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On well posedness of mixed problems for Maxwell's equations

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§ 0. Introduction and results

This paper is concerned with a class of characteristic mixed problems for a strongly hyperbolic system P:

$$(P, B)\begin{cases} Pu=f & \text{in} \quad (0, \infty)\times G, \\ Bu=0 & \text{on} \quad (0, \infty)\times\partial G, \\ u(0, x)=0 & \text{on} \quad G, \end{cases}$$

where

$$P(t, x; D_t, D_x)=D_t+\sum_{j=1}^{n} A_j(t, x) D_j+C(t, x),$$

A_j and C are $m\times m$ matrices, $D_t=-i\partial/\partial t$, $D_j=-i\partial/\partial x_j$ $(j=1, \cdots, n)$, G is an open set of R^n with boundary ∂G and all coefficients of operators and ∂G considered here are assumed to be of the class C^∞. Furthermore the boundary matrix $A_\nu(t, x)=\sum_{j=1}^{n}\nu_j(x) A_j(t, x)$ is assumed to be of constant rank d less than m near $R^1\times\partial G$, where $\nu=(\nu_1, \cdots, \nu_n)$ is the inner unit normal to ∂G. The $B(t, x)$ is an $l\times m$ matrix of constant rank l, where the number l will be determined later (see Lemma 2.7 below).

In treating such kind of mixed problems it is natural to assume the condition that the problem (P, B) is *reflexive*, i. e.,

$$\ker A_\nu(t, x)\subset\ker B(t, x) \qquad \text{for all } x\in\partial G.$$

In fact Lax and Phillips [10] proved that a solution of $Pu=f$ with the boundary condition $Bu=0$ in a weak sense is a solution of the problem in a semi-strong sense provided the problem (P, B) is reflexive. They also showed that, when P is symmetric, the reflexiveness of (P, B) is necessary for the kernel of B to be maximally non-positive for the matrix A_ν. Furthermore Rauch [14] showed that if a mixed problem of constant coefficients in the quarter space $t>0$, $x_1>0$ has a unique strong solution and if it is well posed in the sense of Hersh, then the problem is reflexive.

In the present article, as our first approach to characteristic mixed problems, we shall deal with a reflexive problem (P, B) for Maxwell's equations which is one of typical those. In particular we shall try to clarify the situation occupied by problems with maximally non-positive boundary conditions within all L^2-well posed reflexive ones, on the basis of works of Hersh [5], Agemi and Shirota [2], Sakamoto [15], Agemi [1], Ohkubo and Shirota [12]. (For the definition of the L^2-well posedness see § 1). In doing so the rank l of B may be assumed to equal 2, since the matrix A_ν has exactly two positive eigenvalues. (See Lemma 2.7 below). Then for the Maxwell system P we obtain the following

THEOREM. *Let $G=\{x\in R^3;\ x_1>0\}$. Suppose that B is a constant 2×6 matrix and the problem (P, B) is reflexive.*

(a) *If B is real and (P, B) satisfies Hersh's condition, then the kernel of B is maximally non-positive for the matrix A_1, i.e.,*

$$A_1 u\cdot u \leqq 0 \qquad \textit{for all } u\in(\ker B)\backslash 0 \tag{0.1}$$

and ker B is a maximal subspace obeying the above property (hence (P, B) is L^2-well posed). Here $v\cdot u$ denotes the inner product of v and u in C^6.

(b) *For some non-real matrix B, the problem (P, B) is L^2-well posed, but the kernel of B is not maximally non-positive for A_1 even after any nonsingular transformation of the dependent variables which keeps the system P symmetric.*

In proving Theorem we will transform the system of equations into the form (3.1) below. Then it will be shown that Hersh's condition implies $\|S\|\leqq 1$ if B is real. Here S is the 2×2 matrix in (3.5) below and $\|S\|$ denotes the matrix norm:

$$\sup_{0\neq v\in C^2} |Sv|/|v|\,,$$

where $|v|$ is the usual hermitian norm in C^2.

Although the boundary condition prescribed by (3.5) is not intrinstic (see Majda and Osher [11] for an intrinstic representation of the reflexive boundary condition), we can also prove the following

COROLLARY. *Let G be an open set of R^3 with compact boundary and let $B(t, x)$ be a real 2×6 matrix-valued function. Suppose that the problem (P, B) is reflexive and for every $(t, x)\in R^1\times\partial G$ the frozen problem $(P, B)_{(t,x)}$ in the quarter space with the inner normal $(0, \nu(x))$ on its lateral boundary plane is L^2-well posed. Then (P, B) is also L^2-well posed. More precisely the kernel of $B(t, x)$ is maximally non-positive for the matrix $A_\nu(t, x)$ at each $(t, x)\in R^1\times\partial G$.*

This fact is a direct consequence of Theorem, because of the rotational invariance of the curl operator. In the point of view stated in Corollary certain results have been already obtained by Sato and Shirota [16], Kubota [9] and others.

The assertion (*b*) of Theorem contrasts with the result of Strang [18] for 2×2 systems. For a 3×3 system P, Rauch [13] has already given such an example as ours, where P is strictly hyperbolic but not symmetrizable and which satisfies Kreiss' condition (i. e., the uniform Lopatinskii condition). Nevertheless for the Maxwell system P it can be shown that if a mixed problem (P, B) satisfies the first three hypotheses of Theorem and Kreiss' condition, then the strict inequality in (0. 1) is valid after such a transformation as described there. The proof of this fact will be given elsewhere. (For the case of real boundary conditions see Remark 3. 7 below).

In section 1 we describe assumptions on our problem with terminologies used here and in section 2 we give some preliminary lemmas. In section 3 we prove part (*a*) of Theorem and finally in section 4 we give such an example as described in part (*b*).

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§ 1. Notations and assumptions

In order to develope a general theory for a mixed problem (P, B) used later we shall describe precisely our assumptions on the operators P and B. The coefficients of P are complex $m\times m$ matrix-valued functions defined on $\bar{\Omega}$ and are constant outside a compact subset of $\bar{\Omega}$, where $\Omega=R^1\times G$ and, unless indicated otherwise, G is assumed to be the open half space $\{x=(x_1, x'); x_1>0, x'=(x_2, \cdots, x_n)\in R^{n-1}\}$ $(n\geqq 2)$. The principal symbol P^0 of P is assumed to satisfy the inequality

$$(1.1)\qquad \|(P^0)^{-1}(t, x; \tau, \lambda, \sigma)\|\leqq C|\mathrm{Im}\,\tau|^{-1}$$

for all $(t, x)\in\bar{\Omega}$, $\tau\in\boldsymbol{C}_-$, $\lambda\in R^1$ and $\sigma=(\sigma_2, \cdots, \sigma_n)\in R^{n-1}$, where τ, λ and σ are the covariables of t, x_1 and x' respectively, $\boldsymbol{C}_-$ is the lower open half of the complex plane and C is a positive constant independent of t, x, τ, λ and σ. For a square matrix Q we denote by $\|Q\|$ such matrix norm as defined in the preceding section. Moreover the boundary matrix $A_1(t, x)$ is assumed to be of constant rank $d(<m)$ near the boundary $\partial\Omega$ of Ω so that, without loss of generality, it may be regarded as the block diagonal form

$$(1.2)\qquad A_1=\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \qquad A=\begin{bmatrix} A^+ & 0 \\ 0 & A^- \end{bmatrix} \qquad \text{near } \partial\Omega,$$

where $A^+(A^-)$ is a positive (negative) definite $d^+\times d^+(d^-\times d^-)$ matrix respectively and $d^++d^-=d$.

The $B(t, x)$ is a complex $l\times m$ matrix-valued function defined on $\partial\Omega$ which is of rank l and is constant outside a compact subset of $\partial\Omega$. Furthermore the problem (P, B) is assumed to be reflexive, i. e., for every $(t, x)\in\partial\Omega$ let $B(t, x)$ be of the form

(1. 3) $$B(t, x)=[B_{\mathrm{I}}(t, x), 0],$$

where B_{I} is an $l\times d$ matrix of rank l (see Lemma 2. 1 below). Then a vector $u\in C^m$ will be often written as

$$u=\begin{bmatrix}u_{\mathrm{I}}\\ u_{\mathrm{II}}\end{bmatrix},$$

where u_{I} and u_{II} denote the projections of u on the range of A_1 and the kernel of A_1 respectively. This leads to Green's formula as follows:

(1. 4) $$\begin{aligned}(Pu, v)_{L^2(\Omega)}-(u, P^*v)_{L^2(\Omega)} &= i(A_1u, v)_{L^2(\partial\Omega)}\\ &= i(Au_{\mathrm{I}}, v_{\mathrm{I}})_{L^2(\partial\Omega)},\end{aligned}$$

where P^* is the formal adjoint of P, and the boundary condition $Bu=0$ becomes $B_{\mathrm{I}}u_{\mathrm{I}}=0$.

We also use the following function spaces. For $u\in C_0^\infty(\partial\Omega)$ we define its norm $|u|_{q,\gamma}$ (q, γ are real numbers and $\gamma\neq 0$) by

$$|u|_{q,\gamma}=\|e^{-\gamma t}(\Lambda_\gamma^q u)(t, x')\|_{L^2(\partial\Omega)},$$

where

$$(\Lambda_\gamma^q u)(t, x')=e^{\gamma t}(\gamma^2+D_t^2+\sum_{j=2}^n D_j^2)^{q/2}(e^{-\gamma t}u(t, x')),$$

and by $H_{q,\gamma}(\partial\Omega)$ we denote the completion of $C_0^\infty(\partial\Omega)$ with respect to this norm. By $H_{p,q,\gamma}(\Omega)$ ($p=0, 1, 2, \cdots$) we denote the completion of $C_0^\infty(\bar{\Omega})$ with respect to the norm

$$\|u\|_{p,q,\gamma}=\left[\sum_{j=0}^p\int_0^\infty |(D_1^j u)(\cdot, x_1, \cdot)|^2_{p+q-j,\gamma}dx_1\right]^{1/2}.$$

Moreover we often use the abbreviations $H_{p,\gamma}(\Omega)$ and $\|u\|_{p,\gamma}$ for $H_{p,0,\gamma}(\Omega)$ and $\|u\|_{p,0,\gamma}$ respectively. (See [6] or [8]).

Now let $f\in H_{0,\gamma}(\Omega)$ and $\gamma>0$. Then a function $u\in H_{0,\gamma}(\Omega)$ is said to be a solution of $Pu=f$ with the homogeneous boundary condition $Bu=0$ in a weak sense if the equality

(1. 5) $$(f, v)_{L^2(\Omega)}-(u, P^*v)_{L^2(\Omega)}=0$$

is valid for all $v(t, x)\in C_0^\infty(\bar{\Omega})$ satisfying the homogeneous adjoint boundary condition $v(t, x)\in N'(t, x)$ at each boundary point $(t, x)\in\partial\Omega$. Here, for every $(t, x)\in\partial\Omega$, $N'(t, x)$ stands for the orthogonal complement of $A_1(t, x)N(t, x)$ in C^m, where $N(t, x)=\ker B(t, x)$. Moreover the problem (P, B) is said to be *L^2-well posed* if for every $\gamma\geqq\gamma_0$ and every $f\in H_{0,\gamma}(\Omega)$ there exists uniquely such a solution u as described above and the inequality

$$\|u\|_{0,\gamma}\leqq C_0\gamma^{-1}\|f\|_{0,\gamma}. \tag{1.6}$$

is valid, where γ_0 and C_0 are positive constants independent of f, u and γ. (See Lemma 2.4).

§ 2. Preliminaries concerning reflexive mixed problems

In this section we shall show that the reflexiveness of the problem (P, B) allows us a treatment similar to the one in the case of noncharacteristic boundary. (See particularly Lemmas 2.6, 2.7 and 2.9 below).

We shall first state the following

LEMMA 2.1. *For every $(t, x)\in\partial\Omega$ the assumption* (1.3) *is equivalent to each of the following conditions*:

$$\operatorname{codim} N'(t, x) = \operatorname{rank} A_1(t, x) - \operatorname{rank} B(t, x),$$

$$N''(t, x) = N(t, x),$$

where $N''(t, x)$ is the orthogonal complement of $A_1^*(t, x)N'(t, x)$ in C^m.

PROOF. For simplicity we shall omit the variables t, x in the following matrix and subspaces. We first remark that in general

$$\dim(A_1N) = \dim N - \dim(N\cap\ker A_1), \tag{2.1}$$

$$\dim(A_1^*N') = \dim N' - \dim(N'\cap\ker A_1^*). \tag{2.2}$$

Now (1.3) is equivalent to

$$\dim(N\cap\ker A_1) = \dim\ker A_1.$$

Furthermore, from the definition of N', the first equality in this lemma is written as

$$\dim(A_1N) = \dim N - \dim\ker A_1,$$

which is equivalent to (1.3) according to (2.1).

Next, since $N\subset N''$, to prove the equivalence of the second equality in this lemma and (1.3) we need merely to show

$$\dim N'' - \dim N = \dim\ker A_1 - \dim(N\cap\ker A_1). \tag{2.3}$$

Now we have

$$\dim(N'\cap\ker A_{\mathrm{I}}^*)=\dim\ker A_1,$$

since $\ker A_{\mathrm{I}}^*\subset N'$. Therefore (2. 2) is written as

$$\dim N''=\operatorname{codim} N'+\dim\ker A_1.$$

From this and (2. 1) we obtain (2. 3), since $\operatorname{codim} N'=\dim(A_1N)$.

LEMMA 2. 2. *Let (P, B) be L^2-well posed. Then for every $\gamma\geqq\gamma_0$ we have*

(2. 4) $\|u\|_{0,\gamma}\leqq C_0\gamma^{-1}\|Pu\|_{0,\gamma}$ *for* $u\in C_0^\infty(\bar{\Omega})$ *with* $u(t, x)\in N(t, x)$ *on* $\partial\Omega$

and

(2. 5) $\|v\|_{0,-\gamma}\leqq C_0\gamma^{-1}\|P^*v\|_{0,-\gamma}$ *for* $v\in C_0^\infty(\bar{\Omega})$ *with* $v(t, x)\in N'(t, x)$ *on* $\partial\Omega$,

where γ_0 and C_0 are the same constants as in (1. 6).

PROOF. The inequality (2. 4) follows immediately from (1. 4) and (1. 6). To prove (2. 5) we note that

$$\|v\|_{0,-\gamma}=\sup_{0\neq f\in H_{0,\gamma}(\Omega)}|(f, v)_{L^2(\Omega)}|/\|f\|_{0,\gamma}. \tag{2. 6}$$

Let $f\in H_{0,\gamma}(\Omega)$ and let $u\in H_{0,\gamma}(\Omega)$ be a solution of $Pu=f$ with the boundary condition $Bu=0$ in the weak sence. Then we obtain the desired inequality (2. 5) from (1. 5), (1. 6) and (2. 6).

To derive a priori estimates for derivatives of arbitrary order it is convenient to deal with inhomogeneous boundary conditions.

Let $\{b_1'(t, x), \cdots, b_{m-l}'(t, x)\}$ be an arbitrary smooth basis of $N(t, x)$. Then the adjoint boundary condition $v(t, x)\in N'(t, x)$ is equivalent to

$$[b_1'(t, x), \cdots, b_{m-l}'(t, x)]^*A_{\mathrm{I}}^*(t, x)\,v(t, x)=0.$$

Furthermore according to (1. 3) we can take $b_1'\cdots, b_{m-l}'$ so that

$$[b_1', \cdots, b_{m-l}']^*=\begin{bmatrix} B_{\mathrm{I}}' & 0\\ 0 & I_{m-d}\end{bmatrix},$$

where I_{m-d} is the identity matrix of order $m-d$ and B_{I}' is a $(d-l)\times d$ matrix of rank $d-l$ such that

$$B_{\mathrm{I}}(B_{\mathrm{I}}')^*=0. \tag{2. 7}$$

Now let us define a nonsingular $d\times d$ matrix H by

$$H=\begin{bmatrix} B_{\mathrm{I}}\\ B_{\mathrm{I}}'\end{bmatrix}^*. \tag{2. 8}$$

Then (2. 7) implies

$$B_{\mathrm{I}}=(B_{\mathrm{I}}B_{\mathrm{I}}^{*})\,[I_l, 0]\,H^{-1}. \tag{2.9}$$

Moreover, from (1. 2), the adjoint boundary condition $v(t, x)\in N'(t, x)$ becomes

$$(B_{\mathrm{I}}'A^{*})\,(t, x)\,v_{\mathrm{I}}(t, x)=0.$$

Thus in the same way as in the proof of Lemma 3. 3 in [8] we obtain the following

LEMMA 2. 3. *Let the conclusion of Lemma 2. 2 be valid. Then for every real number q there are positive constants C_q and γ_q such that for $\gamma\geqq\gamma_q$ we have*

$$\|u\|_{0,q,\gamma}\leqq C_q\gamma^{-1}(\|Pu\|_{0,q,\gamma}+|B_{\mathrm{I}}u_{\mathrm{I}}|_{1/2+q,\gamma}) \tag{2.10}$$

for $u\in H_{0,q+1,\gamma}(\Omega)$ with $A_1u\in H_{1,q,\gamma}(\Omega)$, and

$$\|v\|_{0,q,-\gamma}\leqq C_q\gamma^{-1}(\|P^{*}v\|_{0,q,-\gamma}+|B_{\mathrm{I}}'A^{*}v_{\mathrm{I}}|_{1/2+q,-\gamma}) \tag{2.11}$$

for $v\in H_{0,q+1,-\gamma}(\Omega)$ with $A_{\mathrm{I}}^{*}v\in H_{1,q,-\gamma}(\Omega)$.

Using Lemmas 2. 2, 2. 3, (2. 7) and (2. 9) we can now prove the following

LEMMA 2. 4. *Let (P, B) be L^2-well posed and let $q\geqq 0$, $\gamma\geqq\gamma_q>0$. Then for every $f\in H_{0,q,\gamma}(\Omega)$ and $g\in H_{1/2+q,\gamma}(\partial\Omega)\cap\operatorname{range} B_{\mathrm{I}}$ there exists a unique solution $u\in H_{0,q,\gamma}(\Omega)$ with $A_1u\in H_{1,q-1,\gamma}(\Omega)$ of $Pu=f$ satisfying*

$$B_{\mathrm{I}}u_{\mathrm{I}}|_{\partial\Omega}=g \qquad \text{in } \mathscr{D}'(\partial\Omega)$$

and

$$\|u\|_{0,q,\gamma}\leqq C_q\gamma^{-1}(\|f\|_{0,q,\gamma}+|g|_{1/2+q,\gamma}).$$

Here γ_q and C_q are positive constants independent of f, g and γ. Moreover we have

$$u(t, x)=0 \qquad \text{for} \quad t<0$$

if $f(t, x)=0$ for $t<0$ and $g(t, x)=0$ for $t<0$.

The proof may be also carried out in the same way as in §§ 4 and 5 of [8].

LEMMA 2. 5. *Put*

$$\tau+\sum_{j=2}^{n}A_j(t, x)\,\sigma_j=\begin{bmatrix}A_{\mathrm{I\,I}}(t, x;\tau, \sigma) & A_{\mathrm{I\,II}}(t, x;\sigma)\\ A_{\mathrm{II\,I}}(t, x;\sigma) & A_{\mathrm{II\,II}}(t, x;\tau, \sigma)\end{bmatrix}, \tag{2.12}$$

where $A_{\mathrm{I\,I}}$ is a $d\times d$ matrix and $A_{\mathrm{II\,II}}$ a $(m-d)\times(m-d)$ matrix. Then we have

$$\|A_{\mathrm{II\,II}}^{-1}(t, x;\tau, \sigma)\|\leqq C|\operatorname{Im}\tau|^{-1} \tag{2.13}$$

for (t, x) *near* $\partial\Omega$, $\tau\in\boldsymbol{C}_-$ *and* $\sigma\in R^{n-1}$, *where* C *is a positive constant independent of* $(t, x; \tau, \sigma)$.

PROOF. Notice that, under (1.2), P^0 is written as

$$P^0(t, x; \tau, \lambda, \sigma)=\begin{bmatrix} A\lambda+A_{\mathrm{I\,I}} & A_{\mathrm{I\,II}} \\ A_{\mathrm{II\,I}} & A_{\mathrm{II\,II}} \end{bmatrix}. \tag{2.14}$$

For simplicity let us omit the variables t, x in the following matrices. Let $P_{ij}(\tau, \lambda, \sigma)$ and $Q_{ij}(\tau, \sigma)$ $(i, j=1, \cdots, m-d)$ be the $(d+i,\ d+j)$ cofactor of $P^0(\tau, \lambda, \sigma)$ and the (i, j) cofactor of $A_{\mathrm{II\,II}}(\tau, \sigma)$ respectively. Then from (2.14) we have

$$|P_{ij}(\tau, \lambda, \sigma)|=\lambda^d|Q_{ij}(\tau, \sigma)|\cdot|\det A|+O(\lambda^{d-1})$$

as $\lambda\to\infty$, while it follows from (1.1) that

$$\sum_{i,j=1}^{m-d}|P_{ij}(\tau, \lambda, \sigma)|\leqq C|\mathrm{Im}\,\tau|^{-1}|\det P^0(\tau, \lambda, \sigma)| \qquad \text{for } \tau\in\boldsymbol{C}_-.$$

Hence we see that

$$\sum_{i,j=1}^{m-d}|Q_{ij}(\tau, \sigma)|\leqq C|\mathrm{Im}\,\tau|^{-1}|\det A_{\mathrm{II\,II}}(\tau, \sigma)|+O(\lambda^{-1})$$

as $\lambda\to\infty$, since A is nonsingular and

$$|\det P^0(\tau, \lambda, \sigma)|=\lambda^d|\det A|\cdot|\det A_{\mathrm{II\,II}}(\tau, \sigma)|+O(\lambda^{d-1})$$

as $\lambda\to\infty$. Therefore we obtain (2.13), because for a fixed $\sigma\in R^{n-1}$ $A_{\mathrm{II\,II}}(\tau, \sigma)$ is an analytic function of $\tau\in\boldsymbol{C}_-$ and is nonsingular when $|\mathrm{Im}\,\tau|$ is sufficiently large hence $A_{\mathrm{II\,II}}(\tau, \sigma)$ is nonsingular in a dense subset of $\boldsymbol{C}_-$.

The following lemma is also an extension of Theorem 1 in [1] to the case of characteristic boundary.

LEMMA 2.6. *Let* (P, B) *be* L^2-*well posed. Then for every boundary point* $(t^0, x^0)\in\partial\Omega$ *the frozen* (*constant coefficients*) *problem*:

$$(P^0, B)_{(t^0,x^0)}\begin{cases} P^0(t^0, x^0; D_t, D_x)\,u=f & \text{in } (0, \infty)\times G, \\ B(t^0, x^0)u=0 & \text{on } (0, \infty)\times\partial G, \\ u(0, x)=0 & \text{on } G \end{cases}$$

is L^2-*well posed. That is, for every* $\gamma>0$ *and every* $f\in H_{0,\gamma}(\Omega)$ *there exists uniquely a solution* $u\in H_{0,\gamma}(\Omega)$ *of* $P^0(t^0, x^0; D_t, D_x)\,u=f$ *with the boundary condition* $B(t^0, x^0)u=0$ *in the weak sence and the inequality* (1.6) *is valid for the same constant* C_0.

PROOF. The method used here is similar to that of the proof of Lemma 2.1 in [16].

In view of Lemmas 2. 2, 2. 3 and 2. 4 it is enough to prove (2. 4) and (2. 5) for $P^0(t^0, x^0; D_t, D_x)$ and $(P^0)^*(t^0, x^0; D_t, D_x)$ instead of P and P^* respectively and for $\gamma>0$.

Let $w\in C_0^\infty(\bar{\Omega})$ and $B(t^0, x^0)\, w(t, x)=0$ for all $(t, x)\in\partial\Omega$. For $\varepsilon>0$, $\mu>0$ and $(t, x)\in\bar{\Omega}$ we set $\gamma=\varepsilon^{-1}\mu$,

$$u_{\mathrm{I}}(t, x)=H(t, x)\, H^{-1}(t^0, x^0)\, w_{\mathrm{I}}\,(\varepsilon^{-1}(t-t^0),\ \varepsilon^{-1}(x-x^0)),$$

$$u_{\mathrm{II}}(t, x)=w_{\mathrm{II}}(\varepsilon^{-1}(t-t^0), \varepsilon^{-1}(x-x^0))$$

and $u={}^t[{}^tu_{\mathrm{I}}, {}^tu_{\mathrm{II}}]$, where H is the matrix defined by (2. 8). Then we see from (2. 7) and (2. 9) that $B_{\mathrm{I}}(t, x)\, u_{\mathrm{I}}(t, x)=0$ for all $(t, x)\in\partial\Omega$. Therefore applying (2. 4) to u and tending ε to 0 we obtain

$$||w||_{0,\mu}\leqq C_0\mu^{-1}||P^0(t^0, x^0;\ D_t, D_x)\, w||_{0,\mu}\,.$$

Similarly we can prove the desired inequality for $(P^0)^*(t^0, x^0; D_t. D_x)$. This completes the proof.

In what follows we restrict ourselves to the case where (P, B) is of constant coefficients and $P(\tau, \lambda, \sigma)$ is homogeneous in τ, σ and λ.

When $\partial\Omega$ is noncharacteristic, it is known that if (P, B) is L^2-well posed then the rank l of B coincides with the number d^+ of positive eigenvalues of A_1. This is also true in our case, because of the reflexiveness of (P, B). That is, we obtain the following

LEMMA 2. 7. *Let (P, B) be L^2-well posed. Then $l=d^+$.*

To prove this we shall now associate with the problem (P, B) a system of ordinary differential equations in x_1 depending a parameter $(\tau, \sigma)\in \boldsymbol{C}_-\times R^{n-1}$ (see (2. 16) below), as in the case of noncharacteristic boundary. This is carried out formally as follows: Let us consider partial Fourier transform $\hat{u}$ of u with respect to t, x'. Then from (2. 14) we obtain

$$(2.15)\begin{cases}(AD_1+A_{\mathrm{I\,I}}(\tau, \sigma))\,\hat{u}_{\mathrm{I}}(\tau, x_1, \sigma)+A_{\mathrm{I\,II}}(\sigma)\,\hat{u}_{\mathrm{II}}(\tau, x_1, \sigma)=\hat{f}_{\mathrm{I}}(\tau, x_1, \sigma)\,,\ x_1>0\,,\\ A_{\mathrm{II\,I}}(\sigma)\,\hat{u}_{\mathrm{I}}(\tau, x_1, \sigma)+A_{\mathrm{II\,II}}(\tau, \sigma)\,\hat{u}_{\mathrm{II}}(\tau, x_1, \sigma)=\hat{f}_{\mathrm{II}}(\tau, x_1, \sigma)\,,\ x_1>0\,,\\ B_{\mathrm{I}}\hat{u}_{\mathrm{I}}(\tau, 0, \sigma)=0\,,\end{cases}$$

where for instance

$$\hat{u}_{\mathrm{I}}\,(\tau, x_1, \sigma)=\int_{R^1\times R^{n-1}} e^{-i(\tau t+\sigma x')}\, u_{\mathrm{I}}(t, x_1, x')\ dtdx'\,.$$

Now let $\tau\in \boldsymbol{C}_-$ and $\sigma\in R^{n-1}$. Then, since $A_{\mathrm{II\,II}}(\tau, \sigma)$ is nonsingular according to Lemma 2. 5, we may solve the second equation of (2. 15) for $\hat{u}_{\mathrm{II}}$ and insert it into the first. Thus we arrive at

$$\begin{cases}(D_1-M(\tau,\sigma))\,\hat{u}_{\mathrm{I}}(\tau,x_1,\sigma)=F(\tau,x_1,\sigma)\,, & x_1>0\,,\\ B_{\mathrm{I}}\hat{u}_{\mathrm{I}}(\tau,0,\sigma)=0\,,\end{cases}\tag{2.16}$$

where

$$M(\tau,\sigma)=-A^{-1}(A_{\mathrm{I\,I}}-A_{\mathrm{I\,II}}A_{\mathrm{II\,II}}^{-1}A_{\mathrm{II\,I}})(\tau,\sigma)\tag{2.17}$$

and

$$F(\tau,x_1,\sigma)=A^{-1}(\hat{f}_{\mathrm{I}}-A_{\mathrm{I\,II}}A_{\mathrm{II\,II}}^{-1}\hat{f}_{\mathrm{II}})(\tau,x_1,\sigma)\,.$$

From (2.14) and (2.17) we have

$$P(\tau,\lambda,\sigma)=\begin{bmatrix}A & 0\\ 0 & I_{m-d}\end{bmatrix}\begin{bmatrix}\lambda-M(\tau,\sigma) & A^{-1}A_{\mathrm{I\,II}}A_{\mathrm{II\,II}}^{-1}\\ 0 & I_{m-d}\end{bmatrix}\begin{bmatrix}I_d & 0\\ A_{\mathrm{II\,I}} & A_{\mathrm{II\,II}}\end{bmatrix}\tag{2.18}$$

so that

$$\det P(\tau,\lambda,\sigma)=(\det A)(\det(\lambda-M(\tau,\sigma))\det A_{\mathrm{II\,II}}(\tau,\sigma)\,.\tag{2.19}$$

Therefore we find from (1.2), Lemma 2.5 and the hyperbolicity of P that the number of the eigenvalues of $M(\tau,\sigma)$ with positive imaginary parts is independent of $(\tau,\sigma)\in\boldsymbol{C}_-\times R^{n-1}$ and hence is equal to d^+.

For $(\tau,\sigma)\in\boldsymbol{C}_-\times R^{n-1}$ now let $P_{\mathrm{I}}^+(\tau,\sigma)$ denote the projection

$$\frac{1}{2\pi i}\int_{\Gamma^+(\tau,\sigma)}(\lambda-M(\tau,\sigma))^{-1}d\lambda\,,\tag{2.20}$$

where $\Gamma^+(\tau,\sigma)$ is a positively-oriented closed Jordan curve enclosing only the eigenvalues of $M(\tau,\sigma)$ with positive imaginary parts. Then the rank of $P_{\mathrm{I}}^+(\tau,\sigma)$ is equal to d^+. Furthermore to prove Lemma 2.7 we use the following

LEMMA 2.8. *Suppose that*

$\operatorname{rank}(B_{\mathrm{I}}P_{\mathrm{I}}^+)(\tau,\sigma)<d^+$ *for all* (τ,σ) *in an open set of* $\boldsymbol{C}_-\times R^{n-1}$.

Then for every $\gamma>0$ *there is a solution* $u_\gamma\in H_{1,\gamma}(\Omega)\cap C^\infty(\bar{\Omega})$ *of the homogeneous equation* $Pu=0$ *in* Ω *satisfying the boundary condition* $Bu=0$ *on* $\partial\Omega$ *which does not identically vanish.*

PROOF. For $(\tau,\sigma)\in\boldsymbol{C}_-\times R^{n-1}$ we define a projection by

$$P^+(\tau,\sigma)=\frac{1}{2\pi i}\int_{\Gamma^+(\tau,\sigma)}P^{-1}(\tau,\lambda,\sigma)\,A_1d\lambda\,,$$

where $\Gamma^+(\tau,\sigma)$ is the same curve as in (2.20). Then it follows from (2.17) and (2.18) that

$$P^+(\tau,\sigma)=\begin{bmatrix} P_{\mathrm{I}}^+(\tau,\sigma) & 0 \\ -(A_{\mathrm{II}\,\mathrm{II}}^{-1}A_{\mathrm{II}\,\mathrm{I}}P_{\mathrm{I}}^+)(\tau,\sigma) & 0\end{bmatrix}.$$

Moreover from (1.3) we have $BP^+(\tau,\sigma)=[B_{\mathrm{I}}P_{\mathrm{I}}^+(\tau,\sigma),0]$. Therefore denoting by $U^+(\tau,\sigma)$ an $m\times d^+$ matrix consisting of d^+ linearly independent columns of $P^+(\tau,\sigma)$ we find from the hypothesis that there are a point $(\tau^0,\sigma^0)\in \boldsymbol{C}_-\times R^{n-1}$ and a smooth vector-valued function defined on a neighourhood of (τ^0,σ^0) which is a null vector of $BU^+(\tau,\sigma)$ and does not identically vanish. Accordingly we can construct as usual for $\gamma=\gamma^0$ such a solution u_{γ^0} as described in the statement of the lemma, where $\gamma^0=-\mathrm{Im}\,\tau^0$. For arbitrary $\gamma>0$ we now set $u_\gamma(t,x)=u_{\gamma^0}((\gamma^0)^{-1}\gamma t,(\gamma^0)^{-1}\gamma x)$. Then u_γ has the desired property.

Proof of Lemma 2.7. It follows from the previous lemma and the uniqueness of solutions of (P,B) that $l\geqq d^+$. The same argument applied to the adjoint problem of (P,B) gives that $d-l\geqq d^-$, because of Lemmas 2.1 and 2.2. Therefore we conclude that $l=d^+$, since $d=d^++d^-$. Thus we have proved the lemma.

From Lemmas 2.7 and 2.8 we can obtain the following

Lemma 2.9. *Let (P,B) be L^2-well posed. Then $l=d^+$ and the problem satisfies Hersh's condition, i.e.,*

$$\mathrm{rank}\,(B_{\mathrm{I}}P_{\mathrm{I}}^+)(\tau,\sigma)=d^+ \qquad \textit{for all } (\tau,\sigma)\in \boldsymbol{C}_-\times R^{n-1}.$$

For the proof see for instance that of Theorem 3.1 in [2].

Now let $U_{\mathrm{I}}^+(\tau,\sigma)$ be a $d\times d^+$ matrix whose columns form a basis of the range of $P_{\mathrm{I}}^+(\tau,\sigma)$. Then, if $l=d^+$, the function

$$R(\tau,\sigma)=\det\,(B_{\mathrm{I}}U_{\mathrm{I}}^+)(\tau,\sigma) \tag{2.21}$$

is said to be *Lopatinskii determinant* of the problem (P,B) (associated with U_{I}^+). Notice that the zeros of Lopatinskii determinant do not depend on the choice of a basis of P_{I}^+. Therefore Hersh's condition means that $R(\tau,\sigma)\neq 0$ for all $(\tau,\sigma)\in \boldsymbol{C}_-\times R^{n-1}$.

Furthermore let $U_{\mathrm{I}}^-(\tau,\sigma)$ be a $d\times d^-$ matrix whose columns form a basis of the range of the projection $I_d-P_{\mathrm{I}}^+(\tau,\sigma)$. Then, if $\det\,(B_{\mathrm{I}}U_{\mathrm{I}}^+)(\tau,\sigma)\neq 0$, the elements of the $d^+\times d^-$ matrix

$$(B_{\mathrm{I}}U_{\mathrm{I}}^+)^{-1}(B_{\mathrm{I}}U_{\mathrm{I}}^-)(\tau,\sigma) \tag{2.22}$$

are said to be (generalized) *reflection coefficients* of the problem (P,B) (associated with U_{I}^+ and U_{I}^-). This concept will be used in section 4.

Finally we state the following

Lemma 2.10. *If $l=d^+$ and (P,B) satisfies Hersh's condition then the matrix B_{I} in* (1.3) *may be taken to be of the form*

$$B_{\mathrm{I}}=[I_{d^+}, S], \tag{2.23}$$

where S is a constant $d^+\times d^-$ matrix. Moreover if (2.23) *holds then Lopatinskii determinant $R(\tau, \sigma)$ defined by* (2.21) *does not vanish when* $\mathrm{Im}\,\tau<0$ *and* $\sigma=0$.

PROOF. Let $\mathrm{Im}\,\tau<0$. Then from (2.12) and (2.17) we have $M(\tau, 0) = -A^{-1}\tau$. Moreover it may be assumed that the matrix A is diagonal, since P is strongly hyperbolic. Therefore we find from (1.2) and (2.20) that

$$P_{\mathrm{I}}^{+}(\tau, 0)=\begin{bmatrix} I_{d^+} & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.24}$$

Now let $l=d^+$ and $\mathrm{rank}\,(B_{\mathrm{I}}P_{\mathrm{I}}^{+})(\tau, 0)=d^+$. Then (2.24) implies that the left $d^+\times d^+$ block of B_{I} is nonsingular, i.e., the boundary condition $B_{\mathrm{I}}u_{\mathrm{I}}=0$ may be prescribed by a matrix of the form (2.23). This proves the first assertion. Next by (2.24) we can take a basis of the range of $P_{\mathrm{I}}^{+}(\tau, 0)$ so that

$$U_{\mathrm{I}}^{+}(\tau, 0)=\tau\begin{bmatrix} I_{d^+} \\ 0 \end{bmatrix}.$$

Therefore if (2.23) holds then from (2.21) we have $R(\tau, 0)=\tau^{d^+}$, which implies the second assertion.

§ 3. Proof of part (a) of Theorem

Consider Maxwell's equations defined by

$$\left(D_t+\frac{1}{i}\begin{bmatrix} 0 & -\mathrm{curl} \\ \mathrm{curl} & 0 \end{bmatrix}\right)\begin{bmatrix} E \\ H \end{bmatrix}=F,$$

where E is the electric field vector, H is the magnetic field vector and the speed of light is taken as unity. To transform the boundary matrix into the form (1.2) we shall change the dependent varables by

$$u=T_1^{*}\begin{bmatrix} E \\ H \end{bmatrix},$$

where T_1 is the orthogonal matrix:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then the equations become

$$P(D_t, D_x)\, u = (D_t + \sum_{j=1}^{3} A_j D_j)\, u = f, \tag{3.1}$$

where $\quad f = T_1^* F$,

$$A_1 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}, \quad A_2 = \frac{1}{\sqrt{2}} \left(\begin{array}{cccc|cc} & & & & 0 & -1 \\ & & 0 & & 1 & 0 \\ & & & & 1 & 0 \\ & & & & 0 & -1 \\ \hline 0 & 1 & 1 & 0 & & \\ -1 & 0 & 0 & -1 & & 0 \end{array}\right),$$

$$A_3 = \frac{1}{\sqrt{2}} \left(\begin{array}{cccc|cc} & & & & -1 & 0 \\ & & 0 & & 0 & -1 \\ & & & & 0 & 1 \\ & & & & 1 & 0 \\ \hline -1 & 0 & 0 & 1 & & \\ 0 & -1 & 1 & 0 & & 0 \end{array}\right).$$

Since P is symmetric and $\det P(\tau, \lambda, \sigma) = \tau^2(\tau^2 - \lambda^2 - |\sigma|^2)^2$, P is a strongly hyperbolic system of constant multiplicity so that the inequality (1. 1) is valid. Moreover it becomes that $d=4$, $d^+ = d^- = 2$, $A^+ = I_2$, $A^- = -I_2$ in (1. 2) and that for the matrices defined by (2. 12) and (2. 17)

$$\begin{aligned} &A_{\mathrm{I\,I}}(\tau, \sigma) = \tau I_4, \qquad A_{\mathrm{II\,II}}(\tau, \sigma) = \tau I_2, \\ &A_{\mathrm{I\,II}}(\sigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} -T_\sigma \\ T_\sigma J \end{bmatrix}, \quad A_{\mathrm{II\,I}}(\sigma) = A_{\mathrm{I\,II}}^*(\sigma) \end{aligned} \tag{3.2}$$

and

$$M(\tau, \sigma) = -\begin{bmatrix} \tau I_2 & 0 \\ 0 & -\tau I_2 \end{bmatrix} + \frac{1}{2\tau} \begin{bmatrix} |\sigma|^2 I_2 & -T_\sigma J T_\sigma^* \\ T_\sigma J T_\sigma^* & -|\sigma|^2 I_2 \end{bmatrix}. \tag{3.3}$$

Here we have set as follows:

$$T_\sigma = \begin{bmatrix} \sigma_3 & \sigma_2 \\ -\sigma_2 & \sigma_3 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.4}$$

Together with the system P defined by (3. 1) here we may assume that B is a constant 2×6 matrix of the form

$$B = [I_2, S, 0], \text{ i.e., } B_{\mathrm{I}} = [I_2, S] \tag{3.5}$$

and S is a 2×2 matrix. Although it follows from Lemma 2.10 that (3.5) holds under the hypotheses of part (a) of Theorem, we wish to clarify the relation between $\|S\|$ and Hersh's condition and therefore we like to do so.

Now let $\lambda^+(\tau, \sigma)$ denote the root of the equation $\det P(\tau, \lambda, \sigma)=0$ in λ with positive imaginary part when $\mathrm{Im}\ \tau<0$ and in this section assume that $\sigma\in R^2\backslash 0$ (see Lemma 2.10). Then the following two lemmas are valid even if S is non-real. They will be also used in § 4.

LEMMA 3.1. *Let* $\tau\in \boldsymbol{C}_-$ *and* $\sigma\in R^2\backslash 0$. *Then the columns of* the 4×2 *matrix*

$$U_{\mathrm{I}}^+(\tau, \sigma)=\frac{1}{\sqrt{2}\,|\sigma|}\begin{bmatrix}(\tau-\lambda^+(\tau, \sigma))\,T_\sigma\\ -(\tau+\lambda^+(\tau, \sigma))T_\sigma J\end{bmatrix} \tag{3.6}$$

form a basis of the range of the projection $P_{\mathrm{I}}^+(\tau, \sigma)$ *defined by* (2.20). *Moreover Lopatinskii determinant* $R(\tau, \sigma)$ *defined by* (2.21) *is as follows*.

$$R(\tau, \sigma)=(1-\det S)\,\tau^2-(1+\det S)\,\tau\lambda^+(\tau, \sigma)-\Phi(\sigma)/2\,, \tag{3.7}$$

where

$$\Phi(\sigma)=(1-\det S-s_{12}-s_{21})\,\sigma_2^2+2(s_{11}-s_{22})\,\sigma_2\sigma_3+(1-\det S+s_{12}+s_{21})\,\sigma_3^2 \tag{3.8}$$

and

$$S=\begin{bmatrix}s_{11} & s_{12}\\ s_{21} & s_{22}\end{bmatrix}.$$

PROOF. Since $(\lambda^+(\tau, \sigma))^2=\tau^2-|\sigma|^2$ and (3.4) implies

$$T_\sigma^* T_\sigma=T_\sigma T_\sigma^*=|\sigma|^2 I_2\,, \tag{3.9}$$

we find from (3.3) and (3.6) that

$$(\lambda^+(\tau, \sigma)-M(\tau, \sigma))\,U_{\mathrm{I}}^+(\tau, \sigma)=0$$

and rank $U_{\mathrm{I}}^+(\tau, \sigma)=2$. Therefore the first assertion follows from (2.20).

We shall next prove (3.7). From (3.5), (3.6) and (3.9) we have

$$(B_{\mathrm{I}}U_{\mathrm{I}}^+)(\tau, \sigma)=(\sqrt{2}\,|\sigma|)^{-1}T_\sigma((\tau-\lambda^+)\,I_2-(\tau+\lambda^+)\,\hat{S}(\sigma)J)\,, \tag{3.10}$$

where

$$\hat{S}(\sigma)=|\sigma|^{-2}T_\sigma^* ST_\sigma=\begin{bmatrix}\hat{s}_{11} & \hat{s}_{12}\\ \hat{s}_{21} & \hat{s}_{22}\end{bmatrix}(\sigma)\,. \tag{3.11}$$

Hence we obtain

$$\begin{aligned}\det(B_{\mathrm{I}}U_{\mathrm{I}}^+)(\tau, \sigma)&=2^{-1}\det((\tau-\lambda^+)\,I_2-(\tau+\lambda^+)\,\hat{S}(\sigma)\,J)\\ &=\tau^2\,(1-\det\hat{S})-\tau\lambda^+\,(1+\det\hat{S})-|\sigma|^2(1-\det\hat{S}+\hat{s}_{12}+\hat{s}_{21})/2\,.\end{aligned}$$

Moreover it follows from (3. 11) and (3. 4) that det $\hat{S}(\sigma)=\det S$,

$$|\sigma|^2\hat{s}_{12}(\sigma)=(s_{11}-s_{22})\,\sigma_2\sigma_3+s_{12}\sigma_3^2-s_{21}\sigma_2^2\,,$$

$$|\sigma|^2\hat{s}_{21}(\sigma)=(s_{11}-s_{22})\,\sigma_2\sigma_3-s_{12}\sigma_2^2+s_{21}\sigma_3^2\,.$$

Thus we obtain the desired equality (3. 7).

LEMMA 3. 2. *In order that there exists a nonsingular* 6×6 *matrix* W *such that* $W^{-1}A_jW$ *are hermitian for* $j=1, 2, 3$ *and*

$$W^{-1}A_1Wu\cdot u\leqq 0 \qquad \textit{for } u\in(\ker BW)\backslash 0\,, \tag{3. 12}$$

it is necessary and sufficient that there are real numbers α, β *such that* $|\beta|<\alpha$ *and*

$$\alpha(I_2-S^*S)+\beta\left(\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}+S^*\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}S\right)\geqq 0\,. \tag{3. 13}$$

In particular, $A_1u\cdot u\leqq 0$ *for* $u\in\ker B$ *if and only if* $I_2-S^*S\geqq 0$, *i. e.,* $\|S\|\leqq 1$, *and the strict inequality in* (3.12) *is valid if and only if the left side of* (3. 13) *is positive definite.*

PROOF. Necessity. Since it is invariant under unitary transformations that $W^{-1}A_jW$ is hermitian and (3. 12) is valid, we may assume that $W^{-1}A_1W=A_1$ and so W is of the from

$$W=\begin{bmatrix}W_1 & & 0\\ & W_2 & \\ 0 & & W_3\end{bmatrix},$$

where W_j $(j=1, 2, 3)$ are nonsingular 2×2 matrices.

We first find from (3. 1) that $W^{-1}A_2W$ is hermitian if and only if

$$W_1W_1^*=-\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}W_3\,W_3^*\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \tag{3. 14}$$

$$W_2W_2^*=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}W_3\,W_3^*\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{3. 15}$$

and that $W^{-1}A_3W$ is so if and only if

$$W_1W_1^*=W_3W_3^*\,, \tag{3. 16}$$

$$W_2W_2^*=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}W_3\,W_3^*\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}. \tag{3. 17}$$

Now it follows from (3. 14) and (3. 16) that for some real numbers α and β

$$W_1W_1^* = W_3W_3^* = \begin{bmatrix} \alpha & i\beta \\ -i\beta & \alpha \end{bmatrix}. \tag{3.18}$$

Moreover, since $W_3W_3^*$ is positive definite, we have

$$|\beta| < \alpha .$$

From (3.17) and (3.18) we also obtain

$$W_2W_2^* = \begin{bmatrix} \alpha & -i\beta \\ i\beta & \alpha \end{bmatrix}. \tag{3.19}$$

Furthermore we find from (3.5) that, under (3.18) and (3.19) with $|\beta| < \alpha$, (3.12) is equivalent to (3.13). Similarly other assertions of the lemma follow.

Sufficiency. Let

$$\begin{gathered} W = \begin{bmatrix} W_1 & & \\ & W_2 & \\ & & W_3 \end{bmatrix}, \quad W_1 = W_3 = T_0^* \begin{bmatrix} \sqrt{\alpha+\beta} & 0 \\ 0 & \sqrt{\alpha-\beta} \end{bmatrix} T_0 , \\ W_2 = T_0^* \begin{bmatrix} \sqrt{\alpha-\beta} & 0 \\ 0 & \sqrt{\alpha+\beta} \end{bmatrix} T_0, \quad T_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}. \end{gathered} \tag{3.20}$$

Then (3.18) and (3.19) are valid. Therefore we see from the preceding argument that the matrix W thus defined has the desired property, since (3.18) and (3.19) imply (3.15).

From now on in this section the matrix S in (3.5) is assumed to be *real*. Then we obtain the following

LEMMA 3.3. ([5]). *$R(\eta - i\gamma, \sigma) \neq 0$ for all $\gamma > 0$, $\eta \in R^1 \backslash 0$ and $\sigma \in R^2 \backslash 0$ if and only if* $\det S \geqq -1$.

PROOF. Since the proof given at pp. 254-255 in [5] is complicated, we shall give a somewhat simple proof.

Let $\eta \neq 0$ and $\gamma > 0$. For convenience put

$$\lambda^+ (\eta - i\gamma, \sigma) = \alpha + i\beta , \qquad \beta > 0 .$$

Then we have

$$\alpha^2 - \beta^2 = \eta^2 - \gamma^2 - |\sigma|^2 \tag{3.21}$$

and

$$\alpha\beta = -\eta\gamma \tag{3.22}$$

so that $\eta\alpha < 0$. Moreover it follows from (3.7) that

$$\text{(3.23)}\quad \operatorname{Re} R(\eta-i\gamma,\sigma)=(1-\det S)(\eta^2-\gamma^2)-(1+\det S)(\eta\alpha+\gamma\beta)-\Phi(\sigma)/2\,,$$

$$\text{(3.24)}\quad \operatorname{Im} R(\eta-i\gamma,\sigma)=-2\,(1-\det S)\,\eta\gamma-(1+\det S)(\eta\beta-\gamma\alpha)\,.$$

We first show that $\det S \geqq -1$ implies $\operatorname{Im} R(\eta-i\gamma,\sigma)\neq 0$ for $\eta\neq 0$ and $\gamma>0$. When $\det S=-1$, we have from (3.24)

$$\operatorname{Im} R(\eta-i\gamma,\sigma)=-4\eta\gamma\neq 0\,.$$

Now let $\det S>-1$. Then we find from (3.22) and (3.24) that, if $\eta\neq 0$, $\operatorname{Im} R(\eta-i\gamma,\sigma)=0$ is equivalent to

$$\frac{\beta}{\gamma}+\frac{\gamma}{\beta}=-2\,\frac{1-\det S}{1+\det S}\,,$$

which is impossible, because the left hand side $\geqq 2$ and the right one <2 when $\det S>-1$.

Next suppose that $\det S<-1$. Then in the same way as in [5] it can be shown that there is a point (η,γ,σ) such that $R(\eta-i\gamma,\sigma)=0$, $\alpha=-\gamma$ and $\beta=\eta$. In fact, under the last two equalities, (3.22) is valid and (3.21) becomes

$$\text{(3.25)}\quad \eta^2-\gamma^2=|\sigma|^2/2\,.$$

Moreover, according to (3.24), $\operatorname{Im} R(\eta-i\gamma,\sigma)=0$ may be written as

$$\text{(3.26)}\quad \left(\frac{\eta}{\gamma}\right)^2+2\,\frac{1-\det S}{1+\det S}\left(\frac{\eta}{\gamma}\right)+1=0\,.$$

Notice that $(1+\det S)^{-1}(1-\det S)<-1$, since $\det S<-1$. Therefore the equation (3.26) in η/γ has a solution such that $\eta>\gamma>0$. Let us fix such a point (η,γ). Then to complete the proof it is enough to find $\sigma\in R^2$ satisfying (3.25) and $\operatorname{Re} R(\eta-i\gamma,\sigma)=0$. Now it follows from (3.8) and the relation $\eta\alpha+\gamma\beta=0$ that, under (3.25), the equality (3.23) becomes

$$-2\operatorname{Re} R(\eta-i\gamma,\sigma)=(s_{12}+s_{21})(\sigma_3^2-\sigma_2^2)+2(s_{11}-s_{22})\sigma_2\sigma_3\,.$$

The right side of this equation is an indefinite quadratic form so that we can find a desired point σ.

LEMMA 3.4. *Let* $\det S\geqq -1$. *Then* $R(-i\gamma,\sigma)\neq 0$ *for all* $\gamma>0$ *and* $\sigma\in R^2\backslash 0$ *if and only if* $\|S\|\leqq 1$.

PROOF. Let $\gamma>0$. Then $\eta=0$ is equivalent to $\alpha=0$ according to (3.22). Hence it follows from (3.24) that $\operatorname{Im} R(-i\gamma,\sigma)=0$ for $\gamma>0$ and $\sigma\in R^2$. Moreover we may assume in the proof that $\gamma^2+|\sigma|^2=1$, since $R(-i\gamma,\sigma)$ is homogeneous in γ and σ. Therefore from (3.21) and (3.23) we have

$$\text{(3.27)}\quad R(-i\gamma,\sigma)=f(\gamma)-\Phi(\sigma)/2\,,$$

where

$$f(\gamma)=-(1-\det S)\,\gamma^2-(1+\det S)\,\gamma$$

and $\Phi(\sigma)$ is the quadratic from defined by (3. 8). Notice that

(3. 28) $\qquad f(0)=0$ and $f(\gamma)<0$ if $0<\gamma\leqq 1$ and $\det S+1\geqq 0$.

We now claim that $R(-i\gamma,\sigma)\neq 0$ for $\gamma>0$ if and only if $\Phi(\sigma)$ is positive semi-definite. The "if" part follows from (3. 27) and (3. 28). To prove the "only if" part we suppose that there is a point $\sigma^0\in R^2\backslash 0$ satisfying $\Phi(\sigma^0)<0$. Then (3. 27) and (3. 28) imply that $R(0,|\sigma^0|^{-1}\sigma^0)>0$ and $R(-i,0)<0$. Therefore we find another point σ and $\gamma>0$ such that $R(-i\gamma,\sigma)=0$, since $R(-i\gamma,\sigma)$ is real-valued and the set $\{(\gamma,\sigma)\,;\,\gamma^2+|\sigma|^2=1$ and $0<\gamma<1\}$ is connected.

Thus to complete the proof of the lemma it is enough to show that $\Phi(\sigma)$ is positive semi-definite if and only if

(3. 29) $\qquad |\det S|\leqq 1$ and $\det(I_2-S^*S)\geqq 0$,

which is equivalent to $||S||\leqq 1$.

Now from (3. 8) we have

(3. 30) $\qquad$ the discriminant of $\Phi(\sigma)$

$$=(s_{11}-s_{22})^2+(s_{12}+s_{21})^2-(1-\det S)^2$$
$$=-\det(I_2-S^*S)\,.$$

To prove the "only if" part let $\Phi(\sigma)$ be positive semi-definite. Then we find from (3. 8) and (3. 30) that $\det(I_2-S^*S)\geqq 0$ and $\det S\leqq 1$. Hence by hypothesis we obtain (3. 29). Now the "if" part follows from (3. 29) and (3. 30).

From Lemmas 3. 3 and 3. 4 we now obtain the following

LEMMA 3. 5. *$R(\tau,\sigma)\neq 0$ for all $(\tau,\sigma)\in C_-\times(R^2\backslash 0)$ if and only if $||S||\leqq 1$.*

LEMMA 3. 6. *Let $||S||\leqq 1$. Then $||S||<1$ if and only if $R(\eta,\sigma)\neq 0$ for all $(\eta,\sigma)\in R^1\times(R^2\backslash 0)$.*

PROOF. The hypothesis implies (3. 29) so that the quadratic form $\Phi(\sigma)$ defined by (3. 8) is positive semi-definite according to (3. 30).

To prove the "if" part let $R(0,\sigma)\neq 0$ for all $\sigma\neq 0$. Then $\Phi(\sigma)$ is positive definite according to (3. 7), since it is positive semi-definite. Therefore from (3. 8), (3. 29) and (3. 30) we obtain

(3. 31) $\qquad |\det S|<1$ and $\det(I_2-S^*S)>0$,

which is equivalent to $||S||<1$.

To prove the "only if" part suppose that (3.31) is valid hence $\Phi(\sigma)$ is positive definite. If $0<|\eta|<|\sigma|$ then according to (3.7) $\mathrm{Im}\, R(\eta, \sigma)\neq 0$, since $\lambda^+(\eta, \sigma)=i(|\sigma|^2-\eta^2)^{1/2}\neq 0$. If $0<|\sigma|\leqq|\eta|$ then, since $\lambda^+(\eta, \sigma)=-(\mathrm{sgn}\,\eta)(\eta^2-|\sigma|^2)^{1/2}$, we have

$$R(\eta, \sigma)=(1-\det S)\,\eta^2+(1-\det S)|\eta|(\eta^2-|\sigma|^2)^{1/2}-\Phi(\sigma)/2\,.$$

Hence we may assume in the proof that $\eta\geqq|\sigma|>0$. Then it follows that $(\partial R/\partial\eta)(\eta, \sigma)>0$ for $\eta>|\sigma|>0$. Moreover we have from (3.8)

$$2R(|\sigma|, \sigma)=(1-\det S+s_{12}+s_{21})\,\sigma_2^2-2(s_{11}-s_{22})\,\sigma_2\sigma_3+(1-\det S-s_{12}-s_{21})\,\sigma_3^2\,,$$

which is positive definite according to (3.30) and (3.31). Therefore we find that $R(\eta, \sigma)\neq 0$ for $|\eta|\geqq|\sigma|>0$. Finally we have $2R(0, \sigma)=-\Phi(\sigma)<0$ for $\sigma\neq 0$.

PROOF of part (*a*) of THEOREM. Notice that Hersh's condition and the maximal non-positivity are invariant under unitary transformations of the dependent variables, and that if the kernel of B is non-positive for the matrix A_1 in (3.1) then it is maximally non-positive, since rank $B=2$. Therefore the assertion follows from Lemmas 2.10, 3.2 and 3.5. Then for the L^2-well posedness see [10].

REMARK 3.7. The above proof of part (*a*) of Theorem together with Lemma 3.6 implies that if (P, B) satisfies Kreiss' condition, i.e., $R(\tau, \sigma)\neq 0$ for all $(\tau, \sigma)\in(\bar{C}_-\times R^2)\backslash 0$ then the kernel of B is maximally non-positive for A_1 and the strict inequality in (0.1) is valid. The converses of part (*a*) of Theorem and the above assertion also hold by virtue of Lemma 2.9 and the same lemmas used above. Compare with the results in p. 627 of [11].

§ 4. Maxwell's equations with non-real boundary conditions

We shall first derive a necessary and sufficient condition for the problem (P, B) to be L^2-well posed in terms of Lopatinskii determinant and reflection coefficients (see (2.22)), as in the case of noncharacteristic boundary (see [2], [15] or Theorem 4.1 in [12]). Here P is the system defined by (3.1) and B is a constant 2×6 matrix of the form (3.5). Notice that $-\lambda^+(\tau, \sigma)$ is the root of the equation $\det P(\tau, \lambda, \sigma)=0$ in λ with negative imaginary part when $\mathrm{Im}\,\tau<0$. Then we obtain the following

PROPOSITION 4.1. *In order that the problem* (P, B) *is* L^2*-well posed it is necessary and sufficient that* 1) $R(\tau, \sigma)\neq 0$ *if either* $\mathrm{Im}\,\tau<0$ *or* $\mathrm{Im}\,\tau=0$ *and* $|\tau|>|\sigma|$, *and* 2) *for every* $(\eta^0, \sigma^0)\in R^1\times R^2$ *with* $|\eta^0|\leqq|\sigma^0|$ *and* $\sigma^0\neq 0$ *there are a positive constant* $C=C(\eta^0, \sigma^0)$ *and a neighborhood* $U(\eta^0, \sigma^0)$ *in* $\boldsymbol{C}_-\times R^2$ *such that for* $(\tau, \sigma)\in U(\eta^0, \sigma^0)$

$$\|(B_{\mathrm{I}}U_{\mathrm{I}}^{+})^{-1}(B_{\mathrm{I}}U_{\mathrm{I}}^{-})(\tau,\sigma)\| \leqq \begin{cases} C|\operatorname{Im}\tau|^{-1}(\operatorname{Im}\lambda^{+}(\tau,\sigma))\cdot|\lambda^{+}(\tau,\sigma)|, & \textit{if } |\eta^0|=|\sigma^0|, \\ C|\operatorname{Im}\tau|^{-1}, & \textit{if } 0<|\eta^0|<|\sigma^0|, \\ C|\operatorname{Im}\tau|^{-1}|\tau|, & \textit{if } \eta^0=0, \end{cases} \tag{4.1}$$

where U_{I}^{+} *is the matrix given by* (3.6) *and* U_{I}^{-} *is defined as* U_{I}^{+} *resulting from replacing* λ^{+} *by* $-\lambda^{+}$ *in its definition.*

It should be pointed out that the presence of the factor $|\tau|$ in the right side of the last inequality of (4.1) is caused by the unboundedness of the matrix $M(\tau,\sigma)$ (see (3.3)).

To prove Proposition 4.1 we need the following two lemmas.

We first suppose that (P, B) is L^2-well posed. Let $f\in C_0^\infty(\Omega)$ and supp $f\subset\{(t,x);\ t>0\}$. Then it follows from Lemma 2.4 and the homogeneity of P that for every $\gamma>0$ there exists a unique solution $u\in H_{0,1,\gamma}(\Omega)$ of the equation $Pu=f$ satisfying (1.6) such that $u_{\mathrm{I}}\in H_{1,0,\gamma}(\Omega)$, $B_{\mathrm{I}}u_{\mathrm{I}}|_{\partial\Omega}=0$ in $\mathscr{D}'(\partial\Omega)$ and $u(t,x)=0$ for $t<0$. Now let $v\in H_{1,0,\gamma}(\Omega)$ be the solution of the Cauchy problem $Pv=f$ in R^4, $v(t,x)=0$ for $t<0$. Then the inequality (1.6) for v instead of u is valid. Setting $w=u-v$, we see that $w\in H_{0,1,\gamma}(\Omega)$, $w_{\mathrm{I}}\in H_{1,0,\gamma}(\Omega)$,

$$\begin{cases} Pw=0 & \text{in } \Omega, \\ B_{\mathrm{I}}w_{\mathrm{I}}=-B_{\mathrm{I}}v_{\mathrm{I}} & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

and

$$\|w\|_{0,\gamma}\leqq 2\,C_0\gamma^{-1}\|f\|_{0,\gamma},\quad w(t,x)=0 \qquad \text{for } t<0. \tag{4.3}$$

Now the partial Fourier transform $\hat{w}(\tau, x_1, \sigma)$ of $w(t, x_1, x')$ with respect to t, x' may be represented as

$$\hat{w}(\tau,x_1,\sigma)=\int_0^\infty G(\tau,\sigma;\ x_1,y_1)\,\hat{f}(\tau,y_1,\sigma)\,dy_1, \tag{4.4}$$

where G is the matrix defined by (4.12) below. Then we have

LEMMA 4.2. *Let* (P, B) *satisfy Hersh's condition. Then the problem is* L^2*-well posed if and only if for every* $(\eta^0,\sigma^0)\in(R^1\times R^2)\backslash 0$ *there are a positive constant* C *and a neighborhood* $U(\eta^0,\sigma^0)$ *in* $\boldsymbol{C}_{-}\times R^2$ *such that for* $(\tau,\sigma)\in U(\eta^0,\sigma^0)$

$$\|G(\tau,\sigma;\cdot,\cdot)\|_{\mathscr{L}(L^2(0,\infty),L^2(0,\infty))}\leqq C|\operatorname{Im}\tau|^{-1}, \tag{4.5}$$

where $\|\ \|_{\mathscr{L}(L^2(0,\infty),L^2(0,\infty))}$ *denotes the norm as an operator from* $L^2(0,\infty)$ *into itself.*

We shall indicate the proof. If we suppose that for (τ,σ) in a suitable neighborhood $U(\eta^0,\sigma^0)$ the kernel $G(\tau,\sigma;\ x_1,y_1)$ is continuous with respect to τ, σ, x_1 and y_1, then the assertion of Lemma 4.2 can be proved in the same way as in the proof of Theorem 4.1 in [2], since the uniqueness of the solutions is a consequence of Hersh's condition (see Part III in [5]). Therefore we need merely to seek for the kernel G satisfying (4.4) and the above property.

From (4.2), (2.15) and (2.16) we have for almost all $(\tau, \sigma)\in \boldsymbol{C}_-\times R^2$

$$\begin{cases} (D_1-M(\tau,\sigma))\,\hat{w}_{\mathrm{I}}(\tau, x_1, \sigma)=0\,, \quad x_1>0\,, \\ \hat{w}_{\mathrm{II}}(\tau, x_1, \sigma)=-A_{\mathrm{II}}^{-\mathrm{I}}{}_{\mathrm{II}}(\tau,\sigma)A_{\mathrm{II}\,\mathrm{I}}(\sigma)\,\hat{w}_{\mathrm{I}}(\tau, x_1, \sigma), \quad x_1>0 \\ B_{\mathrm{I}}\hat{w}_{\mathrm{I}}(\tau, 0, \sigma)=-B_{\mathrm{I}}\hat{v}_{\mathrm{I}}(\tau, 0, \sigma) \end{cases} \tag{4.6}$$

and from the definition of v

$$(D_1-M(\tau,\sigma))\,\hat{v}_{\mathrm{I}}(\tau, x_1, \sigma)=A^{-1}(\hat{f}_{\mathrm{I}}-A_{\mathrm{I}\,\mathrm{II}}A_{\mathrm{II}}^{-\mathrm{I}}{}_{\mathrm{II}}\hat{f}_{\mathrm{II}})\,(\tau, x_1, \sigma)\,, \quad x_1\in R^1\,. \tag{4.7}$$

Here $M(\tau, \sigma)$ etc. are the matrices defined by (3.2) and (3.3).

In what follows let $(\tau, \sigma)\in U(\eta^0, \sigma^0)$. First let $\sigma^0\neq 0$ and put

$$T(\tau,\sigma)=[U_{\mathrm{I}}^{+}(\tau,\sigma)\,, \quad U_{\mathrm{I}}^{-}(\tau,\sigma)]\,, \tag{4.8}$$

where U_{I}^{+} and U_{I}^{-} denote the matrices in (4.1). Then we have

$$T^{-1}MT=\begin{bmatrix}\lambda^{+}I_2 & 0 \\ 0 & -\lambda^{+}I_2\end{bmatrix}. \tag{4.9}$$

Next let $\sigma^0=0$. Then, since P is a strongly hyperbolic system of constant multiplicity and $\lambda^{+}(\eta^0, \sigma^0)\neq 0$, according to (2.12) and (2.18) we can take the matrix defined by (4.8), so that it is continuous in $U(\eta^0, \sigma^0)$, (4.9) is valid and $T^{-1}(\tau, \sigma)$ is bounded in $U(\eta^0, \sigma^0)$. Therefore, putting

$$\begin{aligned} T^{-1}(\tau,\sigma)\,\hat{w}_{\mathrm{I}}(\tau, x_1, \sigma)&=\begin{bmatrix} w_{\mathrm{I}}^{+} \\ w_{\mathrm{I}}^{-}\end{bmatrix}(\tau, x_1, \sigma)\,, \\ T^{-1}(\tau,\sigma)\,\hat{v}_{\mathrm{I}}(\tau, x_1, \sigma)&=\begin{bmatrix} v_{\mathrm{I}}^{+} \\ v_{\mathrm{I}}^{-}\end{bmatrix}(\tau, x_1, \sigma)\,, \end{aligned} \tag{4.10}$$

where $w_{\mathrm{I}}^{\pm}$, $v_{\mathrm{I}}^{\pm}$ are 2-vectors, we find from the first equality of (4.6), (4.7), (4.9) and the definition of v that

$$w_{\mathrm{I}}^{-}(\tau, x_1, \sigma)=0 \qquad \textit{for}\ x_1>0 \quad \text{and}\ v_{\mathrm{I}}^{+}(\tau, 0, \sigma)=0\,, \tag{4.11}$$

since $f\in C_0^\infty(\Omega)$ and $\mathrm{Im}\ \lambda^{+}(\tau, \sigma)>0$. Moreover from (4.7), (4.9) and (4.10) we have

$$v_{\mathrm{I}}^{-}(\tau, 0, \sigma)=-\int_0^\infty e^{i\lambda^{+}y_1}(T^{-1})_{(2)}\,A^{-1}(\hat{f}_{\mathrm{I}}-A_{\mathrm{I}\,\mathrm{II}}A_{\mathrm{II}}^{-1}{}_{\mathrm{II}}\hat{f}_{\mathrm{II}})\,(\tau, y_1, \sigma)dy_1\,,$$

where $(T^{-1})_{(2)}$ denotes the lower 2×4 block of T^{-1}. Thus from (4. 4), (4. 6) and (4. 8)$\sim$(4. 11) we obtain

(4. 12) $G(\tau, \sigma;\ x_1, y_1)$

$$=\begin{bmatrix} I_4 \\ -A_{\mathrm{II}\,\mathrm{II}}^{-1}A_{\mathrm{II}\,\mathrm{I}} \end{bmatrix} U_{\mathrm{I}}^{+} e^{i\lambda^{+}x_1}(B_{\mathrm{I}}U_{\mathrm{I}}^{+})^{-1}(B_{\mathrm{I}}U_{\mathrm{I}}^{-})\, e^{i\lambda^{+}y_1}(T^{-1})_{(2)}A^{-1}[I_4,\ -A_{\mathrm{I}\,\mathrm{II}}A_{\mathrm{II}\,\mathrm{II}}^{-1}].$$

Now it is clear that the matrix G defined by (4. 12) is continuous with respect to τ, σ, x_1 and y_1 when $\mathrm{Im}\,\tau<0$. Thus we obtain Lemma 4. 2.

LEMMA 4. 3. *Let (P, B) satisfy Hersh's condition and let $(\eta^0, \sigma^0)\in R^1\times(R^2\backslash 0)$. Then* (4. 5) *is equivalent to* (4. 1). *Here, when $|\eta^0|>|\sigma^0|$, the first inequality in* (4. 1) *is taken.*

PROOF. We shall first derive a more concrete representation for the kernel G defined by (4. 12). If follows from (3. 2), (3. 6) and (3. 9) that

$$-(A_{\mathrm{II}\,\mathrm{II}}^{-1}A_{\mathrm{II}\,\mathrm{I}}U_{\mathrm{I}}^{+})(\tau, \sigma)=|\sigma|I_2\,.$$

Moreover from (4. 8) we have

$$T^{-1}(\tau, \sigma)=\frac{\sqrt{2}}{4\tau|\sigma|\lambda^{+}}\begin{bmatrix} -(\tau-\lambda^{+})\,T_{\sigma}^{*} & -(\tau+\lambda^{+})\,JT_{\sigma}^{*} \\ (\tau+\lambda^{+})\,T_{\sigma}^{*} & (\tau-\lambda^{+})\,JT_{\sigma}^{*} \end{bmatrix},$$

which implies

$$(T^{-1})_{(2)}A^{-1}=\frac{\sqrt{2}}{4\tau|\sigma|\lambda^{+}}\,[(\tau+\lambda^{+})\,T_{\sigma}^{*},\ -(\tau-\lambda^{+})\,JT_{\sigma}^{*}]\,.$$

Hence we see from (3. 2) that

$$-(T^{-1})_{(2)}\,A^{-1}A_{\mathrm{I}\,\mathrm{II}}A_{\mathrm{II}\,\mathrm{II}}^{-1}=\frac{|\sigma|}{2\tau\lambda^{+}}\,I_2\,.$$

Thus from (4. 12) we obtain

(4. 13) $G(\tau, \sigma;\ x_1, y_1)$

$$=\frac{1}{4\tau|\sigma|^2\lambda^{+}}\begin{bmatrix} (\tau-\lambda^{+})\,T_{\sigma} \\ -(\tau+\lambda^{+})\,T_{\sigma}J \\ \sqrt{2}\,|\sigma|^2 I_2 \end{bmatrix} e^{i\lambda^{+}x_1}(B_{\mathrm{I}}U_{\mathrm{I}}^{+})^{-1}(B_{\mathrm{I}}U_{\mathrm{I}}^{-})\times$$

$$\times e^{i\lambda^{+}y_1}[(\tau+\lambda^{+})T_{\sigma}^{*},\quad -(\tau-\lambda^{+})\,JT_{\sigma}^{*},\quad \sqrt{2}\,|\sigma|^2 I_2]\,.$$

We next suppose that (4. 5) is valid. Then we find that for $(\tau, \sigma)\in U(\eta^0, \sigma^0)$

$$\int_{\substack{x_1>0\\ y_1>0}} ||G_{\mathrm{II}\,\mathrm{II}}(\tau, \sigma;\ x_1, y_1)||^2\, dx_1dy_1\leqq 4C^2|\mathrm{Im}\,\tau|^{-2},$$

where $G_{\mathrm{II\,II}}$ denotes the lowest right 2×2 block of G and C is the same constant as in (4. 5). (For the proof see § 1 in [15] or § 4 in [12]). Therefore from (4. 13) we obtain (4. 1), since $\sigma^0\neq 0$ and $\mathrm{Im}\,\lambda^+(\eta^0, \sigma^0)>0$ when $|\eta^0|<|\sigma^0|$. Now it is obvious that (4. 1) implies (4. 5).

PROOF of PROPOSITION 4. 1. Necessity. It follows from Lemma 2. 9 that Hersh's condition is satisfied. Moreover $R(\tau, 0)$ may be extended continuously to $\mathrm{Im}\,\tau\leqq 0$ and does not vanish for $\tau\in R^1\backslash 0$, as is seen from the proof of the last assertion of Lemma 2. 10. Therefore in view of Lemmas 4. 2 and 4. 3 we must only prove that $R(\tau, \sigma)\neq 0$ for $(\eta, \sigma)\in R^1\times R^2$ with $|\eta|>|\sigma|>0$, i. e., rank $(B_{\mathrm{I}}U_{\mathrm{I}}^+)(\eta, \sigma)=2$ for such (η, σ), as in [17]. To do it let $(\eta^0, \sigma^0)\in \mathrm{R}^1\times R^2$ and $|\eta^0|>|\sigma^0|>0$. Then it follows from Lemmas 4. 2 and 4. 3 that $\|(B_{\mathrm{I}}U_{\mathrm{I}}^+)^{-1}(B_{\mathrm{I}}U_{\mathrm{I}}^-)(\tau, \sigma)\|$ is bounded on a neighborhood of (η^0, σ^0) in $\boldsymbol{C}_-\times R^2$. Hence there is a sequence $\{(\tau^j, \sigma^j)\}$ such that $\mathrm{Im}\,\tau^j<0$, $(\tau^j, \sigma^j)\rightarrow(\eta^0, \sigma^0)$ and $(B_{\mathrm{I}}U_{\mathrm{I}}^+)^{-1}(B_{\mathrm{I}}U_{\mathrm{I}}^-)(\tau^j, \sigma^j)$ tends to a matrix $Q(\eta^0, \sigma^0)$ as $j\rightarrow\infty$. Therefore we obtain

$$B_{\mathrm{I}}U_{\mathrm{I}}^-(\eta^0, \sigma^0)=(B_{\mathrm{I}}U_{\mathrm{I}}^+)(\eta^0, \sigma^0)\,Q(\eta^0, \sigma^0)\,,$$

which implies

$$\mathrm{rank}\,(B_{\mathrm{I}}\,[U_{\mathrm{I}}^+, U_{\mathrm{I}}^-])(\eta^0, \sigma^0)=\mathrm{rank}\,(B_{\mathrm{I}}U_{\mathrm{I}}^+)(\eta^0, \sigma^0)\,.$$

Thus we find that rank $(B_{\mathrm{I}}U_{\mathrm{I}}^+)(\eta^0, \sigma^0)=2$, since rank $[U_{\mathrm{I}}^+, U_{\mathrm{I}}^-](\eta^0, \sigma^0)=4$ according to the definitions of U_{I}^+ and U_{I}^-.

Sufficiency. In view of Lemmas 4. 2 and 4. 3 it is enough to prove (4. 5) for $(\tau, \sigma)\in U(\eta^0, \sigma^0)$ with $\eta^0\neq 0$ and $\sigma^0=0$. To do it let (η^0, σ^0) be such a point. Then by hypothesis $R(\eta^0, \sigma^0)\neq 0$. Therefore from (4. 12) we obtain the desired inequality (4. 5), since $T^{-1}(\tau, \sigma)$ is bounded in $U(\eta^0, \sigma^0)$, as we have already remarked, and $A_{\mathrm{II\,II}}^{-1}(\tau, \sigma)$ is so according to (3. 2). The proof is complete.

In order to prove part (b) of Theorem we shall now associate with (P, B) another problem (P, B') such that

$$B'=[I_2, S', 0]\,,\qquad S'=\begin{bmatrix} s_{11}' & s_{12}' \\ s_{21}' & s_{22}' \end{bmatrix}, \tag{4. 14}$$

$$s_{11}'=-s_{22}'=(s_{11}-s_{22})/2\,,\quad s_{12}'=s_{21}'=(s_{12}+s_{21})/2\,.$$

Notice that

$$s_{11}'-s_{22}'=s_{11}-s_{22}\,,\quad s_{12}'+s_{21}'=s_{12}+s_{21}\,,\quad s_{11}'+s_{22}'=s_{12}'-s_{21}'=0\,. \tag{4. 15}$$

Then we obtain the following

LEMMA 4. 4. *Suppose that* $|\det S|\neq 1$, $(s_{11}+s_{22})^2+(s_{12}-s_{21})^2=0$ *and that* *the problem* (P, B') *is* L^2*-well posed. Then* (P, B) *is also* L^2*-well posed.*

To prove this we use the following

LEMMA 4.5. *For* $(\tau, \sigma) \in \boldsymbol{C}_- \times (R^2 \backslash 0)$ *with* $R(\tau, \sigma) \neq 0$ *we set*

$$(B_{\mathrm{I}} U_{\mathrm{I}}^+)^{-1} (B_{\mathrm{I}} U_{\mathrm{I}}^-)(\tau, \sigma) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} (\tau, \sigma),$$

where the left side is the matrix in (4.1). *Then*

$$\begin{aligned} 2Rc_{11} &= (1 - \det S)(\tau^2 - (\lambda^+)^2) - \hat{s}_{12}(\tau - \lambda^+)^2 - \hat{s}_{21}(\tau + \lambda^+)^2, \\ 2Rc_{22} &= (1 - \det S)(\tau^2 - (\lambda^+)^2) - \hat{s}_{21}(\tau - \lambda^+)^2 - \hat{s}_{12}(\tau + \lambda^+)^2, \end{aligned} \tag{4.16}$$

$$Rc_{12} = 2\hat{s}_{11}\tau\lambda^+, \quad Rc_{21} = 2\hat{s}_{22}\tau\lambda^+, \tag{4.17}$$

where $\hat{s}_{ij} = \hat{s}_{ij}(\sigma)$ $(i, j = 1, 2)$ *are the functions defined by* (3.11) *with* (3.4). *Moreover we have*

$$R(\tau, \sigma) = (1 - \det S)\tau^2 - (1 + \det S)\tau\lambda^+ - (1 - \det S + \hat{s}_{12} + \hat{s}_{21})|\sigma|^2/2. \tag{4.18}$$

PROOF. From the proof of (3.7) we obtain (4.18) and the equality

$$\det[(\tau - \lambda^+) I_2 - (\tau + \lambda^+)\hat{S}J] = 2R(\tau, \sigma).$$

Moreover we see from (3.9), (3.10) and the definition of U_{I}^- that

$$(B_{\mathrm{I}} U_{\mathrm{I}}^+)^{-1}(B_{\mathrm{I}} U_{\mathrm{I}}^-) = [(\tau - \lambda^+) I_2 - (\tau + \lambda^+)\hat{S}J]^{-1}[(\tau + \lambda^+) I_2 - (\tau - \lambda^+)\hat{S}J].$$

Therefore we obtain (4.16) and (4.17) by (3.4) and the formula for an inverse matrix.

PROOF of LEMMA 4.4. Since (4.14) and the second hypothesis give $\det S' = \det S$, we find from (3.7) and (4.15) that $R(\tau, \sigma)$ is also Lopatinskii determinant of the problem (P, B'). Hence according to Proposition 4.1 the last hypothesis implies that $R(\tau, \sigma) \neq 0$ when either $\mathrm{Im}\,\tau < 0$ or $\mathrm{Im}\,\tau = 0$ and $|\tau| > |\sigma|$. Therefore it remains only to prove (4.1).

From now on $(\tau, \sigma) \in \boldsymbol{C}_- \times R^2$ varies near a given point $(\eta^0, \sigma^0) \in \mathrm{R}^1 \times (R^2 \backslash 0)$. Moreover by C we denote positive constants independent of (τ, σ). Now let $|\eta^0| < |\sigma^0|$. Then we obtain

$$|R(\tau, \sigma)| \geqq C|\mathrm{Im}\,\tau|. \tag{4.19}$$

In fact from (3.7) we have

$$(\partial R/\partial\tau)(\tau, \sigma) = 2(1 - \det S)\tau - (1 + \det S)\, i(\sqrt{|\sigma|^2 - \tau^2} - \tau^2/\sqrt{|\sigma|^2 - \tau^2}),$$

since $\lambda^+(\tau, \sigma) = i\sqrt{|\sigma|^2 - \tau^2}$. Here $\sqrt{}$ denotes the branch such that $\sqrt{1} = 1$. Moreover under the first hypothesis $(1 + \det S)^{-1}(1 - \det S)$ is not pure imaginary. Hence we see that $(\partial R/\partial\tau)(\eta^0, \sigma^0) \neq 0$. Therefore, if $R(\eta^0, \sigma^0) = 0$, we find by the implicit function theorem that R is represented as

$$R(\tau, \sigma) = (\tau - \psi(\sigma))\, R^{(1)}(\tau, \sigma)$$

where $R^{(1)}(\eta^0, \sigma^0) \neq 0$. Moreover Hersh's condition implies that $\mathrm{Im}\ \psi(\sigma) \geqq 0$. Thus we obtain (4. 19).

To prove (4. 1) first let $0 < |\eta^0| < |\sigma^0|$. Then the second inequality of (4. 1) follows from (4. 16), (4. 17) and (4. 19).

Next let $\eta^0 = 0$ and $\sigma^0 \neq 0$. Then we see in the same way as above that $c_{12}(\tau, \sigma)$, $c_{21}(\tau, \sigma)$ and $(c_{11} - c_{22})(\tau, \sigma)$ are estimated by $|\mathrm{Im}\, \tau|^{-1} |\tau|$, since (4. 16) implies

$$R(c_{11} - c_{22}) = 2(\hat{s}_{12} - \hat{s}_{21})\, \tau \lambda^+ . \tag{4. 20}$$

Therefore to obtain the last inequality of (4. 1) we must only prove

$$|(c_{11} + c_{22})(\tau, \sigma)| \leqq C |\mathrm{Im}\, \tau|^{-1} |\tau| . \tag{4. 21}$$

Now (4. 16) gives

$$R(c_{11} + s_{22}) = -2(\hat{s}_{12} + \hat{s}_{21})\, \tau^2 + (1 - \det S + \hat{s}_{12} + \hat{s}_{21}) |\sigma|^2 , \tag{4. 22}$$

since $\tau^2 - (\lambda^+)^2 = |\sigma|^2$. Hence we have from (4. 18)

$$(c_{11} + c_{22})(\tau, \sigma) = -2 + O(R(\tau, \sigma)^{-1} \tau) .$$

From this and (4. 19) we obtain (4. 21).

Finally let $|\eta^0| = |\sigma^0| \neq 0$. Then we see from (4. 17) and (4. 20) that $c_{12}(\tau, \sigma)$, $c_{21}(\tau, \sigma)$ and $(c_{11} - c_{22})(\tau, \sigma)$ are estimated by $|R(\tau, \sigma)|^{-1} |\lambda^+(\tau, \sigma)|$. Moreover we find from (4. 18) and (4. 22) that

$$c_{11} + c_{22} = 2 + 2R^{-1} \lambda^+ \{(1 + \det S) \tau + O(\lambda^+)\} , \tag{4. 23}$$

since $\tau^2 = (\lambda^+)^2 + |\sigma|^2$. Therefore to obtain the first inequality of (4. 1) we need merely to prove

$$|R(\tau, \sigma)|^{-1} \leqq C |\mathrm{Im}\, \tau|^{-1}\, \mathrm{Im}\, \lambda^+(\tau, \sigma) , \tag{4. 24}$$

because (3. 22) gives

$$(\mathrm{Im}\, \lambda^+(\tau, \sigma)) \cdot |\lambda^+(\tau, \sigma)| \geqq C |\mathrm{Im}\, \tau| . \tag{4. 25}$$

Now let c'_{11} and c'_{22} denote the functions defined in Lemma 4. 5 for the problem (P, B') instead of (P, B). Then we find from Proposition 4. 1 that the L^2-well posedness of (P, B′) implies

$$|(c'_{11} + c'_{22})(\tau, \sigma)| \leqq C |\mathrm{Im}\, \tau|^{-1} (\mathrm{Im}\, \lambda^+(\tau, \sigma)) \cdot |\lambda^+(\tau, \sigma)| .$$

Thus we obtain (4. 24) from (4. 25) and the equality (4. 23) applied to (P, B'), since $\eta^0 \neq 0$ and $1 + \det S' = 1 + \det S \neq 0$. This completes the proof of Lemma 4. 4.

PROOF of part (*b*) of THEOREM. Consider the problem (P, B) with

$$S=\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}.$$

We shall first show that (P, B) is L^2-well posed. Since the matrices S' and B' defined by (4. 14) are as follows:

$$S'=\frac{1}{2}\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \qquad \text{and } B'=[I_2, S', 0],$$

it is not difficult to show that $I_2-(S')^*S'$ is positive semi-definite. Hence ker B' is maximally non-positive for A_1 according to Lemma 3. 2. Therefore the problem (P, B') is L^2-well posed. Thus we find from Lemma 4. 4 that (P, B) is also L^2-well posed.

We shall next show that there does not exist such a matrix W as desecribed in the statement of Lemma 3. 2. Since

$$S^*S=\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, \quad S^*\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}S=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

the left side of (3. 13) is as follows:

$$\alpha(I_2-S^*S)+\beta\left(\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}+S^*\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}S\right)=(\alpha+\beta)\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

This is positive semi-definite if and only if $\alpha+\beta=0$. Therefore we obtain the desired assertion from Lemma 3. 2.

REMARK 4. 6. As was pointed out in [13], it is still open the question whether the L^2-well posedness of the problem (P, B) implies the energy inequality:

$$\int_{x_1>0} |u(t, x)|^2 dx \leqq Ce^{\gamma t}\int_{x_1>0} |u(0, x)|^2 dx, \qquad t>0$$

for $\gamma\geqq\gamma_0$ and for solutions u of $Pu=0$ in $t>0$, $x_1>0$ satisfying $Bu=0$ on $t>0$, $x_1=0$, where C and γ_0 are positive constants.

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