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On vanishing contact Bochner curvature tensor

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Recently T. Kashiwada [2][1] has given various necessary and sufficient conditions in order that a Kählerian space has vanishing Bochner curvature tensor. In the present paper, we study some conditions in order that a Sasakian space has vanishing contact Bochner curvature tensor and also give some applications of our results.

In § 1 state fundamental identities for the contact Bochner curvature tensor in a Sasakian space.

§ 2 is devoted to the study of conditions in order that a Sasakian space has vanishing contact Bochner curvature tensor and we give two theorems which are analogous to the results due to T. Kashiwada [2].

T. Sakaguchi [4] has introduced the concept of a complex semi-symmetric metric F-connection in a Kählerian space and in terms of certain properties of the connection he has given a sufficient condition in order that a Kählerian space has vanishing Bochner curvature tensor. On the other hand, in a Sasakian space the concept of a contact conformal connection has been introduced by K. Yano [5]. Corresponding to the study of T. Sakaguchi [4], in § 3 we consider a Sasakian space admitting a contact conformal connection. Then, as an application of our first theorem in § 2, we get a sufficient condition in order that a Sasakian space with a contact conformal connection has vanishing contact Bochner curvature tensor.

M. Matsumoto and G. Chuman [3] have studied a compact Sasakian space with vanishing contact Bochner curvature tensor and have given various conditions for the second Betti number to be zero. In § 4, making use of our second theorem in § 2, we show that the theorem of M. Matsumoto and G. Chuman is valid even if one of the conditions in it is replaced by a weaker one.

§ 1. Preliminaries.

Let M be a m-dimensional Riemannian space covered by a system of coordinate neighborhoods $\{(U; y^h)\}$, where here and in the sequel, the indices

1) Numbers in brackets refer to references at the end of the paper.

$h, i, j, k, \cdots$ run over the range $\{0, 1, \cdots, m-1\}$, and g_{ji} a positive definite Riemannian metric tensor of M. Moreover, let $\left\{ {h \atop j\ i} \right\}$, ∇_j, $K_{kji}{}^h$, K_{ji} and K be the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation induced from $\left\{ {h \atop j\ i} \right\}$, the curvature tensor with respect to ∇_j, the Ricci tensor and the scalar curvature, respectively.

M is called Sasakian if there exists a unit Killing vector field η^h such that

$$\nabla_k \nabla_j \eta_i = \eta_j g_{ki} - \eta_i g_{kj} . \tag{1.1}$$

Throughout this paper we only consider a Sasakian space $M(\eta^h, g_{ji})$.

If we put $\varphi_i{}^h = \nabla_i \eta^h$, $(\varphi_i{}^h, \eta_i, g_{ji})$ give an almost contact metric structure to M and hence M is orientable and m is odd: $m=2n+1$.

Applying the Ricci identity to η_i we have

$$\nabla_k \nabla_j \eta_i - \nabla_j \nabla_k \eta_i = -K_{kji}{}^h \eta_h ,$$

from which it follows that

$$K_{kji}{}^h \eta_h = \eta_k g_{ji} - \eta_j g_{ki}, \quad K_k{}^h \eta_h = (m-1) \eta_k .$$

As (1.1) becomes

$$\nabla_j \varphi_i{}^h = \eta_i \delta_j^h - g_{ji} \eta^h , \tag{1.2}$$

applying the Ricci identity to $\varphi_i{}^h$ we have

$$\nabla_k \nabla_j \varphi_i{}^h - \nabla_j \nabla_k \varphi_i{}^h = K_{kjl}{}^h \varphi_i{}^l - K_{kji}{}^l \varphi_l{}^h ,$$

from which we can get the following formulas:

$$K_{kjlh} \varphi_j{}^l + K_{kjil} \varphi_h{}^l = \varphi_{ki} g_{jh} - \varphi_{ji} g_{kh} + \varphi_{jh} g_{ki} ,$$

$$\frac{1}{2} K_{kjih} \varphi^{ih} = K_{ki} \varphi_j{}^i + (m-2) \varphi_{kj}, \quad K_{ki} \varphi_j{}^i = -K_{ji} \varphi_k{}^i .$$

The contact Bochner curvature tensor $B_{kji}{}^h$ is defined by

$$\begin{aligned} B_{kji}{}^h = & K_{kji}{}^h + (\delta_k^h - \eta_k \eta^h) L_{ji} - (\delta_j^h - \eta_j \eta^h) L_{ki} + L_k{}^h (g_{ji} - \eta_j \eta_i) \\ & - L_j{}^h (g_{ki} - \eta_k \eta_i) + \varphi_k{}^h M_{ji} - \varphi_j{}^h M_{ki} + M_k{}^h \varphi_{ji} - M_j{}^h \varphi_{ki} \\ & - 2(M_{kj} \varphi_i{}^h + \varphi_{kj} M_i{}^h) + (\varphi_k{}^h \varphi_{ji} - \varphi_j{}^h \varphi_{ki} - 2\varphi_{kj} \varphi_i{}^h) , \end{aligned}$$

where

$$L_{ji} = -\frac{1}{m+3} \Big(K_{ji} + (L+3) g_{ji} - (L-1) \eta_j \eta_i \Big) ,$$

$$\begin{aligned} M_{ji} &= -L_{jk} \varphi_i{}^k \\ &= \frac{1}{m+3} \Big(K_{jk} \varphi_i{}^k - (L+3) \varphi_{ji} \Big) , \end{aligned}$$

and

$$L=g^{ji}L_{ji}=-\frac{K+3m+1}{2(m+1)}.$$

The contact Bochner curvature tensor satisfies the following identities:

$$B_{kjih}=B_{ihkj}=-B_{jkih}=-B_{kjhi},\ B_{kjih}+B_{jikh}+B_{ikjh}=0, \tag{1.3}$$

and

$$B_{kjih}\eta^{k}=0. \tag{1.4}$$

§ 2. Main theorems.

In the following, by the word 'φ-basis' we mean an orthonormal basis $\{e_i\}$ such that $e_0=\eta$, $e_{\lambda^*}=e_{\lambda+n}=\varphi e_\lambda$ ($\lambda=1, \cdots, n$). First of all, we state

LEMMA 2.1. *In an m-dimensional Sasakian space M, the contact Bochner curvature tensor vanishes if and only if $B(e_i, e_j, e_i, e_j)=0$ for every φ-basis.*

PROOF. Since the contact Bochner curvature tensor B satisfies the relations (1.3), we have the following equation (cf. Bishop and Goldberg [1]):

$$B(X, Y, X, Y)=\frac{1}{32}\Big(3P(X+\varphi Y)+3P(X-\varphi Y)$$
$$-P(X+Y)-P(X-Y)-4P(X)-4P(Y)\Big),$$

for any vectors X and Y orthogonal to the vector η, where $P(X)=B(X, \varphi X, X, \varphi X)$. Then, taking account of (1.4), we obtain the result of Lemma 2.1.

With respect to a φ-basis $\{\eta, e_\lambda, e_{\lambda^*}\}$, g_{ji}, φ_{ji}, K_{kjih} and K_{ji} have the numerical values as follows:

$$g_{ji}=\delta_{ji},$$
$$\varphi_{\lambda\lambda^*}=-\varphi_{\lambda^*\lambda}=1,\ \varphi_{j\lambda}=0\ (j\neq\lambda^*),\ \varphi_{00}=0,$$
$$K_{kj\lambda\mu^*}+K_{kj\lambda^*\mu}=\varphi_{k\lambda}\delta_{j\mu}-\varphi_{k\mu}\delta_{j\lambda}-\varphi_{j\lambda}\delta_{k\mu}+\varphi_{j\mu}\delta_{k\lambda},\ K_{\lambda00\lambda}=K_{\lambda^*00\lambda^*}=1,$$
$$K_{\lambda^*\mu^*}=K_{\lambda\mu},\ K_{\lambda\mu^*}=-K_{\lambda^*\mu},\ K_{00}=m-1,\ K_{\lambda0}=K_{\lambda^*0}=0.$$

Putting

$$B_{kjih}=K_{kjih}+\frac{1}{m+3}U_{kjih},$$

we have

$$B_{abba}=K_{abba}+\frac{1}{m+3}U_{abba},$$

where

$$(2.1)\qquad \begin{aligned} &U_{abba}=-(K_{aa}+K_{bb}-k+4)\,,\\ &\qquad (|a|\neq|b|,\ |a|=\lambda \text{ for } a=\lambda \text{ or } \lambda^*;\ a,\ b=1,2,\cdots,2n)\\ &U_{\lambda\lambda^*\lambda^*\lambda}=-8K_{\lambda\lambda}+4k+3m-7\,, \end{aligned}$$

k being $(K+m-1)/(m+1)$.

Then we get the following first main theorem which has analogy to the theorem due to Kashiwada [2]:

THEOREM 2.2. *In an $m(\geqq 9)$-dimensional Sasakian space, if*

$$K_{abcd}=0\,,\qquad (|a|,\ |b|,\ |c|,\ |d|\neq)$$

holds for every φ-basis, then the contact Bochner curvature tensor vanishes. The converse is true. (By '$|a|$, $|b|$, $|c|$, $|d|\neq$' we mean that $|a|$, $|b|$, $|c|$, $|d|$ differ from one another.)

PROOF. Let

$$(2.2)\qquad K_{abcd}=0\qquad (|a|,|b|,|c|,|d|\neq)$$

for a φ-basis $\{\eta, e_\lambda, \varphi e_\lambda\}$.

We take another φ-basis

$$(*)\qquad \begin{aligned} &e'_\lambda=te_\lambda+se_\mu\\ &e'_\mu=-se_\lambda+te_\mu\\ &e'_a=e_a\qquad (|a|\neq\lambda,\mu)\ \ (e'_0=e_0) \end{aligned}$$

where t and s are real numbers such that $t^2+s^2=1$ and $ts\neq 0$. As (2.2) holds for this φ-basis, we have

$$0=g\big(K(e'_\lambda, e_a)\,e'_\mu, e_b\big)=-ts(K_{\lambda a\lambda b}-K_{\mu a\mu b})\,,$$

i. e. $$K_{a\lambda\lambda b}=K_{a\mu\mu b}\qquad (|a|,|b|,\lambda,\mu\neq)\,.$$

By replacing e_λ with e_{λ^*}, we have

$$K_{a\lambda^*\lambda^*b}=K_{a\mu\mu b}\qquad (|a|,|b|,\lambda,\mu\neq)\,.$$

So we get

$$(2.3)\qquad K_{a\lambda\lambda b}=K_{a\lambda^*\lambda^*b}\qquad (|a|,|b|,\lambda\neq)\,.$$

Since (2.3) is true for every φ-basis, for φ-basis (*) we know

$$g\big(K(e'_\lambda, e_{\nu^*})\,e_{\nu^*}, e'_\mu\big)=g\big(K(e'_\lambda, e_\nu)\,e_\nu, e'_\mu\big)\qquad (\lambda,\mu,\nu\neq)$$

which implies

(2. 4) $$K_{\lambda\nu^*\nu^*\lambda}-K_{\mu\nu^*\nu^*\mu}=K_{\lambda\nu\nu\lambda}-K_{\mu\nu\nu\mu} \qquad (\lambda, \mu, \nu \neq).$$

Replacing e_μ with e_{μ^*} and adding it to (2. 4), by virtue of (2. 3) we have

(2. 5) $$K_{\lambda\nu^*\nu^*\lambda}=K_{\lambda\nu\nu\lambda} \qquad (\lambda\neq\nu).$$

Since (2. 5) is true for every φ-basis, computing (2. 5) with respect to the φ-basis (*), we have

$$g\left(K(e'_\lambda, e'_{\mu^*})\, e'_{\mu^*}, e'_\lambda\right)=g\left(K(e'_\lambda, e'_\mu)\, e'_\mu, e'_\lambda\right),$$

and we obtain after all,

(2. 6) $$K_{\lambda\lambda^*\lambda^*\lambda}+K_{\mu\mu^*\mu^*\mu}=8K_{\lambda\mu\mu\lambda}-6 \qquad (\lambda\neq\mu).$$

Then we have

$$\sum_{\mu(\neq\lambda)}^{n}(K_{\lambda\lambda^*\lambda^*\lambda}+K_{\mu\mu^*\mu^*\mu})=8\sum_{\mu=1}^{n}K_{\lambda\mu\mu\lambda}-6(n-1)$$

$$(n-2)\,K_{\lambda\lambda^*\lambda^*\lambda}+u=4\left(\sum_{\mu=1}^{n}K_{\lambda\mu\mu\lambda}+\sum_{\mu(\neq\lambda)}^{n}K_{\lambda^*\mu^*\mu\lambda}\right)-6(n-1),$$

or

$$(n+2)\,K_{\lambda\lambda^*\lambda^*\lambda}+u=4K_{\lambda\lambda}-3m+5,$$

i. e.

(2. 7) $$K_{\lambda\lambda^*\lambda^*\lambda}=(4K_{\lambda\lambda}-3m+5-u)/(n+2),$$

where we put $u=\sum_{\mu=1}^{n}K_{\mu\mu^*\mu^*\mu}$ and take account of $K_{\lambda\mu\mu\lambda}=K_{\lambda\mu^*\mu^*\lambda}$ and $K_{\lambda00\lambda}=1$. Taking sum of (2. 7) from $\lambda=1$ to $\lambda=n$, we have

(2. 8) $$u=\left(2K-(1/2)(m-1)(3m-1)\right)/(m+1).$$

So from (2. 7) and (2. 8) we get

(2. 9) $$K_{\lambda\lambda^*\lambda^*\lambda}=(8K_{\lambda\lambda}-4k-3m+7)/(m+3).$$

On the other hand, as

(2. 10) $$K_{\lambda\mu\mu\lambda}=(K_{\lambda\lambda}+K_{\mu\mu}-k+4)/(m+3),$$

(2. 11) $$K_{\lambda^*\mu^*\mu^*\lambda^*}=K_{\lambda\mu\mu\lambda},$$

we obtain from (2. 1), (2. 5) and (2. 9)$\sim$(2. 11)

(2. 12) $$\begin{aligned} B_{abba}&=K_{abba}+\frac{1}{m+3}U_{abba}=0 \qquad (|a|\neq|b|)\\ B_{\lambda\lambda^*\lambda^*\lambda}&=K_{\lambda\lambda^*\lambda^*\lambda}+\frac{1}{m+3}U_{\lambda\lambda^*\lambda^*\lambda}=0.\end{aligned}$$

So by Lemma 2.1, we get

$$B=0.$$

The converse is trivial since $U_{abcd}=0$ for $|a|, |b|, |c|, |d| \neq$.

REMARK: In this proof, we know the property (2.12) depends only on the property (2.6).

Next we have several necessary and sufficient conditions to be $B=0$ in terms of the sectional curvature which is an analogous to that of Kashiwada [2].

THEOREM 2.3. *Let M be an $m(\geqq 9)$-dimensional Sasakian space. Then, the followings are equivalent to one another at every point P of M.*

(1) *The contact Bochner curvature tensor $B(P)=0$.*

(2) *For every φ-basis at P,*

$$H(e_\lambda, e_{\lambda^*})+H(e_\mu, e_{\mu^*})=8H(e_\lambda, e_\mu)-6 \qquad (\lambda\neq\mu),$$

where $H(X, Y)$ means the sectional curvature with respect to the plane spanned by X and Y.

(3) *For each φ-holomorphic 8-plane $W\subset T_P(M)$,*

$$k_P(W, b)=H(e_1, e_2)+H(e_3, e_4)$$

is independent of φ-basis $b=\{e_1, \cdots, e_4, \varphi e_1, \cdots, \varphi e_4\}$ of W.

(4) *For every orthogonal 8 vectors $\{e_1, \cdots, e_4, \varphi e_1, \cdots, \varphi e_4\}$ of $T_P(M)$,*

$$H(e_1, e_2)+H(e_3, e_4)=H(e_1, e_4)+H(e_2, e_3).$$

PROOF. (2)$\Rightarrow$(1) is noted at the last of proof of Theorem 2.2. (1)$\Rightarrow$(2) is trivial since (2.1) and

$$K_{abba}=-\frac{1}{m+3}U_{abba}.$$

(3)$\Rightarrow$(4) is trivial.

(4)$\Rightarrow$(1): Let $\{\eta, e_1, \cdots, e_n, \varphi e_1, \cdots, \varphi e_n\}$ be arbitrary φ-basis of $T_p(M)$. For $\{e_\kappa, e_\lambda, e_\mu, e_\nu, \varphi e_\kappa, \varphi e_\lambda, \varphi e_\mu, \varphi e_\nu\}$, by assumption,

$$K_{\kappa\lambda\lambda\kappa}+K_{\mu\nu\nu\mu}=K_{\kappa\nu\nu\kappa}+K_{\lambda\mu\mu\lambda}.$$

We take another orthonormal vectors $\{e_\kappa, e'_\lambda, e'_\mu, e_\nu, \varphi e_\kappa, \varphi e'_\lambda, \varphi e'_\mu, \varphi e_\nu\}$ such that

$$e'_\lambda=te_\lambda+se_\mu$$

$$e'_\mu=-se_\lambda+te_\mu, \qquad (t^2+s^2=1,\ ts\neq 0).$$

Since $H(e_\kappa, e'_\lambda)+H(e'_\mu, e_\nu)=H(e_\kappa, e_\nu)+H(e'_\lambda, e'_\mu)$, it follows

$$(2.13)\qquad K_{\lambda\kappa\kappa\mu}=K_{\lambda\nu\nu\mu}\,.$$

Since (2.13) is true for every φ-basis, for the above basis,

$$g\left(K(e_\kappa, e'_\lambda)\, e'_\lambda, e_\nu\right)=g\left(K(e_\kappa, e'_\mu)\, e'_\mu, e_\nu\right)$$

which implies

$$K_{\kappa\lambda\mu\nu}+K_{\kappa\mu\lambda\nu}=0\,.$$

Then by Bianchi identity, we get $K_{\kappa\lambda\mu\nu}=0$. Replacing $e_\kappa\to e_{\kappa^*}$, $e_\lambda\to e_{\lambda^*}$, $\cdots$ etc, we obtain $K_{abcd}=0$ ($|a|$, $|b|$, $|c|$, $|d|\neq$). So, by Theorem 2.2, the contact Bochner curvature tensor vanishes.

(1)$\Rightarrow$(3): Let $B=0$. Then, for a φ-basis, it follows

$$K_{abba}=\frac{1}{m+3}(K_{aa}+K_{bb}-k+4)\qquad (|a|\neq|b|)\,.$$

Let $b=\{e_1, e_2, e_3, e_4, \varphi e_1, \varphi e_2, \varphi e_3, \varphi e_4\}$, $b'=\{e'_1, e'_2, e'_3, e'_4, \varphi e'_1, \varphi e'_2, \varphi e'_3, \varphi e'_4\}$, be basis of $W\subset T_P(M)$. We construct two basis of $T_P(M)$ such that

$$f=\{\eta, e_1, \cdots, e_n, \varphi e_1, \cdots, \varphi e_n\}$$

$$f'=\{\eta, e'_1, \cdots, e'_4,\ e_5, \cdots, e_n,\ \varphi e'_1, \cdots, \varphi e'_4,\ \varphi e_5, \cdots, \varphi e_n\}\,.$$

Then we have

$$(2.14)\qquad H(e_1, e_2)+H(e_3, e_4)=\frac{1}{m+3}\left(\sum_{\alpha=1}^{4}K_{\alpha\alpha}-2k+8\right).$$

Let K_{ii}, K'_{ii} be components of Ricci tensor with respect to the basis f and f', respectively. So, as $K=\sum K_{ii}=\sum K'_{ii}$ and $K_{00}=K'_{00}$, $K_{\lambda\lambda}=K'_{\lambda\lambda}$ and $K_{\lambda^*\lambda^*}=K'_{\lambda^*\lambda^*}(\lambda>4)$, we have

$$\sum_{\alpha=1}^{4}K_{\alpha\alpha}=\sum_{\alpha=1}^{4}K'_{\alpha\alpha}\,.$$

Then by virtue of (2.14), we know that $k_P(W, b)$ is independent of b.

We note that the proof of (1)$\Rightarrow$(2) in the above is true too in the case of $m\geqq 5$.

§ 3. Contact conformal connection.

An affine connection $\bar{\nabla}$ is said to be contact conformal connection if its coefficients $\Gamma_{ji}{}^h$ are given by

$$\Gamma_{ji}{}^h=\left\{{h \atop j\ i}\right\}+(\delta_j^h-\eta_j\eta^h)\,p_i+(\delta_i^h-\eta_i\eta^h)\,p_j-(g_{ji}-\eta_j\eta_i)\,p^h$$
$$+\varphi_j{}^h(q_i-\eta_i)+\varphi_i{}^h(q_j-\eta_j)-\varphi_{ji}(q^h-\eta^h)\,,$$

where $p_i=\partial_i p$ for a certain function p, $q_i=-\varphi_i{}^j p_j$ and $\mathscr{L}_\eta p=p_i\eta^i=0$. Then we have

$$q_i\eta^i=0,\ p_i q^i=0,\ \lambda=p_i p^i=q_i q^i\,,$$

and

$$\bar{\nabla}_j\varphi_i{}^h=0,\ \bar{\nabla}_j\eta_i=0,\ \bar{\nabla}_k g_{ji}=p_k(g_{ji}-\eta_j\eta_i)\,.$$

Now, we compute the curvature tensor of $\Gamma_{ji}{}^h$:

$$R_{kji}{}^h=\partial_k\Gamma_{ji}{}^h-\partial_j\Gamma_{ki}{}^h+\Gamma_{kl}{}^h\Gamma_{ji}{}^l-\Gamma_{jl}{}^h\Gamma_{ki}{}^l\,.$$

By a straightforward computation, we find ([5])

$$\begin{aligned} R_{kji}{}^h=&K_{kji}{}^h-(\delta_k^h-\eta_k\eta^h)P_{ji}+(\delta_j^h-\eta_j\eta^h)P_{ki}-P_k{}^h(g_{ji}-\eta_j\eta_i)\\ &+P_j{}^h(g_{ki}-\eta_k\eta_i)-\varphi_k{}^hQ_{ji}+\varphi_j{}^hQ_{ki}-Q_k{}^h\varphi_{ji}+Q_j{}^h\varphi_{ki}\\ &+(\nabla_k q_j-\nabla_j q_k)\varphi_i{}^h+2\varphi_{kj}(q_ip^h-p_iq^h)\\ &+(\varphi_k{}^h\varphi_{ji}-\varphi_j{}^h\varphi_{ki}-2\varphi_{kj}\varphi_i{}^h)\,,\end{aligned}\tag{3.1}$$

where

$$P_{ji}=\nabla_j p_i-p_jp_i+(q_j-\eta_j)(q_i-\eta_j)+\frac{1}{2}\lambda(g_{ji}-\eta_j\eta^i)\,,$$

$$Q_{ji}=-P_{jk}\varphi_i{}^k=\nabla_j q_i-p_j(q_i-\eta_i)-p_i(q_j-\eta_j)+\frac{1}{2}\lambda\varphi_{ji}\,.$$

Then we have

$$R_{kjih}=-R_{jkih}=R_{kjhi}\,.$$

Now, we assume (cf. T. Sakaguchi [4]) that

$$\nabla_j q_i-2q_jp_i+p_j\eta_i+\eta_jp_i+\lambda\varphi_{ji}=0\,.$$

Then, by a direct computation we have

$$R_{kjih}+R_{jikh}+R_{ikjh}=0\,,$$

from which we get

$$R_{kjih}=R_{ihkj}\,.$$

For any φ-holomorphic section $\sigma=(u^h,\varphi_i{}^hu^i)$, φ-holomorphic sectional curvature with respect to $\bar{\nabla}$ is defined by

$$H(\sigma)=H(u^h)=-(R_{kjih}\varphi_t{}^ku^tu^j\varphi_s{}^iu^su^h)/(g_{kj}u^ku^jg_{ih}u^iu^h)\,.$$

Then we can easily see that this $H(\sigma)$ is uniquely determined by the φ-holomorphic section σ and is independent of the choice of u^h on σ. If this

φ-holomorphic sectional curvature is independent of the φ-holomorphic section at each point of M, then a contact conformal connection is said to be of constant φ-holomorphic sectional curvature. If we assume that $\bar{\nabla}$ is of constant φ-holomorphic sectional curvature, then we obtain

$$
\begin{aligned}
R_{kjih} = c\Big(&(g_{kh}-\eta_k\eta_h)(g_{ji}-\eta_j\eta_i)-(g_{jh}-\eta_j\eta_h)(g_{ki}-\eta_k\eta_i) \\
&+\varphi_{kh}\varphi_{ji}-\varphi_{jh}\varphi_{ki}-2\varphi_{kj}\varphi_{ih}\Big)\,,
\end{aligned}
\tag{3.2}
$$

c being a scalar function.

If $m\geqq 9$, then from (3.1) and (3.2) we get

$$K_{abcd}=0\,, \qquad (|a|, |b|, |c|, |d| \neq)$$

for every φ-basis. Thus we have by virtue of Theorem 2.2,

THEOREM 3.1. *In an $m(\geqq 9)$-dimensional Sasakian space M. If M admits a contact conformal connection $\bar{\nabla}$ which satisfies the following*:

(1) $\nabla_j q_i - 2q_j p_i + p_j\eta_i + \eta_j p_i + \lambda\varphi_{ji} = 0$,

(2) *$\bar{\nabla}$ is of constant φ-holomorphic sectional curvature,*

then the contact Bochner curvature tensor of M vanishes.

We note that by Sakaguchi's method Theorem 3.1 holds in the case of $m\geqq 5$.

§ 4. The second Betti number of M.

M. Matsumoto and G. Chuman [3] have studied the contact Bochner curvature tensor and the second Betti number in a compact Sasakian space and had

THEOREM 4.1 ([3]). *Let $M(m\geqq 5)$ be a compact Sasakian space with vanishing contact Bochner curvature tensor. Then the second Betti number $b_2(M)$ of M vanishes, if M satisfies one of the following conditions*:

(1) *$\theta > 2$, where θ denotes the smallest eigenvalue of the Ricci tensor,*

(2) *$H(e_\lambda, e_\mu) + H(e_\lambda, e_{\mu^*}) > -3(2-\delta_{\lambda\mu})/(m-2)$, (especially $\sum_\mu (H(e_\lambda, e_\mu) + H(e_\lambda, e_{\mu^*})) > -3$),*

(3) *M is μ-holomorphically pinched with $\mu > (m-3)/(2(m-1))$.*

In this section, we show that the condition (3) in the above Theorem 4.1 may be replaced by the condition $\mu > 0$.

Now, we assume that H and L defined by

$$H = \sup\Big\{H(X, \varphi, X)\,;\ X\in D_P,\ P\in M\Big\}\,,$$

$$L = \inf\Big\{H(X, \varphi, X)\,;\ X\in D_P,\ P\in M\Big\}\,,$$

exist and $H+3>0$, where D is the distribution defined by the equation $\eta_i dx^i=0$. Then M is said to be μ-holomorphically pinched, μ being $(L+3)/(H+3)$.

Let μ is positive. Then we have $H(X, \varphi X)\geqq L>-3$, for any vector X orthogonal to η. Moreover, if the contact Bochner curvature tensor vanishes, by virtue of Theorem 2.3, we get the inequality

$$8H(e_\lambda, e_\mu)=H(e_\lambda, e_{\lambda^*})+H(e_\mu, e_{\mu^*})+6>0 \qquad (\lambda\neq\mu).$$

Replacing e_μ with e_{μ^*} we also get

$$8H(e_\lambda, e_{\mu^*})>0 \qquad (\lambda\neq\mu).$$

Thus we have

$$H(e_\lambda, e_\mu)+H(e_\lambda, e_{\mu^*})>0 \qquad (\lambda\neq\mu),$$

from which we obtain

$$\sum_\mu \Big(H(e_\lambda, e_\mu)+H(e_\lambda, e_{\mu^*})\Big)>H(e_\lambda, e_{\lambda^*})>-3,$$

and we have the condition (2) in Theorem 4.1. Thus we have

THEOREM 4.2. *Let $M(m\geqq 5)$ be a compact Sasakian space with vanishing contact Bochner curvature tensor. If M is μ-holomorphically pinched with $\mu>0$, then the second Betti number $b_2(M)=0$.*

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