On some variational properties of submanifolds in Riemannian spaces

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§ 1. Introduction. Let V^n be a closed orientable hypersurface in an $(n+1)$-dimensional Euclidean space E^{n+1} and $\bar{V}^n(\varepsilon)$ be a family of admissible hypersurfaces parameterized by the real number ε near $\varepsilon=0$ such that $\bar{V}^n(0)=V^n$. We put

$$J[H_1^c]=\int_{V^n} H_1^c\, d\sigma\,,$$

where c is an arbitrary positive integer, H_1 is the mean curvature of V^n and $d\sigma$ means the volume element of V^n. We denote by δJ the first variation of the functional J:

$$\delta(J[H_1^c])=\left(\frac{\partial}{\partial\varepsilon} J[\bar{H}_1^c(\varepsilon)]\right)_{\varepsilon=0}\,,$$

where $\bar{H}_1(\varepsilon)$ is the mean curvature of $\bar{V}^n(\varepsilon)$.

The normal variation is defined to be the variation such that the direction of the deformation at each point of V^n is in the direction of the normal of V^n. V^n is said to be stable with respect to $J[H_1^c]$ if $\delta(J[H_1^c])=0$ for any normal variation. In particular, when V^n is stable with respect to $J[H_1^n]$, V^n is called the stable hypersurface. B. Y. Chen [1][1)] has proved that a closed orientable hypersurface V^n in E^{n+1} is stable with respect to $J[H_1^c]$ if and only if H_1 and R' satisfy

$$c\Delta H_1^{c-1}+n^2(c-1)H_1^{c+1}+cH_1^{c-1}R'=0\,, \tag{1.1}$$

where Δ denotes the Laplacian with respect to the induced metric on V^n and R' is the scalar curvature of V^n. When $c=1$, we obtain from (1.1), $R'=0$ and this result was given by M. Pinl and H. W. Trapp [2]. If we denote by H_2 the second mean curvature of V^n, from the Gauss equation we get

$$R'=-n(n-1)H_2\,. \tag{1.2}$$

Therofore we can see that if a closed orientable hypersurface V^n in E^{n+1} is stable with respect to $J[H_1]$, then $H_2=0$.

1) Numbers in brackets refer to the references at the end of the paper.

Putting $c=n$ in (1. 1), by virtue of (1. 2) we have

$$\Delta H_1^{n-1} = -H_1^{n-1}\left\{n(n-1)(H_1^2-H_2)\right\}. \tag{1. 3}$$

By means of (1. 3), B. Y. Chen proved that a stable hypersurface V^n in E^{n+1} is a hypersphere for $n=2m+1$ and when $n=2m$ we have the same result under the hypothesis that H_1 does not change its sign.

When V^n is a closed orientable hypersurface in an $(n+1)$-dimensional space form $R^{n+1}(K)$ of curvature K, the first variation of the integral of arbitrary functions with respect to $H_\nu (\nu=0, 1, \cdots, n)$ has been studied by R. C. Reilly [3], where $H_\nu (\nu=1, \cdots, n)$ denotes the ν-th mean curvature of V^n and specially we put $H_0=1$. He showed that if $\delta(J[H_\nu])=0$, then we have

$$(\nu+1)\binom{n}{\nu+1}H_{\nu+1}+K(n-\nu+1)\binom{n}{\nu-1}H_{\nu-1}=0 .$$

If we put $\nu=1$ in the last equation, we get $K=-(n-1)\ H_2$. Thus, we can see that if V^n in $R^{n+1}(K)$ is stable with respect to $J[H_1]$, then H_2=const. Particularly, if $R^{n+1}(K)$ is an Euclidean space E^{n+1}, we have $H_2=0$ and this is the result of M. Pinl and H. W. Trapp.

Recently, the variational properties for the normal variation of a closed orientable hypersurface V^n in a general Riemannian space R^{n+1} have been investigated by T. J. Willmore and C. S. Jhaveri [4]. It was proved that V^n is the stable hypersurface if and only if H_1 satisfies

$$\Delta H_1^{n-1} = -H_1^{n-1}\left\{n(n-1)(H_1^2-H_2)-R_{ij}N^iN^j\right\}, \tag{1. 4}$$

where R_{ij} and N^i denotes the Ricci tensor of R^{n+1} and the unit normal vector of V^n respectively. From (1. 4) we find that if V^n is a stable hypersurface in R^{n+1} and $R_{ij}N^iN^j \leqq 0$ on V^n, then V^n is the minimal or the umbilical hypersurface for $n=2m+1$ and when $n=2m$, we have the same result under the hypothesis that H_1 does not change its sign. Since there exist no closed minimal hypersurface in an Euclidean space (S. B. Myers [5]), when R^{n+1} is E^{n+1}, we get from (1. 4) the result of B. Y. Chen.

The purpose of the present paper is to investigate the variational properties of a closed orientable submanifold V^n of an arbitrary codimension p in a Riemannian space R^{n+p} and give certain generalizations of the above stated results. The terminologies, notations and the basic relations for submanifolds in a Riemannian space are provided in § 2. When the mean curvature H_1 of V^n does not vanish on V^n, the unit normal vector $\underset{E}{N^i}$, which has the same direction with the mean curvature vector, is determined uniquely at each point on V^n (Y. Katsurada, T. Nagai and H. Kôjyô [6]). When $p=1$, the vector $\underset{E}{N^i}$ is the unit normal vector N^i of a closed orientable hypersurface V^n in R^{n+1}. Then, in the present paper the variation in the direction $\underset{E}{N^i}$ is called the normal variation. A submanifold V^n is said to be stable with respect to $J[H_1^c]$ if $\delta(J[H_1^c])=0$ for any normal variation and when V^n is stable with respect to $J[H_1^n]$, we call it the stable submanifold. In § 3 we find the condition for $\delta(J[H_1^c])=0$ with respect to the normal variation and making use of the condition of the case $c=n$ and $c=1$, we study the properties of the stable submanifold and the submanifold which is stable with respect to $J[H_1]$.

The idea of the variation in the direction of a vector field has been introduced by Y. Katsurada [7]. According to this idea, in § 4 we study some variational problems with respect to the variation in the direction ξ^i, where ξ^i is a vector field in R^{n+p}. The condition for $\delta(J[H_1^c])=0$ with respect to the variation in the direction ξ^i is given in § 4. In particular, when ξ^i is the homothetic Killing vector field, we give the properties of V^n which is stable with respect to $J[H_1^c]$ for the variation in the direction ξ^i.

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§ 2. The fundamental equations for submanifolds. Let R^{n+p} $(n\geqq 2, p\geqq 1)$ be an $(n+p)$-dimensional Riemannian space of class C^r $(r\geqq 3)$ and $(x^1, x^2, \cdots, x^{n+p})$ be a local coordinate system of R^{n+p}. Let V^n be an n-dimensional closed orientable submanifold in R^{n+p}, then V^n is expressed locally by the equation

$$x^i = x^i(u^\alpha), \qquad (i=1, 2, \cdots, n+p\,;\ \alpha=1, 2, \cdots, n)^{2)}$$

where $(u^1, u^2, \cdots, u^n)$ is a local coordinate system of V^n and the Jacobian matrix $(\partial x^i/\partial u^\alpha)$ is of rank n. If we denote by g_{ij} the metric tensor of R^{n+p} and put $B^i_\alpha = \partial x^i/\partial u^\alpha$, then the induced metric tensor $g_{\alpha\beta}$ of V^n is given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta\,, \quad (2.1)$$ [3)]

2) In this paper the Latin indices $i, j, k, \cdots$ run from 1 to $n+p$ and the Greek indices $\alpha, \beta, \gamma, \cdots$ run from 1 to n.

3) Throughout this paper we shall use the Einstein convention, that is when the same index appears in any term as an upper index and a lower index, it is understood that this letter is summed for all the values over its range.

and the volume element $d\sigma$ of V^n is given by

$$d\sigma = \sqrt{g}\, du^1 \wedge \cdots \wedge du^n , \tag{2.2}$$

where $g=\det.(g_{\alpha\beta})$.

Let $\underset{P}{N^i}$ $(P=n+1, n+2, \cdots, n+p)$[4] be the contravariant components of p unit vectors which are normal to V^n and mutually orthogonal and the set of $n+p$ vectors

$$(B^i_1, B^j_2, \cdots, B^i_n, \underset{n+1}{N^i}, \underset{n+2}{N^i}, \cdots, \underset{n+p}{N^i}) \tag{2.3}$$

be a positively oriented frame at each point on V^n. Putting

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta , \qquad \underset{P}{N_i} = g_{ij} \underset{P}{N^j} , \tag{2.4}$$

we have

$$\begin{aligned} g^{ij} &= g^{\alpha\beta} B^i_\alpha B^j_\beta + \sum_{P=n+1}^{n+p} \underset{P}{N^i} \underset{P}{N^j} , \\ g_{ij} &= g_{\alpha\beta} B^\alpha_i B^\beta_j + \sum_{P=n+1}^{n+p} \underset{P}{N_i} \underset{P}{N_j} , \end{aligned} \tag{2.5}$$

and we can see that the set of $n+p$ vectors

$$(B^1_i, B^2_i, \cdots, B^n_i, \underset{n+1}{N_i}, \underset{n+2}{N_i}, \cdots, \underset{n+p}{N_i})$$

gives the dual frame of the frame (2.3), where g^{ij} and $g^{\alpha\beta}$ are defined by the equation $g^{ij} g_{jk}=\delta^i_k$ and $g^{\alpha\beta} g_{\beta\gamma}=\delta^\alpha_\gamma$ respectively and δ^i_k and δ^α_γ denote the Kronecker delta.

In the present paper we shall denote by "," and ";" the partial differentiation and the covariant differentiation along V^n due to van der Waerden-Bortolotti respectively. Denoting by $\underset{P}{b_{\alpha\beta}}$ the second fundamental tensor with respect to $\underset{P}{N^i}$ and putting $\underset{P}{b^\gamma{}_\alpha}=g^{\beta\gamma} \underset{P}{b_{\alpha\beta}}$ we get the following fundamental formulas:

$$B^i_{\alpha;\beta} = \sum_{P=n+1}^{n+p} \underset{P}{b_{\alpha\beta}} \underset{P}{N^i} , \qquad \text{(Gauss formula)} \tag{2.6}$$

$$\underset{P}{N^i}{}_{;\alpha} = -\underset{P}{b^\gamma_\alpha} B^i_\gamma + \Gamma''^Q_{P\alpha} \underset{Q}{N^i} , \qquad \text{(Weingarten formula)} \tag{2.7}$$

where

$$\Gamma''^Q_{P\alpha} = (\underset{P}{N^i}{}_{,j} + \Gamma^i_{hj} \underset{P}{N^h}) B^j_\alpha \underset{Q}{N_i} , \tag{2.8}$$

and Γ^i_{hj} are the Christoffel symbols defined by g_{ij}. Since $\underset{P}{N^i} \underset{Q}{N_i}=\delta_{PQ}$, from (2.8) we have

4) In this paper the capital Latin indices $P, Q, R, \cdots$ run from $n+1$ to $n+p$.

$$\Gamma''^{Q}_{P\alpha} + \Gamma''^{P}_{Q\alpha} = 0\,,$$

and if $p=1$, i.e., when V^n is a hypersurface in R^{n+1}, then $\Gamma''^{Q}_{P\alpha}$ vanishes identically.

Putting

$$R^i{}_{jkh} = \Gamma^i_{jh,k} - \Gamma^i_{jk,h} + \Gamma^l_{jh}\Gamma^i_{lk} - \Gamma^l_{jk}\Gamma^i_{lh}\,,$$

$$R'^{\delta}{}_{\alpha\beta\gamma} = \Gamma'^{\delta}_{\alpha\gamma,\beta} - \Gamma'^{\delta}_{\alpha\beta,\gamma} + \Gamma'^{\varepsilon}_{\alpha\gamma}\Gamma'^{\delta}_{\varepsilon\beta} - \Gamma'^{\varepsilon}_{\alpha\beta}\Gamma'^{\delta}_{\varepsilon\gamma}\,,$$

where $\Gamma'^{\alpha}_{\beta\gamma}$ are the Christoffel symbols defined by $g_{\alpha\beta}$, then from the integrability conditions of the Gauss and Weingarten formula we have the following Gauss and Mainardi-Codazzi equations:

$$R_{ihjk}B^i_\delta B^h_\alpha B^j_\beta B^k_\gamma = R'_{\delta\alpha\beta\gamma} - \sum_{P=n+1}^{n+p}\left(\underset{P}{b}_{\delta\beta}\,\underset{P}{b}_{\alpha\gamma} - \underset{P}{b}_{\delta\gamma}\,\underset{P}{b}_{\alpha\beta}\right), \tag{2.9}$$

$$R_{ihjk}\underset{P}{N}^i B^h_\alpha B^j_\beta B^k_\gamma = \underset{P}{b}_{\alpha\gamma;\beta} - \underset{P}{b}_{\alpha\beta;\gamma} + \underset{Q}{b}_{\alpha\gamma}\Gamma''^{Q}_{P\beta} - \underset{Q}{b}_{\alpha\beta}\Gamma''^{Q}_{P\gamma}\,. \tag{2.10}$$

We shall denote by $\underset{P}{H}_\nu$ the ν-th mean curvature of V^n with respect to $\underset{P}{N}^i$. Then we have

$$\underset{P}{H}_1 = \frac{1}{n}\sum_{\alpha=1}^{n}\underset{P}{\kappa}_\alpha = \frac{1}{n}\underset{P}{b}^{\alpha}_{\alpha}\,, \tag{2.11}$$

$$\underset{P}{H}_2 = \frac{2}{n(n-1)}\sum_{\alpha<\beta}\underset{P}{\kappa}_\alpha\,\underset{P}{\kappa}_\beta = \frac{1}{n(n-1)}\left(\underset{P}{b}^{\gamma}_{\gamma}\,\underset{P}{b}^{\delta}_{\delta} - \underset{P}{b}^{\gamma}_{\delta}\,\underset{P}{b}^{\delta}_{\gamma}\right), \tag{2.12}$$

where $\underset{P}{\kappa}_\alpha$ means the principal curvature of V^n for the normal vector $\underset{P}{N}^i$. By means of (2.11) and (2.12) we get

$$\underset{P}{H}_1^2 - \underset{P}{H}_2 = \frac{1}{n^2(n-1)}\sum_{\alpha<\beta}\left(\underset{P}{\kappa}_\alpha - \underset{P}{\kappa}_\beta\right)^2. \tag{2.13}$$

Let H^i be the contravariant component of the mean curvature vector of V^n, then from (2.6) and (2.11) we have

$$H^i = \frac{1}{n}B^i_{\alpha;\beta}\,g^{\alpha\beta} = \frac{1}{n}\sum_{P=n+1}^{n+p}\underset{P}{b}^{\alpha}_{\alpha}\,\underset{P}{N}^i = \sum_{P=n+1}^{n+p}\underset{P}{H}_1\,\underset{P}{N}^i\,, \tag{2.14}$$

and the mean curvature H_1 of V^n is given by

$$H_1 = (g_{ij}H^i H^j)^{1/2}\,. \tag{2.15}$$

When the mean curvature H_1 does not vanish on V^n, we have the unit normal vector $\underset{E}{N}^i$ at each point of V^n. In this case we get $H^i = H_1\underset{E}{N}^i$ and if we take a set of p mutually orthogonal unit normal vectors $\underset{P}{N}^i$ ($P = n+1, n+2, \cdots, n+p$) in such a way that $\underset{n+1}{N}^i = \underset{E}{N}^i$, then from (2.14) and

(2.15) it follows that

$$H_1 = \underset{E}{H_1} = \frac{1}{n}\underset{E}{b_\alpha^\alpha}, \qquad \underset{P}{H_1} = 0 \quad (P=n+2, \cdots, n+p). \tag{2.16}$$

Let C^i be any normal vector of V^n and $(C^i{}_{;\alpha})^N$ be the normal part of $C^i{}_{;\alpha}$. When $(C^i{}_{;\alpha})^N = 0$, the vector C^i is said to be parallel with respect to the connection in the normal bundle. From (2.7), we can see that the vector $\underset{E}{N^i}$ is parallel with respect to the connection in the normal bundle if and only if $\Gamma''^{P}_{E\alpha} = 0$ $(P=n+2, \cdots, n+p,\ \alpha=1, \cdots, n)$.

§3. The normal variation of the integral $J[H_1^c]$. Let V^n be an n-dimensional closed orientable submanifold in an $(n+p)$-dimensional Riemannian space R^{n+p}. In this section we assume that the mean curvature H_1 of V^n does not vanish at each point of V^n. Let

$$\bar{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \rho(u^\alpha)\underset{E}{N^i}(u^\alpha)\,\varepsilon, \tag{3.1}$$

be a normal variation of V^n associated with a function ρ on V^n, where ε is a parameter in a small interval containing 0. Then we have a family of admissible submanifolds $\bar{V}^n(\varepsilon)$ such that $\bar{V}^n(0) = V^n$. When $\bar{\Omega}(\varepsilon)$ be a geometric object on $\bar{V}^n(\varepsilon)$ such that $\bar{\Omega}(0) = \Omega$, we put

$$\delta\Omega = \left(\frac{\partial}{\partial\varepsilon}\bar{\Omega}(\varepsilon)\right)_{\varepsilon=0}.$$

From (3.1) it follows that

$$\bar{B}_\alpha^i = B_\alpha^i + (\rho\underset{E}{N^i})_{,\alpha}\,\varepsilon. \tag{3.2}$$

Since we have

$$\underset{E}{N^i}{}_{;\alpha} = \underset{E}{N^i}{}_{,\alpha} + \Gamma^i_{jk}\underset{E}{N^j}B_\alpha^k, \tag{3.3}$$

by means of (2.7) and (3.2) we get

$$\delta B_\alpha^i = \rho_{;\alpha}\underset{E}{N^i} - \rho(\underset{E}{b_\alpha^\gamma}B_\gamma^i + \Gamma^i_{jk}\underset{E}{N^j}B_\alpha^k - \Gamma''^{P}_{E\alpha}\underset{P}{N^i}). \tag{3.4}$$

By means of (2.1) and (3.4) we get

$$\delta g_{\alpha\beta} = -2\rho\underset{E}{b_{\alpha\beta}}. \tag{3.5}$$

Sinec $\bar{g}^{\alpha\beta}(\varepsilon)\bar{g}_{\beta\gamma}(\varepsilon) = \delta_\gamma^\alpha$, from (3.5) we have

$$\delta g^{\alpha\beta} = 2\rho g^{\alpha\gamma}g^{\beta\delta}\underset{E}{b_{\gamma\delta}}. \tag{3.6}$$

Furthermore, from the relation $\delta\sqrt{g} = \frac{1}{2}\sqrt{g}\,g^{\alpha\beta}\delta g_{\alpha\beta}$ and (3.5) we get

$$\delta d\sigma = -\rho \underset{E}{b}{}^{\alpha}_{\alpha}\, d\sigma = -n\rho H_1\, d\sigma\,. \tag{3.7}$$

From (2.16) it follows that

$$\delta H_1 = \frac{1}{n}\left\{(\delta g^{\alpha\beta})\, \underset{E}{b}{}_{\alpha\beta} + g^{\alpha\beta}\delta \underset{E}{b}{}_{\alpha\beta}\right\}.$$

From the definition of the covariant differentiation along V^n, we have

$$B^i_{\alpha;\beta} = B^i_{\alpha,\beta} + \Gamma^i_{jk}\, B^j_\alpha\, B^k_\beta - \Gamma'^{\gamma}_{\alpha\beta}\, B^i_\gamma\,,$$

and from (3.2) we can see that

$$\frac{\partial}{\partial\varepsilon}(\bar{B}^i_{\alpha,\beta}) = \left(\frac{\partial}{\partial\varepsilon}\bar{B}^i_\alpha\right)_{,\beta} = (\rho \underset{E}{N}{}^i)_{,\alpha,\beta}\,.$$

Then, by means of $\underset{E}{b}{}_{\alpha\beta} = B^i_{\alpha;\beta}\underset{E}{N}{}_i$ we get

$$\begin{aligned} g^{\alpha\beta}\delta \underset{E}{b}{}_{\alpha\beta} = g^{\alpha\beta}\Big\{&(\rho \underset{E}{N}{}^i)_{,\alpha,\beta} + \rho \Gamma^i_{jk,h}\underset{E}{N}{}^h B^j_\alpha B^k_\beta + \Gamma^i_{jk}(\rho \underset{E}{N}{}^j)_{,\alpha} B^k_\beta \\ &+ \Gamma^i_{jk} B^j_\alpha (\rho \underset{E}{N}{}^k)_{,\beta} - \Gamma'^{\gamma}_{\alpha\beta}(\rho \underset{E}{N}{}^i)_{,\gamma}\Big\}\underset{E}{N}{}_i + nH_1 \underset{E}{N}{}^i \delta \underset{E}{N}{}_i\,. \end{aligned} \tag{3.8}$$

Since $\bar{g}^{ij}\underset{E}{\bar{N}}{}_i \underset{E}{\bar{N}}{}_j = 1$, it follows that

$$\underset{E}{N}{}^i \delta \underset{E}{N}{}_i = \rho \Gamma^i_{jk}\underset{E}{N}{}_i \underset{E}{N}{}^j \underset{E}{N}{}^k\,. \tag{3.9}$$

Making use of (3.3) and (3.9), from (3.8) we get

$$g^{\alpha\beta}\delta \underset{E}{b}{}_{\alpha\beta} = g^{\alpha\beta}\left\{(\rho \underset{E}{N}{}^i)_{;\alpha;\beta}\underset{E}{N}{}_i - R_{ijkh}\underset{E}{N}{}^i B^j_\alpha B^k_\beta \underset{E}{N}{}^h\right\}.$$

By virtue of (2.6) and (2.7) we find that

$$g^{\alpha\beta}(\rho \underset{E}{N}{}^i)_{;\alpha;\beta}\underset{E}{N}{}_i = \Delta\rho - \rho(\underset{E}{b}{}^\beta_\alpha \underset{E}{b}{}^\alpha_\beta - g^{\alpha\beta}\Gamma''^{P}_{E\alpha}\Gamma''^{E}_{P\beta})\,.$$

Then we have

$$\delta H_1 = \frac{1}{n}\left\{\rho(\underset{E}{b}{}^\beta_\alpha \underset{E}{b}{}^\alpha_\beta + g^{\alpha\beta}\Gamma''^{P}_{E\alpha}\Gamma''^{E}_{P\beta} - R_{ijkh}\underset{E}{N}{}^i B^j_\alpha B^k_\beta \underset{E}{N}{}^h g^{\alpha\beta}) + \Delta\rho\right\}. \tag{3.10}$$

For any positive integer c we have

$$\delta(J[H_1^c]) = \int_{V^n} cH_1^{c-1}(\delta H_1)\, d\sigma + \int_{V^n} H^c(\delta d\sigma)\,. \tag{3.11}$$

On the other hand, applying the Green's theorem to the closed orientable submanifold V^n, we have

$$\int_{V^n} H_1^{c-1}(\Delta\rho)\, d\sigma = \int_{V^n}(\Delta H_1^{c-1})\,\rho d\sigma\,.$$

Consequently, by means of (3.7) and (3.10) we finally obtain

$$\delta(J[H_1^c]) = \int_{V^n} \rho\Big\{\frac{c}{n}(\Delta H_1^{c-1}) + \frac{c}{n} H_1^{c-1}(\underset{E}{b}{}_\alpha^\beta \underset{E}{b}{}_\beta^\alpha + g^{\alpha\beta}\Gamma''^{P}_{E\alpha}\Gamma''^{E}_{P\beta} - R_{ijkh}\underset{E}{N}{}^i B_\alpha^j B_\beta^k \underset{E}{N}{}^h g^{\alpha\beta} - \frac{n^2}{c}H_1^2)\Big\} d\sigma\,. \tag{3.12}$$

LEMMA 3.1. *Let V^n be a closed orientable submanifold in R^{n+p}, then V^n is stable with respect to $J[H_1^c]$ if and only if*

$$\frac{c}{n}(\Delta H_1^{c-1}) = -\frac{c}{n} H_1^{c-1}(\underset{E}{b}{}_\alpha^\beta \underset{E}{b}{}_\beta^\alpha + g^{\alpha\beta}\Gamma''^{P}_{E\alpha}\Gamma''^{E}_{P\beta} - R_{ijkh}\underset{E}{N}{}^i B_\alpha^j B_\beta^k \underset{E}{N}{}^h g^{\alpha\beta} - \frac{n^2}{c}H_1^2)\,. \tag{3.13}$$

(PROOF) If V^n is stable with respect to $J[H_1^c]$, then we must have $\delta(J[H_1^c])=0$ for any function ρ. Therefore, from (3.12) we have (3.13). The converse is evident. Q.E.D.

THEOREM 3.2. *Let V^n be a closed orientable submanifold in R^{n+p}. If*
(i) *$\underset{E}{N}{}^i$ is parallel with respect to the connection in the normal bundle,*
(ii) *$R_{ijkh}\underset{E}{N}{}^i B_\alpha^j B_\beta^k \underset{E}{N}{}^h g^{\alpha\beta} \leqq 0$ on V^n,*
then every point of V^n is mubilic with respect to $\underset{E}{N}{}^i$.

(PROOF) Putting $c=n$ in (3.13), from (2.12) and our hypothesis (i) we get

$$\Delta H_1^{n-1} = -H_1^{n-1}\{n(n-1)(H_1^2-\underset{E}{H_2}) - R_{ijkh}\underset{E}{N}{}^i B_\alpha^j B_\beta^k \underset{E}{N}{}^h g^{\alpha\beta}\}\,. \tag{3.14}$$

From (2.13) and the hypothesis (ii), we find

$$n(n-1)(H_1^2-\underset{E}{H_2}) - R_{ijkh}\underset{E}{N}{}^i B_\alpha^j B_\beta^k \underset{E}{N}{}^h g^{\alpha\beta} \geqq 0\,. \tag{3.15}$$

On the other hand, since $H_1 \neq 0$ on V^n, from the continuity, H_1 has a fixed sign on V^n. Then from (3.14) we have $\Delta H_1^{n-1} \leqq 0$ on V^n. Consequently, applying the Hopf's theorem we get $\Delta H_1^{n-1}=0$ and since V^n is compact orientable we get $H_1=$const. ($\neq 0$). Then, from (3.14) the left hand member of (3.15) must vanish. This implies that $H_1^2-\underset{E}{H_2}=0$. i.e., every point of V^n is umblic with respect to $\underset{E}{N}{}^i$. Q.E.D.

In particular, when $p=1$, we may put $\underset{E}{N}{}^i=N^i$, where N^i is the unit normal vector of a hypersurface V^n in R^{n+1} and it is determined uniquely at each point on V^n without the assumption $H_1\neq 0$. In this case the hypothesis (i) in Theorem 3.2 is satisfied identically. Furthermore, by means of (2.5) we get

$$R_{ijkh}N^i B_\alpha^j B_\beta^k N^h g^{\alpha\beta} = R_{ih}N^i N^h\,.$$

Then, when $p=1$, Theorem 3.2 gives us the result of T. J. Willmore and C. S. Jhaveri.

THEOREM 3.3 *Let $R^{n+p}(K)$ be a constant curvature space of curvature K and V^n be a closed orientable submanifold in $R^{n+p}(K)$. If*

(i) *$\underset{E}{N}^i$ is parallel with respect to the connection in the normal bundle,*

(ii) *V^n is stable with respect to $J[H_1]$,*

then $\underset{E}{H_2}=const.$

(PROOF) Putting $c=1$ in Lemma 3.1, by virtue of our hypothesis (i) we get

$$\frac{1}{n}\underset{E}{b}{}_\alpha^\beta \underset{E}{b}{}_\beta^\alpha - nH_1^2 - \frac{1}{n}R_{ijkh}\underset{E}{N}^i B_\alpha^j B_\beta^k \underset{E}{N}^h g^{\alpha\beta} = 0. \tag{3.16}$$

Substituting $R_{ijkh}=K(g_{ik}g_{jh}-g_{ih}g_{jk})$ into (3.16), by means of (2.5), (2.11) and (2.12) we obtain $\underset{E}{H_2}=K/(n-1)$. Q.E.D.

In particular, when $p=1$ in Theorem 3.3, we have

$$\frac{1}{n}b_\alpha^\beta b_\beta^\alpha - nH_1^2 - \frac{1}{n}R_{ijkh}N^i B_\alpha^j B_\beta^k N^h g^{\alpha\beta} = 0, \tag{3.16'}$$

without the assumption $H_1 \neq 0$. If R^{n+1} is an Einstein space, we have

$$R_{ijkh}N^i B_\alpha^j B_\beta^k N^h g^{\alpha\beta} = \frac{R}{n+1}.$$

Then, from (3.16)' we have $(n-1)H_2 = -R/n(n+1)$. Therefore, we have

COROLLARY 3.4. *Let V^n be a closed orientable hypersurface in an Einstein space R^{n+1}. If V^n is stable with respect to $J[H_1]$, then $H_2=const.$*

In particular, when R^{n+1} in the above corollary is a constant curvature space, we get the result of R. C. Reilly.

§4. The variation of the integral $J[H_1^c]$ in the direction of a vector field. Let ξ^i be a vector field in R^{n+p} and L_ξ be the operator of Lie derivation with respect to the vector field ξ^i. Then we have (K. Yano [8])

$$L_\xi g_{ij} = \xi_{i;j} + \xi_{j;i}, \tag{4.1}$$

$$\xi^i{}_{;j;k} = L_\xi \Gamma^i_{jk} - R^i{}_{jhk}\xi^h. \tag{4.2}$$

We now consider a variation of a geometrical object in R^{n+p}, defined by

$$\bar{x}^i = x^i + \xi^i(x^j)\varepsilon, \tag{4.3}$$

where ε is a parameter near $\varepsilon=0$. Let V^n be an n-dimensional closed orientable submanifold in R^{n+p} and the local expression of V^n be

$$x^i = x^i(u^\alpha). \tag{4.4}$$

In this section we assume that the submanifold V^n is imbedded in a regular domain with respect to the vector field ξ^i. Then, substituting (4.4) into (4.3) we have

$$\bar{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \xi^i(x^j(u^\alpha))\,\varepsilon, \tag{4.5}$$

and by means of these $n+p$ functions we get a family of admissible submanifolds $\bar{V}^n(\varepsilon)$ parameterized by the real number ε such that $\bar{V}^n(0) = V^n$. From (4.5) it follows that

$$\bar{B}^i_\alpha = B^i_\alpha + \xi^i{}_{,\alpha}\,\varepsilon, \tag{4.6}$$

$$\delta B^i_\alpha = \xi^i{}_{,\alpha}. \tag{4.7}$$

Since we have

$$\xi^i{}_{;\alpha} = \xi^i{}_{,\alpha} + \Gamma^i_{jk}\,\xi^j B^k_\alpha, \tag{4.8}$$

by means of (2.1), (4.1) and (4.7) we have

$$\delta g_{\alpha\beta} = g_{ij}(\xi^i{}_{;\alpha} B^j_\beta + B^i_\alpha \xi^j{}_{;\beta}) = (L_\xi\, g_{ij})\, B^i_\alpha B^j_\beta.$$ [5]

From the last relation we get

$$\delta g^{\alpha\beta} = -g^{\alpha\gamma} g^{\beta\delta}(L_\xi\, g_{ij})\, B^i_\gamma B^j_\delta, \tag{4.9}$$

$$\delta d\sigma = \frac{1}{2} g^{\alpha\beta}(L_\xi\, g_{ij})\, B^i_\alpha B^j_\beta\, d\sigma. \tag{4.10}$$

Let c be a positive integer. Then we have

$$\delta(J[H_1^c]) = \int_{V^n} \frac{c}{2} H_1^{c-2}(\delta H_1^2)\, d\sigma + \int_{V^n} H_1^c(\delta d\sigma), \tag{4.11}$$

for $c \geqq 2$, and (4.11) is valid for $c=1$ under the hypothesis that $H_1 \neq 0$ on V^n.

By means of (2.14) and (2.15) it follows that

$$\delta H_1^2 = \frac{\partial g_{ij}}{\partial x^k} \xi^k H^i H^j + \frac{2}{n}\left\{(\delta B^i_{\alpha;\beta})\, g^{\alpha\beta} + B^i_{\alpha;\beta}(\delta g^{\alpha\beta})\right\} H_i \tag{4.12}$$

From (4.6) we get

$$\frac{\partial}{\partial \varepsilon}(\bar{B}^i_{\alpha,\beta}) = \left(\frac{\partial}{\partial \varepsilon} \bar{B}^i_\alpha\right)_{,\beta} = \xi^i{}_{,\alpha,\beta}.$$

Therefore, by virtue of

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5) This relation, (4 9) and (4.10) have been given by Y. Katsurada [7].

$$B^i_{\alpha;\beta}=B^i_{\alpha,\beta}+\Gamma^i_{jk}\,B^j_\alpha\,B^k_\beta-\Gamma'^\gamma_{\alpha\beta}\,B^i_\gamma\,,$$

we obtain

$$\frac{2}{n}(\delta B^i_{\alpha;\beta})\,g^{\alpha\beta}\,H_i=\frac{2}{n}(\xi^i_{,\alpha,\beta}+\Gamma^i_{hk,p}\,\xi^p\,B^h_\alpha\,B^k_\beta$$
$$+\Gamma^i_{hk}\,\xi^h_{,\alpha}\,B^k_\beta+\Gamma^i_{hk}\,B^h_\alpha\,\xi^k_{,\beta}-\Gamma'^\gamma_{\alpha\beta}\,\xi^i_{,\gamma})\,g^{\alpha\beta}\,H_i\,.$$

On the other hand, by means of (2. 14) we find that

$$\frac{2}{n}\,\Gamma^i_{jk}\,\xi^k\,B^j_{\alpha;\beta}\,g^{\alpha\beta}\,H_i=\frac{\partial g_{ij}}{\partial x^k}\,\xi^k\,H^i\,H^j\,.$$

Then by means of (4. 8) we get

$$\frac{2}{n}(\delta B^i_{\alpha;\beta})\,g^{\alpha\beta}\,H_i=\frac{2}{n}(\xi^i_{;\alpha;\beta}+R^i_{\ hpk}\,\xi^p\,B^h_\alpha\,B^k_\beta)\,g^{\alpha\beta}\,H_i-\frac{\partial g_{ij}}{\partial x^k}\,\xi^k\,H^i\,H^j\,.$$

By means of (2. 14), (4. 1) and (4. 2) we have

$$\tag{4. 13} \begin{aligned}&(\xi^i_{;\alpha;\beta}+R^i_{\ hpk}\,\xi^p\,B^h_\alpha\,B^k_\beta)\,g^{\alpha\beta}\,H_i\\ &=\Big\{(L_\xi\,\Gamma^i_{jk})\,B^j_\alpha\,B^k_\beta\,g^{\alpha\beta}\,H_i+\frac{n}{2}(L_\xi\,g_{ij})\,H^i\,H^j\Big\}\,.\end{aligned}$$

Consequently, from (4. 11) and (4. 12) we have

LEMMA 4. 1. *Let V^n be a closed orientable submanifold in R^{n+p} ($p\geqq 2$). Then, with respect to the variation in the direction of a vector field ξ^i we have*

$$\tag{4. 14} \begin{aligned}\delta(J[H_1^c])=\int_{V^n}\frac{c}{n}\,H_1^{c-2}\Big\{&(L_\xi\,\Gamma^i_{jk})\,B^j_\alpha\,B^k_\beta\,g^{\alpha\beta}\,H_i+\frac{n}{2}(L_\xi\,g_{ij})\,H^i\,H^j\\ &-B^i_{\alpha;\beta}\,g^{\alpha\gamma}\,g^{\beta\delta}\,(L_\xi\,g_{jk})\,B^j_\gamma\,B^k_\delta\,H_i\Big\}\,d\sigma\\ &+\int_{V^n}\frac{1}{2}\,H_1^c\,g^{\alpha\beta}(L_\xi\,g_{ij})\,B^i_\alpha\,B^j_\beta\,d\sigma\end{aligned}$$

for any positive integer c ($\geqq 2$) and (4. 14) *is valid for $c=1$ under the hypothesis that $H_1\neq 0$ on V^n.*

When $p=1$, we have

$$H^i=\frac{1}{n}\,B^i_{\alpha;\beta}\,g^{\alpha\beta}=H_1\,N^i\,,$$

where N^i is the unit normal vector of a hypersurface V^n in R^{n+1}. Then we get for any positive integer c,

$$\tag{4. 15} \begin{aligned}\delta(J[H_1^c])=&\int_{V^n}\frac{c}{n}\,H_1^{c-1}\Big\{(\delta B^i_{\alpha;\beta})\,g^{\alpha\beta}+B^i_{\alpha;\beta}(\delta g^{\alpha\beta})\Big\}\,N_i\,d\sigma\\ &+\int_{V^n}cH_1^c\,N^i(\delta N_i)\,d\sigma+\int_{V^n}H_1^c(\delta d\sigma)\,.\end{aligned}$$

Since $\bar{g}^{ij}\bar{N}_i\bar{N}_j=1$, it follows that

$$N^i\,\delta N_i=\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\xi^k N^i N^j\,.$$

On the other hand, using the same way as in the case of submanifold, we have

$$(4.16)\quad \frac{1}{n}(\delta B^i_{\alpha;\beta})\,g^{\alpha\beta}N_i=\frac{1}{n}(\xi^i{}_{;\alpha;\beta}+R^i_{hpk}\,\xi^p B^h_\alpha B^k_\beta)\,g^{\alpha\beta}N_i-\frac{1}{2}H_1\frac{\partial g_{ij}}{\partial x^k}\xi^k N^i N^j\,.$$

By means of (4.13), (4.15) and (4.16) we have

LEMMA 4.2. *Let V^n be a closed orientable hypersurface in R^{n+1} and c be an arbitrary positive integer. Then, with respect to the variation in the direction of a vector field ξ^i we have*

$$(4.17)\quad \begin{aligned}\delta(J[H_1^c])=&\int_{V^n}\frac{c}{n}H_1^{c-1}\Big\{(L_\xi\Gamma^i_{jk})B^j_\alpha B^k_\beta g^{\alpha\beta}N_i\\&+\frac{n}{2}H_1(L_\xi g_{ij})N^iN^j-b^{\gamma\delta}(L_\xi g_{ij})B^i_\gamma B^j_\delta\Big\}\,d\sigma\\&+\int_{V^n}\frac{1}{2}H_1^c\,g^{\alpha\beta}(L_\xi g_{ij})B^i_\alpha B^j_\beta\,d\sigma\,.\end{aligned}$$

In particular, if ξ^i is a homothetic Killing vector field such that $L_\xi g_{ij}=2\phi g_{ij}$, where $\phi=$const., then we have $L_\xi\Gamma^i_{jk}=0$ and we get the following relation from (4.14) and (4.17):

$$(4.18)\quad \delta(J[H_1^c])=\int_{V^n}(n-c)\,\phi H_1^c\,d\sigma\,.$$

Then we have

THEREM 4.3. *Let ξ^i be a homothetic Killing vector field in R^{n+p} and V^n be a closed orientabl϶ svbmanifold in R^{n+p}. Then $\delta J([H_1^n])=0$ with respect to the variation in the direction of the vector field ξ^i.*

When $c\neq n$, by virtue of Lemma 4.1 and Lemma 4.2 we have

THEOREM 4.4. *Let ξ^i be a homothetic Killing vector field in R^{n+p} and V^n be a closed orientable submanifold in R^{n+p}. If*

(i) *$c(\neq n)$ is an even positive integer,*

(ii) *$\delta(J[H_1^c])=0$ with respect to the variation in the direction of the vector field ξ^i,*

then V^n is the minimal submanifold.

Furthermore, in consequence of Lemma 4.1, we have

THEOREM 4.5. *Let ξ^i be a homothetic Killing vector field in R^{n+p} $(p\geqq 2)$ and V^n be a closed orientable submanifold in R^{n+p}. If*

(i) *$c(\neq n, >1)$ is an odd positive integer,*

(ii) *$\delta(J[H_1^c]=0$ with respect to the variation in the direction of the vector field ξ^i,*

then V^n is the minimal submanifold.

When $p \geqq 2$, by virtue of Lemma 4. 1, we have (4. 18) for $c=1$ under the hypothesis $H_1 \neq 0$. Therefore, Theorem 4. 5 is not valid for the case $c=1$. However, when $p=1$, by virtue of Lemma 4. 2 we get (4. 18) for $c=1$. Then we have

THEOREM 4. 6. *Let ξ^i be a homothetic Killing vector field in R^{n+1} and V^n be a closed orientable hypersurface in R^{n+1}. If*

(i) *$c(\neq n)$ is an odd positive integer,*

(ii) *$\delta(J[H_1^c])=0$ with respect to the variation in the direction of the vector field ξ^i,*

(iii) *H_1 does not change its sign on V^n,*

then V^n is the minimal hypersurface.

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