Hokkaido Mathematical Journal Vol. 7 (1978) p. 122-126

On the Gross' property

By Yukio NAGASAKA

(Received June 14, 1977)

1. Introduction

We consider a meromorphic function $w=f(z)$ in the unit disk $D=\{z\,|\,|z|<1\}$ and study the problem of finding a sufficient condition for $f(z)$ to have the Gross' property. The main results are the following:

(i) *If $f(z)$ has the radial limits $f(e^{i\theta})$ of modulus 1 for all points $e^{i\theta}$ of $\Gamma=\{z\,|\,|z|=1\}$ except for a closed set of logarithmic capacity zero, then $f(z)$ has the Gross' property except $\{w\,|\,|w|=1\}$.*

(ii) *If there exists a spiral path approaching Γ on which $f(z)$ tends to infinity ∞, then $f(z)$ has the Gross' property.*

2. The Gross' property except a closed set

Let R be an open Riemann surface and let $w=f(z)$ be a non-constant meromorphic function on R. We denote by Φ_f the covering Riemann surface generated by the inverse function of $w=f(z)$ over the extended w-plane S. A point of Φ_f which is not an algebraic branch point of Φ_f is called a regular point of Φ_f. Take any regular point $q_0 \in \Phi_f$ lying over the basic point $w_0=f(z_0)\neq\infty$ and consider the longest segment ℓ_θ on Φ_f which starts from q_0, consists of only regular points of Φ_f and lies over the half straight line $\arg(w-w_0)=\theta$ $(0\leqq\theta<2\pi)$ on the finite w-plane. If ℓ_θ has finite length, either the end point of ℓ_θ is an algebraic branch point of Φ_f or ℓ_θ defines an accessible boundary point of Φ_f. The set of all algebraic branch points is countable. Let F be a closed subset of S. We denote by $S(q_0, F)$ the set of all argments θ for which ℓ_θ has finite length and ℓ_θ defines an accessible boundary point B_θ of Φ_f whose projection b_θ is a point of $S-F\cup\{\infty\}$. As a set of points, $H(q_0)=\bigcup_{0\leq\theta<2\pi}\ell_\theta$ is mapped one to one conformally onto a domain $\Omega(q_0)$ in R by $f^{-1}(w)$ and, for every $\theta\in S(q_0, F)$, ℓ_θ is mapped to a path L_θ which starts from z_0 and tends to the the ideal boundary of R.

DEFINITION. We shall say that $f(z)$ *has the Gross' property except F* if the measure m of $S(q_0, F)$ equals zero for every regular point $q_0\in\Phi_f$ whose projection is a point of $S-F\cup\{\infty\}$.

If $F=\phi$ and $f(z)$ has the Gross' property except F, $f(z)$ is said to have the Gross' property (cf. T. Kuroda and A. Sagawa [3]). We set $S(q_0)=S(q_0, \phi)$. W. Gross proved that every meromorphic function in the finite z-plane has the Gross' property (Gross' star theorem).

Since there exists an Evans-Selberg potential on an open Riemann surface belonging to O_G (Z. Kuramochi [2]), we have the next proposition by the same method that M. Tsuji [6] used to extend the Gross' star theorem (cf. Z. Yujobo [7] and p. 12 in K. Noshiro [4]).

PROPOSITION. (i) *Let Ω be a subdomain of R with piecewise analytic relative boundary $\partial\Omega$ such that $\hat{\Omega}\in O_G$, where $\hat{\Omega}$ is the double of Ω along $\partial\Omega$. We set $S_\Omega=\{\theta\in S(q_0)|L_\theta\subset\Omega\}$. Then $m(S_\Omega)=0$.*

(ii) *Let R be a domain in the finite z-plane and K be a compact set of logarithmic capacity zero on ∂R. We denote by S_K the set of all $\theta\in S(q_0)$ such that L_θ terminates at a point of K. Then $m(S_K)=0$.*

3. Theorem 1

Let $w=f(z)$ be a meromorphic function in the unit disk D. For every $e^{i\theta}\in\Gamma$, we define the radial cluster set $C_\rho(f, e^{i\theta})$ of f at $e^{i\theta}$ by

$$C_\rho(f, e^{i\theta})=\bigcap_{n=1}^{\infty}\overline{f(\rho_n(\theta))},$$

where

$$\rho_n(\theta)=\left\{re^{i\theta}\middle|1-\frac{1}{n}<r<1\right\}.$$

THEOREM 1. *Let F be a closed subset of S. If $C_\rho(f, e^{i\theta})\subset F$ for every point $e^{i\theta}\in\Gamma$ except for a closed set E of logarithmic capacity zero, then $f(z)$ has the Gross' property except F.*

The result (i) *in the introduction corresponds to $F=\{w|\ |w|=1\}$ in this theorem.*

PROOF. We set $S_n=\{\theta\in S(q_0, F)|b_\theta\in S-F_n\}$, where F_n is an $\frac{1}{n}$-closed neighborhood of $F\cup\{\infty\}$ in the Riemann sphere S. Since $S(q_0, F)=\bigcup_n S_n$, we have only to show $m(S_n)=0$ for every n.

We fix any n_0 and set $F_0=F_{n_0}$ and $S_0=S_{n_0}$ for simplicity. And we set

$$E_n=\left\{e^{i\theta}\in\Gamma-E\middle|\overline{f(\rho_n(\theta))}\subset F_0\right\}.$$

Since $C_\rho(f, e^{i\theta})\subset F$ for every $e^{i\theta}\in\Gamma-E$, we have $\Gamma=E=\bigcup_n E_n$. We denote by $E_n^{(1)}$ the subset of all points $e^{i\theta}\in E_n$ such that there exist two sequences

$\{e^{i\theta_k}\}_{k=1}^{\infty}$ and $\{e^{i\theta'_k}\}_{k=1}^{\infty}$ in E_n which satisfy $\theta_k<\theta<\theta'_k$ $(k=1, 2, \cdots)$ and $\lim_{k\to\infty}\theta_k=\lim_{k\to\infty}\theta'_k=\theta$. And we set

$$E_n^{(0)}=E_n-E_n^{(1)} \qquad \text{and} \qquad E_0=E\cup(\bigcup_n E_n^{(0)}).$$

Since $E_n^{(0)}$ is a countable set, E_0 is an F_σ-set of logarithmic capacity zero.

We shall show that L_θ terminates at a point of E_0 for every $\theta\in S_0$. Fix any $\theta\in S_0$. Since $\mathscr{l}_\theta$ defines an accessible boundary point of Φ_f, L_θ tends to Γ. Suppose that L_θ has two cluster points α, β $(\alpha\neq\beta)$ on Γ. Then the set of all cluster points of L_θ is a continuum C on Γ. Since Cap $E=0$, $\mathrm{Int}(C)\cap(\Gamma-E)\neq\phi$. Take any point γ of $\mathrm{Int}(C)\cap(\Gamma-E)$. Then there exists a sequence $\{z_k\}_{k=1}^{\infty}$ on $\overline{o\gamma}\cap L_\theta$ such that $\lim_{k\to\infty} z_k=\gamma$, where $\overline{o\gamma}$ is the radius to γ. Then, since $\gamma\in\Gamma-E$, the set A of all cluster points of $\{f(z_k)\}_{k=1}^{\infty}$ is contained in F. But, since $\{z_k\}_{k=1}^{\infty}\subset L_\theta$, we have $A=\{b_\theta\}\in S-F_\theta$. This is a contradiction. Hence we see that L_θ terminates at a point $e^{i\varphi}$ of Γ. Next, suppose $e^{i\varphi}\notin E_0$. Then $e^{i\varphi}\in E_n^{(1)}$ for some n. Then there exists a sequence $\{z'_k\}_{k=1}^{\infty}$ on $L_\theta\cap\rho_n$ such that $\lim_{k\to\infty} z'_k=e^{i\varphi}$, where $\rho_n=\bigcup\{\rho_n(\theta)|\theta\in E_n\}$. Then, since $\{z'_k\}_{k=1}^{\infty}\subset\rho_n$, the set A' of all cluster points of $\{f(z'_k)\}_{k=1}^{\infty}$ is contained in F_0. But, since $\{z'_k\}_{k=1}^{\infty}\subset L_\theta$, we have $A'=\{b_\theta\}\in S-F_0$. This is a contradiction. Hence we have $e^{i\varphi}\in E_0$.

Since E_0 is an F_σ-set of logarithmic capacity zero, by (ii) of Proposition, we have that $f(z)$ has the Gross' property except F. This completes the proof.

If a bounded analytic function $f(z)$ in D has the radial limits $f(e^{i\theta})$ of modulus 1 for all θ except for a set of measure zero, then we call $f(z)$ a function of class (U) in the sence of Seidel. Every function of class (U) has the Iversen's property in $\{w|\,|w|<1\}$ but there exists a function in class (U) which has not the Gross' property in $\{w|\,|w|<1\}$ (Z. Kuramochi [1] and p. 36 in [4]). Let (U^*) be the subclass of $f\in(U)$ such that $f(z)$ has the radial limits $f(e^{i\theta})$ of modulus 1 for all θ except for a closed set of logarithmic capacity zero.

COROLLARY. Every function of class (U^*) has the Gross' property in $\{w|\,|w|<1\}$.

4. Theorem 2

Let $\sigma: z=z(t)$ $(0\leq t<\infty)$ be a piecewise analytic curve on D. If $z(t)$ has the following properties, we use the term "*spiral*": $|z(t)|$ and $\arg z(t)$ are strictly increasing, $\lim_{t\to\infty}|z(t)|=1$ and $\lim_{t\to\infty}\arg z(t)=\infty$.

Let σ be a spiral. For simplicity, we suppose $z(0)=\frac{1}{2}$ and $\arg z(0)=0$. We set

$$\sigma_n=\left\{z(t)\middle|2(n-1)\pi\leq \arg z(t)<2n\pi\right\}.$$

Let $\{J_{n,i}\}_i$ be a finite sequence of subpaths of σ_n such that $J_{n,i}\cap J_{n,j}=\phi$ $(i\neq j)$. We set $J_n=\bigcup_i J_{n,i}$ and $J=\bigcup_{n=1}^{\infty} J_n$.

THEOREM 2. *Let $w=f(z)$ be a meromorphic function in D. If*

$$\lim_{n\to\infty}\operatorname{Cap}(J_n)=0 \qquad \textit{and} \qquad \lim_{\substack{z(t)\in\sigma-J\\ t\to\infty}} f\big(z(t)\big)=\infty\,,$$

then $f(z)$ has the Gross' property.

PROOF. We set $\Omega=D-(\sigma-J)=D-\bigcup_{n=1}^{\infty}(\sigma_n-J_n)$. Let $\hat{\Omega}$ be the double of Ω along $\sigma-J$. We shall show $\hat{\Omega}\in 0_G$.

Let I_n be the closed interval $[a_n, a_{n+1}]$, where a_n is a point of σ such that $\arg a_n=2(n-1)\pi$ and let G_n be a subdomain of Ω whose boundary $(\bigcup_{i=1}^{n-1}(\sigma_i-J_i))\cup\sigma_n\cup I_n$. Set $K=\left\{z\middle| |z|\leqq\frac{1}{4}\right\}$. Let ω_n be the harmonic function on G_n-K which is equal to zero on ∂K and to 1 on $J_n\cup I_n$ and whose normal derivative vanishes on $\bigcup_{i=1}^{n}(\sigma_i-J_i)$, and let ω'_n be the harmonic function on $G'_n=\{z|\,|z|<2\}-K\cup J_n\cup I_n$ which is equal to zero on $\partial K\cup\{z|\,|z|=2\}$ and to 1 on $J_n\cup I_n$. We denote by $D(\ )$ the Dirichlet integral. By Dirichlet principle, we have

$$D_{G_n-K}(\omega_n)\leqq D_{G'_n}(\omega'_n)\,.$$

Since

$$\lim_{n\to\infty}\operatorname{Cap}(J_n\cup I_n)\leqq\lim_{n\to\infty}\operatorname{Cap}(J_n)+\lim_{n\to\infty}\operatorname{Cap}(I_n)=0\,,$$

we have $\lim_{n\to\infty}D_{G'_n}(\omega'_n)=0$. Then $\lim_{n\to\infty}D_{G_n-K}(\omega_n)=0$ and we have $\hat{\Omega}\in O_G$. Next, we set

$$\Omega_n=D-\bigcup_{k=n}^{\infty}(\sigma_k-J_k)\,.$$

Similary, we get $\hat{\Omega}_n\in 0_G$.

Take any regular point $q_0\in\Phi_f$ with the projection $w_0=f(z_0)$ and consider $S(q_0)$. We set

$$S_n=\left\{\theta\in S(q_0)\middle|\,|b_\theta-w_0|<n\right\}$$

for every n. Fix any n. Since $\lim_{\substack{z(t)\in\sigma-J \\ t\to\infty}} f(z(t))=\infty$, there exists an m_0 such that $|f(z(t))|>|w_0|+n$ for all $z(t)\in \bigcup_{m=m_0}^{\infty}(\sigma_m-J_m)$. Then we have that

$$\bigcup\left\{L_\theta|\theta\in S_n\right\}\subset\Omega_{m_0}.$$

Since $\hat{\Omega}_{m_0}\in O_G$, by (i) of Proposition, we have $m(S_n)=0$. Since $S(q_0)=\bigcup_{n=1}^{\infty}S_n$, we have $m(S(q_0))=0$. This completes the proof.

COROLLARY 1. *If there exists a spiral path on which $f(z)$ tends to infinity, then $f(z)$ has the Gross' property.*

Let (V) be the class of holomorphic and unbounded function on D with the property that it remains bounded on some spiral path in D. G. Variron proved that if $f(z)$ is a function of (V), there exists another spiral path on which $f(z)$ tends to infinity (cf. W. Seidel [5]). From this Variron's theorem, we have the following:

COROLLARY 2. *Every function of (V) has the Gross' property.*

References

[1] Z. KURAMOCHI: On the behaviour of analytic functions on abstract Riemann surfaces, Osaka Math. J. 7 (1955), 109–127.

[2] Z. KURAMOCHI: Evans's theorem on abstract Riemann surface with null-boundaries, Proc. Japan Acad. 32 (1956), 1–9.

[3] T. KURODA and A. SAGAWA: Remark on the Gross property, Tohoku Math. J. 20 (1968), 394–399.

[4] K. NOSHIRO: Cluster sets, Springer-Verlag, 1960.

[5] W. SEIDEL: Holomorphic functions with spiral asypmtotic paths, Nagoya Math. J. 14 (1959), 159–171.

[6] M. TSUJI: On the behavior of a meromorphic function in the neighborhood of a closed set of capacity zero, Proc. Japan Acad. 18 (1942), 213–219.

[7] Z. YUJOBO: On the Riemann surface, no Green function of which exists, Mathematica Japonicae 2 (1951), 61–68.

Department of Mathematics,
Hokkaido University