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Convolutions of measures on some thin sets

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1. Introduction.

Let G be a LCA group with dual $\hat{G}$, and E a compact subset of G. Following Rudin [4], we say that E is a Kronecker set if, to each $f\in C(E)$ with $|f|=1$ and $\varepsilon>0$, there exists $\gamma\in\hat{G}$ such that $\|f-\gamma\|_E<\varepsilon$. Let also $T=\{|z|=1\}$ be the circle group. Then in [2; Lemma 6.10], T. W. Körner proved that there exist two Kronecker sets K_1 and $K_2\subset T$, both homeomorphic to $D_2=\{0,1\}^{\infty}$, and two nonzero nonnegative Borel measures μ_1 on K_1 and μ_2 on K_2, such that $K_1+K_2=T$, and $\mu_1*\mu_2\in C^{\infty}(T)$.

In this paper, we prove analogs to the above result for general LCA groups. I thank Professor S. Saeki for his useful advices.

2. Notations.

Throughout this paper, G is used to denote a nondiscrete LCA group. We shall respectively denote by $A(G)=(L^1(\hat{G}))^{\wedge}$ and $C_c(G)$, the Fourier algebra of G, and the set of all continuous functions with compact support. Also we shall respectively denote by $M(G)$, $M_c(G)$, and $M_0(G)$, the measure algebra of all bounded regular measures on G, the set of all continuous measures in $M(G)$, and the set of those measures $\mu\in M(G)$ whose Fourier-Stieltjes transforms vanish at infinity. For $\mu\in M(G)$, $\|\mu\|$ will denote the total variation norm. The symbols $C_c^+(G)$, $M^+(G)$, and $M_c^+(G)$, will designate the set of all nonnegative functions in $C_c(G)$, the set of all nonnegative measures in $M(G)$, and the set of all nonnegative measures in $M_c(G)$, respectively. For μ, $\nu\in M(G)$, we write $\mu\perp\nu$ if μ and ν are mutually singular. We set $M_0^{\perp}(G)=\{\mu\in M(G)|\mu$ is singular with each $\nu\in M_0(G)\}$. For the other notation, we refer to Rudin [4].

DEFINITION 1. Let 0 be the unit of G. $q(G)=\sup\{s|$every neighborhood $0\in G$ contains an element of order $\geq s\}$.

DEFINITION 2. Let $q\geq 2$ be an integer, and E a totally disconnected compact subset of G. We say that E is a K_q-set if, to each $f\in C(E)$ with $f^q=1$, there exists $\gamma\in\hat{G}$ such that $f=\gamma$ on E.

3. Statements of the results.

THEOREM A. *Let G be a metrizable LCA group with $q(G)=\infty$. Let also $E\subset G$ be a compact set, and $N\geq 2$ an integer. Then there exist disjoint Kronecker sets $K_1, \cdots, K_N$, all homeomorphic to $D_2=\{0, 1\}^{\infty}$, and nonzero measures $\mu_1\in M_c^+(K_1), \cdots, \mu_N\in M_c^+(K_N)$, such that*

(i) *the union of any $N-1$ sets of the K_j's is a Kronecker set,*

(ii) *$\mu_1*\cdots*\mu_N$ is absolutely continuous with respect to Haar measure on G, and its Radon-Nikodym derivative is in $A(G)$ and positive on E.*

THEOREM B. *Let G be a nondiscrete metrizable LCA group with $q=q(G)<\infty$, and $N\geq 2$ an integer. Then there exists an open neighborhood U of 0 such that for each compact set $E\subset U$ with $0\notin E$, the conclusion of Theorem A is satisfied with K_q-set in place of Kronecker set.*

REMARK. In the above result, if $N\neq 2$ or $q\neq 2$, the condition $0\notin E$ is unnecessary.

COROLLARY 1. *Let G be a nondiscrete LCA group. Then there exist nonzero measures μ_1, $\mu_2\in M_c^+(G)\cap M_0^{\perp}(G)$ such that*

(i) $\|\mu_i\| = \overline{\lim_{\gamma\to\infty}}\,|\hat{\mu}_i(\gamma)|$ $\quad (i=1, 2)$,

(ii) $\mu_1*\mu_2\in A(G)\cap C_c^+(G)$.

COROLLARY 2 ([1]). *Let G be a nondiscrete LCA group. Then $M_0^{\perp}(G)$ is not an algebra.*

4. Some lemmas for the proof of Theorem A.

LEMMA 1. *Let f, g be in $A(G)\cap C_c^+(G)$ with $\|f\|_{L^1(G)}=\|g\|_{L^1(G)}=1$, and $q(G)=\infty$. Let also $0<\eta<1$ be given. Then there exists $\nu\in M^+(G)$ with $\|\nu\|=1$, such that*

(i) $\operatorname{supp}\nu\subset \operatorname{int}(\operatorname{supp} f)$,

(ii) $\operatorname{supp}\nu$ *is a finite Kronecker set,*

(iii) $\|f*g-\nu*g\|_{A(G)}<\eta$.

PROOF. Let $\eta'>0$ be given. Since f, g are in $C_c^+(G)$, there exists $\{V_i\}$ a finite partition of supp f, such that

$$\operatorname{int} V_i\cap \operatorname{int}(\operatorname{supp} f)\neq\phi\,, \tag{4.1.1}$$

$$|f(x)-f(y)|<\eta',\ |g(x)-g(y)|<\eta',\quad \text{for any } z\in G,\ x,\ y\in z+V_i\,. \tag{4.1.2}$$

Then there exists $\{y_i\}$ a finite Kronecker set such that $y_i\in V_i\cap \operatorname{int}(\operatorname{supp} f)$. Put

$$\nu' = \sum_i \left(\int_{V_i} f(y_i)\,dy\right)\delta_{y_i}\,.$$

From (4. 1. 2), we see that

$$(4.1.3)\qquad \begin{aligned} &| \|\nu'\| - 1 | \\ &= |\sum_i \int_{V_i} f(y_i)\, dy - \sum_i \int f(y)\, \chi_{V_i}(y)\, dy| \\ &\le \eta' \cdot m(\operatorname{supp} f), \end{aligned}$$

where m denotes Haar measure of G, and from (4. 1. 2), (4. 1. 3) that for each $x \in G$

$$(4.1.4)\qquad \begin{aligned} &|f*g(x) - \nu'*g(x)| \\ &\le \sum_i \left\{ \int_{V_i} |g(x-y)|\, |f(y) - f(y_i)| dy \right. \\ &\qquad \left. + \int_{V_i} |g(x-y) - g(x-y_i)|\, |f(y_i)| dy \right\} \\ &\le \eta' \|g\|_{L^1(G)} + \eta' \cdot \left(1 + \eta' \cdot m(\operatorname{supp} f)\right). \end{aligned}$$

Putting $\nu = \dfrac{\nu'}{\|\nu'\|}$, we have by (4. 1. 3)

$$(4.1.5)\qquad \begin{aligned} \|\nu - \nu'\| &\le |1 - \|\nu'\| | \\ &\le \eta' \cdot m(\operatorname{supp} f). \end{aligned}$$

Also by (4. 1. 4) and (4. 1. 5), we have

$$(4.1.6)\qquad \begin{aligned} &|f*g(x) - \nu*g(x)| \\ &\le |f*g(x) - \nu'*g(x)| + |\nu'*g(x) - \nu*g(x)| \\ &\le \eta' \|g\|_{L^1(G)} + \eta' \cdot \left(1 + \eta' \cdot m(\operatorname{supp} f)\right) + \left(\eta' \cdot m(\operatorname{supp} f)\right) \cdot \|g\|_\infty . \end{aligned}$$

Now we can complete the proof as follows. First choose $\hat{F}$ a compact subset of $\hat{G}$ such that

$$(4.1.7)\qquad \int_{\hat{F}^c} |\hat{g}(\gamma)| d\gamma < \frac{\eta}{4}.$$

By (4. 1. 1), (4. 1. 2) and (4. 1. 6), there exists ν, satisfying (i) ond (ii), such that

$$(4.1.8)\qquad \|f*g - \nu*g\|_\infty < \left[m(\operatorname{supp} f + \operatorname{supp} g)\right]^{-1} \cdot \left[\hat{m}(\hat{F})\right]^{-1} \cdot \frac{\eta}{2},$$

where $\hat{m}$ denotes Haar measure of $\hat{G}$. Then by (4. 1. 7) and (4. 1. 8), we have

$$\begin{aligned} \|f*g - \nu*g\|_{A(G)} \le &\int_{\hat{F}} |\hat{f}(\gamma)\, \hat{g}(\gamma) - \hat{\nu}(\gamma)| d\gamma \\ &+ \int_{\hat{F}^c} |\hat{f}(\gamma)\, g(\gamma) - \hat{\nu}(\gamma)\, \hat{g}(\gamma)| d\gamma \end{aligned}$$

$$\|f*g-\nu*g\|_{A(G)} \leq \left[\hat{m}(\hat{F})\right]\cdot\int|f*g(x)-\nu*g(x)|dx$$
$$+2\int_{\hat{F}^c}|\hat{g}(\gamma)|d\gamma$$
$$\leq \eta .$$

LEMMA 2. *Let E be a compact subset of G, ε a positive number, and let f_1, f_2, f_3 in $A(G)\cap C_c^+(G)$ satisfy*

$$\|f_1\|_{L^1(G)} = \|f_2\|_{L^1(G)} = \|f_3\|_{L^1(G)} = 1 \text{ and}$$
$$\mathrm{int}\,(\mathrm{supp}\, f_1)+\mathrm{int}\,(\mathrm{supp}\, f_2)+\mathrm{int}\,(\mathrm{supp}\, f_3)\supset E.$$

Then there exist ν_1, ν_2 in $M^+(G)$ with $\|\nu_1\|=\|\nu_2\|=1$, $N\geq 2$ an integer, and $\{a_{1i}, \cdots, a_{Ni}\}\subset \mathrm{int}\,(\mathrm{supp}\, f_i)$ $(i=1, 2)$, such that

(i) $\bigcup_{j=1}^{N}\left(a_{j1}+a_{j2}+\mathrm{int}\,(\mathrm{supp}\, f_3)\right)\supset E$,

(ii) $\mathrm{supp}\,\nu_i=\{a_{1i}, \cdots, a_{Ni}\}$ $(i=1, 2)$, $\mathrm{supp}\,\nu_1\cap \mathrm{supp}\,\nu_2=\phi$, *and* $(\mathrm{supp}\,\nu_1)\cup(\mathrm{supp}\,\nu_2)$ *is a Kronecker set,*

(iii) $\|f_1*f_2*f_3-\nu_1*\nu_2*f_3\|_{A(G)}<\varepsilon$.

PROOF. Since E is a compact set, it follows from the hypothesis that there exist $N'\geq 2$ an integer, and $\{a_{1i}, \cdots, a_{N'i}\}\subset \mathrm{int}\,(\mathrm{supp}\, f_i)$ $(i=1, 2)$, such that

(4. 2. 1) $\bigcup_{j=1}^{N'}\left(a_{j1}+a_{j2}+\mathrm{int}\,(\mathrm{supp}\, f_3)\right)\supset E$, and

(4. 2. 2) $\{a_{11}, \cdots, a_{N'1}\,;\, a_{12}, \cdots, a_{N'2}\}$ is Kronecker set.

Then by Lemma 1, we have $\nu_1\in M^+(G)$, $\|\nu_1\|=1$, and Card $(\mathrm{supp}\,\nu_1)\geq N'$, such that

(4. 2. 3) Supp $\nu_1\cup\{a_{12}, \cdots, a_{N'2}\}$ is a finite Kronecker set,

(4. 2. 4) $\mathrm{int}\,(\mathrm{supp}\, f_1)\supset \mathrm{supp}\,\nu_1\supset\{a_{11}, \cdots, a_{N'1}\}$, and

(4. 2. 5) $\|f_1*f_2*f_3-\nu_1*f_2*f_3\|_{A(G)}<\dfrac{\varepsilon}{2}$.

In the same way, we have $\nu^2\in M^+(G)$, $\|\nu_2\|=1$ such that

(4. 2. 6) $(\mathrm{supp}\,\nu_1)\cup(\mathrm{supp}\,\nu_2)$ is a finite Kronecker set, and $(\mathrm{supp}\,\nu_1)\cap(\mathrm{supp}\,\nu_2)=\phi$,

(4. 2. 7) int $(\mathrm{supp}\, f_2)\supset \mathrm{supp}\,\nu_2\supset\{a_{12}, \cdots, a_{N'2}\}$, Card $(\mathrm{supp}\,\nu_2)\geq$ Card $(\mathrm{supp}\,\nu_1)$, and

(4. 2. 8) $\|f_2*f_3-\nu_2*f_3\|_{A(G)}<\dfrac{\varepsilon}{2}$.

Without loss of generality, we may assume Card (supp ν_1) = Card (supp ν_2) (if necessary, replace ν_1 by an appropriate ν_1').

Put N = Card (supp ν_1) = Card (supp ν_2). We have (i), (ii) in Lemma 2 by (4. 2. 1) and (4. 2. 7). By (4. 2. 5) and (4. 2. 8), we have

$$\begin{aligned}
&\|f_1*f_2*f_3-\nu_1*\nu_2*f_3\|_{A(G)}\\
&\quad\leq \|f_1*f_2*f_3-\nu_1*f_2*f_3\|_{A(G)}\\
&\qquad\quad + \|\nu_1*f_2*f_3-\nu_1*\nu_2*f_3\|_{A(G)}\\
&\quad\leq \frac{\varepsilon}{3}+\|f_2*f_3-\nu_2*f_3\|_{A(G)}\\
&\quad<\varepsilon .
\end{aligned}$$

which establishes part (iii).

LEMMA 3. *Under the notation in Lemma* 2, *let* η_1 *and* η_2 *be positive numbers. Then there exist* W *a compact neighborhood of* 0 *and* $\hat{K}$ *a finite subset of* $\hat{G}$, *such that*

(i) *diam* $(W)<\eta_1$,

(ii) $\{a_{ji}+W\}_{j=1,\cdots,N;\, i=1,2}$ *are disjoint sets,* $a_{ji}+W\subset \operatorname{int}(\operatorname{supp} f_i)$ $(j=1,\cdots, N;\ i=1,2)$, *and* $\bigcup_{j=1}^{N}\left(x_{j1}+x_{j2}+\operatorname{int}(\operatorname{supp} f_3)\right)\supset E$ *for any* $x_{si}\in a_{si}+W$ $(s=1,\cdots, N;\ i=1,2)$,

(iii) *given* α_{js} *reals* $(j=1,\cdots,N;\ s=1,2)$, *there exists* $\gamma\in\hat{K}$ *satisfyng*

$$|\gamma(x)-\exp(i\alpha_{js})|<\eta_2 \text{ for } x\in a_{js}+W \qquad (j=1,\cdots,N;\ s=1,2).$$

PROOF. By (i) in Lemma 2, and [5; Lemma 2], we have W_1 a compact neighborhood of 0 satisfying (i) and (ii). On the other hand, by (ii) in Lemma 2, there exists $\hat{K}\subset\hat{G}$ a finite set, such that for any α_{js} reals $(j=1,\cdots,N;\ s=1,2)$, we can take $\gamma\in\hat{K}$ satisfying

$$(*)\qquad |\gamma(a_{js})-\exp(i\alpha_{js})|<\frac{\eta_2}{2} \qquad (j=1,\cdots,N;\ s=1,2).$$

After all, by (*), it is sufficient to choose $W\subset W_1$ a compact neighborhood of 0 such that

$$|\gamma(x)-\gamma(a_{js})|<\frac{\eta_2}{2}$$
$$\text{for any } \gamma\in\hat{K} \text{ and } x\in a_{js}+W \qquad (j=1,\cdots,N;\ s=1,2).$$

LEMMA 4. *Under the notation in Lemma* 3, *there exist* $f_1', f_2'\in A(G)\cap C_c^+(G)$ *with* $\|f_1'\|_{L^1(G)}=\|f_2'\|_{L^1(G)}=1$, *such that*

(i) $\operatorname{int}(\operatorname{supp} f_1')+\operatorname{int}(\operatorname{supp} f_2')+\operatorname{int}(\operatorname{supp} f_3)\supset E$,

(ii) $\operatorname{supp} f_i'\subset \bigcup_{j=1}^{N}(a_{ji}+W) \qquad (i=1,2)$,

(iii) $\|f_1*f_2*f_3-f_1'*f_2'*f_3\|_{A(G)}<2\varepsilon$.

PROOF. First, let $\hat{F}$ be a compact subset of $\hat{G}$ such that

$$\int_{\hat{F}^c} \hat{f}_3(\gamma)|d\gamma<\frac{\varepsilon}{8}. \tag{4. 4. 1}$$

Next, for W in Lemma 3, there exists $h\in A(G)\cap C_c^+(G)$ with supp $h\subset W$ and $\|h\|_{L^1(G)}=1$, such that

$$\int_{\hat{F}} |\hat{f}_3(\gamma)-\hat{h}(\gamma)\hat{f}_3(\gamma)|d\gamma<\frac{\varepsilon}{4}. \tag{4. 4. 2}$$

Put $f_1'=\nu_1*h$, $f_2'=\nu_2*h$ where ν_1, ν_2 are as in Lemma 2. Then, we have

$$\|f_i'\|_{L^1(G)}=1,\ f_i'\in A(G)\cap C_c^+(G),\ \text{and}$$
$$\operatorname{supp} f_i\subset \bigcup_{j=1}^{N}(a_{ji}+W) \qquad (i=1, 2).$$

So we have (i) and (ii) immediately. Also by Lemma 2, we have

$$\begin{aligned}
&\|f_1*f_2*f_3-f_1'*f_2'*f_3\|_{A(G)}\\
&\leq \|f_1*f_2*f_3-\nu_1*\nu_2*f_3\|_{A(G)}\\
&\quad+\|\nu_1*\nu_2*f_3-f_1'*f_2'*f_3\|_{A(G)}\\
&\leq \varepsilon+2\|f_3-h*f_3\|_{A(G)}.
\end{aligned}$$

By (4. 4. 1) and (4. 4. 2), we have

$$\|f_3-h*f_3\|_{A(G)}<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}.$$

5. The proofs of Theorem A and Theorem B.

Tne proof of Theorem A. We prove Theorem A only for $N=3$, since the proof for the general case is similar.

First, choose g_1, g_2, $g_3\in A(G)\cap C_c^+(G)$ with $\|g_1\|_{L^1(G)}=\|g_2\|_{L^1(G)}=\|g_3\|_{L^1(G)}=1$, such that

$$\operatorname{int}(\operatorname{supp} g_1)+\operatorname{int}(\operatorname{supp} g_2)+\operatorname{int}(\operatorname{supp} g_3)\supset E,$$

and

$$g_1*g_2*g_3\geq\delta>0 \text{ on } E.$$

Next, suppose that for some $r\geq 0$, we constructed

$$g_{3r+1},\ g_{3r+2},\ g_{3r+3}\in A(G)\cap C_c^+(G)$$

with $\|g_{3r+1}\|_{L^1(G)}=\|g_{3r+2}\|_{L^1(G)}=\|g_{3r+3}\|_{L^1(G)}=1$, such that

$$\mathrm{int}\,(\mathrm{supp}\ g_{3r+1})+\mathrm{int}\,(\mathrm{supp}\ g_{3r+2})+\mathrm{int}\,(\mathrm{supp}\ g_{3r+3})\supset E,$$

and

$$g_{3r+1}*g_{3r+2}*g_{3r+3}\geq\delta(1-10^{-1}-\cdots-10^{-r})\ \text{on}\ E.$$

We now describe the construction in the $(r+1)^{\text{th}}$ step. We use Lemmas 2 and 3 for $f_i=g_{3r+i}$ $(i=1, 2, 3)$, and positive numbers ε, η_1, η_2 which tend to 0 rapidly depending on r. Then we get $g'_{3r+1}=f'_1$, $g'_{3r+2}=f'_2$, $g_{3r+3}=f_3$ as in Lemma 4. Again putting $f_1=g'_{3r+2}$, $f_2=g_{3r+3}$, $f_3=g'_{3r+1}$, we get similarly g''_{3r+1}, g'_{3r+3}, g'_{3r+1} in place of f'_1, f'_2, f_3 as in Lemma 4. In the third step, putting $f_1=g'_{3r+3}$, $f_2=g'_{3r+1}$, $f_3=g''_{3r+2}$, we get g''_{3r+3}, g''_{3r+1}, g''_{3r+2} in place of f'_1, f'_2, f'_3 as in Lemma 4.

$$\text{Put}\ g_{3(r+1)+i}=g''_{3r+i}\qquad (i=1, 2, 3).$$

We can demand that

(5. 1. 1) $\quad g_{3(r+1)+i}\in A(G)\cap C_c^+(G),\ \|g_{3(r+1)+i}\|_{L^1(G)}=1\qquad (i=1, 2, 3),$

(5. 1. 2) $\quad \mathrm{int}\,(\mathrm{supp}\ g_{3(r+1)+1}+\mathrm{int}\,(\mathrm{supp}\ g_{3(r+1)+2})+\mathrm{int}\,(\mathrm{supp}\ g_{3(r+1)+3})\supset E,$

and

(5. 1. 3) $\quad \|g_{3r+1}*g_{3r+2}*g_{3r+3}-g_{3(r+1)+1}*g_{3(r+1)+2}*g_{3(r+1)+3}\|_{A(G)}\leq\dfrac{\delta}{10^{r+1}}.$

From (5. 1. 3), we have

$$g_{3(r+1)+1}*g_{3(r+1)+2}*g_{3(r+1)+3}\geq\delta(1-10^{-1}-\cdots-10^{-(r+1)})\ \text{on}\ E.$$

This completes our inductive construction.

Now let

$$L_{3r+i}\ \text{be}\ \mathrm{supp}\ g_{3r+i}\qquad (i=1, 2, 3),$$

and

$$K_i=\bigcap_{r=0}^{\infty}L_{3r+i}\qquad (i=1, 2, 3).$$

Then if we carefully choose $\{g_{3r+i}\}_{r\geq0}$, $i=1, 2, 3$ recalling the proofs of Lemma 3 and Lemma 4, we can demand that the K_i's $(i=1, 2, 3)$ are disjoint Kronecker sets, homeomorphic to Cantor set such that the union of any two sets in K_1, K_2, K_3 is a Kronecker set and $K_1+K_2+K_3\supset E$. (cf [4; 5. 2. 4])

By (5. 1. 1), we can find an increasing sequence $\{r_n\}$ of natural numbers such that $\{g_{3r_n+i}\}_{n\geq1}$ has a limit μ_i in the weak* topology of $M(G)$ for $i=1, 2, 3$.

Then it is easy to see that $\mu_i \in M_c^+(K_i)$ and $\|\mu_i\|=1$ $(i=1, 2, 3)$.

On the other hand, by (5. 1. 3), $(g_{3r+1}*g_{3r+2}*g_{3r+3})_{r\geq 0}$ is a Cauchy sequence in $A(G)$.

Putting g the limit of $g_{3r+1}*g_{3r+2}*g_{3r+3}$ as r tends to ∞, we have

$$g \in A(G) \cap C_c^+(G),\ g \geq \frac{\delta}{100} > 0 \text{ on } E,$$

and

$$d(\mu_1*\mu_2*\mu_3) = gdm\,.$$

We get the desired result.

The proof of Theorem B. If we use the next Lemma 5, Theorem B can be proved in the same way as Theorem A. Because we can use as $\eta_2=0,\ \alpha_{js} \in \left\{\frac{2\pi}{q}, \frac{4\pi}{q}, \cdots, \frac{2q\pi}{q}\right\}$ in Lemma 3.

LEMMA 5 ([3 ; chap XIII 3. 7]). *Let G be a nondiscrete LCA group, and $q=q(G)<\infty$. Then there exists U an open neighborhood of* 0, *such that for any $k \geq 1$ integer and $V_1, \cdots, V_k \subset U$ disjoint, nonempty open sets, there exist $x_i \in V_i$, all with order q, such that $\{x_1, \cdots, x_k\}$ is an independent set.*

6. The proofs of Corollaries.

Since Corollary 2 is clear from Corollary 1, it is sufficient to show Corollary 1.

The proof of Corollary 1. Since G is a nondiscrete LCA group, $\hat{G}$ is a noncompact set. Thus there exists Y an open subgroup of G, which is σ-compact, but is not compact. Put $H=Y^{\perp}$.

We have that G/H is a nondiscrete, metrizable group, and

$$H \text{ is a compact group}\,. \tag{6. 1}$$

By Theorem A and Theorem B, there exist ν_i nonzero measures in $M^+(G/H)$, such that

$$\operatorname{supp} \nu_i \text{ is a compact subset of some } K_q\text{-set } (i=1, 2), \tag{6. 2}$$

and

$$\nu_1*\nu_2 \in A(G/H)\,. \tag{6. 3}$$

Then From [4 ; 5. 5. 3] and [4 ; 5. 6. 10], it is easy to see

$$\nu_i \in M_0^{\perp}(G/H) \qquad (i=1, 2)\,. \tag{6. 4}$$

By (6. 2), recalling $\nu_i \geq 0$ $(i=1, 2)$, we get

$$\|\nu_i\| = \overline{\lim_{\gamma\to\infty}} |\hat{\nu}_i(\gamma)| \qquad (i=1, 2)\,. \tag{6. 5}$$

Let m_H be the normalized Haar measure on H. We define $\mu_i \in M^+(G)$ by setting

$$\int_G f(x)\,d\mu_i = \int_{G/H}\int_H f(x+y)\,dm_H(y)\,d\nu_i \qquad (i=1,2) \tag{6.6}$$

for $f \in C_0(G)$, where $C_0(G)$ is the completion of $C_c(G)$ in the supremum norm. By (6. 1) and (6. 2), supp μ_i is compact ($i=1, 2$). Then for any bounded continuous function f, we have (6. 6). In particular for $\gamma \in \widehat{G}$, we get

$$\hat{\mu}_i(\gamma) = \begin{cases} \hat{\nu}_i(\gamma) & \text{for } \gamma \in Y \\ 0 & \text{for } \gamma \notin Y. \end{cases} \tag{6.7}$$

We claim that μ_1 and μ_2 have the required properties. Recalling the choice of Y, (6. 3), (6. 5), and (6. 6), we get

$$\|\mu_i\| = \overline{\lim_{\gamma\to\infty}}|\hat{\mu}_i(\gamma)|, \quad \mu_i \in M_c^+(G) \qquad (i=1, 2),$$

and

$$\mu_1 * \mu_2 \in A(G).$$

To complete the proof, it is sufficient to show $\mu_1 \in M_0^\perp(G)$. Suppose that μ_1 is not in $M_0^\perp(G)$. Then there exists μ' a nonzero nonnegative measure in $M_0(G)$ such that μ' is absolutely continuous with respect to μ_1. Hence putting $\mu = \mu' * m_H$, we get that μ is absolutely continuous with respect to μ_1. We define $\nu \in M(G/H)$ by

$$\int_G f(x)\,d\mu = \int_{G/H}\int_H f(x+y)\,dm_H(y)\,d\nu \text{ for } f \in C_0(G).$$

Then μ being absolutely continuous with respect to μ_1 and $\mu \in M_0(G)$, we get that ν is absolutely continuous with respect to ν_1, and nonzero measure in $M_0(G/H)$. This contradicts (6. 4).

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