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Cluster sets on Riemann surfaces

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1. Introduction

In the theory of cluster sets of meromorphic functions defined in a domain in the z-plane, the next Theorem A and Theorem B are well-known. Let $w=f(z)$ be a meromorphic function in a domain D in the z-plane. Let b_0 be a point of boundary B of D and let E be a subset of B such that $b_0\in E$ and $b_0\in\overline{B-E}$ (the closure— is taken in the w-sphere). We denote by $C(f, b_0)$ the cluster set of $f(z)$ at b_0 and denote by $C_{B-E}(f, b_0)$ the boundary cluster set of $f(z)$ at b_0 modulo E. When D is the unit disk $\{z; |z|<1\}$, we denote by $C_{R-E}(f, b_0)$ the radial boundary cluster set of $f(z)$ at b_0 modulo E. Then there exists the next relation between the boundary $\partial C(f, b_0)$ of $C(f, b_0)$ and $C_{B-E}(f, b_0)$ (or $C_{R-E}(f, b_0)$) (cf. E. F. Collingwood and A. J. Lohwater [1] and K. Noshiro [7]).

THEOREM A. (*M. Tsuji* [10]) *If E is a compact set of capacity zero, then $\partial C(f, b_0)\subset C_{B-E}(f, b_0)$, that is $\Omega=C(f, b_0)-C_{B-E}(f, b_0)$ is an open set. And if $\Omega\neq\phi$, then every value of Ω is assumed infinitely often by $f(z)$ in any neighborhood of b_0 with the possible exception of a set of capacity zero.*

THEOREM B. (*M. Ohtsuka* [8]) *Let D be the unit disk. If E is a set of linear measure zero, then $\partial C(f, b_0)\subset C_{R-E}(f, b_0)$, that is $\Omega'=C(f, b_0)-C_{R-E}(f, b_0)$ is an open set. And if $\Omega'\neq\phi$, then every value of Ω' is assumed by $f(z)$ in any neighborhood of b_0 with the possible exception of a set of capacity zero.*

In this paper we study these theorems for meromorphic functions defined in an open Riemann surface. For this purpose, it is necessary to consider appropriate compactifications of Riemann surfaces. Z. Kuramochi [5] considered compactifications of Riemann surfaces with regular metrics and extended Theorem A to the case of Riemann surfaces (Theorem 1 in § 2). Our results in this paper are Theorem 2, Theorem 3 and Theorem 5 in § 2. Theorem 3 is an extention of Theorem B to the case of Riemann surfaces. We apply Kuramochi's method in [5] to prove these theorems.

2. Definitions and theorems

2.1. *Compactifications.* Let R be an open Riemann surface. We consider a metrizable compactification R^* of R such that $R^* \succ R_S^*$. Here R_S^* is the Stoilow's compactification of R and $R^* \succ R_S^*$ means that there exists a continuous mapping π of R^* onto R_S^* such that $\pi|R$ is the identity and $\pi^{-1}(R)=R$. We set $\Delta=R^*-R$. For any subset A of R^* we denote by $\bar{A}^*$ the closure of A in R^* and denote by $\mathrm{Int}^*(A)$ the interior of A in R^*. We shall use the following fact:

LEMMA 1. *Let G be a subregion of R such that the relative boundary ∂G is compact in R. If $R^* \succ R_S^*$, then $\bar{G}^* \cap \overline{R-G}^* \cap \Delta=\phi$ and so $\bar{G}^* \cap \Delta \subset$ $\mathrm{Int}^*(\bar{G}^*)$.*

PROOF. We set $\Delta_S=R_S^*-R$. Suppose $\bar{G}^S \cap \overline{R-G}^S \cap \Delta_S \neq \phi$, where the closure $-S$ is taken in R_S^*. Let $e \in \bar{G}^S \cap \overline{R-G}^S \cap \Delta_S$. We denote by $\{G_n\}$ a determinating sequence of e. Since $G_n \cap G \neq \phi$ for every n, there exists some n_0 such that $G_n \cap G=G_n$ for every $n \geqq n_0$. Then $G_n \cap (R-G)=G_n \cap G \cap (R-G)=\phi$ for every $n \geqq n_0$. This contradicts $e \in \overline{R-G}^S$. Hence $\bar{G}^S \cap \overline{R-G}^S \cap \Delta_S=\phi$. Since $\pi(\bar{A}^*)=\bar{A}^S$ for every subset A of R and $\pi(\Delta)=\Delta_S$, we have $\bar{G}^* \cap \overline{R-G}^* \cap \Delta=\phi$. Next suppose $b \in \bar{G}^* \cap \Delta$. Since $b \notin \overline{R-G}^*$, there exists a neighborhood $N(b)$ of b such that $N(b) \cap R \subset G$. Then $N(b) \subset \bar{G}^*$. Hence we have $\bar{G}^* \cap \Delta \subset \mathrm{Int}^*(\bar{G}^*)$.

Z. Kuramochi [5] defined a regular metic on R^*. By Lemma 1, we have the following: If $R^* \succ R_S^*$, then any metric which is induced by R^* is a regular metric.

2.2. *Boundary cluster set.* Let $w=f(z)$ be a meromorphic function on R which maps into the w-sphere. We denote by d a metric induced by R^*. Let b_0 be a point of Δ and let E be a subset of Δ such that $b_0 \in E$ and $b_0 \in \overline{\Delta-E}^*$. We fix a decreasing sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers and we set

$$V_n=\left\{b \in R^* ;\ d(b, b_0) \leqq r_n\right\}$$

$$U_n=V_n \cap R \quad \text{and} \quad \Gamma_n=\partial U_n .$$

We shall define six kinds of cluster sets (1)-(6) by means of the sequence $\{r_n\}_{n=1}^{\infty}$. But, except for (3), these cluster sets do not depend on the choice of such a sequence. We define the cluster set $C(f, b_0)$ of $f(z)$ at b_0 by

$$C(f, b_0)=\bigcap_{n=1}^{\infty} \overline{f(U_n)} \tag{1}$$

and define the boundary cluster set $C_{\Delta-E}(f, b_0)$ of $f(z)$ at b_0 modulo E by

(2) $$C_{\Delta-E}(f, b_0) = \bigcap_{n=1}^{\infty} \overline{M_n^{(1)}}\,, \text{ where}$$
$$M_n^{(1)} = \cup \left\{C(f, b)\,;\ b \in V_n \cap \Delta - E\right\}.$$

A subregion G of R is called an SO_{HB}-region if every HB function on G with vanishing continuous boundary values on ∂G reduces to constant zero.

THEOREM 1. (*Z. Kuramochi* [5]) *Suppose that E has the following properties*: (i) *Every subregion G of R such that $\overline{G}^* \cap \Delta \subset E$ is an SO_{HB}-region,* (ii) *$\overline{\Gamma}_n^* \cap E = \phi$ for every n. Then*

$$\Omega = C(f, b_0) - C_{\Delta-E}(f, b_0)$$

is an open set. And if $\Omega \neq \phi$, then every value of Ω is assumed infinitely often by $f(z)$ in any neighborhood of b_0 with the possible exception of a set of capacity zero.

We fix an exhaustion $\{R_n\}$ of R. For any subset F of R we set $H(f, F) = \bigcap_{n=1}^{\infty} \overline{f(F-R_n)}$. Then $H(f, F)$ does not depend the choice of an exhaustion. We define another boundary cluster set $C_{\{\Gamma_n\}}(f, b_0)$ of $f(z)$ at b_0 by

(3) $$C_{\{\Gamma_n\}}(f, b_0) = \bigcap_{n=1}^{\infty} \overline{N}_n\,, \text{ where } N_n = \bigcup_{i=n}^{\infty} H(f, \Gamma_i)\,.$$

THEOREM 2. *If E has the property* (i) *of Theorem* 1, *then*

$$\Omega_1 = C(f, b_0) - C_{\Delta-E}(f, b_0) - C_{\{\Gamma_n\}}(f, b_0)$$

is an open set. And if $\Omega_1 \neq \phi$, then every value of Ω_1 is assumed infinitely often by $f(z)$ in any neighborhood of b_0 with the possible exception of a set of capacity zero.

In Theorem 2, if E has the property (ii) of Theorem 1 besides the property (i), then $H(f, \Gamma_m) \subset M_n^{(1)}$ for every $m \geqq n$ and so $C_{\{\Gamma_n\}}(f, b_0) \subset C_{\Delta-E}(f, b_0)$. Hence we see that Theorem 2 implies Theorem 1.

2.3. *Radial boundary cluster set.* We suppose that R is a hyperbolic Riemann surface. Let $g(z, z_0)$ be the Green function of R with pole at $z_0 \in R$. We refer to Chapter III. 6 in L. Sario and M. Nakai [9] for the definition and properties of Green lines. We consider Green lines issuing from z_0. Then the set $\mathscr{L}$ of all Green lines admits the Green measure m. A Green line ℓ such that $\inf_{z \in \ell} g(z, z_0) = 0$ is called a regular Green line. Any regular Green line tends to the ideal boundary of R as $g(z, z_0) \to 0$. We denote by $\mathscr{L}_r$ the set of all regular Green lines. It is known that

$m(\mathscr{L}-\mathscr{L}_r)=0$. For any subset F of R, we denote by $\mathscr{L}(F)$ the set of all regular Green lines ℓ such that $\ell\cap F$ is not relatively compact in R, that is $\inf_{z\in\ell\cap F} g(z, z_0)=0$.

Let b_0 be a point of Δ and let $\mathcal{E}$ be a subset of $\mathscr{L}_r$ such that $\{\ell\in\mathscr{L}_r;\ b_0\in\overline{\ell^*}\}\subset\mathcal{E}$ and $\mathscr{L}(U_n)-\mathcal{E}\neq\phi$ for every n. Then we define the radial boundary cluster set $C_{\mathscr{L}-\mathcal{E}}(f, b_0)$ of $f(z)$ at b_0 modulo $\mathcal{E}$ by

$$(4)\qquad C_{\mathscr{L}-\mathcal{E}}(f, b_0)=\bigcap_{n=1}^{\infty}\overline{M_n^{(2)}},\ \text{where}$$

$$M_n^{(2)}=\cup\left\{H(f, \ell\cap U_n);\ \ell\in\mathscr{L}(U_n)-\mathcal{E}\right\}.$$

THEOREM 3. *If* $m(\mathcal{E})=0$, *then*

$$\Omega_2=C(f, b_0)-C_{\mathscr{L}-\mathcal{E}}(f, b_0)-C_{\{\Gamma_n\}}(f, b_0)$$

is an open set. And if $\Omega_2\neq\phi$, *then every value of* Ω_2 *is assumed infinitely often* $f(z)$ *in any neighborhood of* b_0 *with the possible exception of a set of capacity zero.*

2.4. *Fine boundary cluster set.* We suppose that R is a hyperbolic Riemann surface. Let R_M^* be the Martin compactification of R. We refer to § 13 and § 14 in C. Constantinescu and A. Cornea [2] for the definition and properties of R_M^*. It is known that $R_M^*\succ R_S^*$ and R_M^* is metrizable. Let $k_b(z)$ be the Martin function of R with pole at $b\in R_M^*$ and let Δ_1 be the set of all minimal boundary points of $\Delta_M=R_M^*-R$. For every $b\in\Delta_1$, we denote by $\mathfrak{G}_b$ the family of all open subset G of R such that $(k_b)_{R-G}(z)\not\equiv k_b(z)$ on R. Then we define the fine boundary cluster set $f^{\wedge}(b)$ of $f(z)$ at $b\in\Delta_1$ by

$$(5)\qquad f^{\wedge}(b)=\cap\left\{\overline{f(G)};\ G\in\mathfrak{G}_b\right\}.$$

Let b_0 be a point of Δ_M and let E be a subset of Δ_M such that $b_0\in E$ and $b_0\in\overline{\Delta_1-E}^M$ (the closure $-M$ is taken in R_M^*). Then we define the fine boundary cluster set $C^{\wedge}_{\Delta_M-E}(f, b_0)$ of $f(z)$ at b_0 modulo E by

$$(6)\qquad C^{\wedge}_{\Delta_M-E}(f, b_0)=\bigcap_{n=1}^{\infty}\overline{M_n^{(3)}},\ \text{where}$$

$$M_n^{(3)}=\cup\left\{f^{\wedge}(b);\ b\in V_n\cap\Delta_1-E\right\}.$$

THEOREM 4. (*T. Fuji'i'e* [3]) *Suppose that* b_0 *is a minimal and regular* (*with respect to Dirichlet problem*) *point of* Δ_M *and that* E *is a set of the harmonic measure zero on* Δ_M. *Then*

$$\Omega'=C(f, b_0)-C^{\wedge}_{\Delta_M-E}(f, b_0)$$

is an open set. And if $\Omega' \neq \phi$, then every value of Ω' is assumed infinitely often by $f(z)$ in any neighborhood of b_0 with the possible exception of a set of capacity zero.

In the next theorem, we suppose neither the minimality nor the regularity of b_0.

THEOREM 5. *Let E be a set of harmonic measure zero on Δ_M. Then*

$$\Omega_3 = C(f, b_0) - C^{\frown}_{\Delta_M - E}(f, b_0) - C_{\{\Gamma_n\}}(f, b_0)$$

is an open set. If $\Omega_3 \neq \phi$, then every value of Ω_3 is assumed infinitely often by $f(z)$ in any neighborhood of b_0 with the possible exception of a set of capacity zero.

REMARK. Theorem 5 does not always imply Theorem 4.

3. SO_{HB}-region

We state properties of SO_{HB}-regions which need to prove the theorems in § 2. Let $w=f(z)$ be a moromorphic function on R. For any subset A of R and any point w of the w-sphere, we denote by $n(w, f|A)$ the number of points in $f^{-1}(w) \cap A$ with the multiple points counted repeatedly. For any subset B of the w-sphere we set $n_B(f|A) = \sup\{n(w, f|A); w \in B\}$. Let G' be an open disk in the w-sphere and let G be a connected component of $f^{-1}(G')$. If G is an SO_{HB}-region, the mapping $f|G: G \to G'$ is of type Bl. Hence the next lemma follows from Heins' theorem (cf. Satz 10. 5 in [2]).

LEMMA 2. *If G is an SO_{HB}-region, then $n(w, f|G) = n_{G'}(f|G)$ for every $w \in G'$ except for a set of capacity zero.*

We suppose that R is a hyperbolic Riemann surface. We denote by $\underline{m}$ the inner measure induced by the Green measure m. Then we have the following:

LEMMA 3. *Let G be a subregion of R. If $\underline{m}(\mathscr{L}(G))=0$, then G is an SO_{HB}-region.*

PROOF. Let R_W^* be the Wiener compactification of R, let Γ_W be the harmonic boundary of $\Delta_W = R_W^* - R$ and let μ_z^W be the harmonic measure on Δ_W with respect to $z \in R$. Suppose $G \notin SO_{HB}$. Then we have $(\overline{G}^W - \overline{\partial G}^W) \cap \Gamma_W \neq \phi$, where the closure $-W$ is taken in R_W^* (Satz 9. 12 in [2]). Since $\overline{G}^W - \overline{\partial G}^W$ is an open set in R_W^* (Satz 9. 9 in [2]), there exists an open neighborhood $N(\xi)$ of a point ξ of Γ_W such that $N(\xi) \subset \overline{G}^W - \overline{\partial G}^W$. We set $N_1 = N(\xi) \cap \Delta_W$ and $\tilde{N}_1 = \{\ell \in \mathscr{L}_r; \overline{\ell}^W \cap N_1 \neq \phi\}$. Then we have $\mu_{z_0}^W(N_1) \leqq \underline{m}(\tilde{N}_1)$ (Theorem 1 in Y. Nagasaka [6]). Since the support of $\mu_{z_0}^W$ equals Γ_W we have $\mu_{z_0}^W(N_1) > 0$ and so $\underline{m}(\tilde{N}_1) > 0$. Suppose $\tilde{N}_1 - \mathscr{L}(G) \neq \phi$. Let $\ell \in \tilde{N}_1 -$

$\mathscr{L}(G)$. By $\ell \notin \mathscr{L}(G)$, there exists some n_0 such that $\ell \cap G \subset R_{n_0}$. And by $\ell \in \tilde{N}_1$, there exists a neighborhood $N(\xi_1)$ of a point ξ_1 of N_1 such that $N(\xi_1) \cap R_{n_0+1} = \phi$, $N(\xi_1) \subset N(\xi)$ and $N(\xi_1) \cap R \cap \ell \neq \phi$. But, since $N(\xi) \cap R \subset G$, we have

$$N(\xi_1) \cap R \cap \ell \subset (R - R_{n_0+1}) \cap G \cap \ell \subset (R - R_{n_0+1}) \cap R_{n_0} = \phi .$$

This is a contradiction. Hence we have $\tilde{N}_1 \subset \mathscr{L}(G)$ and so $\underline{m}(\mathscr{L}(G)) > 0$. This complete the proof.

We suppose that R is a hyperbolic Riemann surface. Let μ^M be the harmonic measure on Δ_M. We set $\Delta_1(G) = \{b \in \Delta_1 ; G \in \mathfrak{G}_b\}$. It is known that $\Delta_1(G)$ is an F_σ-set. Let G be a subregion of R. We denote by w_n the bounded harmonic function in $G \cap R_n$ which takes the boundary values 0 on $\partial G \cap R_n$ and 1 on $G \cap \partial R_n$. Then w_n decreases to a harmonic function w^G in G. Then we have the following equality,

$$w^G(z) = 1 - 1_{R-G}(z) = \int \left\{ k_b(z) - (k_b)_{R-G}(z) \right\} d\chi(b) ,$$

where χ is the canonical measure of 1. By Brelot's theorem (cf. Satz 13. 4 in [2]), $\mu^M(E) = 0$ is equivalent to $\chi(E) = 0$ for every Borel set E of Δ^M. Hence we have the next lemma.

LEMMA 4. *A subregion G of R is an SO_{HB}-region if and only if* $\mu^M(\Delta_1(G)) = 0$.

4. The proof of theorems

In § 4, we give at the same time the proofs of Theorem 2, Theorem 3 and Theorem 5 by the same method as Z. Kuramochi used to prove Theorem 1.

Let α be an arbitrary point of Ω_i $(i = 1, 2, 3)$. Since the boundary cluster sets (2), (3), (4) and (6) are closed sets, we have only to show $\alpha \in \mathrm{Int}(C(f, b_0))$. Since $\alpha \notin \overline{M^{(i)}_{n_0}} \cup \overline{N}_{n_0}$ for some n_0, there exists a disk $D(\alpha, t_1) = \{w ; |w - \alpha| < t_1\}$ such that $D(\alpha, t_1) \cap (\overline{M^{(i)}_{n_0}} \cup \overline{N}_{n_0}) = \phi$. By $\alpha \notin \overline{N}_{n_0}$, $\alpha \notin \overline{f(\Gamma_{n_0} - R_k)}$ for some k. Then the number of points in $f^{-1}(\alpha) \cap \Gamma_{n_0}$ is finite. Hence, by a slight deformation of V_{n_0}, we can find a neighborhood V'_{n_0} of b_0 such that $V'_{n_0} - K = V_{n_0} - K$ for some compact set K in R and that $\alpha \notin \overline{f(\partial(V'_{n_0} \cap R))}$. Set $U'_{n_0} = V'_{n_0} \cap R$ and $\Gamma'_{n_0} = \partial U'_{n_0}$. Here we note $H(f, \Gamma'_{n_0}) = H(f, \Gamma_{n_0})$, $\mathscr{L}(U'_{n_0}) = \mathscr{L}(U_{n_0})$ and $H(f, \ell \cap U'_{n_0}) = H(f, \ell \cap U_{n_0})$ for every $\ell \in \mathscr{L}(U'_{n_0})$. Since $\alpha \notin \overline{f(\Gamma'_{n_0})}$, $D(\alpha, t_2) \cap f(\Gamma'_{n_0}) = \phi$ for some $t_2 > 0$. We fix a $t_0 : 0 < t_0 < \min(t_1, t_2)$ and write $D_0 = D(\alpha, t_0)$ for simplicity. By $\alpha \in C(f, b_0)$, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in U'_{n_0} such that $f(z_n) \in D_0$ for every n and

$\lim_{n\to\infty} f(z_n)=\alpha$. We fix this sequence. For every n there exists a connected component G_n of $f^{-1}(D_0)$ which contains the point z_n of $\{z_n\}_{n=1}^{\infty}$. Since $\bar{D}_0\cap f(\Gamma'_{n_0})=\phi$, we have $G_n\subset U'_{n_0}-\Gamma'_{n_0}$. Then G_n may coincide with other G_m. We shall show that every G_n is an SO_{HB}-region and that $b_0\notin \mathrm{Int}^*(\bar{G}_n^*)$ in the case of $\alpha\in\Omega_i$ $(i=1, 2)$ and $b_0\notin \mathrm{Int}^M(\bar{G}_n^M)$ in the case of $\alpha\in\Omega_3$ respectively.

(i) The case of $\alpha\in\Omega_1$. Suppose $\bar{G}_n^*\cap\Delta-E\neq\phi$. Let $a\in\bar{G}_n^*\cap\Delta-E$. Since $\overline{f(G_n)}\subset\bar{D}_0$, $C(f, a)\cap\bar{D}_0\neq\phi$. But, since $\bar{G}_n^*\subset V'_{n_0}$, $a\in V'_{n_0}\cap\Delta-E$ and so $C(f, a)\subset M_{n_0}^{(1)}\subset D(\alpha, r_1)^c$. This is a contradiction. Hence we have $\bar{G}_n^*\cap\Delta\subset E$. Therefore, by the property of E of Theorem 2, we see that G_n is an SO_{HB}-region. And since $b_0\in\overline{\Delta-E^*}$, we see $b_0\notin\mathrm{Int}^*(\bar{G}_n^*)$.

(ii) The case of $\alpha\in\Omega_2$. Suppose $\mathscr{L}(G_n)-\mathcal{E}\neq\phi$. Let $\ell\in\mathscr{L}(G_n)-\mathcal{E}$. Since $G_n\subset U'_{n_0}$, we have $\ell\in\mathscr{L}(U'_{n_0})-\mathcal{E}$ and so

$$\begin{aligned}\phi\neq H(f, \ell\cap G_n)&\subset H(f, \ell\cap U'_{n_0})\\ &=H(f, \ell\cap U_{n_0})\subset M_{n_0}^{(2)}\subset D(\alpha, r_1)^c.\end{aligned}$$

But this contradicts $H(f, \ell\cap G_n)\subset\bar{D}_0$. Hence we have $\mathscr{L}(G_n)\subset\mathcal{E}$. Then, by the assumption of Theorem 3, we have $m(\mathscr{L}(G_n))=0$. Therefore we see that G_n is an SO_{HB}-region by Lemma 3. If $b_0\in\mathrm{Int}^*(\bar{G}_n^*)$, $U_k\subset G_n$ for some $k>n_0$ and so $\mathscr{L}(U_k)-\mathcal{E}\subset\mathscr{L}(G_n)\subset\mathcal{E}$. This contradicts $\mathscr{L}(U_k)-\mathcal{E}\neq\phi$. Hence we have $b_0\notin\mathrm{Int}^*(\bar{G}_n^*)$.

(iii) The case of $\alpha\in\Omega_3$. Suppose $\Delta_1(G_n)-E\neq\phi$. Let $a\in\Delta_1(G_n)-E$. Since $\Delta_1(G_n)\subset\bar{G}_n^M\subset V'_{n_0}$, we have $a\in V'_{n_0}\cap\Delta_1-E$ and so $f^{\wedge}(a)\subset M_{n_0}^{(3)}\subset D(\alpha, r_1)^c$. But this contradicts $f^{\wedge}(a)\subset\overline{f(G_n)}\subset\bar{D}_0$. Hence we have $\Delta_1(G_n)\subset E$. Then, by the assumption of Theorem 5, we have $\mu^M(\Delta_1(G_n))=0$. Hence we see that G_n is an SO_{HB}-region by Lemma 4. If $b_0\in\mathrm{Int}^M(\bar{G}_n^M)$, $U_k\subset G_n$ for some $k>n_0$ and so $V_{k+1}\cap\Delta_1\subset\Delta_1(G_n)\subset E$. This contradicts $b_0\in\overline{\Delta_1-E^M}$. Hence we have $b_0\notin\mathrm{Int}^M(\bar{G}_n^M)$.

Therefore we have $n(w, f|G_n)=n_{D_0}(f|G_n)$ for all $w\in D_0$ except for a set of capacity zero by Lemma 2 and in particular $\overline{f(G_n)}=\bar{D}_0$.

First we treat the case where there is an infinite number of distinct components G_n. In this case, for simplicity, we suppose $G_n\cap G_m=\phi$ if $n\neq m$. If the number of n such that $G_n\cap\Gamma_k\neq\phi$ for some $k(>n_0)$ is infinite, the level curves $|f(z)-\alpha|=r_0$ clusters at a point of $\bar{\Gamma}_k^*\cap\Delta$ (or $\bar{\Gamma}_k^M\cap\Delta_M$) and so $H(f, \Gamma_k)\cap\partial D_0\neq\phi$. But this contradicts $H(f, \Gamma_k)\subset N_{n_0}\subset D(\alpha, r_1)^c$. Hence the number is finite and so G_n converges to b_0. This shows that $C(f, b_0)\supset\bigcap_{n=1}^{\infty}\overline{f(G_n)}=\bar{D}_0$ and so $b_0\in\mathrm{Int}(C(f, b_0))$ and that every value of D_0 is assumed infinitely often by $f(z)$ in any neighborhood of b_0 except for a set of capacity zero.

Accordingly, to prove the theorems, it suffices to consider the only case where the number of components of $f^{-1}(D(\alpha, r))$ containing at least one point of $\{z_n\}_{n=1}^{\infty}$ is finite for every $0<r\leqq r_0$. Then there exists at least a component G_0 of $f^{-1}(D_0)$ containing a subsequence of $\{z_n\}_{n=1}^{\infty}$. We set $N_0=n_{D_0}(f|G_0)$ for simplicity. We shall show $N_0=\infty$. Suppose $N_0<\infty$. Then the set $\{w\in D_0\,;\, n(w, f|G_0)\leqq N_0-1\}$ is closed relative to D_0. By Lemma 2, this set is of capacity zero. Since a compact set of capacity zero is totally disconnected, there exists a number $0<r'<r_0$ such that $n(w, f|G_0)=N_0$ for every $w\in\partial D(\alpha, r')$. Then there exists a component $G_0'(\subset G_0)$ of $f^{-1}(D(\alpha, r'))$ which contains a subsequence of $\{z_n\}_{n=1}^{\infty}$. Then, by routine method, we see that $\partial G_0'$ is compact in R. Since $b_0\in\overline{G}_0'^*\cap\Delta$ (or $b_0\in\overline{G}_0'^M\cap\Delta_M$), we have $b_0\in\mathrm{Int}^*(\overline{G}_0'^*)$ (or $b_0\in\mathrm{Int}^M(\overline{G}_0'^M)$) by Lemma 1. But this is a contradiction. Hence we have $N_0=\infty$. Finally we shall show $\bigcap_{n=n_0+1}^{\infty}\overline{f(U_n\cap G_0)}\supset\overline{D}_0$. Take an arbitrary number $n\geqq n_0+1$. If $\Gamma_n\cap G_0$ is not relatively compact in R, then $H(f, \Gamma_n)\cap\overline{D}_0\neq\phi$. But this contradicts $H(f, \Gamma_n)\subset N_{n_0}\subset D(\alpha, r_1)^c$. Therefore $\Gamma_n\cap G_0$ is compact and so $\Gamma_n\cap G_0\subset R_k$ for some k. Hence, by a slight deformation of V_n, we can find a neighborhood V_n' of b_0 which satisfies the following conditions. (i) $V_n'\subset V_n$, (ii) $V_n'-R_k=V_n-R_k$ for the above k, (iii) $\alpha\notin f(\partial(V_n'\cap R)\cap G_0)$, (iv) $\partial(V_n'\cap R)\cap G_0$ consists of a finite number of analytic curves $\gamma_1, \cdots, \gamma_l$ such that $D_0-f(\bigcup_{i=1}^{l}\gamma_i)$ is composed of a finite number of components $D_1, \cdots, D_m$. We set $U_n'=V_n'\cap R-\partial(V_n'\cap R)$ and $N_i=\sup\{n(w, f|U_n'\cap G_0),\ w\in D_i\}$. Then, by Lemma 2, we have $n(w, f|U_n'\cap G_0)=N_i$ for all $w\in D_i$ except for a set of capacity zero. Since the number of points in $f^{-1}(w)\cap U_n'\cap G_0$ jumps only a finite number when w crosses $f(\gamma_i)$, we have the following: If $N_i<\infty$ for some i, then $N_j<\infty$ for every j such that ∂D_i adjoins∂D_j and so $N_j<\infty$ for all $j=1, \cdots, m$. Now let D_{i_0} be a component containing α in $\{D_i\}_{i=1}^{m}$. Take a disk $D(\alpha, r)\subset D_{i_0}$. Since $n_{D(\alpha,r)}(f|G_\alpha)=\infty$ for some component $G_\alpha(\subset U_n'\cap G_0)$ of $f^{-1}(D(\alpha, r))$ containing a subsequence of $\{z_n\}_{n=1}^{\infty}$, we see $N_{i_0}=\infty$. Hence we see $N_1=\cdots=N_m=\infty$. Therefore we see that $\overline{f(U_n)}\supset\overline{f(U_n'\cap G_0)}\supset\overline{D}_0$ and so $C(f, b_0)\supset\overline{D}_0$ and that every value of D_0 is assumed infinitely often by $f(z)$ in any neighborhood of b_0 except for a set of capacity zero. This completes the proofs.

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