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A family of hypersurfaces in S^{n+1} defined by the harmonic conjugate relation

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Introduction.

In the present paper, the author will study certain hypersurfaces of the $(n+1)$-dimensional unit sphere S^{n+1} which are defined by the harmonic conjugate relation with respect to it as a quadratic hypersurface in R^{n+2}. Here, the harmonic conjugate relation is used in the sense as follows: a point x in R^{n+2} is called *harmonic conjugate* to a point y in R^{n+2} with respect to S^{n+1}, if x is on the polar hyperplane of y with respect to S^{n+1}.

The motivation of the introduction of such hypersurfaces of S^{n+1} is due to his work [3] in which he has investigated minimal hypersurfaces of S^{n+1} with three principal curvature fields and tried to find out examples of such hypersurfaces. He succeeded in constructing such hypersurfaces of special type (Theorem 4 in [3]) under certain conditions for three tangent vector fields of them determined corresponding to these principal curvature fields. In order to find out examples of such minimal hypersurfaces of S^{n+1} without the above mentioned conditions, the author payed his attention to the family of hypersurfaces dealed with in the present paper which were perceived through the properties of the examples in [3] and will try to find out minimal hypersurfaces of this kind out of these families. And, he will show naturality of the examples in [3] by Theorem 6 in some sense. However he will not succeed in finding out minimal hypersurfaces expected to be in these families.

§ 1. The harmonic conjugate relation

For a subset $A\subset R^{n+1}$, we denote by $[A]$ the smallest linear subspace containing A in the following. For $A_1, A_2, \cdots, A_m\subset R^{n+1}$, let

$$[A_1,, A_2, \cdots, A_m] := [A_1\cup A_2\cup\cdots\cup A_m] .$$

Let S^n be the unit n-sphere in R^{n+1} given by

$$\sum_{i=1}^{n+1} x_i^2 = 1 \tag{1.1}$$

and P_y the polar hyperplane of a point y of R^{n+1} with respect to S^n given by

$$\sum_{i=1}^{n+1} y_i x_i = 1, \tag{1.2}$$

where $y=(y_1, \cdots, y_{n+1})$. For $A \subset R^{n+1}$, we define

$$\text{h-conj}\, A := \cap_{x \in A} P_x .$$

We can easily prove the following

LEMMA 1. *For A, $B \subset R^{n+1}$, we have*

i) $$\text{h-conj}\, A = \text{h-conj}\, [A],$$

ii) $$\dim(\text{h-conj}\, A) = n - \dim [A]$$

iii) $$\text{h-conj}\, A \supset B \rightarrow \text{h-conj}\, B \supset A .$$

Now, we call A is harmonic conjugate to B with respect to S^n, if h-conj $B \supset A$. By Lemma 1, we can say A and B are mutually harmonic conjugate (with respect to S^n).

LEMMA 2. *Let A and B be mutually harmonic conjugate linear subspaces of R^{n+1} with respect to S^n, then for any point y of S^n, $y \in A \cup B$, the spheres $S_A = [A, y] \cap S^n$ and $S_B = [B, y] \cap S^n$ are orthogonal at y.*

PROOF. Putting $\dim A = p \geqq 1$ and $\dim B = q \geqq 1$, we choose $(p+1)$ points $a_\alpha = (a_{\alpha i})$, $\alpha = 0, 1, \cdots, p$, spanning A and $(q+1)$ points $b_\lambda = (b_{\lambda i})$, $\lambda = 0, 1, \cdots, q$, spanning B. Then, we have

$$(a_\alpha, b_\lambda) := \sum_{i=1}^{n+1} a_{\alpha i}\, b_{\lambda i} = 1 . \tag{1.3}$$

Since any point $x \in [A, y]$ can be written as

$$x = \sum_\alpha u_\alpha a_\alpha + ty, \tag{1.4}$$

where

$$\sum_\alpha u_\alpha + t = 1, \tag{1.5}$$

we have

$$(x, x) = \sum_{\alpha,\beta} (a_\alpha, a_\beta)\, u_\alpha u_\beta + 2 \sum_\alpha (a_\alpha, y)\, u_\alpha t + t^2 = 1 . \tag{1.6}$$

For y, we have $(u_0, \cdots, u_p, t) = (0, \cdots, 0, 1)$. Hence for any differential at y along S_A we have from (1.5) and (1.6)

$$\sum_\alpha du_\alpha + dt = 0, \qquad \sum_\alpha (a_\alpha, y)\, du_\alpha + dt = 0,$$

that is

$$dt = -\sum_\alpha du_\alpha, \tag{1.7}$$

$$\sum_\alpha \left\{(a_\alpha, y)-1\right\} du_\alpha = 0 . \tag{1.8}$$

Therefore, any tangent vector X of S_A at y can be expressed as

$$X = \sum_\alpha \xi_\alpha (a_\alpha - y), \tag{1.9}$$

where ξ_α satisfy the condition:

$$\sum_\alpha \left\{(a_\alpha, y)-1\right\} \xi_\alpha = 0 . \tag{1.10}$$

Analogously, any tangent vector Y of S_B at y can be expressed as

$$Y = \sum_\lambda \eta_\lambda (b_\lambda - y), \tag{1.11}$$

where η_λ satisfy the condition:

$$\sum_\lambda \left\{(b_\lambda, y)-1\right\} \eta_\lambda = 0 . \tag{1.12}$$

Now, we compute the inner product $\langle X, Y\rangle$ of X and Y. By means of (1.9), (1.11), (1.10), (1.12) and (1.3), we have

$$\begin{aligned}\langle X, Y\rangle &= \sum_{\alpha,\lambda} (a_\alpha - y,\ b_\lambda - y)\, \xi_\alpha \eta_\lambda \\ &= \sum_{\alpha,\lambda} \left\{(a_\alpha, b_\lambda)-(a_\alpha, y)-(b_\lambda, y)+1\right\} \xi_\alpha \eta_\lambda \\ &= -\sum_\lambda \eta_\lambda \sum_\alpha \left\{(a_\alpha, y)-1\right\} \xi_\alpha - \sum_\alpha \xi_\alpha \sum_\lambda \left\{(b_\lambda, y)-1\right\} \eta_\lambda = 0,\end{aligned}$$

which shows that

$$T_y S_A \perp T_y S_B . \qquad \text{Q.E.D.}$$

Let A and B be linear subspaces as in Lemma 2. If $A \cap B \neq \phi$, for any point $x \in A \cap B$ it must be $(x, x)=1$, hence x is a point of S^n. Furthermore we have $A \subset P_x$ and $B \subset P_x$. Since P_x is the tangent hyperplane of S^n at x, A and B must be tangent to S^n and orthogonal to each other. If $A \cap B = \phi$, there exists no direction which is parallel to A and B, because we have the same situation in the hyperplane at ∞ with respect to the induced polarity from S^n. Thus, we obtain easily the following

LEMMA 3. *Let A and B two linear subspaces of R^{n+1} which are not tangent to S^n at the same point and harmonic conjugate to each other, then A and B are mutually independent,* i.e.

$$\dim [A, B] = \dim A + \dim B + 1 .$$

LEMMA 4. *If A is harmonic conjugate to B_1 and B_2, then so is A to $[B_1, B_2]$.*

PROOF. By the assumption, we have h-conj $A \supset B_1$ and B_2 hence

$$\text{h-conj}\, A \supset [B_1, B_2] . \qquad \text{Q. E. D.}$$

§ 2. The definition of $M^n(x_0; L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$

In the following, we suppose that $L_i^{m_i}$, $i=1, 2, 3$, are m_i-dimensional linear subspaces of R^{n+2} respectively such that they are *mutually harmonic conjugate to each other with respect to the unit* $(n+1)$*-sphere*:

$$\sum_{i=1}^{n+2} x_i^2 = 1 , \tag{2.1}$$

not tagent to S^{n+1} and

$$m_1 + m_2 + m_3 = n . \tag{2.2}$$

LEMMA 5. We have

$$\text{h-conj}\, L_i^{m_i} = [L_j^{m_j}, L_k^{m_k}] \quad (i, j, k: \textit{distinct}) , \tag{2.3}$$

$$[L_1^{m_1}, L_2^{m_2}, L_3^{m_3}] = R^{n+2} . \tag{2.4}$$

PROOF. By Lemma 3, $L_1^{m_1}$ and $L_2^{m_2}$ are mutually independent and so

$$\dim [L_1^{m_1}, L_2^{m_2}] = m_1 + m_2 + 1 .$$

By Lemma 4, $[L_1^{m_1}, L_2^{m_2}]$ is harmonic conjugate to $L_3^{m_3}$. Once more by Lemma 3, $[L_1^{m_1}, L_2^{m_2}]$ and $L_3^{m_3}$ are mutually independent and hence

$$\text{h-conj}\, L_3^{m_3} = [L_1^{m_1}, L_2^{m_2}] .$$

Since $\Big[[L_1^{m_1}, L_2^{m_2}], L_3^{m_3}\Big] = [L_1^{m_1}, L_2^{m_2}, L_3^{m_3}]$, it must be

$$[L_1^{m_1}, L_2^{m_2}, L_3^{m_3}] = R^{n+2} . \qquad \text{Q. E. D.}$$

Now, we take a fixed point x_0 of S^{n+1} *not contained in each* $L_i^{m_i}$, $i=1, 2, 3$, and $P_{x_0} \not\supset L_i^{m_i}$, $i=1, 2, 3$. We choose $u_i \in L_i^{m_i}$, $i=1, 2, 3$, such that $\dim [x_0, u_1, u_2, u_3] = 3$. Let us put

$$\begin{aligned}
&[x_0, u_i] \cap S^{n+1} - \{x_0\} = x_i ; \\
&[x_i, u_j] \cap S^{n+1} - \{x_i\} = x_{ij}, \; i \neq j ; \\
&[x_{ij}, u_k] \cap S^{n+1} - \{x_{ij}\} = x_{ijk}, \; k \neq i, \; k \neq j ; \\
&\qquad\qquad i, j, k = 1, 2, 3 .
\end{aligned}$$

Here we have used the convention that for $x \in S^{n+1}$, $u \in R^{n+2}$, $u \neq x$, $[x, u] \cap S^{n+1} - \{x\} = x$ if the straight line $[x, u]$ is tangent to S^{n+1}.

LEMMA 6. $x_{ij} = x_{ji}$ and $x_{ijk} = x_{ikj}$ for $i \neq j$, $k \neq i$, $k \neq j$.

PROOF. We may put $i=1$, $j=2$, $k=3$. Since u_1 and u_2 are harmonic conjugate to each other with respect to the circle:

$$S^1 = [u_1, u_2, x_0] \cap S^{n+1} .$$

Hence, by the well-known fact in projective geometry as shown in Fig. 1, we have

$$x_{12} = x_{21} .$$

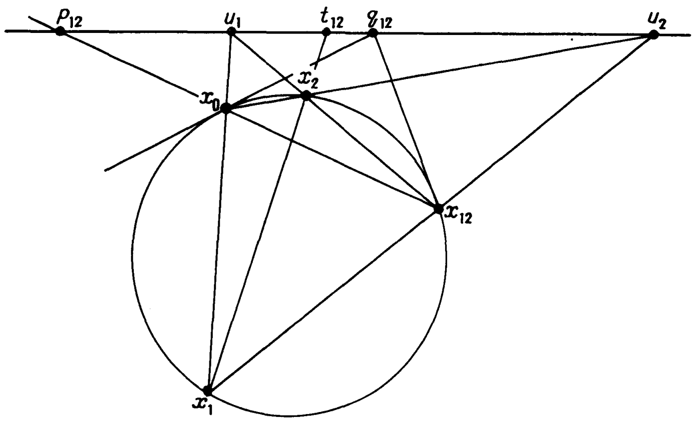

Fig. 1.

Next, since $\{x_1, x_{12}, x_{13}\}$ satisfies analogous conditions to those of $\{x_0, x_1, x_2\}$ we obtain easily $x_{123}=x_{132}$. Q. E. D.

By virtue of this lemma, we denote the point x_{123} by $x=x(u_1, u_2, u_3)$

Now, for $i \neq j$ we put

$$\begin{aligned} p_{ij} &= [x_0, x_{ij}] \cap [u_i, u_j] = p_{ji}, \\ t_{ij} &= [x_i, x_j] \cap [u_i, u_j] = t_{ji}, \\ q_{ij} &= P_{x_0} \cap [u_i, u_j] = q_{ji} \end{aligned}$$

then we see easily that

i) the pair $\{u_i, u_j\}$ is harmonic conjugate to the pair $\{p_{ij}, t_{ij}\}$,

ii) p_{ij} and q_{ij} are harmonic conjugate to each other with respect to S^{n+1}.

Since $p_{12} \in [u_1, u_2]$, p_{12} and u_3 are harmonic conjugate to each other with respect to S^{n+1}. Regarding as

$$[x_0, p_{12}] \cap S^{n+1} - \{x_0\} = x_{12} ,$$

we have

$$[x_0, x_{123}] \cap [p_{12}, u_3] = [x_0, x] \cap [u_1, u_2, u_3] ,$$

which we denote $p=p(u_1, u_2, u_3,)$.

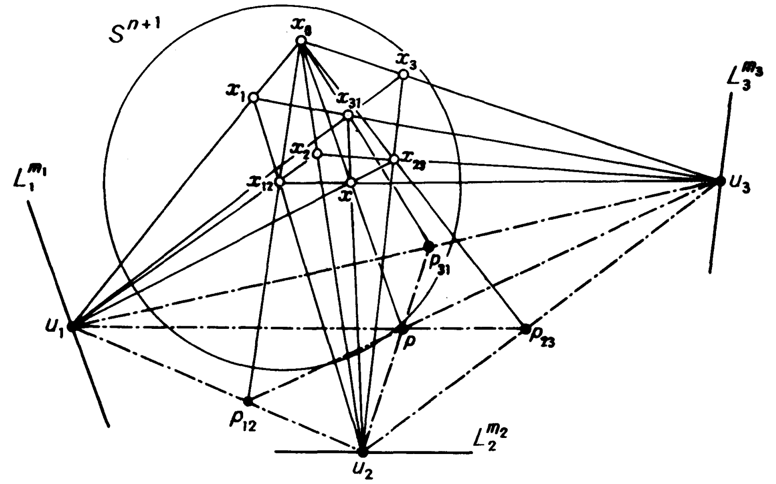

Fig. 2.

Using the fact that 3 lines $[u_1, p_{23}]$, $[u_2, p_{31}]$ and $[u_3, p_{12}]$ are concurrent at p and the above mentioned fact i), we obtain easily the following

LEMMA 7. *3 points t_{23}, t_{31} and t_{12} are on the line m which is harmonic conjugate to the point p with respect to the triangle $u_1u_2u_3$ and 3 points q_{23}, q_{31} and q_{12} are on the line $l=P_{x_0}\cap[u_1, u_2, u_3]$.*

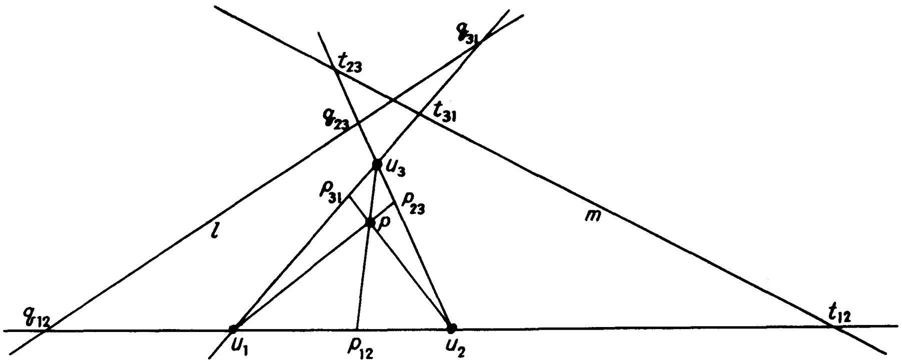

Fig. 3.

DETINITION. Let $L_i^{m_i}$, $i=1, 2, 3$, and x_0 be the above mentioned linear subspaces of R^{n+2} and a point of S^{n+1}. We denote the set of points $x(u_1, u_2, u_3)$ for $u_i\in L_i^{m_i}$, $i=1, 2, 3$, such that

i) $\dim[x_0, u_1, u_2, u_3]=3$,

ii) *at most one of* $\{u_1, u_2, u_3\}$ *belongs to* P_{x_0},

iii) $[u_1, u_2, u_3]$ *does not tangent to* S^{n+1} *at one of* $\{u_1, u_2, u_3\}$,

by $M^n=M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$.

LEMMA 8. *For a point $x(u_1, u_2, u_3)$ as in the above definition, we have*

$$x_{23}\overline{\in} L_1^{m_1},\ x_{31}\overline{\in} L_2^{m_2},\ x_{12}\overline{\in} L_3^{m_3}.$$

PROOF. Supposing $x_{12}\in L_3^{m_3}$, we obtain immediately

$$[L_1^{m_1}, L_2^{m_2}]\subset P_{x_{12}},$$

hence $P_{x_{12}}\ni u_1, u_2$, which implies $x_{12}=x_1=x_2=x_0$ and so

$$u_1\in P_{x_0} \text{ and } u_2\in P_{x_0}.$$

This fact contradicts to the condition ii). Q. E. D.

LEMMA 9. *$S_i^{m_i}(u_j, u_k):=[L_i^{m_i}, x_{jk}]\cap S^{n+1}$, where (i, j, k) is one of $(1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$, is an m_i-dimensional sphere.*

PROFF. By Lemma 8, $[L_3^{m_3}, x_{12}]$ is an m_3+1 dimensional linear subspace of R^{n+2}. Since we have

$$T_{x_{12}}S_3^{m_3}(u_1, u_2) = P_{x_{12}}\cap[L_3^{m_3}, x_{12}],$$

it is sufficient to prove that $P_{x_{12}}\not\supset[L_3^{m_3}, x_{12}]$.

Now, we suppose $P_{x_{12}}\supset L_3^{m_3}$. Then, we have $x_{12}\in$h-conj$\,L_3^{m_3}=[L_1^{m_1}, L_2^{m_2}]$ by Lemma 5. If $x_{12}=x_0$, we have $P_{x_0}\ni u_1, u_2, u_3$, which contradicts to the above condition ii) for x. Therefore, we have $x_{12}\neq x_0$. If $x_{12}\neq p_{12}$, it must be $x_0\in[L_1^{m_1}, L_2^{m_2}]$ and hence $P_{x_0}\supset$h-conj$\,[L_1^{m_1}, L_2^{m_2}]=L_3^{m_3}$, which contradicts to the way of choice of the point x_0. Therefore, we have $x_{12}=p_{12}$. In the following we divide our argument into the two cases:

$$\alpha)\quad p_{12}\neq u_1 \text{ and } u_2; \qquad \beta)\quad p_{12}=u_1 \text{ or } u_2.$$

Case α). It must be $x_1=u_1$ and $x_2=u_2$, which is impossible, because S^{n+1} is a sphere.

Case β). If $p_{12}=u_1$, then $x_1=u_1$ and $[u_1, u_2]$ is tangent to S^{n+1} at u_1. If $S^{n+1}\cap[u_1, u_2, u_3]$ is a circle, then it is impossible that the triangle $u_1u_2u_3$ is self-conjugate with respect to this circle. Hence, the plane $[u_1, u_2, u_3]$ is tangent to S^{n+1} at u_1, and this is also impossible by the condition iii) for x. Thus, we see that $P_{x_{12}}\not\supset L_3^{m_3}$. Q. E. D.

By virtue of Lemma 9, setting

(2.5) $E_1^{m_1}(x):=T_xS_1^{m_1}(u_2, u_3)$, $E_2^{m_2}(x):=T_xS_2^{m_2}(u_3, u_1)$, $E_2^{m_3}(x):=T_xS_3^{m_3}(u_1, u_2)$,

we have

$$E_i^{m_i}(x)\perp E_j^{m_j}(x) \qquad \text{for } i\neq j$$

by Lemma 2, because we can prove

(2.6) $$x\bar{\in} L_i^{m_i},\ i=1, 2, 3.$$

For if $x\in L_i^{m_i}$, it must be $u_i=x$, therefore $[u_1, u_2, u_3]$ must be tangent to S^{n+1} at u_i by the analogous argument to the proof of Lemma 9.

Therefore $E_i^{m_i}(x)$ makes an m_i-dimensional distribution of M^n for $i=1$, 2, 3 and these are mutually orthogonal to each other.

§ 3. The normal vector of M^n in S^{n+1}

In this section, we shall determine the normal unit vector N_x of M^n at x.

At the beginning, for fixed $i=1, 2, 3$, we denote the hyperplane containing $L_i^{m_i}$ and parallel to h-conj $L_i^{m_i}$ by P_i^{n+1}. Then, by means of Lemma 5, we can express a point $x\in R^{n+2}$ uniquely as

$$x = p_1\hat{u}_1+p_2\hat{u}_2+p_3\hat{u}_3, \tag{3.1}$$

where

$$p_1+p_2+p_3=1, \quad \hat{u}_i\in L_i^{m_i}, \quad i=1, 2, 3.$$

We call a point x of M^n a *general point* provided $x\notin \bigcup_{i=1}^{3} P_i^{n+1}$.

LEMMA 10. *On the expression* (3.1) *for a general point x of M^n we have*

$$N_x \,||\, [\hat{u}_1, \hat{u}_2, \hat{u}_3]\cap P_x.$$

PROOF. By the argument at the end of § 2, we have

$$E_i^{m_i}(x)=[L_i^{m_i}, x]\cap P_x=[L_i^{m_i}, x_{jk}]\cap P_x, \; i\neq j, \; i\neq k, \tag{3.2}$$

which are orthogonal subspaces in P_x.

Now, we shows that $x\notin[L_i^{m_i}, L_j^{m_j}]$ for $i\neq j$. Suppose that $x\in[L_1^{m_1}, L_2^{m_2}]$, then $P_x\supset L_3^{m_3}$, which implies $x=x_{12}$. The fact $x_{12}\in[L_1^{m_1}, L_2^{m_2}]$ implies $x_0\in[L_1^{m_1}, L_2^{m_2}]$ and hence $P_{x_0}\supset L_3^{m_3}$. This is impossible from the way of choice of x_0.

Therefore $[L_i^{m_i}, x]\cap P_x=E_i^{m_i}(x)$ and $[L_j^{m_j}, L_k^{m_k}, x]\cap P_x$ are mutually orthogonal complements in P_x by means of Lemma 2. Hence, we obtain the fact:

$$N_x \,||\, [L_j^{m_j}, L_k^{m_k}, x] \qquad \text{for } j\neq k.$$

Next, we obtain from (3.1) for $x=x(u_1, u_2, u_3)$

$$\sum_{i=1}^{3} p_i(\hat{u}_i-x)=0.$$

For the point x, it is clear that $p_i\neq 0$, for $i=1, 2, 3$, since $x\notin[L_i^{m_i}, L_j^{m_j}]$ for $i\neq j$. By the expression

$$p_1(\hat{u}_1-x)=-p_2(\hat{u}_2-x)-p_3(\hat{u}_3-x),$$

we see that

$$\hat{u}_1-x \,||\, \Big[[L_2^{m_2}, x], [L_3^{m_3}, x]\Big]=[L_2^{m_2}, L_3^{m_3}, x].$$

We have also

$$\hat{u}_1 - x \| [L_3^{m_3}, L_1^{m_1}, x] \text{ and } \hat{u}_1 - x \| [L_1^{m_1}, L_2^{m_2}, x].$$

Hence we obtain

$$\hat{u}_1 - x \| \cap_{i=1}^{3} [\text{h-conj } L_i^{m_i}, x].$$

Analogously we obtain the relations

$$\hat{u}_j - x \| \cap_{i=1}^{3} [\text{h-conj } L_i^{m_i}, x], \text{ for } j = 1, 2, 3,$$

from which, using the fact $x \in [\hat{u}_1, \hat{u}_2, \hat{u}_3]$, we get

$$(3.3) \qquad [\hat{u}_1, \hat{u}_2, \hat{u}_3] \subset \cap_{i=1}^{3} [\text{h-conj } L_i^{m_i}, x].$$

Therefore, we obtain

$$[\hat{u}_1, \hat{u}_2, \hat{u}_3] \cap P_x \subset \cap_{i=1}^{3} \Big([\text{h-conj } L_i^{m_i}, x] \cap P_x \Big),$$

of which the right hand side has the common direction of the orthogonal complements of $E_i^{m_i}(x)$ in P_x, that is the normal direction of $T_x M^n$ in P_x. Hence we have

$$N_x \| [\hat{u}_1, \hat{u}_2, \hat{u}_3] \cap P_x. \qquad \text{Q. E. D.}$$

Now, we suppose *that* $x_0 \overline{\in} \cup_{i=1}^{3} P_i^{n+1}$ *in the following*. Then, by (3.1) we can put

$$(3.4) \qquad \begin{aligned} x_0 &= p_1^0 \hat{u}_1^0 + p_2^0 \hat{u}_2^0 + p_3^0 \hat{u}_3^0, \\ p_1^0 &+ p_2^0 + p_3^0 = 1, \quad \hat{u}_i^0 \in L_i^{m_i}, \ i = 1, 2, 3. \end{aligned}$$

LEMMA 12. *On* (3.4), *we have* $p_i^0 \neq 0$, $i = 1, 2, 3$.

PROOF. Supposing $p_1^0 = 0$, we have $x_0 \in [L_2^{m_2}, L_3^{m_3}]$, hence $L_1^{m_1} \subset P_{x_0}$ which contradicts to the way of choice of the point x_0 given in § 2. Hence $p_1^0 \neq 0$. Analogously we have $p_2^0 \neq 0$ and $p_3^0 \neq 0$. Q. E. D.

LEMMA 13. *When* $m_3 \geqq 2$, *the normal lines of* M^n *along* $S_3^{m_3}(u_1, u_2)$ *in* R^{n+2} *form locally an* (m_3+1) *dimensional right cone.*

PROOF. Fixing u_1 and u_2, we regard u_3 as a variable. By the definition of $x = x(u_1, u_2, u_3)$, we can put

$$(3.5) \qquad \begin{aligned} x_{12} &= x_0 + q_1(u_1 - x_0) + q_2(u_2 - x_0) \\ &= (1 - q_1 - q_2) x_0 + q_1 u_1 + q_2 u_2 \end{aligned}$$

and

$$(3.6) \qquad x = (1 - \rho) x_{12} + \rho u_3,$$

where ρ is regarded as a real valued function of u_3. Substituting (3.4) and

(3. 5) into (3. 6), we obtain

$$x=(1-\rho)\Big\{(1-q_1-q_2)\sum p_i^0\hat{u}_i^0+q_1u_1+q_2u_2\Big\}+\rho u_3$$

$$\begin{aligned}
&=(1-\rho)\Big\{(1-q_1-q_2)\,p_1^0\hat{u}_1^0+q_1u_1\Big\}\\
&+(1-\rho)\Big\{(1-q_1-q_2)\,p_2^0\hat{u}_2^0+q_2u_2\Big\}\\
&+(1-\rho)(1-q_1-q_2)\,p_3^0\hat{u}_3^0+\rho\, u_3\,.
\end{aligned} \tag{3. 7}$$

Setting

$$\begin{cases} p_1=(1-\rho)\Big\{(1-q_1-q_2)\,p_1^0+q_1\Big\}\,,\\ p_2=(1-\rho)\Big\{(1-q_1-q_2)\,p_2^0+q_2\Big\}\,,\\ p_3=(1-\rho)(1-q_1-q_2)\,p_3^0+\rho\,, \end{cases} \tag{3. 8}$$

we can easily see that

$$p_1+p_2+p_3=1\,. \tag{3. 9}$$

By means of (2. 6), $x\neq u_3$, hence we have

$$\rho\neq1\,. \tag{3. 10}$$

We have also

$$q_1+q_2\neq1\,. \tag{3. 11}$$

Otherwise, from (3. 5) we get $x_{12}=q_1u_1+(1-q_1)\,u_2$, which implies

i) $x_1=u_1$, $x_2=u_2$, $x_{12}\neq x_1$ and x_2;

or

ii) $x_1=u_1=x_{12}$; or iii) $x_2=u_2=x_{12}$.

i) is impossible for S^{n+1} and ii) and iii) are also impossible since the triangle $u_1u_2u_3$ is self-conjugate with respect to the circle $[u_1, u_2, u_3]\cap S^{n+1}$.

We consider the case

$$p_i\neq0 \qquad \text{for } i=1, 2, 3\,. \tag{3. 12}$$

This condition is equivalent to the following:

$$\begin{cases} (1-q_1-q_2)\,p_1^0+q_1\neq0\,,\\ (1-q_1-q_2)\,p_2^0+q_2\neq0\,,\\ (1-\rho)(1-q_1-q_2)\,p_3^0+\rho\neq0 \end{cases} \tag{3. 13}$$

by (3. 10). Then, we can set

$$(3.14)\qquad \begin{cases} \hat{u}_1=\dfrac{(1-q_1-q_2)\,p_1^0\hat{u}_1^0+q_1u_1}{(1-q_1-q_2)\,p_1^0+q_1}, \\ \hat{u}_2=\dfrac{(1-q_1-q_2)\,p_2^0\hat{u}_2^0+q_2u_2}{(1-q_1-q_2)\,p_2^0+q_2}, \\ \hat{u}_3=\dfrac{(1-\rho)(1-q_1-q_2)\,p_3^0\hat{u}_3^0+\rho u_3}{(1-\rho)(1-q_1-q_2)\,p_3^0+\rho}, \end{cases}$$

and (3.7) can be written as

$$(3.15)\qquad x=\sum_{i=1}^{3}p_i\hat{u}_i,\ p_1+p_2+p_3=1,\ \hat{u}_i\in L_i^{m_i},\ i=1,2,3.$$

The first two of (3.14) shows that the points $\hat{u}_1$ and $\hat{u}_2$ are also fixed.

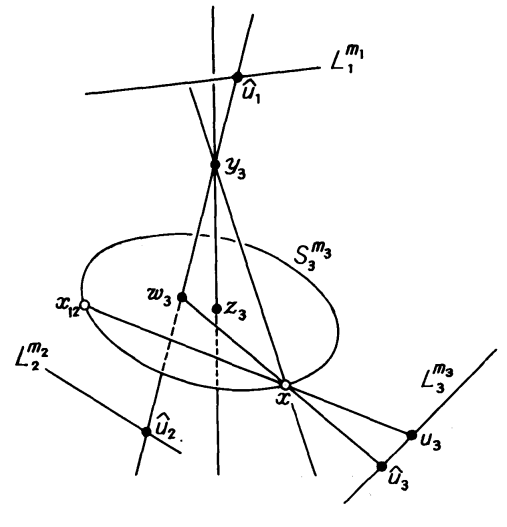

Fig. 4.

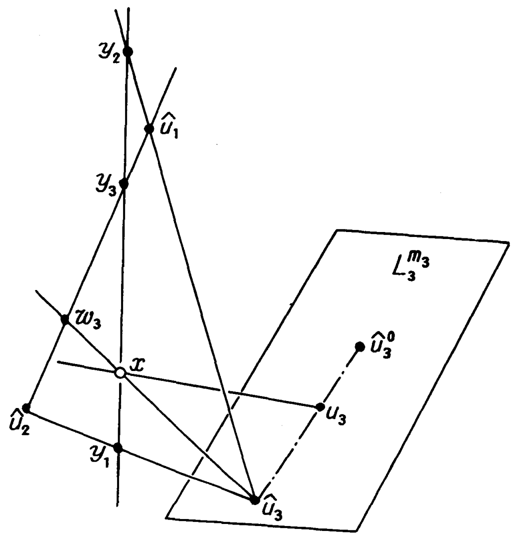

Fig. 5.

By Lemma 10, the normal line of M^n at x is $[\hat{u}_1, \hat{u}_2, \hat{u}_3]\cap P_x$ and we denote its intersection with $[\hat{u}_1, \hat{u}_2]$ by y_3. It is clear that the point $w_3=[x, \hat{u}_3]\cap[\hat{u}_1, \hat{u}_2]$ is a fixed point on the (m_3+1)-plane $[x, L_3^{m_3}]=[x_{12}, L_3^{m_3}]$ and dim $[\hat{u}_1, \hat{u}_2, x_{12}, L_3^{m_3}]=m_3+2$. Therefore, as is shown in Fig. 4, $S_3^{m_3}(u_1, u_2)$, $\hat{u}_1$, $\hat{u}_2$, the normal line at x are all in a fixed (m_3+2)-dimensional linear space.

Now, noticing $m_3\geqq 2$, we can prove that y_3 is on the line in this linear space which passes the centor z_3 of $S_3^{m_3}(u_1, u_2)$ and is perpendicular to $[x_{12}, L_3^{m_3}]$. Otherwise, let $y_3'\neq z_3$ be the orthogonal projection of y_3 onto $[x_{12}, L_3^{m_3}]$. Then, 3 points y_3', z_3 and x are collinear. Accordingly, the moving point x must be on the great circle which is the intersection of S^{n+1} and the plane determined by z_3 and the orthogonal projection of the line $[\hat{u}_1, \hat{u}_2]$ onto $[x_{12}, L_3^{m_3}]$. This contradicts to $m_3\geqq 2$.

We see easily that the point y_3 is a fixed point on $[\hat{u}_1, \hat{u}_2]$. This shows that the normal lines of M^n along $S_3^{m_3}(u_1, u_2)$ make locally a right cone.

Q. E. D.

THEOREM 1. *When $m_i \geqq 2$, $i=1, 2, 3$, $M^n = M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ is a hypersurface of S^{n+1} with 3 principal curvatures of multiplicities m_1, m_2 and m_3 respectively.*

PROOF. On each $L_i^{m_i}$, $i=1, 2, 3$, we choose an orthonormal cartesian coordinates

u^{α_1}, $\alpha_1=1, \cdots, m_1$; u^{α_2}, $\alpha_2=m_1+1, \cdots, m_1+m_2$; u^{α_3}, $\alpha_3=m_1+m_2+1, \cdots$, n

and denote the tanget vector fields on M^n corresponding to $\partial/\partial u^{\alpha_1}$ on $L_1^{m_1}$; $\partial/\partial u^{\alpha_2}$ on $L_3^{m_2}$; $\partial/\partial u^{\alpha_3}$ on $L_3^{m_3}$, through the projections from x_{23}, x_{31}, x_{12} by X_{α_1} ; X_{α_2} ; X_{α_3}, respectively.

Using the local coordinates $u^1, \cdots, u^n$, we denote the line element of M^n by

(3.16) $$ds^2 = \sum_{i,j=1}^{n} g_{ij}(u)\, du^i\, du^j,$$

then we have

$$g_{ij} = \langle X_i, X_j \rangle = (X_i, X_j),$$

where $\langle\ ,\ \rangle$ denotes the Riemannian innerproduct of M^n and $(\ ,\ )$ the Euclidean inner product in R^{n+2}. By means of Lemma 2, we have

(3.17) $$g_{\alpha_i \alpha_j} = 0 \qquad \text{for } i \neq j\,.$$

Next, we set the components of the 2nd fundamental form of M^n :

(3.18) $$h_{ij} := \langle \nabla_{X_i} X_j, N \rangle = h_{ji},\ i, j = 1, 2, \cdots, n,$$

where ∇ denotes the covarint differentiation of S^{n+1}. At each point x of M^n, we assign a vector $\xi_3(x)$ which is the unit outer normal vector of $S_3^{m_3}(u_1, u_2)$ at x. Then, we decompose $N = N_x$ as

(3.19) $$N = \langle N, \xi_3 \rangle \xi_3 + \eta_3\,.$$

We see easily that

$$\eta_3 \perp [x_{12}, L_3^{m_3}]\,.$$

By means of Lemma 13, $\langle N, \xi_3 \rangle$ is constant and η_3 is parallel along $S_3^{m_3}(u_1, u_2)$. Therefore, from (3.19) we obtain

$$\frac{\partial N}{\partial u^{\alpha_3}} = \langle N, \xi_3 \rangle \frac{\partial \xi_3}{\partial u^{\alpha_3}}\,,$$

and

$$\left\langle \frac{\partial x}{\partial u^{\alpha_i}}, \frac{\partial N}{\partial u^{\alpha_3}} \right\rangle = \langle N, \xi_3 \rangle \cdot \left\langle \frac{\partial x}{\partial u^{\alpha_i}}, \frac{\partial \xi_3}{\partial u^{\alpha_3}} \right\rangle.$$

On the other hand, we have $x-z_3=|x-z_3|\xi_3$, where $z_3=z_3(u_1, u_2)$ is the centor of $S_3^{m_3}(u_1, u_2)$ and so

$$\frac{\partial x}{\partial u^{\alpha_3}} = |x-z_3| \frac{\partial \xi_3}{\partial u^{\alpha_3}} .$$

Hence, we obtain from (3.18)

$$h_{\alpha_i\alpha_3} = -\langle \frac{\partial x}{\partial u^{\alpha_i}}, \frac{\partial N}{\partial u^{\alpha_3}} \rangle = -\frac{\langle N, \xi_3\rangle}{|x-z_3|} \langle \frac{\partial x}{\partial u^{\alpha_i}}, \frac{\partial x}{\partial u^{\alpha_3}} \rangle ,$$

i. e.

$$h_{\alpha_i\alpha_3} = -\frac{\langle N, \xi_3\rangle}{|x-z_3|} g_{\alpha_i\alpha_3} .$$

Considering analogously $z_1(u_2, u_3)$, $\xi_1(u_2, u_3)$ for $S_1^{m_1}(u_2, u_3)$ and $z_2(u_3, u_1)$, $\xi_2(u_3, u_1)$ for $S_2^{m_2}(u_3, u_1)$, we obtain the following:

$$\text{(3.20)} \qquad \begin{cases} h_{\alpha_i\alpha_j} = 0, \ i \neq j ; \\ h_{\alpha_i\beta_i} = -\dfrac{\langle N, \xi_i\rangle}{|x-z_i|} g_{\alpha_i\beta_i}, \ i, j = 1, 2, 3, \end{cases}$$

which shows that

$$\text{(3.21)} \qquad \mu_i = -\frac{\langle N, \xi_i\rangle}{|x-z_i|} , \ i = 1, 2, 3,$$

is a principal curvature of M^n of multiplicity m_i and the corresponding eigen space is $E_i^{m_i}(x)$. Q. E. D.

LEMMA 14. *For M^n as in Theorem 1, we have*

$$\text{(3.22)} \qquad \sum_{i=1}^{3} m_i\mu_i = \sum_{i=1}^{3} \frac{m_i}{\overline{xy_i}} ,$$

where $\overline{xy_i}$ denotes the length with sign measured by N on the normal line of M^n at x.

PROOF. Along $S_3^{m_3}(u_1, u_2)$, let θ_3 be the angle as is shown in Fig. 6 determined by

$$\langle N, -\xi_3\rangle = \cos\theta_3 .$$

Then, we have easily

$$\overline{xy_3} \cos\theta_3 = |x-z_3| ,$$

and hence from (3.21)

$$\mu_3 = \frac{\cos\theta_3}{|x-z_3|} = \frac{1}{\overline{xy_3}} .$$

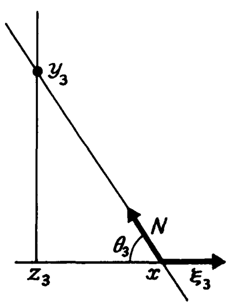

Fig. 6.

We shall obtain analogous formulas for μ_1 and μ_2 and so (3. 22).

Q. E. D.

Now, we compute the right-hand side of (3.22). Let $x\in M^n$ be a general point, then by (3. 1) we can put

$$x=p_1\hat{u}_1+p_2\hat{u}_2+p_3\hat{u}_3\,,$$

where

$$p_1+p_2+p_3=1,\quad \hat{u}_i\in L_i^{m_i},\quad i=1,2,3\,.$$

Foe any point y on the normal line in $[\hat{u}_1,\hat{u}_2,\hat{u}_3]$, we put

$$y=q_1\hat{u}_1+q_2\hat{u}_2+q_3\hat{u}_3,\quad q_1+q_2+q_3=1\,.$$

Then, we have

$$\begin{aligned}0=(x,y-x)&=\sum_{i,j=1}^{3}p_i(q_j-p_j)(\hat{u}_i,\hat{u}_j)\\&=\sum_{i=1}^{3}p_i(q_i-p_i)\left((\hat{u}_i,\hat{u}_i)-1\right)+\sum_{i=1}^{3}p_i\sum_{j=1}^{3}(q_j-p_j)\,,\end{aligned}$$

hence

$$\sum_{i=1}^{3}b_ip_i(q_i-p_i)=0\,,$$

where

$$b_i:=(\hat{u}_i,\hat{u}_i)-1,\quad i=1,2,3\,. \tag{3. 23}$$

We have also

$$\sum_{i=1}^{3}(q_i-p_i)=0\,.$$

Therefore, we obtain the equalities

$$\frac{b_2p_2-b_3p_3}{q_1-p_1}=\frac{b_3p_3-b_1p_1}{q_2-p_2}=\frac{b_1p_1-b_2p_2}{q_3-p_3}\,. \tag{3. 24}$$

THEOREM 2. *Let x be a general point of $M^n=M^n(x_0,L_1^{m_1},L_2^{m_2},L_3^{m_3})$ with $m_i\geqq 2$, $i=1,2,3$, and represent it as*

$$x=\sum_{i=1}^{3}p_i\hat{u}_i,\quad p_1+p_2+p_3=1,\quad \hat{u}_i\in L_i^{m_i},\quad i=1,2,3\,.$$

Then, M^n is minimal at x, if and only if

$$\frac{m_1}{p_1}(b_2p_2-b_3p_3)+\frac{m_2}{p_2}(b_3p_3-b_1p_1)+\frac{m_3}{p_3}(b_1p_1-b_2p_2)=0\,. \tag{3. 25}$$

PROOF. First, we notice that $p_i\neq 0$, $i=1,2,3$, which was shown in the proof of Lemma 10. Using (3.24), we put

$$y_3 = x+\rho_3 \sum_{i=1}^{3} r_i\hat{u}_i,$$

where

$$r_1 = b_2p_2-b_3p_3, \quad r_2 = b_3p_3-b_1p_1, \quad r_3 = b_1p_1-b_2p_2 .$$

Since $y_3 \in [\hat{u}_1, \hat{u}_2]$, it must be $p_3+\rho_3 r_3=0$, hence we have

$$y_3-x = -\frac{p_3}{r_3}\sum_{i=1}^{3} r_i\hat{u}_i .$$

By the way of measuring length on the normal line of M^n at x, we have

$$\overline{xy_3} = -\frac{p_3}{r_3}\left(\sum_{i=1}^{3} r_i\hat{u}_i, N\right).$$

We obtain analogous formulas for y_1 and y_2. Hence, from (3.22) and these formulas we obtain the following:

$$\sum_{i=1}^{3} m_i\mu_i = -\frac{1}{\left(\sum_{i=1}^{3} r_i\hat{u}_i, N\right)}\times \tag{3.26}$$
$$\times\left\{\frac{m_1}{p_1}(b_2p_2-b_3p_3)+\frac{m_2}{p_2}(b_3p_3-b_1p_1)+\frac{m_3}{p_3}(b_1p_1-b_2p_2)\right\},$$

which implies immediately the statement of this theorem. Q. E. D.

§ 4. The conditions in order that M^n is minimal

In this section, we shall investigate the conditions that $M^n = M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ with $m_i \geqq 2$, $i=1, 2, 3$, is a minimal hypersurface in S^{n+1}.

Using the notation in the proof of Theorem 2, since we have

$$1 = (x, x) = \sum_{i=1}^{3} p_i^2(\hat{u}_i,\hat{u}_i)+2(p_2p_3+p_3p_1+p_1p_2)$$

for a general point x of M^n, we obtain the equality:

$$\sum_{i=1}^{3} b_ip_i^2 = 0, \tag{4.1}$$

where

$$b_i = (\hat{u}_i, \hat{u}_i)-1, \quad i = 1, 2, 3, \tag{4.2}$$

$$p_1+p_2+p_3 = 1. \tag{4.3}$$

If we consider b_i as constants and p_i as variables, (4.1) represents in general an ellipse or hyperbola. In fact, considering (p_1, p_2, p_3) as homogeneous coordinates in the projective plane $P^2(R)$, (b_2b_3, b_3b_1, b_1b_2) is the pole of the projective line $p_1+p_2+p_3=0$. Using this fact, p_1, p_2 and p_3 satisfying (4.3) can be written as

$$\tag{4.4} \begin{cases} p_1 = \dfrac{b_2b_3}{B} - \xi + b_2\eta, \\ p_2 = \dfrac{b_3b_1}{B} + \xi + b_1\eta, \\ p_3 = \dfrac{b_1b_2}{B} - (b_1+b_2)\eta, \end{cases}$$

where $B=b_2b_3+b_3b_1+b_1b_2$. Substituting (4.4) into (4.1), we obtain

$$\tag{4.5} \xi^2 + B\eta^2 = -\frac{b_1b_2b_3}{(b_1+b_2)B}.$$

Now, we fix u_1 and u_2 in $x=x(u_1, u_2, u_3)$. Then, $\hat{u}_1$ and $\hat{u}_2$ given by (3.14) and b_1 and b_2 given by (4.2) are all fixed. Suppose u_3 moves along a curve in $L_3^{m_3}$ and we denote the derivatives with respect to the parameter of this curve by "$\prime$". Setting

$$\tag{4.6} p_i' = q_i, \quad i=1, 2, 3,$$

we obtain from (4.3) and (4.1)

$$\tag{4.7} q_1+q_2+q_3=0,$$

$$\tag{4.8} \sum_{i=1}^{3} b_ip_iq_i + \frac{b_3'}{2}p_3^2 = 0.$$

On the other hand, by means of (3.24), the equality $\langle x', N_x\rangle=0$ implies

$$\begin{aligned} \Big\langle x', \sum_{i=1}^{3} r_i\hat{u}_i\Big\rangle &= \Big\langle \sum_{i=1}^{3} q_i\hat{u}_i + p_3\hat{u}_3', \sum_{j=1}^{3} r_j\hat{u}_j\Big\rangle \\ &= \sum_{i=1}^{3} b_ir_iq_i + \frac{b_3'}{2}p_3r_3 = 0, \quad \text{i.e.} \end{aligned}$$

$$\tag{4.9} \sum_{i=1}^{3} b_ir_iq_i + \frac{b_3'}{2}p_3r_3 = 0.$$

Since we have easily

$$p_1r_3 - r_1p_3 = -b_2p_2, \quad p_2r_3 - r_2p_3 = b_1p_1$$

by (4.1) and (4.3), we get from (4.8) and (4.9)

$$\sum_{i=1}^{3} b_i(p_ir_3 - r_ip_3)q_i = b_1b_2(p_1q_2 - p_2q_1) = 0.$$

Now, in general we may suppose that $b_1 b_2 \neq 0$, i. e.

(4. 10) $$\hat{u}_1 \overline{\in} S^{n+1} \text{ amd } \hat{u}_2 \overline{\in} S^{n+1},$$

taking note of the right-hand side of (3. 14). Therefore, from the above computation we may put

(4. 11) $$q_1 = \rho p_1, \quad q_2 = \rho p_2, \quad q_3 = -\rho(p_1+p_2).$$

Then, substituting (4. 11) into (4. 8) we obtain

(4. 12) $$\rho\left\{b_1 p_1^2 + b_2 p_2^2 - b_3 p_3 (p_1+p_2)\right\} + \frac{p_3^2}{2} b_3' = 0, \text{ i. e.}$$
$$b_3' = 2\rho \frac{b_3}{p_3}$$

by (4. 1) and (4. 2).

Finally, we compute the derivatives of the quantities in (3. 25), using the formulas obtained above. Since

$$\left(\frac{b_2p_2 - b_3p_3}{p_1}\right)' = -\frac{q_1}{p_1^2}(b_2p_2 - b_3p_3) + \frac{1}{p_1}(b_2q_2 - b_3q_3 - b_3'p_3) = -\rho\frac{b_3}{p_1},$$
$$\left(\frac{b_3p_3 - b_1p_1}{p_2}\right)' = -\frac{q_2}{p_2^2}(b_3p_3 - b_1p_1) + \frac{1}{p_2}(b_3q_3 - b_1q_1 + b_3'p_3) = \rho\frac{b_3}{p_2},$$
$$\left(\frac{b_1p_1 - b_2p_2}{p_3}\right)' = -\frac{q_3}{p_3^2}(b_1p_1 - b_2p_2) + \frac{1}{p_3}(b_1q_1 - b_2q_2) = \rho\frac{b_1p_1 - b_2p_2}{p_3^2},$$

we have

$$\left(\sum_{i=1}^{3}\frac{m_i r_i}{p_i}\right)' = \rho b_3\left(-\frac{m_1}{p_1} + \frac{m_2}{p_2}\right) + \frac{\rho m_3}{p_3^2}(b_1p_1 - b_2p_2)$$
$$= \frac{\rho}{p_3}\sum_{i=1}^{3}\frac{m_i r_i}{p_i} - \frac{\rho}{p_3}\left\{\frac{m_1(b_2p_2 - b_3p_3)}{p_1} + \frac{m_2(b_3p_3 - b_1p_1)}{p_2}\right\} + \rho b_3\left(-\frac{m_1}{p_1} + \frac{m_2}{p_2}\right),$$

i. e.

(4. 13) $$\left(\sum_{i=1}^{3}\frac{m_i r_i}{p_i}\right)' = \frac{\rho}{p_3}\sum_{i=1}^{3}\frac{m_i r_i}{p_i} + \frac{\rho\, m_1 m_2}{p_1 p_2 p_3}\left(\frac{b_1 p_1^2}{m_1} - \frac{b_2 p_2^2}{m_2}\right).$$

We obtain easily

(4. 14) $$\left(\frac{b_1 p_1^2}{m_1} - \frac{b_2 p_2^2}{m_2}\right)' = 2\rho\left(\frac{b_1 p_1^2}{m_1} - \frac{b_2 p_2^2}{m_2}\right).$$

THEOREM 3. *$M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ with $m_i \geqq 2$, $i=1, 2, 3$, can not be minimal in S^{n+1}.*

PROOF. It is clear that on $M^n = M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ almost all points are general points in the sense stated at the beginning of § 3. Therefore, we can use the argument above. Hence, from (4. 13) and (4. 14) and Theorem 2, the condition in order that M^n is minimal is that there exist general points such that

$$\text{(i)} \qquad \frac{b_1 p_1^2}{m_1} = \frac{b_2 p_2^2}{m_2} = \frac{b_3 p_3^2}{m_3},$$

$$\text{(ii)} \qquad \frac{m_1}{p_1}(b_2p_2 - b_3p_3) + \frac{m_2}{p_2}(b_3p_3 - b_1p_1) + \frac{m_3}{p_3}(b_1p_1 - b_2p_2) = 0 .$$

From (i), we get easily

$$\frac{b_i p_i^2}{m_i} = \frac{\sum_{j=1}^{3} b_j p_j^2}{n} = 0 \qquad \text{for } i = 1, 2, 3,$$

and so $p_1 = p_2 = p_3 = 0$. This is impossible. Q. E. D.

§ 5. The limiting case $L_3^{m_3} \subset P_\infty^{n+1}$

In this section, we shall consider the case in which $L_3^{m_3}$ in $M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ goes into the hyperplane at infinity of R^{n+2} which we denote by P_∞^{n+1}. Then, we have

$$\text{h-conj } L_3^{m_3} \ni \text{origin } O \text{ of } R^{n+2}$$

hence

$$\text{h-conj } L_3^{m_3} = [L_1^{m_1}, L_2^{m_2}] : = \tilde{E}_3^{m_1+m_2+1} \perp [O, L_3^{m_3}],$$

where $[O, L_3^{m_3}]$ denotes the (m_3+1)-dimensional plane through the origin O of R^{n+2} with the direction $L_3^{m_3} \subset P_\infty^{n+1}$. Therefore,

$$\tilde{S}_3^{m_1+m_2} : = S^{n+1} \cap \tilde{E}_3^{m_1+m_2+1}$$

is an (m_1+m_2)-dimensional great sphere of S^{n+1} and $L_1^{m_1}$ and $L_2^{m_2}$ are harmonic conjugate with respect to $\tilde{S}_3^{m_1+m_2}$.

We define M^n by an analogous way to the definition of M^n described in § 2, but we represent $u_3 \in L_3^{m_3}$ by a unit vector $v \perp \tilde{E}_3^{m_1+m_2+1}$ whose direction corresponds to u_3. Let P_i^{n+1} be the hyperplane through $L_i^{m_i}$ and parallel to $[L_j^{m_j}, L_3^{m_3}]$ $(i, j=1. 2, i \neq j)$.

We take a fixed point x_0 of S^{n+1} not contained in $P_1^{n+1} \cup P_2^{n+1} \cup \tilde{E}_3^{m_1+m_2+1}$, and $P_{x_0} \not\supset L_i^{m_i}$, $i=1, 2, 3$. For $u_i \in L_i^{m_i}$, $i=1, 2, 3$, such that i) $\dim [x_0, u_1, u_2, u_3] = 3$, ii) *at most one of* $\{u_1, u_2, u_3\}$ *belongs to* P_{x_0}, iii) $[u_1, u_2, u_3]$ *does not tangent to* S^{n+1} *at* u_1 *or* u_2, we construct the points x_i, $x_{ij} (i \neq j)$, $x_{ijk} (i \neq j, k \neq i, k \neq j)$ as before. The point $x = x_{123} = x(u_1, u_2, u_3)$ not contained in $P_1^{n+1} \cup P_2^{n+1} \cup \tilde{E}_3^{m_1+m_2+1}$ is called a general point of $M^n = M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ and it can be written as

(5. 1) $$x = p_1\hat{u}_1 + p_2\hat{u}_2 + p\hat{v},$$

where

$$\hat{u}_i \in L_i^{m_i},\ i=1,2;\ \hat{v} \perp E_3^{m_1+m_2+1},\ |\hat{v}| = 1;$$

(5. 2) $$p_1 + p_2 = 1.$$

Since $x \in S^{n+1}$, we get from (5. 1)

(5. 3) $$b_1p_1^2 + b_2p_2^2 + p^2 = 0,$$

where

(5. 4) $$b_i = (\hat{u}_i, \hat{u}_i) - 1, \quad i = 1, 2.$$

We can express a vector with the direction of normal vector N_x at $x \in M^n = M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ as $r_1\hat{u}_1 + r_2\hat{u}_2 + s\hat{v}$, $r_1 + r_2 = 0$, and get from the equality

$$0 = \langle x, r_1\hat{u}_1 + r_2\hat{u}_2 + s\hat{v}\rangle = b_1p_1r_1 + b_2p_2r_2 + ps$$
$$= (b_1p_1 - b_2p_2)\, r_1 + ps,$$

(5. 5) $$N_x \,||\, -p\hat{u}_1 + p\hat{u}_2 + (b_1p_1 - b_2p_2)\,\hat{v}.$$

Now, we assume that $m_i \geqq 2$, $i=1, 2, 3$. Then, Lemma 14 is true with slight modifications owing to the fact $L_3^{m_3} \subset P_\infty^{n+1}$ and so we have

(5. 6) $$\sum_{i=1}^{3} m_i\mu_i = \sum_{i=1}^{3} \frac{m_i}{\overline{xy_i}},$$

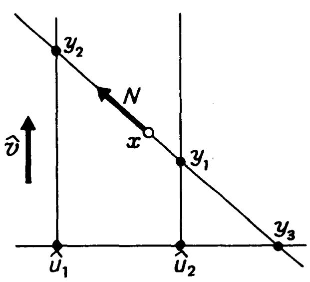

Fig. 7.

Setting

$$y_i = x + \rho_i \left\{p\,(\hat{u}_2 - \hat{u}_1) + (b_1p_1 - b_2p_2)\,\hat{v}\right\},$$

we have the following

$$p_1 - \rho_1 p = 0,\ p_2 + \rho_2 p = 0,\ p + \rho_3(b_1p_1 - b_2p_2) = 0,$$

from which we see that the equality $\sum_{i=1}^{3} m_i\mu_i = 0$ is equivalent to the equality:

$$\left(\frac{m_1}{p_1} - \frac{m_2}{p_2}\right)p - m_3\,\frac{b_1p_1 - b_2p_2}{p} = 0.$$

Therefore, we have the following

THEOREM 4. *Let x be a general point of $M^n = M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ with $m_i \geqq 2$, $i = 1, 2, 3$, and $L_3^{m_3} \subset P_\infty^{n+1}$ and represent it as*

$$x=p_1\hat{u}_1+p_2\hat{u}_2+p\hat{v},\quad p_1+p_2=1,\quad \hat{u}_i\in L_i^{m_i},\ i=1,2,$$
$$\hat{v}\perp[L_1^{m_1}, L_2^{m_2}],\quad |\hat{v}|=1.$$

Then, M^n is minimal at x, if and only if

$$\left(\frac{m_1}{p_1}-\frac{m_2}{p_2}\right)p-\frac{m_3}{p}(b_1p_1-b_2p_2)=0\,, \tag{5.7}$$

where $$b_i=\langle\hat{u}_i, \hat{u}\rangle-1.$$

First, fixing u_1 and u_2 in $x=x(u_1, u_2, v)$, by an analogous consideration to that in § 3, $\hat{u}_1$ and $\hat{u}_2$, so b_1 and b_2 are all fixed. Let move v along a curve and put

$$p_i'=q_i,\quad i=1,2,\ \text{and}\ p'=q\,. \tag{5.8}$$

Since we have $q_1+q_2=0$, from (5. 3) we obtain easily

$$(b_1p_1-b_2p_2)\,q_1+pq=0\,. \tag{5.9}$$

The equality $\langle x', N_x\rangle=0$ and (5. 5) imply

$$\langle q_1\hat{u}_1-q_1\hat{u}_2+q\hat{v}+p\hat{v}', -p\hat{u}_1+p\hat{u}_2+(b_1p_1-b_2p_2)\,\hat{v}\rangle$$
$$=-pq_1(b_1+b_2)+(b_1p_1-b_2p_2)q=0,\ \text{i. e.}$$
$$(b_1+b_2)\,pq_1-(b_1p_1-b_2\,p_2)\,q=0\,. \tag{5.10}$$

Regarding (5. 9) and (5. 10) as linear equations of q_1 and q, we have

$$(b_1p_1-b_2p_2)^2+(b_1+b_2)\,p^2$$
$$=(b_1p_1-b_2p_2)^2+(b_1+b_2)\,p^2-(b_1+b_2)\,(b_1p_1^2+bp_2^2+p^2)$$
$$=-b_1b_2\,(p_1+p_2)^2=-b_1b_2\,.$$

Since we may suppose that $b_1b_2\neq0$ as before, we get from (5. 9) and (5. 10)

$$q_1=q_2=q=0, \tag{5.11}$$

hence

$$x'=p\hat{v}'\,.$$

Therefore, we obtain in this case

$$\left[\left(\frac{m_1}{p_1}-\frac{m_2}{p_2}\right)p-\frac{m_3}{p}\left(b_1p_1-b_2p_2\right)\right]'=0\,. \tag{5.12}$$

Second, fixing u_1 and v in $x=x(u_1, u_2, v)$, let move u_2 along a curve. In this case, x_1, x_3 and x_{13} are fixed. Using (5. 8) and noting b_1 is constant, we obtain from (5. 3)

$$(b_1p_1-b_2p_2)\,q_1+pq+\frac{b_2'}{2}\,p_2{}^2=0\,. \tag{5.13}$$

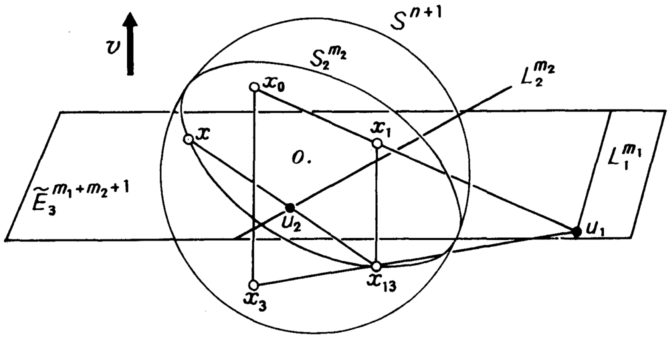

Fig. 8.

The equality $\langle x', N_x\rangle=0$ and (5. 5) imply

$$\langle q_1\hat{u}_1-q_1\hat{u}_2+q\hat{v}+p_2\hat{u}_2', -p\hat{u}_1+p\hat{u}_2+(b_1p_1-b_2p_2)\hat{v}\rangle$$
$$=-pq_1(b_1+b_2)+(b_1p_1-b_2p_2)\,q+\frac{b_2'}{2}pp_2=0, \text{ i. e.}$$

$$-(b_1+b_2)\,pq_1+(b_1p_1-b_2p_2)\,q+\frac{b_2'}{2}pp_2=0\,. \tag{5. 14}$$

Eliminating b_2' from (5. 13) and (5. 14), we get

$$\Big\{b_1p_1-b_2p_2+(b_1+b_2)\,p_2\Big\}\,pq_1+\Big\{p^2-(b_1p_1-b_2p_2)\,p_2\Big\}\,q=b_1(pq_1-p_1q)=0\,.$$

We may suppose in general that $b_1b_2\neq 0$ as before. Hence, we can put

$$q_1=\rho p_1,\quad q_2=-\rho p_1,\quad q=\rho p. \tag{5. 15}$$

Substituting (5. 15) into (5.14), we get

$$\rho\Big\{-(b_1+b_2)\,pp_1+(b_1p_1-b_2p_2)\,p\Big\}+\frac{b_2'}{2}pp_2=0, \text{ i. e.}$$
$$-\rho b_2p+\frac{b_2'}{2}pp_2=0$$

For a general point, we may suppose $p\neq 0$ and $p_2\neq 0$. Hence

$$b_2'=2\rho\frac{b_2}{p_2}\,. \tag{5. 16}$$

By (5. 15) and (5. 16), we have

$$\left[\left(\frac{m_1}{p_1}-\frac{m_2}{p_2}\right)p-\frac{m_3}{p}(b_1p_1-b_2p_2)\right]'=-\left(\frac{m_1}{p_1^2}q_1-\frac{m_2}{p_2^2}q_2\right)p$$
$$+\left(\frac{m_1}{p_1}-\frac{m_2}{p_2}\right)q+\frac{m_3q}{p_2}(b_1p_1-b_2p_2)-\frac{m_3}{p}(b_1p_1-b_2q_2-b_2'p_2)$$

$$= \rho\left[-\left(\frac{m_1}{p_1^2}+\frac{m_2}{p_2^2}\right)p_1p+\left(\frac{m_1}{p_1}-\frac{m_2}{p_2}\right)q+\frac{m_3}{p}(b_1p_1-b_2p_2)\right.$$

$$\left.-\frac{m_3}{p}(b_1+b_2)p_1+\frac{2m_3b_2}{p}\right]=\rho\left(-\frac{m_2}{p_2^2}p+\frac{m_3}{p}b_2\right),\ \text{i.e.}$$

$$\left[\left(\frac{m_1}{p_1}-\frac{m_2}{p_2}\right)p-\frac{m_3}{p}(b_1p_1-b_2p_2)\right]'=\rho\left(-\frac{m_2p}{p_2^2}+\frac{m_3b_2}{p}\right). \tag{5.17}$$

We have laso

$$\left(\frac{b_2p_2^2}{m_2}-\frac{p^2}{m_3}\right)'=\frac{2b_2p_2q_2}{m_2}+\frac{b_2'p_2^2}{m_2}-\frac{2pq}{m_3}$$

$$=-2\rho\left(\frac{b_2p_1p_2}{m_2}-\frac{b_2p_2}{m_2}+\frac{p^2}{m_3}\right)=2\rho\left(\frac{b_2p_2^2}{m_2}-\frac{p^2}{m_3}\right),\ \text{i.e.}$$

$$\left(\frac{b_2p_2^2}{m_2}-\frac{p^2}{m_3}\right)'=2\rho\left(\frac{b_2p_2^2}{m_2}-\frac{p^2}{m_3}\right). \tag{5.18}$$

By means of (5.18), we see that if there exists a general point x of M^n such that

$$\frac{b_2p_2^2}{m_2}-\frac{p^2}{m_3}=0, \tag{5.19}$$

then on the m_2-sphere $S_2^{m_2}(u_1, v)=S^{n+1}\cap[x, L_2^{m_2}]$ this equality holds identically.

Third, fixing u_2 and v in $x=x(u_1, u_2, v)$ and moving u_1 along a curve, we obtain

$$\left[\left(\frac{m_1}{p_1}-\frac{m_2}{p_2}\right)p-\frac{m_3}{p}(b_1p_1-b_2p_2)\right]'=\rho\left(-\frac{m_1p}{p_1^2}+\frac{m_3b_1}{p}\right) \tag{5.20}$$

and

$$\left(\frac{b_1p_1^2}{m_1}-\frac{p^2}{m_3}\right)'=2\rho\left(\frac{b_1p_1^2}{m_1}-\frac{p^2}{m_3}\right) \tag{5.21}$$

by an analogous computation. Hence, we see that if there exists a general point x of M^n such that

$$\frac{b_1p_1^2}{m_1}-\frac{p^2}{m_3}=0, \tag{5.22}$$

then on the m_1-sphere $S_1^{m_1}(u_2, v)=S^{n+1}\cap[x, L_1^{m_1}]$ this equality holds identically.

THEOREM 5. *$M^n(x_0, L_1, L_2, L_3)$ with $m_i\geqq 2$, $i=1, 2, 3$, and $L_3^{m_3}\subset P_\infty^{n+1}$ can not be minimal in S^{n+1}.*

PROOF. If $M^n=M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ is minimal in S^{n+1}, then (5.7) holds on it by Theorem 4. Hence, from (5.17) and (5.20) we obtain

$$\frac{b_1p_1^2}{m_1}=\frac{b_2p_2^2}{m_2}=\frac{p^2}{m_3}=\frac{b_1p_1^2+b_2p_2^2+p^2}{n}=0,$$

and so $p_1=p_2=p=0$. This is impossible. Therefore, M^n can not be minimal in S^{n+1}. Q. E. D.

§ 6. The limiting case $L_2^{m_2}, L_3^{m_3}\subset P_\infty^{n+1}$

In this section, we shall consider the case in which $L_2^{m_2}$ and $L_3^{m_3}$ in $M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ go into the hyperplane at infinity P_∞^{n+1} of R^{n+2}. $L_2^{m_2}$, $L_3^{m_3}\subset P_\infty^{n+1}$ implies

$$\text{h-conj}\, L_2^{m_2} := \tilde{E}_2^{m_3+m_1+1}\ni O, \quad \text{h-conj}\, L_3^{m_3} := \tilde{E}_3^{m_1+m_2+1}\ni O$$

and hence

$$L_1^{m_1}=\text{h-conj}\, L_2^{m_2}\cap \text{h-conj}\, L_3^{m_3}=\tilde{E}_2^{m_3+m_1+1}\cap\tilde{E}_3^{m_1+m_2+1}$$

is a linear space through the origin O of R^{n+2}, and $L_1^{m_1}$, $[L_2^{m_2}, O]$ and $[L_3^{m_3}, O]$ are mutually orthogonal to others at O.

As in the case of $L_3^{m_3}\subset P_\infty^{n+1}$ treated in § 5, we represent u_2 and u_3 in $x=x(u_1, u_2, u_3)$ by unit vectors v_2 and v_3 whose directions correspond to u_2 and u_3 respectively.

We take a fixed point x_0 of S^{n+1} *not contained in* $\tilde{E}_2^{m_3+m_1+1}\cup\tilde{E}_3^{m_1+m_2+1}$, and $P_{x_0}\not\supset L_i^{m_i}$, $i=1,2,3$. For $u_i\in L_i^{m_i}$, $i=1,2,3$, such that

i) $\dim\,[x_0, u_1, u_2, u_3]=3$,

ii) *at most one of* $\{u_1, u_2, u_3\}$ *belongs to* P_{x_0} *in* $P^{n+2}=R^{n+2}\cup P_\infty^{m+1}$.

iii) $[u_1, u_2, u_3]$ *does not tangent to* S^{n+1} *at* u_1,

we construct the point $x=x(u_1, u_2, u_3)$ as before. Then, the point x can be written as

$$x=\hat{u}_1+p_2\hat{v}_2+p_3\hat{v}_3, \tag{6.1}$$

where

$$\hat{u}_1\in L_1^{m_1},\quad \hat{v}_2\in[O, L_2^{m_2}],\quad \hat{v}\in[O, L_3^{m_3}],\quad |\hat{v}_2|=|\hat{v}_3|=1.$$

Since $x\in S^{n+1}$, we get from (6.1)

$$b_1+p_2^2+p_3^2=0, \tag{6.2}$$

where

$$b_1=(\hat{u}_1, \hat{u}_1)-1. \tag{6.3}$$

LEMMA 15. *On the expression* (6.1) *of* x, *we have*

$$N_x \,||\, \frac{1}{p_2}\hat{v}_2-\frac{1}{p_3}\hat{v}_3.$$

PROOF. For simplicity, we set $L_1^{m_1}=R_1^{m_1}$, $[O, L_2^{m_2}]=R_2^{m_2+1}$, $[O, L_3^{m_3}]=R_3^{m_3+1}$ and we have $R^{n+2}=R_1^{m_1}\times R_2^{m_2+1}\times R_3^{m_3+1}$. Then, we put

$$x_0=(\hat{u}_1^0,\, p_2^0\hat{v}_2^0,\, p_3^0\hat{v}_3^0) \tag{6.4}$$

and we have

$$b_1^0+(p_2^0)^2+(p_3^0)^2=0, \quad b_1^0=(\hat{u}_1^0,\hat{u}_1^0)-1. \tag{6.5}$$

Writing $x_2=x_2(u_2)$ as

$$x_2=x_0+\lambda v_2, \quad |v_2|=1,$$

we have

$$x_2=(\hat{u}_1^0,\, p_2^0\hat{v}_2^0+\lambda v_2,\, p_3^0\hat{v}_3^0).$$

From $1=(x_2, x_2)$, we have

$$b_1^0+|p_2^0\hat{v}_2^0+\lambda v_2|^2+(p_3^0)^2=0$$

and $\lambda+2p_2^0\langle\hat{v}_2^0, v_2\rangle=0$ by (6.5). Hence, we get

$$x_2=\left(\hat{u}_1^0,\, p_2^0(\hat{v}_2^0-2\langle\hat{v}_2^0, v_2\rangle v_2),\, p_3^0\hat{v}_3^0\right). \tag{6.6}$$

Next, writing $x_{23}=x_{23}(u_2, u_3)$ as

$$x_{23}=x_2+\lambda v_3, \quad |v_3|=1,$$

we have

$$x_{23}=\left(\hat{u}_1^0,\, p_2^0(\hat{v}_2^0-2\langle\hat{v}_2^0, v_2\rangle v_2),\, p_3^0\hat{v}_3^0+\lambda v_3\right).$$

From $1=(x_{23}, x_{23})$, we have

$$b_1^0+(p_2^0)^2+|p_3^0\hat{v}_3^0+\lambda v_3|^2=0$$

and $\lambda+2p_3^0\langle\hat{v}_3^0, v_3\rangle=0$ by (6.5). Hence, we get

$$x_{23}=\left(\hat{u}_1^0,\, p_2^0(\hat{v}_2^0-2\langle\hat{v}_2^0, v_2\rangle v_2),\, p_3^0(\hat{v}_3^0-2\langle\hat{v}_3^0, v_3\rangle v_3)\right). \tag{6.7}$$

Then, writing $x=x(u_1, u_2, u_3)$ as

$$x=\lambda x_{23}+(1-\lambda)u_1,$$

from $(x, x)=1$ we have

$$\begin{aligned} 1&=|\lambda\hat{u}_1^0+(1-\lambda)u_1|^2+\lambda^2\left((p_2^0)^2+(p_3^0)^2\right)\\ &=\lambda^2(b_1^0+1)+2\lambda(1-\lambda)\langle\hat{u}_1^0, u_1\rangle\\ &\quad+(1-\lambda)^2(b_1+1)-\lambda^2 b_1^0, \text{ i.e.} \end{aligned}$$

$$(\lambda-1)\left[\lambda+1-2\lambda\langle\hat{u}_1^0, u_1\rangle+(\lambda-1)(b_1+1)\right]=0,$$

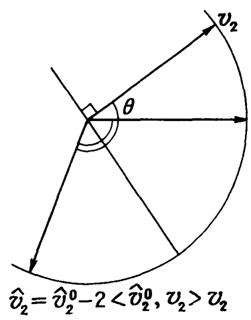

Fig. 9.

where $b_1=(u_1, u_1)-1$. In general, we may put $\lambda\neq 1$, we get

$$\lambda=\frac{b_1}{b_1+2-2\langle \hat{u}_1^0, u_1\rangle}, \quad b_1=\langle u_1, u_1\rangle-1. \tag{6.8}$$

Hence, regarding the expression (6.1) for x, we have

$$\begin{cases} \hat{u}_1=\lambda\hat{u}_1^0+(1-\lambda)\,u_1=u_1+\lambda\,(\hat{u}_1^0-u_1) \\ \hat{v}_2=\hat{v}_2^0-2\langle \hat{v}_2^0, v_2\rangle v_2, \quad \hat{v}_3=\hat{v}_3^0-2\langle \hat{v}_3^0, v_3\rangle v_3, \\ p_2=\lambda p_2^0, \quad p_3=\lambda p_3^0, \quad \text{where } \lambda \text{ is given by (6.8)}. \end{cases} \tag{6.9}$$

Now, putting $N_x=(\xi_1, \xi_2, \xi_3)$, $\xi_1\in R_1^{m_1}$, $\xi_2\in R_2^{m_2+1}$, $\xi_3\in R_3^{m_3+1}$, we have

$$\langle x, N_x\rangle=\langle \hat{u}_1, \xi_1\rangle+p_2\langle \hat{v}_2, \xi_2\rangle+p_3\langle \hat{v}_3, \xi_3\rangle=0.$$

Fixing v_2 and v_3 and moving u_1 along a curve in $L_1^{m_1}$, we have from (6.9)

$$\begin{aligned} x' &= \hat{u}_1'+p_2'\hat{v}_2+p_3'\hat{v}_3=\hat{u}_1'+\lambda'\,(p_2^0\hat{v}_2+p_3^0\hat{v}_3) \\ &=(1-\lambda)\,u_1'+\lambda'\,(\hat{u}_1^0-u_1+p_2^0\hat{v}_2+p_3^0\hat{v}_3) \\ &=(1-\lambda)\,u_1'+\frac{\lambda'}{\lambda}(x-u_1), \end{aligned}$$

and hence

$$0=\langle x', N_x\rangle=\langle (1-\lambda)\,u_1'+\frac{\lambda'}{\lambda}(x-u_1), N_x\rangle, \text{ i.e.}$$

$$\langle (1-\lambda)\,u_1'-\frac{\lambda'}{\lambda}u_1, \xi_1\rangle=0. \tag{6.10}$$

On the other hand, we get from (6.8)

$$1-\lambda=\frac{2(1-\langle \hat{u}_1^0, u_1\rangle)}{b_1+2-2\langle \hat{u}_1^0, u_1\rangle}\neq 0$$

and so (6.10) implies $\xi_1=0$. Therefore, N_x is of the form

$$N_x=(0, \xi_2, \xi_3).$$

Next, fixing u_1 and v_3 and moving v_2 along a curve in the unit m_2-sphere of $R_2^{m_2+1}$ with centor at the origin. Then, $\lambda, p_2, p_3, \hat{u}_1$ and $\hat{v}_3$ are all fixed by (6.9). Hence we have

$$x'=p_2\hat{v}_2'=-2p_2\,(\langle \hat{v}_2^0, v_2'\rangle v_2+\langle \hat{v}_2^0, v_2\rangle v_2')$$

and hence from $\langle x', N_x\rangle=0$

$$\langle \hat{v}_2^0, v_2'\rangle\,\langle v_2, \xi_2\rangle+\langle \hat{v}_2^0, v_2\rangle\,\langle v_2', \xi_2\rangle=0.$$

Taking first $v_2'\perp v_2$ and $\hat{v}_2^0$, we get $\langle v_2', \xi_2\rangle=0$, because we may put $\langle \hat{v}_2^0, v_2\rangle$

$\neq 0$ in general. Hence ξ_2 linearly depends on v_2 and $\hat{v}_2^0$. Second, we put in the above equality as

$$v_2' = \{\hat{v}_2^0 - \langle \hat{v}_2^0, v_2\rangle v_2\}/\sqrt{1-\langle \hat{v}_2^0, v_2\rangle^2}$$

and we obtain

$$\frac{\langle \xi_2, v_2\rangle}{-\langle \hat{v}_2^0, v_2\rangle} = \frac{\langle \xi_2, v_2'\rangle}{\langle \hat{v}_2^0, v_2'\rangle},$$

which shows that $\xi_2 || \hat{v}_2$. Analogously, we obtain $\xi_3 || \hat{v}_3$. Therefore, we can write N_x as $N_x = (0, r_2\hat{v}_2, r_3\hat{v}_3)$. Then, from $\langle x, N_x\rangle = p_2\langle \hat{v}_2, \xi_2\rangle + p_3\langle \hat{v}_3, \xi_3\rangle = p_2r_2 + p_3r_3 = 0$, we get

$$N_x || \frac{1}{p_2}\hat{v}_2 - \frac{1}{p_3}\hat{v}_3 . \qquad \text{Q. E. D.}$$

By means of Lemma 15, we see that M^n has also 3 principal curvatures μ_i with multiplicity m_i, $i=1, 2, 3$, as in the previous cases and that

$$\text{(6. 11)} \quad \mu_1 = 0, \quad \mu_2 = \frac{1}{\overline{xy_2}}, \quad \mu_3 = \frac{1}{\overline{xy_3}},$$

where y_2 and y_3 are the points on the normal line at x as is shown in Fig. 10. Therefore, we have

$$\text{(6. 12)} \quad \sum_{i=1}^{3} m_i\mu_i = \frac{m_2}{\overline{xy_2}} + \frac{m_3}{\overline{xy_3}} .$$

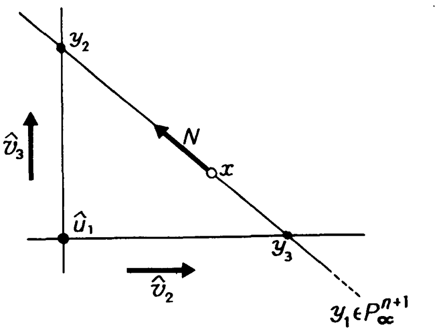

Fig. 10.

Setting

$$y_i = x + \rho_i\left(\frac{1}{p_2}\hat{v}_2 - \frac{1}{p_3}\hat{v}_3\right), \quad i = 2, 3,$$

we have

$$p_2 + \frac{\rho_2}{p_2} = 0 \text{ and } p_3 - \frac{\rho_3}{p_3} = 0,$$

from which we see that the equality $\sum_{i=1}^{3} m_i\mu_i = 0$ is equivalent to the equality

$$\frac{m_2}{p_2^2} - \frac{m_3}{p_3^2} = 0 .$$

Now, first fixing u_2 and u_3 in $x = x(u_1, u_2, u_3)$, i. e. v_2 and v_3, let move u_1 along a curve in $L_1^{m_1}$ and put $p_2' = q_2$, $p_3' = q_3$. Then, from (6. 2) we obtain

$$\frac{1}{2}b_1' + p_2q_2 + p_3q_3 = 0 .$$

The equality $\langle x', N_x\rangle=0$ and Lemma 15 imply

$$\langle \hat{u}_1'+q_2\hat{v}_2+q_3\hat{v}_3, \frac{1}{p_2}\hat{v}_2-\frac{1}{p_3}\hat{v}_3\rangle = \frac{q_2}{p_2}-\frac{q_3}{p_3}=0,$$

because $\hat{v}_2$ and $\hat{v}_3$ are also fixed by (6. 9). Hence, we can put

(6. 13) $$q_2=\rho p_2,\quad q_3=\rho p_3,\quad b_1'=-2\rho(p_2^2+p_3^2)=2\rho b_1.$$

Therefore, we have

(6. 14) $$\left(\frac{m_2}{p_2^2}-\frac{m_3}{p_3^2}\right)'=-2\rho\left(\frac{m_2}{p_2^2}-\frac{m_3}{p_3^2}\right),$$

from which we see that if there exists a general point x of M^n such that

(6. 15) $$\frac{m_2}{p_2^2}-\frac{m_3}{p_3^2}=0,$$

then on the m_1-sphere $S_1^{m_1}(u_2, u_3)=S^{n+1}\cap[x_{23}, L_1^{m_1}]$ this equality holds identically.

Second, fixing u_1 and u_3 in $x=x(u_1, u_2, u_3)$, i. e. u_1 and v_3, let move v_2 along a curve in the unit m_2-sphere of $R_2^{m_2+1}$. Then, $x_1, x_3, x_{13}, \hat{u}_1, \hat{v}_3, p_2$ and p_3 are all fixed as easily seen from (6. 9). Therefore, we have

(6. 16) $$\left(\frac{m_2}{p_2^2}-\frac{m_3}{p_3^2}\right)'=0,$$

from which we see that the analogous fact holds in this case. And, we have the same fact for the case in which u_1 and u_2 in $x(u_1, u_2, u_2)$ are fixed and u_3 is moved.

In the present case, a point x of M^n is called a general point if $x\overline{\in}\tilde{E}_2^{m_3+m_1+1}\cup\tilde{E}_3^{m_1+m_2+1}$. From the argument above, we obtain the following

LEMMA 16. *$M^n=M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ with $m_i\geqq 2$, $i=1, 2, 3$, and $L_2^{m_2}\subset P_\infty^{n+1}$, $L_3^{m_2}\subset P_\infty^{n+1}$, is minimal in S^{n+1}, if there exists a general point x of M^n such that $m_2/p_2^2=m_3/p_3^2$.*

THEOREM 6. *$M^n=M^n(x_0, L_1^{m_1}, L_2^{m_2}, L_3^{m_3})$ with $m_i\geqq 2$, $i=1, 2, 3$,* and *$L_2^{m_2}\subset P_\infty^{n+1}$, $L_3^{m_3}\subset P_\infty^{n+1}$, is minimal in S^{n+1} by suitable choice of the point x_0.*

PROOF. It is sufficient to prove this theorem to show that there exist points x_0 satisfing the condition stated in Lemma 16.

First of all, putting $p_2=\rho\varepsilon_2\sqrt{m_2}$, $p_3=\rho\varepsilon_3\sqrt{m_3}$, $\varepsilon_2=\pm 1$, $\varepsilon_3=\pm 1$, we get from (6. 2)

$$(\hat{u}_1, \hat{u}_1)=1+b_1=1-p_2^2-p_3^2=1-\rho^2(m_2+m_3).$$

Therefore, taking a number $\rho>0$ such that $\rho<1/\sqrt{m_2+m_3}$, we take a point

x for $\hat{u}_1, \hat{v}_2, \hat{v}_3$ given by

$$x = \hat{u}_1 + p_2\hat{v}_2 + p_3\hat{v}_3 ,$$

where

$$\hat{u}_1 \in L_1^{m_1} \text{ and } |\hat{u}_1| = \sqrt{1-\rho^2(m_2+m_3)} ;$$
$$\hat{v}_2 \perp \mathrm{R}_2^{m_3+m_1+1}, \ |\hat{v}_2| = 1 ; \ \hat{v}_3 \perp R_3^{m_1+m_2+1}, \ |\hat{v}_3| = 1 .$$

It is obvious that we can choose u_1, $\hat{u}_1^0$, v_2, $\hat{v}_2^0$, v_3, $\hat{v}_3^0$ so that they satisfy (6. 9) for the given point x above. In fact, taking $\hat{v}_2^0$, $\hat{v}_3^0$, u_1 and setting $\hat{u}_1^0 = u_1 + (1/\lambda)(\hat{u}_1 - u_1)$, where λ is now considered as a variable to be determined, we substitute this into (6. 8). Then, we obtain easily

$$\lambda = \frac{\langle u_1, u_1\rangle + 1 - 2\langle \hat{u}_1, u_1\rangle}{\langle u_1, u_1\rangle - 1} .$$

Using this value of λ, $\hat{v}_1^0$ is determined by the above equality. By virtue of this process, we obtain x_0 given by the equality

$$x_0 = \hat{u}_1^0 + \frac{p_2}{\lambda}\hat{v}_2^0 + \frac{p_3}{\lambda}\hat{v}_3^0 . \qquad \text{Q. E. D.}$$

References

[1] T. OTSUKI: A theory of Riemannian manifolds, Kôdai Math. Sem. Rep., 20 (1968), 282–295.

[2] T. OTSUKI: Minimal hypersurfaces in a Riemannian manifold of constant curvature, American J. Math., 112 (1970), 145–173.

[3] T. OTSUKI: Minimal hypersurfaces with three principal curvature fields in S^{n+1}, to appear in Kôdai Math. Sem. Rep..

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