_n}{\partial \xi_n} \right)(x', 0, \xi', 0) = -1.$$

First we shall assume $k_j=1$. Then we have by (2.7) $\partial \eta_n / \partial \xi_n(x', 0, \xi', 0) \neq 0$ or $\partial y_n / \partial \xi_n(x', 0, \xi', 0) \neq 0$. The implicit function theorem (c.f [13]) implies

$$(2.8) \quad (\eta_n - i y_n \eta_{n-1})(x, \xi) = (\xi_n - i A_j(x, \xi')) Q'_j(x, \xi).$$

Here $Q'_j(x, \xi)$ is non-zero positively homogeneous of degree 0 and $A_j(x, \xi')$ is positively homogeneous of degree 1. Comparing the null sets of the

both sides in (2.8), we get $A_j(x', 0, \xi') = 0$. Hence we can write $A_j(x, \xi') = x_n a_j(x, \xi')$. We have by (2.8) that

$$\begin{aligned} (2.9) \quad 2i\eta_{n-1} &= \{\eta_n - iy_n \eta_{n-1}, \eta_n + iy_n \eta_{n-1}\} (y', 0, \eta', 0) \\ &= |Q_j|^2 \{\xi_n - ix_n a_j, \xi_n + ix_n \bar{a}_j\} (x', 0, \xi', 0) \\ &= 2i(|Q_j|^2 \operatorname{Re} a_j) (x', 0, \xi', 0). \end{aligned}$$

Thus $\operatorname{Re} a_j(x, \xi') \neq 0$ and the sign of $\operatorname{Re} a_j(x, \xi')$ is equal to that of $\sigma(P_j)$. Hence we have D by setting $Q_j = \bar{Q}_j Q'_j$.

Now let $k_j = k_1 (j=1, \dots, N_1)$. From the assumption (2.1)

$$\begin{aligned} &\{p_{1,m_1}, p_{j,m_j}\} \\ &= \{(\xi_n - ix_n^{k_1} \xi_{n-1}) \bar{Q}_1, (\eta_n - iy_n^{k_j} \eta_{n-1}) \bar{Q}_j\} (x', 0, \xi', 0) \\ &= (\bar{Q}_1 \bar{Q}_j \partial \eta_n / \partial x_n) (x', 0, \xi', 0) = 0 \end{aligned}$$

By (2.7) it follows that $\partial \eta_n / \partial \xi_n (x', 0, \xi', 0) \neq 0$. Thus by the implicit function theorem we can write for any $j (1 \leq j \leq N_1)$

$$\begin{aligned} B_j(x, \xi) &= (\eta_n - iy_n^{k_j} \eta_{n-1}) (x, \xi) \\ &= (\xi_n - ix_n A_1^j(x, \xi')) Q'_j(x, \xi), \end{aligned}$$

where the first equality is definition and $A_1^j(x, \xi')$, $Q'_j(x, \xi)$ are positively homogeneous of degree 1, 0 respectively. Moreover $Q'_j(x, \xi)$ is non-zero function. We shall show that $A_1^j(x', 0, \xi') = 0$. Apply (2.3) to $H_{v_{1,m_1}} p_{j,m_j}$ as $q_1(x, \xi) = \xi_n - ix_n^{k_1} \xi_{n-1}$, $R_1(x, \xi) = Q_1(x, \xi)$, $q_2(x, \xi) = \xi_n - ix_n A_1^j(x, \xi')$ and $R_2(x, \xi) = \bar{Q}_j Q'_j(x, \xi)$. Then by (2.1), (2.4) it implies that $i(\bar{Q}_1 \bar{Q}_j Q'_j) A_1^j(x', 0, \xi', 0) = 0$. Therefore for any $j (1 \leq j \leq N_1)$ we can write

$$B_j(x, \xi) = (\xi_n - ix_n^2 A_2^j(x, \xi')) Q'_j(x, \xi),$$

where $A_2^j(x, \xi')$ is positively homogeneous of degree 1. Inductively we assume that for all $j (1 \leq j \leq N_1)$ we can write

$$B_j(x, \xi) = (\xi_n - ix_n^l A_l^j(x, \xi')) Q'_j(x, \xi),$$

where $l < k_1$. Set $P_{j(l)}(x, \xi) = (\xi_n - ix_n^l b_l(x, \xi')) R_l(x, \xi) (l=1, \dots, l-1)$. We apply (2.3) to $H_{p_{j(l)}} \dots H_{p_{j(l-1)}} H_{v_{1,m_1}} p_{j,m_j}$ as $R_l = R_l(x, \xi)$, $q_l = \xi_n - ix_n^l b_l(x, \xi') (l=1, \dots, l-1)$, $R_l = \bar{Q}_1(x, \xi)$, $q_l = (\xi_n - ix_n^{k_1} \xi_{n-1})$, $R_{l+1} = \bar{Q}_j Q'_j(x, \xi)$ and $q_{l+1} = (\xi_n - ix_n^l A_l^j(x, \xi'))$. Then by (2.1) and (2.4) it implies that

$$il! (R_1 \dots R_{l-1} \bar{Q}_1 \bar{Q}_j Q'_j A_l^j) (x', 0, \xi', 0) = 0.$$

Thus we can write for all $j (1 \leq j \leq N_1)$

$$B_j(x, \xi) = (\xi_n - i x_n^{l+1} A_{l+1}^j(x, \xi')) Q_j'(x, \xi).$$

By the induction with respect to l we obtain the statement D) except $Rea_j \neq 0$.

let $k_j = k_{N_1+1}$ ($j = N_1 + 1, \dots, N_2$). Since for any j ($j = N_1 + 1, \dots, N_2$) and $l < k_j$ we assume $p_{j(i)} = p_{i, m_i}(x, \xi)$ with $k_i > k_j$, in this case $p_{i, m_i}(x, \xi)$ ($i = 1, \dots, N_1$) acts $p_{j, m_j}(x, \xi)$ ($j = N_1 + 1, \dots, N_2$) like as $p_{1, m_1}(x, \xi)$ does $p_{i, m_i}(x, \xi)$ ($i = 1, \dots, N_1$). Repeating the same argument as that when $j = 1, \dots, N_1$ we can show that for all j ($j = N_1 + 1, \dots, N_2$)

$$B_j(x, \xi) = (\xi_n - i x_n^{k_j} a_j(x, \xi')) Q_j'(x, \xi).$$

Here $a_j(x, \xi')$ and $Q_j'(x, \xi)$ are positively homogeneous of degree 1 and 0 respectively and $Q_j'(x, \xi) \neq 0$. Similarly we can prove the statement D) when $k_j < k_{N_2}$ without $Rea_j \neq 0$.

We shall show that $Rea_j(x, \xi') \neq 0$. If we put $q_j(x, \xi) = \xi_n - i x_n^{k_j} a_j(x, \xi')$, by (2.3) and (2.4) we have

$$(2.10) \quad ((H_{B_j})^{k_j} \bar{B}_j)(x', 0, \xi', 0) = ((Q_j')^{k_j} \bar{Q}_j') ((H_{q_j})^{k_j} \bar{q}_j)(x', 0, \xi', 0).$$

Since $((H_{q_j})^{k_j} \bar{q}_j)(x', 0, \xi', 0) = 2ik_j! Rea_j(x', 0, \xi')$, (2.10) gives that

$$\eta_{n-1}(x', 0, \xi', 0) = ((Q_j')^{k_j} \bar{Q}_j' Rea_j)(x', 0, \xi', 0).$$

This shows that $Rea_j(x, \xi')$ is non-zero. If we set $\tilde{Q}_j Q_j' = Q_j$, $\lambda = \lambda_1$ then we have the statement D).

If k_j is odd, then the sign of η_{n-1} is equal to that of $\sigma(P_j)$. We shall consider the Poisson bracket such that $\{B_j, \{\bar{B}_j, \{\dots, \{\bar{B}_j, \{B_j, \bar{B}_j\}\} \dots\}\}$ where the numbers of B_j and \bar{B}_j in the bracket are $(k_j+1)/2$ respectively. By (2.3) and (2.4) we see that

$$\eta_{n-1}(x', 0, \xi', 0) = |Q_j'|^{k_j+1} (Rea_j)(x', 0, \xi', 0).$$

This shows that the sign of Rea_j is equal to that of $\sigma(P_j)$. This completes the proof.

Now we shall compute the full symbol of operator $L(x, D)$. To do so we use the following terminology. For the operators A and B form $\mathcal{D}'(X_1)$ to $\mathcal{D}'(X_2)$, where X_j are paracompact C^∞ -manifolds, $A \equiv B$ at $(\rho_1, \rho_2) \in (T^*(X_1 \times X_2) \setminus 0)$ means that there exists an open conic neighbourhood V_j of ρ_j in $T^*(X_j)$ such that $WF'(A-B) \cap (V_1 \times V_2) = \emptyset$. It is well-known (c.f Section 6 in [15]) that for any canonical transformation χ from a conic neighbourhood ρ to a conic neighbourhood of $\chi(\rho) \in T^*(R^n) \setminus 0$ there exists a Fourier integral operator U belongs to $I^0(R^n \times X, \Gamma')$, where Γ is a closed

conic subset of the graph of χ containing $(\rho, \chi(\rho))$ as a interior point and $\Gamma' = \{(x, \xi, y, -\eta); (x, \xi, y, \eta) \in \Gamma\}$. It satisfies that

$$(2.11) \quad UU^* \equiv I \text{ at } (\chi(\rho), \chi(\rho)) \text{ and } U^*U \equiv I \text{ at } (\rho, \rho).$$

Moreover for all $P \in L_c^m(R^n)$ with the principal symbol p_m , we have $UPU^* \in L_c^m(R^n)$ and it's principal symbol is equal to $p_m \circ \chi^{-1}$ in a conic neighbourhood of $\chi(\rho)$.

Now we shall consider the operator

$$L(x, D) = (P \circ Q)(x, D) + R(x, D).$$

where $P \in M^{m_1, k}(\Sigma, X)$, $Q \in M^{m_2, k}(\Sigma, X)$, $R \in M^{m_1+m_2-1, k-1}(\Sigma, X)$. Here we suppose that k is odd. Moreover $H_{p_{m_1}, q_{m_2}}^l = 0$ on Σ for $l=1, \dots, k-1$ and $H_{p_{m_1}, r_{m_1+m_2-1}}^l = 0$ on Σ for $l=1, \dots, k-2$ when $k > 1$. From (2.11) we get

$$\tilde{L}(x, D) = ULU^* \equiv UPU^*UQU^* + URU^* \text{ at } (\chi(\rho), \chi(\rho)),$$

where χ is the canonical transformation denoted in Proposition 2.2. From Proposition 2.2 we have

$$(2.12) \quad UPU^*(x, \xi) \sim e(x, \xi) \left((\xi_n - ix_n^k a(x, \xi')) + p_0(x, \xi) + p_{-1}(x, \xi) \right),$$

$$(2.13) \quad UQU^*(x, \xi) \sim f(x, \xi) \left((\xi_n - ix_n^k b(x, \xi')) + q_0(x, \xi) + q_{-1}(x, \xi) \right),$$

$$(2.14) \quad UPU^*(x, \xi) \sim g(x, \xi) \left((\xi_n - ix_n^{k-1} c(x, \xi')) + r_0(x, \xi) + r_{-1}(x, \xi) \right).$$

Here e, f and g are non-zero positively homogeneous of degree m_1-1 , m_2-1 and m_1+m_2-2 respectively, p_0, q_0 and r_0 are positively homogeneous of degree 0 and p_{-1}, q_{-1} and r_{-1} belong to $S_c^{-1}(X)$. Moreover a, b and c are positively homogeneous of degree 1 and if k is odd, the sign of Rea and Reb are equal to $\sigma(P)$ and $\sigma(Q)$ respectively. Using this fact we get the following

LEMMA 2.3. *There exists a properly supported elliptic pseudo-differential operator E , which belongs to $L^{m_1+m_2-2}(R^n)$, such that the principal symbol of E is equal to $(ef)(x, \xi)$ and*

$$\tilde{L}(x, D) \equiv E(x, D) \circ M(x, D) \text{ at } (\chi(\rho), \chi(\rho)).$$

Here $M(x, D)$ is a classical pseudo-differential operator belonging to $L_c^2(R^n)$ such that in a conic neighbourhood of $\chi(\rho)$

$$\begin{aligned} M(x, \xi) \sim & \xi_n^2 - ix_n^k(a+b)(x, \xi') \xi_n - x_n^{2k}(ab)(x, \xi') \\ & - kx_n^{k-1}b(x, \xi') - ix_n^{k-1}(e^{-1}f^{-1}gc)(x, \xi) \\ & + x_n^k A(x, \xi) + B(x, \xi) \xi_n + C(x, \xi), \end{aligned}$$

where A and B are positively homogeneous of degree 1 and 0 respectively and $C(x, \xi) \in S_c^0(R^n)$.

PROOF. Let $h(x, \xi)$ be $C^\infty(R^n \times R^n)$ non-zero function which is equal to $(ef)(x, \xi)$ when (x, ξ) belongs to some conic neighbourhood of $\chi(\rho)$ and $|\xi| > 1$. By using $h(x, \xi)$ we define a pseudo-differential operator $E(x, D)$. Now we shall seek an operator $M(x, D)$ with a symbol

$$M(x, \xi) \sim \sum_{j=0}^{\infty} M_{2-j}(x, \xi),$$

where $M_{2-j}(x, \xi)$ is positively homogeneous of degree $2-j$. By the composition formula for pseudo-differential operators (c.f [12]), we have

$$(E \circ M)(x, \xi) \sim \sum_{\alpha, j} ((iD_\xi)^\alpha h)(D_x^\alpha M_{2-j})(x, \xi).$$

From (2.12) and (2.13) we first get

$$M_2(x, \xi) = \xi_n^2 - ix_n^k(a+b)(x, \xi')\xi_n - x_n^{2k}(ab)(x, \xi').$$

Since $hM_1 + \sum_{|\alpha|=1} ((iD_\xi)^\alpha h)(D_x^\alpha M_2) = (ef)(-kx_n^{k-1}b(x, \xi') - ix_n^{k-1}(e^{-1}f^{-1}gc)(x, \xi) + x_n^k A_1(x, \xi) + B_1(x, \xi))$ if $|\xi| > 1$, where A_1 and B_1 are positively homogeneous of degree 1 and 0 respectively. Hence, we can write

$$\begin{aligned} M_1(x, \xi) = & -kx_n^{k-1}b(x, \xi') - ix_n^{k-1}(e^{-1}f^{-1}gc)(x, \xi) \\ & + x_n^k A(x, \xi) + B(x, \xi)\xi_n. \end{aligned}$$

Since $h(x, \xi) \neq 0$, we obtain $M_{2-j}(x, \xi)$ in turn by the induction with respect to j . This completes the proof.

3. The proof of Theorem 1.2.

In this section, by using a discussion of vector valued pseudo-differential operators we shall prove Theorem 1.2.

For any pair of Hilbert spaces H_1 and H_2 we denote by $\mathcal{L}(H_1, H_2)$ the Banach space of bounded linear operators from H_1 to H_2 . We define $S^m(R^{n-1}; H_1, H_2)$ as the space of C^∞ functions $p(x', \xi')$ on $R^{n-1} \times R^{n-1}$ with values in $\mathcal{L}(H_1, H_2)$ such that for all $K \ni R^{n-1}$ and multi-indices α and β there is a constant C , depending on K , α and β such that

$$(3.1) \quad \|D_x^\alpha D_\xi^\beta p(x', \xi')\|_{\mathcal{L}(H_1, H_2)} \leq C(1 + |\xi'|)^{m-|\beta|} \quad \text{for } (x', \xi') \in K \times R^{n-1}.$$

Let $L^m(R^n; H_1, H_2)$ be the space of vector valued pseudo-differential operators $C_0^\infty(R^{n-1}; H_1) \rightarrow C^\infty(R^{n-1}; H_2)$, given by

$$P(x', D')u = \int e^{i\langle x' - y', \xi' \rangle} p(x', \xi') u(y') d\xi' dy',$$

where $p(x', \xi') \in S^m(R^{n-1}; H_1, H_2)$ and $u \in C_0^\infty(R^{n-1}; H_1)$. In particular, in this section we shall take as H_1 or H_2 the space $V_{\xi'}^k(R)$, given by the norm

$$\|u\|_{V_{\xi'}^k(R)}^2 = \sum_{\substack{\alpha \leq 2 \\ \beta \leq (2-\alpha)k}} (1 + |\xi'|^2)^{(2-\alpha+\beta)/(k+1)} \|y^\beta D^\alpha u\|_{L^2(R)}^2.$$

In this case the norm in (3.1) depends on ξ' , but all the calculi remain valid because we have the inequality

$$\|u\|_{V^k(R)} \leq \|u\|_{V_{\xi'}^k(R)} \leq (1 + |\xi'|)^2 \|u\|_{V^k(R)}.$$

Here $V^k(R)$ is a Hilbert space normed by

$$\|u\|_{V^k(R)}^2 = \sum_{\substack{\alpha \leq 2 \\ \beta \leq (2-\alpha)k}} \|y^\beta D^\alpha u\|_{L^2(R)}^2.$$

For instance if $P \in L^m(R^{n-1}; H_1, V_{\xi'}^k(R))$ then $P \in L^{m+2}(R_{n-1}; H_1, V^k(R))$.

As to the wave front sets of vector valued pseudo-differential operators, we have the following

PROPOSITION 3.1. *Let $P \in L^m(R^{n-1}; H_1, H_2)$, where H_1 and H_2 are equal to $V_{\xi'}^k(R)$ or $L^2(R)$. Then, considering P as an operator $C_0^\infty(R^n) \rightarrow \mathcal{D}'(R^n)$, we have*

- i) *If $(x, \xi, y, \eta) \in WF'(P)$ and $|\xi'| + |\eta'| \neq 0$, then $(x', \xi') = (y', \eta')$,*
- ii) *If $m = -\infty$, then $WF'(P) \subset \{((x', x_n, 0, \xi_n), (y', y_n, 0, \eta_n)) \in T^*(R^n \times R^n) \setminus \{0\}\}$.*

The above i) is Proposition 4.4 in [15] and ii) is easily shown from the proof of the same proposition.

By Lemma 2.3, we shall investigate the pseudo-differential operator $M(x, D)$ with the symbol

$$(3.2) \quad \begin{aligned} & \xi_n^2 - ix_n^k(a+b)(x, \xi') \xi_n - x_n^{2k}(ab)(x, \xi') - kx_n^{k-1}b(x, \xi') \\ & + x_n^{k-1}\tilde{c}(x, \xi) + x_n^k A(x, \xi) + B(x, \xi) \xi_n + C(x, \xi), \end{aligned}$$

where $\tilde{c} = -i(e^{-1}f^{-1}gc)(x, \xi)$. Now we extend a, b and \tilde{c} on $R^n \times R^{n-1}$ as the $C^\infty(R^n \times R^{n-1})$ functions preserving a homogeneity when $|\xi'| > 1$. The functions \tilde{c}, A, B and C are also extended over $R^n \times R^n$ as the $C^\infty(R^n \times R^n)$ functions preserving a homogeneity if $|\xi| > 1$. Extended symbols a, b, \tilde{c}, A, B and C satisfy the following condition. For all multi-indices α, β

$$|D_x^\alpha D_\theta^\beta p(x, \theta)| \leq C(1 + |\theta|)^{m-|\alpha|}, (x, \theta) \in R^n \times R^N,$$

where $N = n-1$ or n and $m = 0$ or 1 . Then we have the following

LEMMA 3.2. *Let $p(x, \xi)$ be an element of $S^m(R^n \times R^n)$ such that $m \leq 1$ and for all multi-indices α, β*

$$(3.3) \quad \left| D_x^\alpha D_\xi^\beta p(x, \xi) \right| \leq C (1 + |\xi|)^{m-|\beta|} (x, \xi) \in R^n \times R^n.$$

Then for all multi-indices μ, ν and $l \leq 2k$ the operator norm of $x_n^l \circ (D_x^\mu D_\xi^\nu p)(x, \xi', D_n)$ in $\mathcal{L}(V_{\xi'}^k(R), L^2(R))$ is bounded by $C(1 + |\xi'|)^{m-|\nu|-(k+l+2)/(k+1)}$ if $p(x, \xi', 0) \equiv 0$. If $p(x, \xi', 0) \not\equiv 0$, then the norm is bounded by $C(1 + |\xi'|)^{m-|\nu|-(l+2)/(k+1)}$.

PROOF. It is known in Proposition 3.1 of [8] that if $p(x, \xi)$ satisfies the condition (3.3) and $m \leq 0$, then the operator norm of $p(x, \xi', D_n)$ in $\mathcal{L}(L^2(R), L^2(R))$ is bounded by

$$(3.4) \quad C \sum_{\alpha+\beta \leq 2} \sup (1 + |\xi_n|)^\beta \left| D_{x_n}^\alpha D_{\xi_n}^\beta p(x, \xi) \right|.$$

By Taylor's formula we can write $p(x, \xi) = p(x, \xi', 0) + p_1(x, \xi) \xi_n$, where $p_1(x, \xi)$ belongs to $S^{m-1}(R^n \times R^n)$ and satisfies the condition (3.3). By a simple computation it follows that

$$\begin{aligned} x_n^l (D_x^\mu D_\xi^\nu p)(x, \xi', D_n) &= (D_x^\mu D_\xi^\nu p)(x, \xi', 0) \cdot x_n^l \\ &+ \sum_{l' \leq l} (-1)^{l-l'} \binom{l}{l'} (D_x^\mu D_\xi^\nu D_{\xi_n}^{l-l'} p_1)(x, \xi', D_n) \cdot (x_n^{l'} D_n). \end{aligned}$$

It is clear that

$$(3.5) \quad \begin{aligned} &\left\| (D_x^\mu D_\xi^\nu p)(x, \xi', 0) \cdot x_n^l u \right\|_{L^2(R)} \\ &\leq C (1 + |\xi'|)^{m-|\nu|-(l+2)/(k+1)} \|u\|_{V_{\xi'}^k(R)}. \end{aligned}$$

From (3.4), $(1 + |\xi|)^{-\beta} \leq (1 + |\xi_n|)^{-\beta}$ and $(1 + |\xi|)^{m-1-|\nu|-(l-l')/(k+1)} \leq (1 + |\xi'|)^{m-1-|\nu|-(l-l')/(k+1)}$ we get the following inequality;

$$(3.6) \quad \begin{aligned} &\left\| (D_x^\mu D_\xi^\nu D_{\xi_n}^{l-l'} p_1)(x, \xi', D_n) (x_n^{l'} D_n u) \right\|_{L^2(R)} \\ &\leq C (1 + |\xi'|)^{m-|\nu|-(k+l+2+k(l-l'))/(k+1)} \|u\|_{V_{\xi'}^k(R)}. \end{aligned}$$

This completes the proof.

We define the properly supported pseudo-differential operator $M_0(x, D)$ with the symbol

$$(3.7) \quad \begin{aligned} &\xi_n^2 - i x_n^k (a + b)(x', 0, \xi') \xi_n - x_n^{2k} (ab)(x', 0, \xi') \\ &- k x_n^{k-1} b(x', 0, \xi') + x_n^{k-1} \bar{c}(x', 0, \xi', 0). \end{aligned}$$

Then we have the following

LEMMA 3.3. Let $M(x, D)$ be the properly supported pseudo-differential operator with the symbol (3.2) and $M_0(x, D)$ be the properly supported pseudo-differential operator with the symbol (3.7). Then there exists a properly supported pseudo-differential operator $M_1(x, D)$ such that

- i) $M_*(x, \xi', D_n)$ belongs to $S^0(R^{n-1}; V_{\xi'}^k(R), L^2(R))$.
- ii) $(M - M_0)(x, \xi) = M_*(x, \xi)$ in a small conic neighbourhood of $\chi(\rho)$.
- iii) For any $\varepsilon > 0$ there exists a conic neighbourhood V' of (x'_0, ξ'_0) , where $\chi(\rho) = (x'_0, 0, \xi'_0, 0)$, such that

$$\|M_*(x, \xi', D_n)\|_{\mathcal{L}(V_{\xi'}^k(R), L^2(R))} \leq \varepsilon,$$

when $(x', \xi') \in V'$ and $|\xi'|$ is sufficiently large.

PROOF. Let $\varphi(x)$ be a $C_0^\infty(R^n)$ function such that $\text{supp } \varphi \subset \{x \in R^n; |x - (x'_0, 0)| < 1\}$ and $\varphi(x) = 1$ in a neighbourhood of $(x'_0, 0)$. We define $M_*(x, D)$ with the symbol

$$(3.8) \quad \begin{aligned} & -i((a+b)(x, \xi') - (a+b)(x', 0, \xi')) x_n^k \xi_n - \varphi(x/\delta) \\ & \times ((ab)(x, \xi') - (ab)(x', 0, \xi')) x_n^{2k} - k(b(x, \xi') \\ & - b(x', 0, \xi')) x_n^{k-1} + x_n^{k-1}(\bar{c}(x, \xi) - \bar{c}(x', 0, \xi', 0)) \\ & + x_n^k A(x, \xi) + B(x, \xi) \xi_n + C(x, \xi), \end{aligned}$$

where δ is a positive number. We shall show that $M_*(x, \xi', D_n)$ belongs to $S^0(R^{n-1}; V_{\xi'}^k(R), L^2(R))$. It is clear that the vector valued pseudo-differential operators defined by the first, second and third terms of (3.8) belong to $L^0(R^{n-1}; V_{\xi'}^k(R), L^2(R))$. By Talor's formula, we get $\bar{c}(x, \xi) - \bar{c}(x', 0, \xi', 0) = \bar{c}_0(x, \xi) \xi_n + x_n \bar{c}_1(x, \xi)$, where $\bar{c}_0(x, \xi)$ and $\bar{c}_1(x, \xi)$ are positively homogeneous of degree 0 and 1 respectively. From Lemma 3.2 it follows that $x_n^{k-1} \bar{c}_0(x, \xi', D_n) D_n + x_n^k \bar{c}_1(x, \xi', D_n) + x_n^k A(x, \xi', D_n) + B(x, \xi', D_n) D_n + C(x, \xi', D_n)$ belongs to $S^0(R^{n-1}; V_{\xi'}^k(R), L^2(R))$. ii) is clear by the definition of $M_*(x, \xi)$. We shall prove iii). By Taylor's formula we obtain

$$(3.9) \quad \varphi(x/\delta) x_n^{2k} ((ab)(x, \xi') - (ab)(x', 0, \xi')) = x_n \varphi(x/\delta) x_n^{2k} f(x, \xi'),$$

where $f(x, \xi')$ is positively homogeneous of degree 2. Since $\sup |\varphi(x/\delta) x_n| < \varepsilon$ if δ is sufficiently small, the operator defined by (3.9) has a small norm in $\mathcal{L}(V_{\xi'}^k(R), L^2(R))$ if δ is sufficiently small. We can write

$$(3.10) \quad x_n^k ((a+b)(x, \xi') - (a+b)(x', 0, \xi')) D_n = x_n^{k+1} (a_1 + b_1)(x, \xi') D_n$$

$$(3.11) \quad x_n^{k-1} (b(x, \xi') - b(x', 0, \xi')) = x_n^k b_1(x, \xi'),$$

where a_1 and b_1 are positively homogeneous of degree 1. By the definition of the norm $V_{\xi'}^k(R)$ we see that the norms of the operators defined by (3.10) and (3.11) are estimated by $(1 + |\xi'|)^{-1/(k+1)}$. From Lemma 3.2 we have

$$\|x_n^k ((\bar{c}_1 + A)(x, \xi', D_n))\|_{\mathcal{L}(V_{\xi'}^k(R), L^2(R))} \leq C(1 + |\xi'|)^{-1/(k+1)},$$

$$\begin{aligned} \|x_n^{k-1} \tilde{c}_0(x, \xi', D_n) D_n\|_{\mathcal{L}(V_{\xi'}^k(R), L^2(R))} &\leq C(1 + |\xi'|)^{-k/(k+1)}, \\ \|B(x, \xi', D_n) D_n + C(x, \xi', D_n)\|_{\mathcal{L}(V_{\xi'}^k(R), L^2(R))} &\leq C(1 + |\xi'|)^{-1/(k+1)}, \end{aligned}$$

which complete the proof of lemma.

We can show the following

PROPOSITION 3.4. *Let $L(x, D)$ be the properly supported pseudo-differential operator defined by (1.1), where k is odd and $\sigma(P)\sigma(Q) < 0$. Suppose that $L(x, D)$ satisfies the condition i) or ii) in Theorem 1.2, then $M_0(x, \xi', D_n)$ in (3.7) generated by L is an isomorphic operator from $V_{\xi'}^k(R)$ to $L^2(R)$ when (x', ξ') belongs to a small conic neighbourhood of (x'_0, ξ'_0) and $|\xi'|$ is sufficiently large.*

PROOF. We make a change of variables. We put

$$\begin{aligned} t &= |\xi'|^{1/(k+1)} x_n, \quad \tilde{u}(t) = |\xi'|^{2/(k+1)-1/2(k+1)} u(|\xi'|^{-1/(k+1)} t), \\ \tilde{f}(t) &= |\xi'|^{-1/2(k+1)} f(|\xi'|^{-1/(k+1)} t). \end{aligned}$$

Then the equation $M_0(x', x_n, \xi', D_n) u(x_n) = f(x_n)$ transforms

$$\begin{aligned} M_0(x', t, \xi'/|\xi'|, D_t) \tilde{u}(t) \\ = (D_t - it^k a(x', 0, \xi'/|\xi'|)) (D_t - it^k b(x', 0, \xi'/|\xi'|)) u(t) \\ + t^{k-1} \tilde{c}(x', 0, \xi'/|\xi'|, 0) \tilde{u}(t) = \tilde{f}(t). \end{aligned}$$

The linear mapping $u(x_n) \rightarrow \tilde{u}(t)$ and $f(x_n) \rightarrow \tilde{f}(t)$ are isomorphic from $V_{\xi'}^k(R)$ to $V^k(R)$ and from $L^2(R)$ to $L^2(R)$ respectively. From this fact we have only to show that $M_0(x', t, \xi'/|\xi'|, D_t)$ is isomorphic from $V^k(R)$ to $L^2(R)$ if (x', ξ') belongs to a small conic neighbourhood of (x'_0, ξ'_0) and $|\xi'|$ is sufficiently large. We apply Proposition A.3 in Appendix to $M_0(x', t, \xi'/|\xi'|, D_t)$. We use the notation of (2.12), (2.13) and (2.14). From (2.3) and (2.4) we get

$$(3.12) \quad (H_{p_{m_1}}^{k-1} r_{m_1+m_2-1})(\rho) = -i(k-1)! (e^{k-1} g \tilde{c})(x', 0, \xi', 0),$$

$$(3.13) \quad (H_{p_{m_1}}^k q_{m_2})(\rho) = ik! (e^k f(a-b))(x', 0, \xi', 0),$$

where $\rho \in \Sigma$ and $\chi(\rho) = (x', 0, \xi', 0)$. By (3.12), (3.13) it implies that

$$\frac{\tilde{c}}{a-b}(x', 0, \xi', 0) = ik (H_{p_{m_1}}^{k-1} r_{m_1+m_2-1}) / (H_{p_{m_1}}^k q_{m_2})(\rho).$$

Therefore if $L(x, D)$ satisfies the condition i) or ii) of Theorem 1.2, by Proposition A.3 in Appendix it implies that $M_0(x', t, \xi'/|\xi'|, D_t)$ is isomorphic from $V^k(R)$ to $L^2(R)$. This completes the proof.

Now we shall construct a local parametrix, which means the following.

PROPOSITION 3.5. *Let $\widetilde{M}(x', D')$ be $M_0(x', D') + M_*(x', D')$, where $M_*(x', D')$ is a vector valued pseudo-differential operator defined by the symbol $M_*(x, \xi', D_n)$. Then there exists a vector valued pseudo-differential operator $E_0(x', D') \in L^0(R^{n-1}; L^2(R), V_{\xi'}^k(R))$ such that*

$$(3.4) \quad I - E_0 \widetilde{M} \in L^{-\infty}(R^{n-1}; V_{\xi'}^k(R), V_{\xi'}^k(R)),$$

$$(3.5) \quad I - \widetilde{M} E_0 \in L^{-\infty}(R^{n-1}; L^2(R), L^2(R)).$$

PROOF. Since $\|M_*(x, \xi', D_n)\|_{\mathcal{S}(V_{\xi'}^k(R), L^2(R))} < \varepsilon$, if ε is small enough, it implies by Proposition 3.4 that $\widetilde{M}(x, \xi', D_n)$ has a uniformly bounded inverse operator $E_0^0(x, \xi', D_n)$ in a conic neighbourhood of (x'_0, ξ'_0) when $|\xi'|$ is sufficiently large. We show that $E_0^0(x', \xi')$ is actually a symbol. Differentiation of the equation $M(x', \xi') E_0^0(x', \xi') = I$ gives

$$\begin{aligned} M(x', \xi') (D_x^{\alpha'} D_{\xi'}^{\beta'} E_0^0)(x', \xi') \\ = \sum C_{\alpha', \beta'} (D_x^{\alpha'} D_{\xi'}^{\beta'} M)(x', \xi') (D_x^{\alpha' - \alpha'} D_{\xi'}^{\beta' - \beta'} E_0^0)(x', \xi'), \end{aligned}$$

when $|\xi'|$ is large. Here the summation extended over multi-indices α' and β' with $\alpha' \leq \alpha$, $\beta' \leq \beta$ not both zero. If we multiply it from the left by $E_0^0(x', \xi')$ it follows by the induction with respect to $|\alpha + \beta|$ that $E_0^0(x', \xi') \in S^0(R^{n-1}; L^2(R), V_{\xi'}^k(R))$. The construction of $E_0(x', D')$ is therefore formally same as that of a parametrix of an elliptic operator in the scalar case (c.f Proposition 2.5.1 in [12]). This completes the proof.

Now we shall start the direct proof of Theorem 1.2. Let $\bar{\psi}, \psi$ be properly supported pseudo-differential operators such that the symbols of $\bar{\psi}, \psi$ belong to $S^0(R^n \times R^n)$ and these supports are contained in a small conic neighbourhood of $\chi(\rho)$. Furthermore these symbols are equal to 1 in a small conic neighbourhood of $\chi(\rho)$. Set

$$E = \bar{\psi} \circ E_0(x', D') \circ \psi.$$

Then we have the following

PROPOSITION 3.6.

- i) $WF'(E) \subset \text{diag}(T^*(R^n) \setminus 0)$ and $(\chi(\rho), \chi(\rho)) \in WF'(E)$.
- ii) For all $s \in \mathbb{R}$ E is continuous as $H_s^{\text{loc}}(R^n) \rightarrow H_{s+2/(k+1)}^{\text{loc}}(R^n)$.

PROOF. Let $\bar{\psi}_1, \psi_1$ be properly supported pseudo-differential operators in $L^0(R^n)$ such that $\bar{\psi}_1(x, \xi) = 1$ and $\psi_1(x, \xi) = 1$ in $\text{supp}(\bar{\psi}(x, \xi))$ and $\text{supp}(\psi(x, \xi))$ respectively. These supports are contained in a small conic neighbourhood of $\chi(\rho)$. From (3.14), (3.15) and ii) of Proposition 3.1 it follows

that

$$(3.16) \quad \tilde{\varphi}_1 E_0 \phi_1 \tilde{M} \equiv \tilde{M} \tilde{\varphi}_1 E_0 \phi_1 \equiv I \quad \text{at} \quad (\chi(\rho), \chi(\rho)).$$

Let $((x, \xi), (y, \eta)) \in WE'(\tilde{\varphi}_1 E_0 \phi_1)$ and $(x, \xi), (y, \eta)$ belong to small conic neighbourhood of $\chi(\rho)$. If $(x, \xi) \neq (y, \eta)$, then from i) of Proposition 3.1, it implies that $x_n \neq 0$ or $\xi_n \neq 0$ or $y_n \neq 0$ or $\eta_n \neq 0$ when $|\xi'| + |\eta'| \neq 0$. On the other hand if $|\xi'| + |\eta'| = 0$ then $\xi_n \neq 0$ or $\eta_n \neq 0$. Therefore we may assume (x, ξ) or (y, η) is an elliptic point of $\tilde{M}(x, D)$. This implies that $((x, \xi), (y, \eta)) \in WF'(\tilde{\varphi}_1 E_0 \phi_1 \tilde{M}) \cup WF'(\tilde{M} \tilde{\varphi}_1 E_0 \phi_1)$, which contradicts to (3.16). Since $\text{supp } \tilde{\varphi}$ and $\text{supp } \phi$ are sufficiently small, we get $WF'(E) \subset \text{diag}(T^*(R^n) \setminus 0)$. It is clear that $(\chi(\rho), \chi(\rho)) \in WF'(E)$, because $\tilde{M}E \equiv E\tilde{M} \equiv I$ at $(\chi(\rho), \chi(\rho))$. We have

$L^0(R^{n-1}; L^2(R), V_{\xi'}^k(R)) \subset L^{-2/(k+1)}(R^{n-1}; L^2(R), L^2(R))$ and $(WF(\tilde{\varphi}) \cup WF(\phi)) \cap \{(x, \xi) \in T^*(R^n) \setminus 0; \xi' = 0\} = \emptyset$. It implies ii) by proposition A.2 in [16]. This complies the proof.

Since $\tilde{M} \equiv M$ at $(\chi(\rho), \chi(\rho))$ and $WF'(E) \subset \text{diag } V$ where V is small conic neighbourhood of $\chi(\rho)$, we see that $ME \equiv \tilde{M}E$ and $EM \equiv E\tilde{M}$ at $(\chi(\rho), \chi(\rho))$. Let us $F_{\chi(\rho)} = E \circ E'$, where E' is a parametrix of $E(x, D)$ defined in Lemma 2.3. Then

$$F_{\chi(\rho)} \tilde{L}(x, D) \equiv \tilde{L}(x, D) F_{\chi(\rho)} \equiv I \quad \text{at} \quad (\chi(\rho), \chi(\rho)).$$

For any point $\rho \in \Sigma$ we shall define the operator $F_\rho = U^* F_{\chi(\rho)} U$. If $\rho \notin \Sigma$, then since ρ is an elliptic point of $L(x, D)$, we can define the operator $F_\rho = L'_\rho$, where L'_ρ is a local parametrix of L at ρ . Let $\phi_j \in L^0(X)$, $j \in J$, be a locally finite collection of properly supported pseudo-differential operators such that for $\rho_j \in T^*(X) \setminus 0$ we have

$$\sum \phi_j \equiv I, \quad WF'(\phi_j) \subset V_{\rho_j},$$

where V_{ρ_j} is a small conic neighbourhood of ρ_j . Set $F = \sum \phi_j F_{\rho_j}$, then it is clear that $WF'(F) \subset \text{diag}(T^*(X) \setminus 0)$. If $F_\rho L \equiv L F_\rho \equiv I$ at (ρ', ρ') then $F_\rho \equiv F_{\rho'}$ at (ρ', ρ') , which implies $WF'(F) = \text{diag}(T^*(X) \setminus 0)$. To complete the proof of Theorem 1.2, we may assume that F is properly supported after adding operator with C^∞ kernel. Therefore F is a desired operator in Theorem 1.2.

4. Other results.

In this section we shall state theorems when k is even case or k is odd and $\sigma(P)\sigma(Q)=1$. First we shall show a theorem when k is even.

THEOREM 4.1. Let k be an even integer and $P(x, D), Q(x, D)$ and $R(x, D)$ be elements of $M^{m_1, k}(\Sigma, X)$, $M^{m_2, k}(\Sigma, X)$ and $L^{m_1+m_2-1}(X)$ respectively. We consider the operator $L(x, D) = (P \circ Q) + R$. Assume that $L(x, D)$ satisfies the following condition A) or B)

A) $R(x, D)$ belongs to $M^{m_1+m_2-1, j}(\Sigma, X)$ ($j \geq k$); otherwise $R(x, D)$ belongs to $L^{m_1+m_2-2}(X)$.

B) $R(x, D)$ belongs to $M^{m_1+m_2-1, k-1}(\Sigma, X)$ and $H_{p_{m_1}}^{k-1}(r_{m_1+m_2-1} - i\{p_{m_1}, q_{m_2}\}) = 0$ on Σ .

Here when $R(x, D) \in M^{m_1+m_2-1, l}(\Sigma, X)$ we assume that P, Q and R satisfy the conditions of Proposition 2.2. Then there exists a properly supported operator $F; \mathcal{D}'(X) \rightarrow \mathcal{D}'(X)$ which is continuous; $H_s^{\text{loc}}(X) \rightarrow H_{s+m_1+m_2-2k/(k+1)}^{\text{loc}}(X)$ for all $s \in \mathbb{R}$ such that

$$FL(x, D) \equiv L(x, D)F \equiv I, \quad WF'(F) = \text{diag}(T^*(X) \setminus \{0\}).$$

PROOF. When $L(x, D)$ satisfies the condition A) then $M_0(x, \xi', D_n)$ of (3.7) is equal to

$$(D_n - ix_n^k a(x', 0, \xi'))(D_n - ix_n^k b(x', 0, \xi')).$$

If L satisfies the condition B) then $M_0(x, \xi', D_n)$ is equal to

$$(D_n - ix_n^k b(x, 0, \xi'))(D_n - ix_n^k a(x, 0, \xi')).$$

From Lemma A.3 in Appendix $M_0(x, \xi', D_n)$ is isomorphic from $V_{\xi'}^k(R)$ to $L^2(R)$. Thus the proof of this theorem is similar to that of Theorem 1.2.

COLLORARY 4.2. Let $L(x, D)$ be the operator treated in Theorem 4.1. Then $L(x, D)$ is locally solvable at every point of X and strictly hypoelliptic. For any compact set of X the following estimate holds for all $u \in C_0^\infty(K)$

$$\|u\|_{m_1+m_2-2k/(k+1)} \leq C(K) (\|Lu\|_0 + \|u\|_{m_1+m_2-2}).$$

In the next case, we assume that k is odd and $\chi(P)\chi(Q)=1$, which implies that the index of $M_0(x, \xi', D_n)$ is equal to ± 2 (see Proposition A.2). Therefore using the analogous argument of Section 5 in [15], we can easily verify the following theorem. To avoid confusion we omit the proof. In the following theorem adjoints will be taken with respect to L^2 inner product on $C_0^\infty(X)$ defined by some strictly positive density on X .

THEOREM 4.3. We assume $P(x, D) \in M^{m_1, k}(\Sigma, X)$, $Q(x, D) \in M^{m_2, k}(\Sigma, X)$ and $R(x, D)$ belongs to $M^{m_1+m_2-1, j}(\Sigma, X)$ or $L_c^{m_1+m_2-2}(X)$. Here k is odd and $j \geq k-1$. Let $L(x, D) = P \circ Q + R$, where P, Q and R satisfy the condition A) in Proposition 2.2. Then we have the following.

i) If $\sigma(P)=1$, $\sigma(Q)=1$ then there exist properly supported operators $F, F^+ : \mathcal{D}'(X) \rightarrow \mathcal{D}'(X)$ such that F is continuous: $H_s^{\text{loc}}(X) \rightarrow H_{s+m_1+m_2-2k/(k+1)}^{\text{loc}}(X)$, F^+ is continuous: $H_s^{\text{loc}}(X) \rightarrow H_s^{\text{loc}}(X)$ for all $s \in \mathbb{R}$,

$$F^+ + FL \equiv I, \quad LF \equiv I, \quad (F^+)^* \equiv F^+,$$

$$WF'(F) = \text{diag}(T^*(X) \setminus 0) \quad \text{and} \quad WF'(F^+) = \text{diag}(\Sigma),$$

ii) If $\sigma(P)=-1$ and $\sigma(Q)=-1$, then there exist properly supported operators $F, F^- : \mathcal{D}'(X) \rightarrow \mathcal{D}'(X)$ such that F is continuous $H_s^{\text{loc}}(X) \rightarrow H_{s+m_1+m_2-2k/(k+1)}^{\text{loc}}(X)$, F^- is continuous: $H_s^{\text{loc}}(X) \rightarrow H_s^{\text{loc}}(X)$ for all $s \in \mathbb{R}$,

$$FL \equiv I, \quad F^- + LF \equiv I, \quad (F^-)^* \equiv F^-,$$

$$WF'(F) = \text{diag}(T^*(X) \setminus 0) \quad \text{and} \quad WF'(F^-) = \text{diag}(\Sigma).$$

From this theorem we can obtain the information with respect to local solvability and hypoellipticity.

COLLORARY 4.4. If $\sigma(P)=1$ and $\sigma(Q)=1$, then the operator $L(x, D)$ is locally solvable at every point of X . If $\sigma(P)=-1$ and $\sigma(Q)=-1$, then $L(x, D)$ is strictly hypoelliptic. Moreover for all compact set of X there exists a constant $C(K)$ such that for all $u \in C_0^\infty(K)$ we have

$$\|u\|_{m_1+m_2-2k/(k+1)} \leq C(K) (\|Lu\|_0 + \|u\|_{m_1+m_2-2}).$$

REMARK. In Cardoso and Treves [5], they considered the operator $L(x, D)$, which essentially satisfies the condition ii) in Theorem 4.3. In their case they proved non-solvability of L . The above Collorary 4.3 does not contradict to the result of Gilioli and Treves [8]. In their case, the characteristic set has two component. $\sigma(P)$ and $\sigma(Q)$ are positive in one component and $\sigma(P)$ and $\sigma(Q)$ are negative in the other component.

Appendix.

In the present Appendix, we investigate the index of the following ordinary differential operator

$$(A.1) \quad L(t, D_t) = (D_t - iat^k)(D_t - ibt^k) + ct^{k-1}.$$

Here a, b and c are complex numbers. We assume $\text{Re}a \neq 0$ and $\text{Re}b \neq 0$ throughout this Appendix. We shall regard $L(t, D_t)$ as an operator on the Hilbert space $V^k(\mathbb{R})$, which is normed by

$$\|u\|_{V^k(\mathbb{R})}^2 = \sum_{\substack{\alpha \leq 2 \\ \beta \leq (2-\alpha)k}} \|t^\beta D_t^\alpha u\|_{L^2(\mathbb{R})}^2.$$

In this situation, we note the following

PROPOSITION A. 1.

- i) $L(t, D_t)$ is a continuous operator from $V^k(R)$ to $L^2(R)$.
- ii) $L(t, D_t)$ is a Neotherian operator from $V^k(R)$ to $L^2(R)$.
- iii) If $u \in L^2(R)$ and $Lu \in \mathcal{S}(R)$, then u belongs to $\mathcal{S}(R)$.

The statement i) is clear by definition. ii) and iii) are the special case of Theorem 2.1 in [8] and Proposition 1.10 in [10] respectively. By ii) we can take of the index of $L(t, D_t): V^k(R) \rightarrow L^2(R)$, which we denote by $\text{ind}(L)$.

PROPOSITION A. 2. Let $L(t, D_t)$ be an ordinary differential operator defined in (A. 1). Then we have

$$\text{ind}(L) = 0 \text{ if } k \text{ is even and}$$

$$\text{ind}(L) = \text{Rea}/|\text{Rea}| + \text{Reb}/|\text{Reb}| \text{ if } k \text{ is odd.}$$

PROOF. In view of Proposition A. 1 iii) it suffices to show that the operator ct^{k-1} is a compact operator from $V^k(R)$ to $L^2(R)$. Let $A(r) \in C^\infty(R)$ be a monotonously increasing function such that $A(r) = 1$ if $r < 1$, $A(r) = r$ if $r > 2$. The transformation $t \rightarrow s = tA(|t|^k)$ being a diffeomorphism from R to R , so we denote its inverse transformation by $t = \varphi(s)$. For any $f(t) \in L^2(R)$ and $u(t) \in V^k(R)$ we put $i_1(f) = A^{-1/2}(\varphi(s))f(\varphi(s))$ and $i_2(u)(s) = A^{3/2}(\varphi(s))u(\varphi(s))$, respectively. Since the operator i_1 and i_2 are isomorphisms from $L^2(R)$ to $L^2(R)$ and $V^k(R)$ to $H^2(R)$ respectively. Here $H^2(R)$ is a usual Sobolev space. Hence we show that the multiplication operator $T(s) = i_1 \circ (ct^{k-1}) \circ i_2^{-1}$ is compact operator from $H^2(R)$ to $L^2(R)$. By a simple computation we get

$$(A. 2) \quad T(s) = |c| |s|^{-1} \text{ if } |s| > 2^k.$$

Let $\chi(s)$ be a $C_0^\infty(R)$ function such that $\text{supp } \chi \subset \{s; |s| < 1\}$ and $\chi = 1$ in a neighbourhood of 0. For arbitrary $\varepsilon > 0$, the operator $\chi(\varepsilon s)T$ is continuous operator from $H^2(R)$ to $H^2(B_\varepsilon)$, where $B_\varepsilon = \{s; |s| \leq 1/\varepsilon\}$. By Rellich's theorem, we see that for any $\varepsilon > 0$, $\chi(\varepsilon s)T$ is a compact operator from $H^2(R)$ to $L^2(R)$. From (A. 2) it is clear that $\|(1 - \chi(\varepsilon s))T\|_{\mathcal{L}(H^2(R), L^2(R))} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore T is a compact operator from $H^2(R)$ to $L^2(R)$. This completes the proof.

When the index of L is zero, we can give a detailed information about L .

PROPOSITION A. 3. Let $L(t, D_t)$ be an ordinary differential operator defined in (A. 1). Then we have

- i) When k is even and $c = 0$, then $L(t, D_t)$ is isomorphism from $V^k(R)$ to $L^2(R)$.

ii) We assume that k is odd and $(\text{Re}a)(\text{Re}b) < 0$. If $L(t, D_t)$ satisfies the following condition A) or B), then $L(t, D_t)$ is isomorphic from $V^k(R)$ to $L^2(R)$.

A) If $\text{Re}a > 0$, there is no positive integer n such that

$$\frac{c}{a-b} = 1 - n(k+1) \quad \text{or} \quad -n(k+1),$$

B) If $\text{Re}b > 0$, there is no positive integer n such that

$$\frac{c}{a-b} = (n-1)(k+1) \quad \text{or} \quad 1 + (n-1)(k+1).$$

PROOF. i) is clear from iii) of Proposition A.1. We shall verify ii). Since $L(t, D_t)$ is a Notherian operator, the range of L is closed is $L^2(R)$. Thus if we show that the dense set $C_0^\infty(R)$ of $L^2(R)$ is contained in the range of L , we have that $L(t, D_t)$ is isomorphism, because the index of L is equal to 0 and the domain of L is $V^k(R)$. In Gilioli and Treves [7] (pp. 371-374), they proved that if $L(t, D_t)$ satisfies the condition A) or B) then for arbitrary $f \in C_0^\infty(R)$ there exists $u \in C^2(R) \cap L^2(R)$ such that $L(t, D_t)u = f$. By iii) of Proposition A.1 it follows that u belongs to $V^k(R)$. This completes the proof.

References

- [1] BEALS, R. and FEFFERMAN, C.: On local solvability of linear partial differential equations, *Ann. of Math.*, 97 (1973), 483-498.
- [2] BOUTET DE MONVEL, L. and TREVES, F.: On a class of pseudodifferential operators with double characteristics, *Invent. Math.*, 24 (1974), 1-34.
- [3] BOUTET DE MONVEL, L. and TREVES, F.: On a class of system of pseudodifferential operators with double characteristics, *Comm. Pure Appl. Math.*, 27 (1974), 59-89.
- [4] BOUTET DE MONVEL, L.: Hypoelliptic operators with double characteristics and related pseudo-differential operators, *Comm. Pure Appl. Math.*, 27 (1974), 585-639.
- [5] CARDOSO, F. and TREVES, F.: A necessary condition of local solvability for pseudo-differential equations with double characteristics, *Ann. Inst. Fourier, Grenoble*, 24 (1974), 225-292.
- [6] DUISTERMAAT, J. J. and HÖRMANDER, L.: Fourier integral operator II, *Acta Math.*, 128 (1972), 183-269.
- [7] GILIOLI, A. and TREVES, F.: An example in the solvability theory of linear PDE'S, *Amer. J. Math.*, 96 (1974), 366-384.
- [8] GRUŠIN, V. V.: On a class of hypoelliptic operators, *Math. USSR Sobornik*, 12 (1970), 458-476.
- [9] GRUŠIN, V. V.: On a class of elliptic operators degenerate on a submanifold,

- Math. USSR Sobornik, 13 (1971), 155-185.
- [10] GRUŠIN V. V.: Pseudodifferential operators on R^n with bounded symbols, *Functional Anal. Appl.*, 4 (1970), 202-212.
 - [11] GRUŠIN, V. V.: Hypoelliptic differential equations and pseudodifferential operators with operator valued symbols, *Math. USSR Sobornik*, 17 (1972), 497-514.
 - [12] HÖRMANDER, L.: Fourier integral operators I, *Acta Math.*, 127 (1971), 79-183.
 - [13] MATHER, J. N.: On Nirenberg's proof of Malgrange's preparation theorem, *Lecture Notes in Math.*, 192 (1971), 116-120.
 - [14] NIRENBERG, L. and TREVES, F.: On local solvability of linear partial differential equations, Part I and II, *Comm. Pure Appl. Math.* 23 (1970), 1-38 and 459-510.
 - [15] SJÖSTRAND, J.: Parametrices for pseudodifferential operator with multiple characteristics. *Arkiv för Math.*, 12 (1974), 85-130.
 - [16] SJÖSTRAND, J.: Operator of principal type with interior boundary conditions. *Acta Math.*, 130 (1973), 1-51.
 - [17] TREVES, F.: A new method of proof of the subelliptic estimates, *Comm. Pure Appl. Math.*, 24 (1971), 71-115.
 - [18] TREVES, F.: Concatenations of second order evolution equations applies to local solvability and hypoellipticity, *Comm. Pure Appl. Math.*, 26 (1973), 201-250.
 - [19] YAMAMOTO, K.: Parametrices for pseudo-differential equations with double characteristics II, (to appear).

Department of Mathematics
Hokkaido University.