Hokkaido Mathematical Journal Vol. 10 (1981) p. 93-123

Regularity of solutions to hyperbolic mixed problems with uniformly characteristic boundary

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(Received April 22, 1980; Revised July 16, 1980)

§ 0. Introduction and results

Let G be a domain in $\boldsymbol{R}^n$ with smooth boundary ∂G. We consider the following mixed problem for a hermitian hyperbolic system P:

$$(P, B)\qquad \begin{cases} Pu=f & \text{in}\quad [t_1, t_2]\times G\,, \\ Bu=g & \text{on}\quad [t_1, t_2]\times\partial G\,, \\ u(t_1, x)=h & \text{for}\quad x\in G\,, \end{cases}$$

where $t_1<t_2$,

$$(0.1)\qquad P(t, x\,;\, D_t, D_x)=D_t+\sum_{j=1}^{n}A_j(t, x)\,D_j+C(t, x)\,,$$

A_j and C are $m\times m$ matrices, $A_j=A_j^*$, $D_t=-i\dfrac{\partial}{\partial t}$ and $D_j=-i\dfrac{\partial}{\partial x_j}$. The $B(t, x)$ is an $l\times m$ matrix of constant rank l and all of A_j, C and B are smooth and constant for large $|t|+|x|$.

For the sake of simplicity of description, throughout in the present paper we may suppose that G is the open half space $\{x_n>0\}$. Furthermore it is assumed that ∂G is uniformly characteristic for P, *i.e.*, the boundary matrix A_n is of constant rank d less than m near ∂G. This article is concerned with the L^2-well possedness for (P, B) and the regularity of solutions to (P, B) under the L^2-well posedness. In particular, we are here interested in the problem whose solution u satisfies the estimates of the type

$$\begin{aligned}(0.2)\qquad &\sum_{i=0}^{p}\Big\{e^{-rt}\big|D_t^i u(t)\big|_{p-i,r}^2+\gamma\int_{t_1}^{t}e^{-rs}\big|D_s^i u(s)\big|_{p-i,r}^2\,ds\Big\}\\ &+\sum_{i+j\le p}\gamma\int_{t_1}^{t}e^{-rs}\big\langle D_s^i D_n^j A_n u(s)\big\rangle_{p-\frac{1}{2}-i-j,r}^2\,ds\\ &\le C_p\sum_{i=0}^{p}\Big\{e^{-rt_1}\big|D_t^i u(t_1)\big|_{p-i,r}^2\\ &+\gamma^{-1}\int_{t_1}^{t}e^{-rs}\Big(\big|D_s^i f(s)\big|_{p-i,r}^2+\big\langle D_s^i g(s)\big\rangle_{p+\frac{1}{2}-i,r}^2\Big)\,ds\Big\},\quad t_1<t<t_2\,,\end{aligned}$$

for $\gamma \geq \gamma_p$, where $p \geq 0$ are integers and γ_p, $C_p > 0$ are constants independent of t_1, t_2, f, g and h (see § 1 for notations).

Without loss of generality, we may take A_n to be block diagonal:

$$(0.3)\qquad A_n = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A^+ & 0 \\ 0 & A^- \end{bmatrix},$$

where $A^+(A^-)$ is a positive (negative) definite $d^+ \times d^+ (d^- \times d^-)$ matrix respectively and $d^+ + d^- = d$. Furthermore, it is natural to assume that $l = d^+$ and (P, B) is reflexive, *i.e.*, $\ker A_n \subset \ker B$ on ∂G. So, under (0.3) we may take B to be of the form

$$(0.4)\qquad B = (B_{\mathrm{I}}, 0), \qquad B_{\mathrm{I}} = (I_{d^+}, S),$$

where I_i denotes the $i \times i$ unit matrix and S is a $d^+ \times d^-$ matrix (cf. Kubota and Ohkubo [3], Lemmas 2.9 and 2.10). Let $P_0(t, x; \tau, \sigma, \lambda)$ be the principal symbol of P, where τ, $\sigma = (\sigma_1, \cdots, \sigma_{n-1})$, λ are covariables of t, $x' = (x_1, \cdots, x_{n-1})$, x_n respectively. Then we have

THEOREM 1. *Let $P_0(t, x; \tau, \sigma, \lambda)$ be of constant multiplicity in τ. Suppose that the kernel of B is maximally non-positive for P on ∂G, i.e.,*

$$(0.5)\qquad A_n u \cdot u \leq 0 \quad \textit{for} \quad u \in \ker B \quad \textit{on} \quad \partial G$$

and $\ker B$ *is a maximal subspace obaying the above property. Then, for every t_1, t_2 and every $f \in C^0([t_1, t_2]; L^2(G))$, $g \in C^0([t_1, t_2]; H^{\frac{1}{2}}(\partial G))$ and $h \in L^2(G)$ there exists a unique strong solution $u \in C^0([t_1, t_2]; L^2(G))$ to (P, B) satisfying inequality* (0.2) *with $p = 0$.*

To get higher order estimates (0.2) with $p \geq 1$ which are analogous to those under Kriess' condition (cf. Majda and Osher [6]), we must assume additional conditions besides those of Theorem 1 (cf. Tsuji [12]). Let us write $P_0(t, x; 0, \sigma, 0)$ as

$$(0.6)\qquad \sum_{j=1}^{n-1} A_j(t, x)\sigma_j = \begin{bmatrix} A_{\mathrm{I\,I}} & A_{\mathrm{I\,II}} \\ A_{\mathrm{II\,I}} & A_{\mathrm{II\,II}} \end{bmatrix} (t, x; \sigma),$$

where $A_{\mathrm{I\,I}}$ and $A_{\mathrm{II\,II}}$ are $d \times d$ and $(m-d) \times (m-d)$ matrices respectively. We now make the following assumption (A) on P_0:

$$(\text{A-1})\qquad (A_{\mathrm{II\,I}} A^{-1} A_{\mathrm{I\,I}} - A_{\mathrm{II\,II}} A_{\mathrm{II\,I}} A^{-1})(t, x; \sigma) = 0 \qquad \text{for} \quad \sigma \in \boldsymbol{R}^{n-1},$$

$$(\text{A-2})\qquad (A_{\mathrm{II\,I}} A^{-1} A_{\mathrm{I\,II}})(t, x; \sigma) = 0 \qquad \text{for} \quad \sigma \in \boldsymbol{R}^{n-1}.$$

Then we have

THEOREM 2. *Suppose the conclusion of Theorem* 1 *and in addition the condition* (A). *For each integer $p \geq 1$, if the data f, g, h belong* *to* $\bigcap_{i=0}^{p} C^i([t_1, t_2]; H^{p-i}(G))$, $\bigcap_{i=0}^{p} C^i([t_1, t_2]; H^{p+\frac{1}{2}-i}(\partial G))$, $H^p(G)$ *respectively and fulfill compatibility conditions of order* $p-1$ *(see Definition* 5.1 *below), then the solution* u *belongs to* $\bigcap_{i=0}^{p} C^i([t_1, t_2]; H^{p-i}(G))$ *and satisfies inequality* (0.2) *together with the following inequality which yields an estimate for the first term on the right side in* (0.2):

$$\sum_{i=0}^{p}\left|D_t^i u(t_1)\right|_{p-i,r}^2 \leq C_p\left(|h|_{p,r}^2+\sum_{i=0}^{p-1}\left|D_t^i f(t_1)\right|_{p-1-i,r}^2\right). \tag{0.7}$$

Estimates (0.2) are as sharp as those in [6]. Furthermore the condition (A) with respect to P is satisfied by important physical examples such as the curl operator, Maxwell system and the linearized shallow water equations, which are treated also in [6]. But the conditions in [6], satisfied by these examples, seem to be too complicated to compare with our conditions (cf. § 6).

Here we shall explain briefly why (A) is required to obtain higher order estimates (0.2). Let us write $u={}^t({}^tu_{\rm I}, {}^tu_{\rm II})\in \boldsymbol{C}^m$ where $u_{\rm I}$ and $u_{\rm II}$ are d- and $(m-d)$-vectors respectively, and denote by $A_{\rm JK}$ operators with symbols $A_{\rm JK}(t, x; \sigma)$ (J, K=I, II). Then by (0.3) and (0.6) the equation $Pu=f$ is rewritten as

$$\begin{aligned} D_n u_{\rm I} &= A^{-1}\left(f_{\rm I}-D_t u_{\rm I}-A_{\rm I\,I}u_{\rm I}-A_{\rm I\,II}u_{\rm II}-(Cu)_{\rm I}\right), \\ (D_t+A_{\rm I\,II}+C_{\rm II\,II})\, u_{\rm II} &= f_{\rm II}-A_{\rm II\,I}u_{\rm I}-C_{\rm II\,I}u_{\rm I}, \end{aligned} \tag{0.8}$$

where $C_{\rm II\,I}$ and $C_{\rm II\,II}$ are the lowest left $(m-d)\times d$ and right $(m-d)\times(m-d)$ blocks of C respectively. Unlike the noncharacteristic case, these do not yield directly an estimate of $D_n u_{\rm II}$ in terms of f and tangential derivatives of u. So we apply D_n to the second equation of the above to get

$$\begin{aligned} &(D_t+A_{\rm II\,II}+C_{\rm II\,II})\,(D_n u_{\rm II}) \\ &\quad = D_n f_{\rm II}-A_{\rm II\,I}(D_n u_{\rm I})-C_{\rm II\,I}D_n u_{\rm I}+K_1 u, \end{aligned}$$

where (and in what follows) K_i stand for first order (tangential) differential operators not containing D_n. Substituting (0.8) into the second term on the right and rearranging yield

$$\begin{aligned} &(D_t+A_{\rm II\,II}+C_{\rm II\,II})\,(D_n u_{\rm II}) \\ &\quad = D_n f_{\rm II}-A_{\rm II\,I}A^{-1}f_{\rm I}-C_{\rm II\,I}D_n u_{\rm I}+K_2 u \\ &\qquad + A_{\rm II\,I}A^{-1}A_{\rm I\,I}u_{\rm I}+A_{\rm II\,I}A^{-1}A_{\rm I\,II}u_{\rm II}+A_{\rm II\,I}A^{-1}D_t u_{\rm I}. \end{aligned}$$

Rewriting the last term of the above as $D_t A_{\rm II\,I}A^{-1}u_{\rm I}+K_3 u_{\rm I}$ we get for $x_n\geq 0$

(0. 9) $$(D_t+A_{\mathrm{II\,II}}+C_{\mathrm{II\,II}})(D_n u_{\mathrm{II}}-A_{\mathrm{II\,I}}A^{-1}u_{\mathrm{I}})$$
$$=D_n f_{\mathrm{II}}-A_{\mathrm{II\,I}}A^{-1}f_{\mathrm{I}}-C_{\mathrm{II\,I}}D_n u_{\mathrm{I}}+K_4 u$$
$$+(A_{\mathrm{II\,I}}A^{-1}A_{\mathrm{I\,I}}-A_{\mathrm{II\,II}}A_{\mathrm{II\,I}}A^{-1})u_{\mathrm{I}}+A_{\mathrm{II\,I}}A^{-1}A_{\mathrm{I\,II}}u_{\mathrm{II}}.$$

Note that $(D_t+A_{\mathrm{II\,II}}+C_{\mathrm{II\,II}})$ is a hermitian hyperbolic operator which does not contain D_n. Then, under (A), we can obtain an estimate of $(D_n u_{\mathrm{II}}-A_{\mathrm{II\,I}}A^{-1}u_{\mathrm{I}})$ and hence $D_n u_{\mathrm{II}}$ in terms of first order derivatives of f and u_{I} and tangential derivatives of u_{II} (c. f. Lemma 4. 1). Thus, using estimates of $D_n u_{\mathrm{I}}$ following from (0. 8) and of tangential derivatives of u which are obtained from (0. 2) with $p=0$ as in the noncharacteristic case, we can raise the differentiability of u to obtain (0. 2) with $p=1$. The analogous procedures together with differentiations of (0. 8) and (0. 9) will yield (0. 2) with $p\geq 2$.

Section 2 is concerned with the proof of Theorem 1 and L^2-estimates of u are derived from the assumptions of Theorem 1. Sections 3 and 4 are concerned with the proof of Theorem 2, or higher order estimates. In section 3, assuming the conclusion of Theorem 1 we mainly discuss tangential estimates of u, and in section 4, assuming further the condition (A) we do normal estimates of u_{II}. The proofs of Theorems 1 and 2 are accomplished in section 5. In section 6 we give examples of (P, B) for which (A) is satisfied.

§ 1. Norms and notations

Let $H^q(\boldsymbol{R}^{n-1})$ be the usual Sobolev space of vector valued functions $v(x')=(v_1, \cdots, v_k)$, $(q\in\boldsymbol{R}^1)$. We use the following norm with $\gamma>0$ of $H^q(\boldsymbol{R}^{n-1})$:

(1. 1) $$\langle v\rangle_{q,\gamma}=\langle\Lambda_\gamma^q v\rangle_0,$$

where $\langle\cdot\rangle_0$ is the standard norm of $L^2(\boldsymbol{R}^{n-1})$ and

$$(\Lambda_\gamma^q v)(x')=(2\pi)^{-(n-1)}\int_{\boldsymbol{R}^{n-1}}\left(\gamma^2+|\sigma|^2\right)^{q/2}\hat{v}(\sigma)\,e^{ix'\sigma}d\sigma,$$
$$\hat{v}(\sigma)=\int_{\boldsymbol{R}^{n-1}}e^{-ix'\sigma}v(x')\,dx'.$$

As the norm of the Sobolev space $H^p(G)$ $(p=0, 1, \cdots)$ of vector valued functions $u(x)=(u_1, \cdots, u_k)$ we use

(1. 2) $$|u|_{p,\gamma}=\left(\sum_{k=0}^{p}|\Lambda_\gamma^{p-k}D_n^k u|_0^2\right)^{\frac{1}{2}}$$

which is equivalent to $(\sum_{i+|\alpha|\leq p}|\gamma^i D^\alpha u|_0^2)^{\frac{1}{2}}$, $|\cdot|_0$ being the standard norm of $L^2(G)$.

Let X be a Banach space and $C^j([t_1, t_2]\,;\,X)$ be the space of X-valued functions of $t\in[t_1, t_2]$ which are j times continuously differentiable. For $u(t)\in\bigcap_{j=0}^{p} C^j([t_1, t_2]\,;\, H^{p-j}(G))$ and $v(t)\in\bigcap_{j=0}^{[q]} C^j([t_1, t_2]\,;\, H^{q-j}(\boldsymbol{R}^{n-1}))$ with $q\geq 0$ we define the norms by

$$\text{(1.3)}\qquad \left\|\left|u(t)\right|\right\|_{p,\gamma}^2=\sum_{j=0}^{p}\left|D_t^j u(t)\right|_{p-j,\gamma}^2\,,$$

$$\langle\langle v(t)\rangle\rangle_{q,\gamma}^2=\sum_{j=0}^{[q]}\langle D_t^j v(t)\rangle_{q-j,\gamma}^2\,,$$

respectively. Then we see from (1.2) and (1.1) that

$$\text{(1.4)}\qquad \left\|\left|u(t)\right|\right\|_{p,\gamma}^2\approx\sum_{j+k\leq p}\left|\Lambda_\gamma^{p-j-k}D_t^j D_n^k u(t)\right|_0^2\,,$$

$$\text{(1.5)}\qquad \langle\langle v(t)\rangle\rangle_{p,\gamma}^2\approx\sum_{j\leq[q]}\langle\Lambda_\gamma^{q-j}D_t^j v(t)\rangle_0^2\,,$$

where $\approx$ stands for the equivalence of norms. Hence putting

$$\text{(1.6)}\qquad \left\|u(t)\right\|_{p,\gamma}^2=\sum_{j\leq p}\left|\Lambda_\gamma^{p-j}D_t^j u(t)\right|_0^2\,,$$

$$\text{(1.7)}\qquad \langle\langle\langle u(t)\rangle\rangle\rangle_{q,\gamma}^2=\sum_{j+k\leq[q]}\langle\Lambda_\gamma^{q-j-k}D_t^j D_n^k u(t)\rangle_0^2\,,$$

we get

$$\text{(1.8)}\qquad \left\|\left|u(t)\right|\right\|_{p,\gamma}^2\approx\sum_{k=0}^{p}\left\|D_n^k u(t)\right\|_{p-k,\gamma}^2\,,$$

$$\langle\langle\langle u(t)\rangle\rangle\rangle_{q,\gamma}^2\approx\sum_{k=0}^{[q]}\langle\langle D_n^k u(t)\rangle\rangle_{q-k,\gamma}^2\,.$$

Now, for real γ ($\gamma\neq 0$) we set

$$\langle\!\langle v\rangle\!\rangle_{q,\gamma}^2=\int_{-\infty}^{\infty}\langle e^{-\gamma t}(\Lambda_\tau^q v)(t)\rangle_0^2 dt\,,$$

$$\text{(1.9)}\qquad \|u\|_{q,\gamma}^2=\int_{-\infty}^{\infty}\left|e^{-\gamma t}(\Lambda_\tau^q u)(t)\right|_0^2 dt\,,$$

$$|||u|||_{p,\gamma}^2=\sum_{k=0}^{p}\|D_n^k u\|_{p-k,\gamma}^2\,.$$

Here $\Lambda_\tau^q=\Lambda_\tau^q(D_t, D_{x'}\,;\,\gamma)$ is the pseudo-differential operator with the symbol $(\gamma^2+\eta^2+|\sigma|^2)^{q/2}=(|\tau^2|+|\sigma|^2)^{q/2}$, where $\tau=\eta-i\gamma$ and a pseudo-differential operator $\beta(t, x\,;\, D_t, D_{x'}\,;\,\gamma)$ with the symbol $\beta(t, x\,;\,\eta, \sigma\,;\,\gamma)=\beta(t, x\,;\,\tau, \sigma)$ is defined by

$$\text{(1.10)}\qquad \beta(t, x\,;\, D_t, D_{x'}\,;\,\gamma)\cdot v(t, x')$$

$$=(2\pi)^{-n}e^{\gamma t}\int_{R^n}\beta(t, x\,;\,\eta-i\gamma, \sigma)\,\widehat{e^{-\gamma t}v}(\eta, \sigma)\,e^{i\eta t+i\sigma x'}d\eta d\sigma$$

(cf., for instance, Ohkubo and Shirota [8]). We denote by $H_{q,\gamma}(\boldsymbol{R}^1\times\boldsymbol{R}^{n-1})$, $H_{0,q;\gamma}(\boldsymbol{R}^1\times G)$ and $H_{p,0;\gamma}(\boldsymbol{R}^1\times G)$ the completions of $C_0^\infty(\boldsymbol{R}^1\times\boldsymbol{R}^{n-1})$, $C_0^\infty(\boldsymbol{R}^1\times\bar{G})$ and $C_0^\infty(\boldsymbol{R}^1\times\bar{G})$ with respect to the norms in (1. 9), respectively (cf. Kubota [2]). We finally remark that for $\gamma>0$

$$\begin{aligned}
&\langle\!\langle v\rangle\!\rangle_{q,\pm\gamma}^2\approx\int_{-\infty}^{\infty}e^{\mp 2\gamma t}\big\langle\!\big\langle v(t)\big\rangle\!\big\rangle_{q,\gamma}^2dt\,, \qquad q\geq 0\,,\\
(1.11)\qquad &\|u\|_{p,\pm\gamma}^2\approx\int_{-\infty}^{\infty}e^{\mp 2\gamma t}\big\|u(t)\big\|_{p,\gamma}^2dt\,,\\
&|\!|\!|u|\!|\!|_{p,\pm\gamma}^2\approx\int_{-\infty}^{\infty}e^{\mp 2\gamma t}\big|\!\big|\!\big|u(t)\big|\!\big|\!\big|_{p,\gamma}^2dt\,.
\end{aligned}$$

§ 2. L^2-estimates

In this section, under the assumptions of Theorem 1, we first discuss the L^2-well posedness for the following boundary value problem $(P, B)_0$:

$$(P,B)_0\qquad\begin{cases}Pu=f & \text{in } \boldsymbol{R}^1\times G\,,\\ Bu=g & \text{on } \boldsymbol{R}^1\times\partial G\,,\end{cases}$$

and next derive a priori estimate (0. 2) with $p=0$.

Hereafter in all inequalities let γ be sufficiently large and denote by C suitable positive constants independent of γ, t_1, t_2 and functions appeared there, unless especially stated.

PROPOSITION 2. 1. *Let $P_0(t, x\,;\,\tau,\sigma,\lambda)$ be of constant multiplicity in τ. Then there exist constants C, $\gamma_0>0$ such that for every $\gamma\geq\gamma_0$ and $u\in C_0^\infty(\boldsymbol{R}^1\times G)$*

$$(2.1)\qquad \langle\!\langle\Lambda_\gamma^{-\frac{1}{2}}u_{\mathrm{I}}\rangle\!\rangle_{0,\gamma}^2\leq C\Big(\|u\|_{0,\gamma}^2+\gamma^{-2}\|f\|_{0,\gamma}^2+\gamma^{-1}\langle\!\langle g\rangle\!\rangle_{0,\gamma}^2\Big)\,,$$

the norms being defined by (1. 9).

PROOF. From (0. 4) and [3], Lemma 2. 10 there exists a small $\varepsilon>0$ such that for all $(t, x', \tau, \sigma)\in\boldsymbol{R}^1\times\partial G\times((\bar{\boldsymbol{C}}_-\times\boldsymbol{R}^{n-1})\backslash 0)$ with $|\sigma|\leq\varepsilon|\tau|$

$$(2.2)\qquad R(t, x'\,;\,\tau,\sigma)\neq 0\,,$$

where $\bar{\boldsymbol{C}}_-=\{\tau\in\bar{\boldsymbol{C}}^1\,;\,\mathrm{Im}\,\tau\leq 0\}$ and R is a Lopatinskii determinant for $(P, B)_0$.

Let $\beta_1(\tau,\sigma)$ and $\beta_2(\tau,\sigma)$ be positively homogeneous scalar valued C^∞-functions of degree 0 in (η,σ,γ) with $(\eta-i\gamma,\sigma)\in(\bar{\boldsymbol{C}}_-\times\boldsymbol{R}^{n-1})\backslash 0$ satisfying $\beta_1+\beta_2=1$ whose supports are in $\{(\tau,\sigma)\,;\,|\sigma|\geq(\varepsilon/2)|\tau|\}$ and $\{(\tau,\sigma)\,;\,|\sigma|\leq\varepsilon|\tau|\}$ respectively. Let $\beta_i(D_t, D_{x'}\,;\,\gamma)$ $(i=1, 2)$ be the pseudo-differential operators as in (1. 10); noting $u_{\mathrm{I}}=\beta_1(D_t, D_{x'}\,;\,\gamma)\,u_{\mathrm{I}}+\beta_2(D_t, D_{x'}\,;\,\gamma)\,u_{\mathrm{I}}$ we shall estimate β_1u_{I} and β_2u_{I} (cf. Miyatake [7]).

Since on the support of $\beta_1(\tau, \sigma)$

$$\left(\gamma^2+|\sigma|^2\right)^{-1} \leq 8\varepsilon^{-2}\left(|\tau|^2+|\sigma|^2\right)^{-1} \qquad \text{for} \quad \gamma=-\operatorname{Im}\tau>0,$$

we see from (1. 9) that for every $u \in C_0^\infty(\boldsymbol{R}^1 \times G)$

$$\langle\!\langle \Lambda_\tau^{-\frac{1}{2}} \beta_1 u_{\mathrm{I}} \rangle\!\rangle_{0,\gamma}^2 \leq C_\varepsilon \langle\!\langle \Lambda_\tau^{-\frac{1}{2}} u_{\mathrm{I}} \rangle\!\rangle_{0,\gamma}^2,$$

where $C_\varepsilon>0$ is independent of γ and u_{I}. Since $\langle\!\langle \Lambda_\tau^{-\frac{1}{2}} u_{\mathrm{I}} \rangle\!\rangle_{0,\gamma}^2 = 2\operatorname{Im}(e^{-\gamma t}(D_n u_{\mathrm{I}}), e^{-\gamma t}\Lambda_\tau^{-1} u_{\mathrm{I}})_{L^2(\boldsymbol{R}^1\times G)}$, from (0. 8)

$$\langle\!\langle \Lambda_\tau^{-\frac{1}{2}} u_{\mathrm{I}} \rangle\!\rangle_{0,\gamma}^2 \leq C\left(\|u\|_{0,\gamma}^2+\gamma^{-2}\|f\|_{0,\gamma}^2\right).$$

Therefore (2. 1) follows if we prove under (2. 2) that for every $u \in C_0^\infty(\boldsymbol{R}^1 \times G)$

$$\langle\!\langle \beta_2 u_{\mathrm{I}} \rangle\!\rangle_{0,\gamma}^2 \leq C\left(\gamma^{-1}\|u\|_{0,\gamma}^2+\gamma^{-1}\|f\|_{0,\gamma}^2+\langle\!\langle g \rangle\!\rangle_{0,\gamma}^2+\langle\!\langle \Lambda_\tau^{-1} u_{\mathrm{I}} \rangle\!\rangle_{0,\gamma}^2\right). \tag{2. 3}$$

To establish this we shall localize and reduce $(P, B)_0$ to a noncharacteristic problem for $\beta_2 u_{\mathrm{I}}$. Set

$$\begin{bmatrix} Q_{\mathrm{I\,I}} & Q_{\mathrm{I\,II}} \\ Q_{\mathrm{II\,I}} & Q_{\mathrm{II\,II}} \end{bmatrix}(t, x;\ \tau, \sigma) = P_0(t, x;\ \tau, \sigma, 0) = \tau I_m + \sum_{j=1}^{n-1} A_j \sigma_j, \tag{2. 4}$$

where $Q_{\mathrm{I\,I}}$, $Q_{\mathrm{II\,II}}$ are $d \times d$, $(m-d)\times(m-d)$ matrices respectively. Then $(P, B)_0$ is rewritten as

$$\begin{cases} (D_n + A^{-1}Q_{\mathrm{I\,I}})\, u_{\mathrm{I}} + A^{-1}Q_{\mathrm{I\,II}} u_{\mathrm{II}} = A^{-1}(f_{\mathrm{I}}-(Cu)_{\mathrm{I}}), \\ Q_{\mathrm{II\,II}} u_{\mathrm{II}} = f_{\mathrm{II}} - Q_{\mathrm{II\,I}} u_{\mathrm{I}} - (Cu)_{\mathrm{II}} \qquad \text{in} \quad \boldsymbol{R}^1 \times G, \\ B_{\mathrm{I}} u_{\mathrm{I}} = g \qquad \text{on} \quad \boldsymbol{R}^1 \times \partial G, \end{cases}$$

where the last equality is due to (0. 4).

In what follows, for J, K $=$ I, II we denote again by $Q_{\mathrm{JK}}(t, x;\ \tau, \sigma)$ an appropriate positively homogeneous C^∞-extention to $(\bar{C}_- \times \boldsymbol{R}^{n-1})\backslash 0$ of $Q_{\mathrm{JK}}(t, x;\ \tau, \sigma)$ restricted to the support of $\beta_2(\tau, \sigma)$ such that for any $(\tau, \sigma) \in (\bar{C}_- \times \boldsymbol{R}^{n-1})\backslash 0$ there exists a $(\tau_0, \sigma_0) \in \operatorname{supp}\beta_2$ satisfying $Q_{\mathrm{JK}}(t, x;\ \tau, \sigma) = Q_{\mathrm{JK}}(t, x;\ \tau_0, \sigma_0)$ and $|\tau_0|^2+|\sigma_0|^2=1$ (cf. [8] for instance). Since the degrees of β_2 and Q_{JK} are 0 and 1 respectively, it follows from the above equations that

$$\begin{cases} (D_n + A^{-1}Q_{\mathrm{I\,I}})\, \beta_2 u_{\mathrm{I}} + A^{-1}Q_{\mathrm{I\,II}}(\beta_2 u_{\mathrm{II}}) = \beta_2 A^{-1} f_{\mathrm{I}} + \tilde{K}_1 u, \\ Q_{\mathrm{II\,II}} \beta_2 u_{\mathrm{II}} = -Q_{\mathrm{II\,I}} \beta_2 u_{\mathrm{I}} + \beta_2 f_{\mathrm{II}} + \tilde{K}_2 u \qquad \text{in} \quad \boldsymbol{R}^1 \times G, \end{cases} \tag{2. 5}$$

$$B_{\mathrm{I}}(\beta_2 u_{\mathrm{I}}) = \beta_2 g + [B_{\mathrm{I}}, \beta_2]\, u_{\mathrm{I}} \qquad \text{on} \quad \boldsymbol{R}^1 \times \partial G, \tag{2. 6}$$

where (and in what follows) $\tilde{K}_i$ stand for pseudo-differential operators of degree 0.

Since from (2. 4) $\det Q_{\mathrm{II\,II}}(t, x;\ \pm 1, 0) = (\pm 1)^{m-d} \neq 0$, we have

$$\det Q_{\mathrm{II}\,\mathrm{II}}(t, x; \tau, \sigma) \neq 0 \qquad \text{on} \quad (\bar{C}_- \times \boldsymbol{R}^{n-1})\backslash 0,$$

if ε is taken sufficiently small. Hence there exists an inverse $Q_{\mathrm{I}}^{-1}{}_{\mathrm{II}}$ of $Q_{\mathrm{II}\,\mathrm{II}}$ whose principal symbol is $Q_{\mathrm{I}}^{-1}{}_{\mathrm{II}}(t, x; \tau, \sigma)$ of degree -1. Therefore we get from the second equation of (2. 5)

$$\beta_2 u_{\mathrm{II}} = -Q_{\mathrm{II}}^{-1}{}_{\mathrm{II}} Q_{\mathrm{II}\,\mathrm{I}} \beta_2 u_{\mathrm{I}} + Q_{\mathrm{II}}^{-1}{}_{\mathrm{II}} \beta_2 f_{\mathrm{II}} + \tilde{K}_3 \Lambda_\tau^{-1} u.$$

Inserting the above into the first of (2. 5) yields

$$\begin{aligned}(2.7) \qquad & \big(D_n - M(t, x; D_t, D_{x'}; \gamma)\big)(\beta_2 u_{\mathrm{I}}) \\ &= (\beta_2 A^{-1} f_{\mathrm{I}} - A^{-1} Q_{\mathrm{I}\,\mathrm{II}} Q_{\mathrm{II}}^{-1}{}_{\mathrm{II}} \beta_2 f_{\mathrm{II}}) + \tilde{K}_4 u \qquad \text{in} \quad \boldsymbol{R}^1 \times G,\end{aligned}$$

where M denotes the $d \times d$ matrix

$$(2.8) \qquad M(t, x; \tau, \sigma) = -A^{-1}(Q_{\mathrm{I}\,\mathrm{I}} - Q_{\mathrm{I}\,\mathrm{II}} Q_{\mathrm{II}}^{-1}{}_{\mathrm{II}} Q_{\mathrm{II}\,\mathrm{I}})(t, x; \tau, \sigma).$$

Let us recall that R is a Lopatinskii determinant for the noncharacteristic problem (2. 7) with boundary condition (2. 6) with respect to $\beta_2 u_{\mathrm{I}}$ (see [3], (2. 20) and (2. 21)). Therefore, applying Kreiss' methods we can derive (2. 3) from (2. 2) and the following lemma (see, for instance, [8], Lemma 5. 1 and subsection 8. 1):

LEMMA 2. 2. *Under the assumption of Proposition* 2. 1, *there exist constants* ε, $C>0$, *nonsingular* $d \times d$ C^∞-*matrix* $S_{\mathrm{I}}(t, x; \tau, \sigma)$ *and* C^∞-*functions* $\lambda_i(t, x; \tau, \sigma)$ $(i=1, \cdots, d)$ *such that for* $|\sigma| \leq \varepsilon|\tau|$ *and* $|\tau|^2 + |\sigma|^2 = 1$

$$S_{\mathrm{I}}^{-1} M S_{\mathrm{I}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix} \qquad \textit{and} \qquad |\mathrm{Im}\,\lambda_i| \geq C|\mathrm{Im}\,\tau|.$$

PROOF. Let τ_i be the roots in τ of $\det P_0(t, x; \tau, \sigma, \lambda) = 0$ and m_i their multiplicities. Then from the assumption, all $\tau_i(t, x; \sigma, \lambda)$ are infinitely differentiable over $\boldsymbol{R}^1 \times \bar{G} \times \boldsymbol{R}^n$ and analytic, real and distinct for $(\sigma, \lambda) \in \boldsymbol{R}^n \backslash 0$. Furthermore by (0. 3) we may assume that

$$\begin{aligned} &\tau_i(t, x; 0, \lambda) = a_i(t, x)\lambda \qquad && \text{for} \quad i \leq d', \\ &\tau_i(t, x; 0, \lambda) = 0 && \text{for} \quad i \geq d'+1, \end{aligned}$$

where $a_i \neq 0$ are the eigenvalues of $-A$ and d' is the number of distinct ones. Hence we get $\dfrac{\partial \tau_i}{\partial \lambda}(t, x; 0, \lambda) \neq 0$ for $i \leq d'$ and $\sum_{i \leq d'} m_i = d$. So, according to the implicit function theorem, there exist C^∞-functions $\lambda_i(t, x; \tau, \sigma)$ which are real and analytic in $\tau \in \boldsymbol{R}^1$ such that

$$(2.9) \qquad \tau - \tau_i(t, x; \sigma, \lambda) = \big(\lambda - \lambda_i(t, x; \tau, \sigma)\big)\, c_i(t, x; \tau, \sigma, \lambda) \qquad i \leq d',$$

with $c_i(t, x; \tau, 0, \lambda) \neq 0$. Since (2. 9) implies $\dfrac{\partial \lambda_i}{\partial \tau} \neq 0$, $|\mathrm{Im}\, \lambda_i| \geq C|\mathrm{Im}\, \tau|$ holds for small $|\sigma|$ and some $C>0$.

Now, since $\sum_{j=1}^{n-1} A_j \sigma_j + A_n \lambda$ is hermitian and has eigenvalues $-\tau_i$, there exist linearly independent associated C^∞-eigenvectors $\{\tilde{h}_i^j\}_{j=1,\cdots,m_i}$, so that $P_0(t, x; \tau_i, \sigma, \lambda)\, \tilde{h}_i^j(t, x; \sigma, \lambda)=0$. Set $h_i^j(t, x; \tau, \sigma)=\tilde{h}_i^j(t, x; \sigma, \lambda_i(t, x; \tau, \sigma)) \in C^\infty$ for $i \leq d'$, $j=1, \cdots, m_i$ and for small $|\sigma|$, then it follows from (2. 9) that h_i^j are null vectors of $P_0(t, x; \tau, \sigma, \lambda_i)$. Therefore we get from (2. 4) and (0. 3)

$$(h_i^j)_{\mathrm{II}} = -Q_{\mathrm{II}\,\mathrm{II}}^{-1} Q_{\mathrm{II}\,\mathrm{I}} (h_i^j)_{\mathrm{I}}\,.$$

So putting

$$\begin{bmatrix} S_{\mathrm{I}} \\ -Q_{\mathrm{II}\,\mathrm{II}}^{-1} Q_{\mathrm{II}\,\mathrm{I}} S_{\mathrm{I}} \end{bmatrix} = [h_1^1, \cdots, h_1^{m_1}, \cdots, h_{d'}^1, \cdots, h_{d'}^{m_{d'}}]\,,$$

where $S_{\mathrm{I}} \in C^\infty$ is the upper $d \times d$ matrix, we see from the linear independence of $\{h_i^j\}_{i,j}$ that $\det S_{\mathrm{I}} \neq 0$. Furthermore since

$$P_0(t, x; \tau, \sigma, \lambda_i) = \begin{bmatrix} A & 0 \\ 0 & I_{m-d} \end{bmatrix} \begin{bmatrix} \lambda_i I_d - M & A^{-1} Q_{\mathrm{I}\,\mathrm{II}} Q_{\mathrm{II}\,\mathrm{II}}^{-1} \\ 0 & I_{m-d} \end{bmatrix} \begin{bmatrix} I_d & 0 \\ Q_{\mathrm{II}\,\mathrm{I}} & Q_{\mathrm{II}\,\mathrm{II}} \end{bmatrix}$$

according to (2. 4), (2. 8) and (0. 3), we conclude that $M(h_i^j)_{\mathrm{I}} = \lambda_i (h_i^j)_{\mathrm{I}}$. This implies the desired diagonalization of M by S_{I}.

Using Proposition 2. 1 essentially we will obtain

PROPOSITION 2. 3. *Suppose the conditions of Theorem* 1. *Then for any integer p there exists constants C_p, $\gamma_p > 0$ such that for every $\gamma \geq \gamma_p$, $f \in H_{0,p;\gamma}(\boldsymbol{R}^1 \times G)$ and g with $\Lambda_\gamma^{\frac{1}{2}} g \in H_{p,\gamma}(\boldsymbol{R}^1 \times \partial G)$ the problem $(P, B)_0$ has a unique solution $u \in H_{0,p;\gamma}(\boldsymbol{R}^1 \times G)$ satisfying the inequality*

$$\text{(2. 10)} \qquad ||u||_{p,\gamma}^2 + \langle\!\langle \Lambda_\gamma^{-\frac{1}{2}} u_{\mathrm{I}} \rangle\!\rangle_{p,\gamma}^2 \leq C_p \gamma^{-2} \left(||f||_{p,\gamma}^2 + \langle\!\langle \Lambda_\gamma^{\frac{1}{2}} g \rangle\!\rangle_{p,\gamma}^2 \right).$$

Moreover, for any real t_0, if $f=g=0$ for all $t<t_0$ then $u=0$ for all $t<t_0$.

In order to prove Proposition 2. 3, we first discuss the maximal non-positiveness of (P, B). Let $u_{\mathrm{I}} = {}^t({}^t u^+, {}^t u^-)$ where $u^\pm$ are $d^\pm$-vectors respectively. Then from (0. 3) we have

$$\text{(2. 11)} \qquad A_n u \cdot u = A^+ u^+ \cdot u^+ + A^- u^- \cdot u^-\,.$$

Since by (0. 4) $u \in \ker B$ implies $u^+ = -S u^-$, substitute this into (2. 11). Then (0. 5) becomes

$$\text{(2. 12)} \qquad (A^- + S^* A^+ S)\, u^- \cdot u^- \leq 0 \qquad \text{for} \quad u^- \in \boldsymbol{C}^{d^-} \quad \text{on} \quad \partial G\,.$$

Now, for convenience, we put $\mathscr{H}^1=\{u\,;\,u, D_t u, D_j u (j=1,\cdots,n-1)$ and $D_n u_{\mathrm{I}}\in L^2([t_1,t_2]\times G)\}$; then $H^1([t_1,t_2]\times G)\subset\mathscr{H}^1\subset C^0([t_1,t_2]\,;\,L^2(G))$. For $u\in\mathscr{H}^1$ it follows from (0. 1) and (0. 3) that

$$\frac{d}{dt}\left|u(t)\right|_0^2=-2\operatorname{Im}\Big(D_t u(t),u(t)\Big)$$
$$\leq\Big\{\langle A_n u,u\rangle-2\operatorname{Im}(f,u)+C|u|_0^2\Big\}(t)\,,$$

where $C>0$ and $(\cdot,\cdot)$, $\langle\cdot,\cdot\rangle$ stand for the inner products in $L^2(G)$, $L^2(\partial G)$ respectively. Set $g=Bu|_{\partial G}$. Then we have from (0. 4) and (2. 11)

$$\langle A_n u,u\rangle(t)=\Big\{\Big\langle(A^-+S^*A^+S)\,u^-,u^-\Big\rangle$$
$$+\langle A^+g,g\rangle-\langle A^+g,Su^-\rangle-\langle A^+Su^-,g\rangle\Big\}(t)\,.$$

Therefore, we find (2. 12) implies that for any positive $\delta\leq1$

$$\frac{d}{dt}\left|u(t)\right|_0^2\leq\gamma\left|u(t)\right|_0^2+C\gamma^{-1}\left|f(t)\right|_0^2$$
$$+C\Big(\delta^{-1}\gamma^{-1}\Big\langle\Lambda_\gamma^{1/2}g(t)\Big\rangle_0^2+\delta\gamma\Big\langle\Lambda_\gamma^{-1/2}u_{\mathrm{I}}(t)\Big\rangle_0^2\Big)\,,$$

i. e.,

$$\text{(2. 13)}\qquad\frac{d}{dt}\Big(e^{-\gamma t}\left|u(t)\right|_0^2\Big)\leq Ce^{-\gamma t}\Big\{\gamma^{-1}\Big(\left|f(t)\right|_0^2$$
$$+\delta^{-1}\Big\langle\Lambda_\gamma^{\frac12}g(t)\Big\rangle_0^2\Big)+\delta\gamma\Big\langle\Lambda_\gamma^{-\frac12}u_{\mathrm{I}}(t)\Big\rangle_0^2\Big\}\qquad u\in\mathscr{H}^1\,,$$

where δ is independent of C, γ, t_1, t_2 and u (c. f. Taniguchi [11]).

LEMMA 2. 4. *Let $h(t)\geq0$ be an integrable function and $\gamma>0$. Then*

$$\int_a^t ds\int_a^s e^{-\gamma(s+s')}\,h(s')\,ds'\leq\gamma^{-1}\int_a^t e^{-2\gamma s'}\,h(s')\,ds'\,,\qquad t>a\,.$$

Using this lemma which is verified directly, we obtain

LEMMA 2. 5. *Assume the conditions of Theorem 1. Then for any integer p there exist constants C_p, $\gamma_p>0$ such that for every $\gamma\geq\gamma_p$ and $u,v\in C_0^\infty(\boldsymbol{R}^1\times G)$*

$$\text{(2. 14)}\qquad||u||_{p,\gamma}^2+\langle\!\langle\Lambda_\gamma^{-1/2}u_{\mathrm{I}}\rangle\!\rangle_{p,\gamma}^2\leq C_p\gamma^{-2}\Big(||Pu||_{p,\gamma}^2+\langle\!\langle\Lambda_\gamma^{1/2}Bu\rangle\!\rangle_{p,\gamma}^2\Big)\,,$$

$$\text{(2. 15)}\qquad||v||_{p,-\gamma}^2+\langle\!\langle\Lambda_\gamma^{-1/2}v_{\mathrm{I}}\rangle\!\rangle_{p,-\gamma}^2\leq C_p\gamma^{-2}\Big(||P^*v||_{p,-\gamma}^2+\langle\!\langle\Lambda_\gamma^{1/2}B'v\rangle\!\rangle_{p,-\gamma}^2\Big)\,,$$

where

$$
(2.16)\qquad
\begin{aligned}
&P^*(t, x\,;\ D_t, D_x) = D_t + \sum_{j=1}^{n} A_j^*(t, x)\, D_j + \sum_{j=1}^{n} \Big(D_j A_j^*(t, x)\Big) + C^*(t, x)\,,\\
&B'(t, x) = \Big(B_{\mathrm{I}}'(t, x)\, A^*(t, x), 0\Big)\,, \qquad B_{\mathrm{I}}(B_{\mathrm{I}}')^* = 0
\end{aligned}
$$

and B_{I}', B' are $(d-l)\times d$, $(d-l)\times m$ matrices of rank $d-l$ respectively.

PROOF. Since (P, B) is maximally non-positive, (2.13) holds for $u\in C_0^\infty(\boldsymbol{R}^1\times G)$; integrating it from $-\infty$ to t we get

$$
\begin{aligned}
e^{-rt}\Big|u(t)\Big|_0^2 \le C\Big\{\delta\gamma\int_{-\infty}^{t} e^{-rs}\Big\langle \Lambda_\gamma^{-1/2} u_{\mathrm{I}}(s)\Big\rangle_0^2 ds\\
+\gamma^{-1}\int_{-\infty}^{t} e^{-rs}\Big(\Big|f(s)\Big|_0^2 + \delta^{-1}\Big\langle \Lambda_\gamma^{1/2} g(s)\Big\rangle_0^2\Big)\, ds\Big\}\,.
\end{aligned}
$$

Multiplying the above by e^{-rt} and integrating from $-\infty$ to ∞, we see from Lemma 2.4 and (1.11) that

$$
||u||_{0,r}^2 \le C\Big\{\delta\langle\!\langle \Lambda_\gamma^{-1/2} u_{\mathrm{I}}\rangle\!\rangle_{0,r}^2 + \gamma^{-2}\Big(||f||_{0,r}^2 + \delta^{-1}\langle\!\langle \Lambda_\gamma^{1/2} g\rangle\!\rangle_{0,r}^2\Big)\Big\}\,.
$$

This with $\delta\ll 1$ and (2.1) lead to (2.14) with $p=0$.

Next, apply Λ_τ^p to the both sides of $Pu=f$ and $Bu|_{\partial G}=g$ and use (2.14) with $p=0$. Then by the relations (0.8) and

$$
\langle\!\langle \Lambda_\tau^{p-1}\Lambda_\gamma^{1/2} u_{\mathrm{I}}\rangle\!\rangle_{0,r}^2 \le C||\Lambda_\tau^p u_{\mathrm{I}}||_{0,r}^2\,,
$$

we obtain (2.14) for each integer p.

Now according to [5], Lemma 3.2, if $N\equiv\ker B$ is maximally non-positive for A_n then $N'\equiv\ker B'=(A_n(N))^\perp$ is non-positive for $(-A_n)$. Furthermore (0.4) implies $\operatorname{codim} N'=d-d^+=d^-$ (see [3], Lemma 2.1). So $\ker B'$ is maximally non-positive for $(-A_n)$. Hence applying the same arguments to the problem

$$
(2.17)\qquad
\begin{cases}
P^*(-t, x\,;\ -D_t, D_x)\, v(-t, x) = f'(-t, x) & \text{in } \boldsymbol{R}^1\times G\,,\\
B'(-t, x)\, v(-t, x) = g'(-t, x) & \text{on } \boldsymbol{R}^1\times\partial G\,,
\end{cases}
$$

as we derived (2.14), we obtain (2.15).

LEMMA 2.6. *For an integer p let $\tilde\Lambda_\tau^p=\tilde\Lambda_\tau^p(D_t, D_{x'}\,;\ \gamma)$ be a pseudo-differential operator whose symbol is $(\tau+|\sigma|)^p=(\eta-i\gamma+|\sigma|)^p$ where $\gamma\neq 0$. Then:*

(i) $\langle\!\langle v\rangle\!\rangle_{p,r}\approx\langle\!\langle\tilde\Lambda_\tau^p v\rangle\!\rangle_{0,r}$ *for* $v\in H_{p,r}(\boldsymbol{R}^1\times\boldsymbol{R}^{n-1})$.

(ii) *Let $\gamma>0$ and $p<0$. Then, for any real t_0, v belongs to $H_{p,r}(\boldsymbol{R}^1\times\boldsymbol{R}^{n-1})$ and vanishes identically for $t<t_0$ if and only if $\tilde\Lambda_\tau^p v$ belongs to $H_{0,r}(\boldsymbol{R}^1\times\boldsymbol{R}^{n-1})$ and vanishes identically for $t<t_0$.*

PROOF. The equivalence (i) follows from the inequality

$$\left|(\tau+|\sigma|)^p\right|^2 \leq C_1\left(\gamma^2+\eta^2+|\sigma|^2\right)^p \leq C_2\left|(\tau+|\sigma|)^p\right|^2,$$

where C_i do not depend on τ and σ.

To show (ii) it suffices, in view of (i), to prove that, for $v \in C_0^\infty(\boldsymbol{R}^1 \times \boldsymbol{R}^{n-1})$ $v=0$ $(t<t_0)$ is equivalent to $\tilde{\Lambda}_\tau^p \tau=0$ $(t<t_0)$. Set $w=\tilde{\Lambda}_\tau^p v$, then $v=\tilde{\Lambda}_\tau^{-p} w$ and w does not depend on γ since $(\tau+|\sigma|)^p$ is analytic in τ with $\gamma>0$. First let $w=0$ $(t<t_0)$, then $v=0$ $(t<t_0)$ since $(\tau+|\sigma|)^{-p}$ is entire analytic in $\tau \in \boldsymbol{C}^1$ for every fixed $\sigma \in \boldsymbol{R}^1$. Next since $|(\tau+|\sigma|)^{-p}|^2 \geq 1$ foy $\gamma \geq 1$ we have

$$\langle\!\langle w \rangle\!\rangle_{0,\gamma}^2 \leq \langle\!\langle \tilde{\Lambda}_\tau^{-p} w \rangle\!\rangle_{0,\gamma}^2 = \langle\!\langle v \rangle\!\rangle_{0,\gamma}^2 .$$

Therefore we can derive $w=0$ $(t<t_0)$ from $v=0$ $(t<t_0)$ by the standard technique: According to (1.9) $v=0$ $(t<t_0)$ implies that the right side of the above is dominated by $Ce^{-2\gamma t_0}$, where C does not depend on γ. So making γ sufficiently large lead to $w=0$ $(t<t_0)$.

PROOF OF PROPOSITION 2.3. The former statement of the proposition follows from (2.14) and (2.15) (see [3]). Since the latter statement for $p \geq 0$ follows from (2.10) with $p=0$ as usual, we shall prove that for $p<0$. (2.10) and Lemma 2.6 (i) imply that the solution $u \in H_{0,p;\gamma}(\boldsymbol{R}^1 \times G)$ satisfies

$$\|\tilde{\Lambda}_\tau^p u\|_{0,\gamma}^2 \leq C\gamma^{-2}\left(\|\tilde{\Lambda}_\tau^p f\|_{0,\gamma}^2 + \langle\!\langle \tilde{\Lambda}_\tau^p \Lambda_\gamma^{\frac{1}{2}} g \rangle\!\rangle_{0,\gamma}^2\right).$$

Let $f=g=0$ $(t<t_0)$. Then from Lemma 2.6 (ii) we have $\tilde{\Lambda}_\tau^p f=\tilde{\Lambda}_\tau^p \Lambda_\gamma^{\frac{1}{2}} g=0$ $(t<t_0)$. This and the above inequality yield $\tilde{\Lambda}_\tau^p u=0$ $(t<t_0)$, which implies $u=0$ $(t<t_0)$ by Lemma 2.6 (ii).

PROPOSITION 2.7. *Under the assumptions of Theorem* 1, *there exist constants* C_0, $\gamma_0>0$ *such that for every* $\gamma \geq \gamma_0$ *and* $u \in \mathscr{H}^1$

$$e^{-\gamma t}\left|u(t)\right|_0^2+\gamma\int_{t_1}^t e^{-\gamma s}\left(\left|u(s)\right|_0^2+\left\langle u_{\mathrm{I}}(s)\right\rangle_{-\frac{1}{2},\gamma}^2\right) ds \tag{2.18}$$
$$\leq C_0\left\{e^{-\gamma t_1}|h|_0^2+\gamma^{-1}\int_{t_1}^t e^{-\gamma s}\left(\left|f(s)\right|_0^2+\left\langle g(s)\right\rangle_{\frac{1}{2},\gamma}^2\right) ds\right\},$$

where $\mathscr{H}^1$ *is the function space defined in the first line* of p. 102.

PROOF. According to Proposition A.1 in the Appendix below we have (A.1). Moreover, (2.13) holds since (P, B) is maximally non-positive. Integrate (2.13) with $\delta=1$ from t_1 to t and apply (A.1) to the resulting last term. Then we get

$$e^{-\gamma t}\left|u(t)\right|_0^2 \tag{2.19}$$
$$\leq C\left\{e^{-\gamma t_1}|h|_0^2+\gamma^{-1}\int_{t_1}^t e^{-\gamma s}\left(\left|f(s)\right|_0^2+\left\langle g(s)\right\rangle_{\frac{1}{2},\gamma}^2\right) ds\right\}.$$

Multiply (2. 19) by $e^{-\gamma t}$, integrate from t_1 to t and use Lemma 2. 4. Then after replacing γ by $\gamma/2$ we obtain

$$(2.20)\qquad \gamma\int_{t_1}^{t} e^{-\gamma s}\left|u(s)\right|_0^2 ds$$
$$\leq C\left\{e^{-\gamma t_1}|h|_0^2+\gamma^{-1}\int_{t_1}^{t} e^{-\gamma s}\left(\left|f(s)\right|_0^2+\langle g(s)\rangle_{\frac{1}{2},\gamma}^2\right) ds\right\}.$$

(A. 1), (2. 19) and (2. 20) imply (2. 18).

§ 3. Tangential estimates

Throughout this and the following sections we suppose $p\geq 1$ and, for $u\in H^{p+1}([t_1, t_2]\times G)$, set $f=Pu$, $g=Bu|_{\partial G}$ and $h=u(t_1)$. We first list in two lemmas some inequalities following from only the relations $Pu=f$ and $u(t_1)=h$, which include (0. 7) and estimates of normal derivatives of u_{I}. Then under the conclusion of Theorem 1, we will derive estimates of tangential derivatives of u, an estimate for the last term on the left side in (0. 2) and the differentiability in tangential directions of the solution to $(P, B)_0$.

LEMMA 3. 1. *There exists a constant $C_p>0$ such that for every $\gamma>0$ and $u\in H^{p+1}([t_1, t_2]\times G)$*

$$(3.1)\qquad \left\|u(t_1)\right\|_{p,\gamma}^2\leq C_p\left(|h|_{p,\gamma}^2+\left\|f(t_1)\right\|_{p-1,\gamma}^2\right),$$

$$(3.2)\qquad \left\|f(t_1)\right\|_{p-1,\gamma}^2\leq C_p\left\|u(t_1)\right\|_{p,\gamma}^2.$$

LEMMA 3. 2. *Let $1\leq k\leq p$. Then there exists constants C_p, $\gamma_p>0$ such that for every $\gamma\geq\gamma_p$ and $u\in H^{p+1}([t_1, t_2]\times G)$*

$$(3.3)\qquad e^{-\gamma t}\left\|D_n^k u_{\mathrm{I}}(t)\right\|_{p-k,\gamma}^2+\gamma\int_{t_1}^{t} e^{-\gamma s}\left\|D_n^k u_{\mathrm{I}}(s)\right\|_{p-k,\gamma}^2 ds$$
$$\leq C_p\left\{\left(e^{-\gamma t_1}\left\|u(t_1)\right\|_{p,\gamma}^2+\gamma^{-1}\int_{t_1}^{t} e^{-\gamma s}\left\|f(s)\right\|_{p,\gamma}^2 ds\right)\right.$$
$$+\left(e^{-\gamma t}\left\|u(t)\right\|_{p,\gamma}^2+\gamma\int_{t_1}^{t} e^{-\gamma s}\left\|u(s)\right\|_{p,\gamma}^2 ds\right)$$
$$\left.+\sum_{j=1}^{k-1}\left(e^{-\gamma t}\left\|D_n^j u_{\mathrm{II}}(t)\right\|_{p-j,\gamma}^2+\gamma\int_{t_1}^{t} e^{-\gamma s}\left\|D_n^j u_{\mathrm{II}}(s)\right\|_{p-j,\gamma}^2 ds\right)\right\}$$

and

$$(3.4)\qquad \left\|D_n u_{\mathrm{I}}(t)\right\|_{p-1,\gamma}^2\leq C_p\left(\left\|u(t)\right\|_{p,\gamma}^2+\left\|f(t)\right\|_{p-1,\gamma}^2\right).$$

PROOF OF LEMMA 3. 1. Let $1\leq j\leq p$. Then, since $f=Pu$ it follows from (0. 1) that

$$D_t^j u(t_1) = D_t^{j-1} f(t_1) - D_t^{j-1}\Big(\sum_{i=1}^n A_i D_i + C\Big) u(t_1)\,.$$

Hence by (1.2) and the relation $|\cdot|_{k,r} \le |\cdot|_{l,r}$ for $k \le l$ we have for some $C \ge 0$

$$\Big|D_t^j u(t_1)\Big|_{p-j,r}^2 \le \Big|D_t^{j-1} f(t_1)\Big|_{p-j,r}^2 + C\sum_{i=0}^{j-1}\Big|D_t^i u(t_1)\Big|_{p-i,r}^2\,.$$

This together with (1.3) implies (3.1), *i.e.*, (0.7), and (3.2) follows similarly.

To prove Lemma 3.2 we use

LEMMA 3.3. *It holds for every* $\gamma > 0$ *and* $f \in H^1([t_1, t_2] \times G)$ *that*

$$\Big|f(t)\Big|_0^2 \le 2\Big(\Big|f(t_1)\Big|_0^2 + \gamma^{-1}\int_{t_1}^t e^{\gamma(t-s)}\Big|\frac{\partial f}{\partial s}(s)\Big|_0^2 ds\Big), \qquad t_1 < t < t_2\,.$$

PROOF. This is a direct consequence of the equality

$$f(t) = \int_{t_1}^t (e^{-(\gamma/2)(t-s)})\Big(e^{(\gamma/2)(t-s)}\frac{\partial f(s)}{\partial s}\Big) ds + f(t_1)$$

and Schwarz' inequality.

PROOF OF LEMMA 3.2. Let $1 \le k \le p$. Then it follows from (0.8) that

$$D_n^k u_{\mathrm{I}} = D_n^{k-1} A^{-1}\Big(f_{\mathrm{I}} - D_t u_{\mathrm{I}} - A_{\mathrm{I\,I}} u_{\mathrm{I}} - A_{\mathrm{I\,II}} u_{\mathrm{II}} - (Cu)_{\mathrm{I}}\Big)\,. \tag{3.5}$$

By this, (1.6) and the relation $||\cdot||_{i,r} \le ||\cdot||_{j,r}$ for $i \le j$ we have for some $C \ge 0$

$$\Big\|D_n^k u_{\mathrm{I}}(t)\Big\|_{p-k,r}^2 \le C\sum_{j=0}^{k-1}\Big(\Big\|D_n^j f_{\mathrm{I}}(t)\Big\|_{p-1-j,r}^2 + \Big\|D_n^j u(t)\Big\|_{p-j,r}^2\Big)\,. \tag{3.6}$$

Since $k \le p$, (1.8) implies that the first terms in the brackets on the right are dominated by $|||f_{\mathrm{I}}(t)|||_{p-1,r}^2$; rewriting the second terms as $||D_n^j u_{\mathrm{I}}(t)||_{p-j,r}^2 + ||D_n^j u_{\mathrm{II}}(t)||_{p-j,r}^2$, we inductively deduce that for $k = 1, \cdots, p$

$$\Big\|D_n^k u_{\mathrm{I}}(t)\Big\|_{p-k,r}^2 \le C\Big(\Big|\Big|\Big|f_{\mathrm{I}}(t)\Big|\Big|\Big|_{p-1,r}^2 + \Big\|u(t)\Big\|_{p,r}^2 + \sum_{j=1}^{k-1}\Big\|D_n^j u_{\mathrm{II}}(t)\Big\|_{p-j,r}^2\Big)\,.$$

Moreover, since $f_{\mathrm{I}} \in H^p([t_1, t_2] \times G)$ Lemma 3.3 and (1.4) give

$$\Big|\Big|\Big|f_{\mathrm{I}}(t)\Big|\Big|\Big|_{p-1,r}^2 \le C\Big(\Big|\Big|\Big|f_{\mathrm{I}}(t_1)\Big|\Big|\Big|_{p-1,r}^2 + \gamma^{-1}\int_{t_1}^t e^{\gamma(t-s)}\Big|\Big|\Big|f_{\mathrm{I}}(s)\Big|\Big|\Big|_{p,r}^2 ds\Big)\,.$$

Hence, by (3.2)

$$\begin{aligned} &e^{-\gamma t}\Big\|D_n^k u_{\mathrm{I}}(t)\Big\|_{p-k,r}^2 \\ &\quad\le C\Big\{e^{-\gamma t_1}\Big|\Big|\Big|u(t_1)\Big|\Big|\Big|_{p,r}^2 + \gamma^{-1}\int_{t_1}^t e^{-\gamma s}\Big|\Big|\Big|f(s)\Big|\Big|\Big|_{p,r}^2 ds \\ &\qquad + e^{-\gamma t}\Big(\Big\|u(t)\Big\|_{p,r}^2 + \sum_{j=1}^{k-1}\Big\|D_n^j u_{\mathrm{II}}(t)\Big\|_{p-j,r}^2\Big)\Big\} \end{aligned}$$

since $t > t_1$. This means that the first term on the left in (3. 3) is estimated by the right. The estimate of the second term is obtained from the above inequality as we derived (2. 20) from (2. 19). Thus (3. 3) is established, and (3. 4) follows from (3. 6) with $k=1$.

LEMMA 3. 4. *Under the conclusion of Theorem* 1, *there exist constants* C_p, $\gamma_p > 0$ *such that for every* $\gamma \geq \gamma_p$ *and* $u \in H^{p+2}([t_1, t_2] \times G)$

$$
(3.7) \qquad e^{-\gamma t}\left\|u(t)\right\|_{p,\gamma}^2 + \gamma \int_{t_1}^{t} e^{-\gamma s}\left(\left\|u(s)\right\|_{p,\gamma}^2 + \left\langle\left\langle u_{\mathrm{I}}(s)\right\rangle\right\rangle_{p-\frac{1}{2},\gamma}^2\right) ds
$$
$$
\leq C_p\left\{e^{-\gamma t_1}\left\|u(t_1)\right\|_{p,\gamma}^2 + \gamma^{-1}\int_{t_1}^{t} e^{-\gamma s}\left(\left\|f(s)\right\|_{p,\gamma}^2 + \left\langle\left\langle g(s)\right\rangle\right\rangle_{p+\frac{1}{2},\gamma}^2\right) ds\right\}.
$$

PROOF. Let $0 \leq i \leq p$ and $u \in H^{p+2}([t_1, t_2] \times G)$; then $P(D_t^i \Lambda_\gamma^{p-i} u) \in H^1([t_1, t_2] \times G) \subset C^0([t_1, t_2]\,;\, L^2(G))$ and $B(D_t^i \Lambda_\gamma^{p-i} u)|_{\partial G} \in C^0([t_1, t_2]\,;\, H^{\frac{1}{2}}(\partial G))$. Hence, by the assumption we can apply a priori estimate (0. 2) with $p=0$ to $D_t^i \Lambda_\gamma^{p-i} u$; noting that $\langle D_t^i \Lambda_\gamma^{p-i} u_{\mathrm{I}}(t)\rangle_{-\frac{1}{2},\gamma}^2 \leq C\langle A_n D_t^i \Lambda_\gamma^{p-i} u(t)\rangle_{-\frac{1}{2},\gamma}^2$, we get

$$
(3.8) \qquad e^{-\gamma t}\left|D_t^i \Lambda_\gamma^{p-i} u(t)\right|_0^2 + \gamma \int_{t_1}^{t} e^{-\gamma s}\left(\left|D_t^i \Lambda_\gamma^{p-i} u(s)\right|_0^2 + \left\langle D_t^i \Lambda_\gamma^{p-i} u_{\mathrm{I}}(s)\right\rangle_{-\frac{1}{2},\gamma}^2\right) ds
$$
$$
\leq C\left\{e^{-\gamma t_1}\left|D_t^i \Lambda_\gamma^{p-i} u(t_1)\right|_0^2\right.
$$
$$
\left. + \gamma^{-1}\int_{t_1}^{t} e^{-\gamma s}\left(\left|P(D_t^i \Lambda_\gamma^{p-i} u(s))\right|_0^2 + \left\langle B(D_t^i \Lambda_\gamma^{p-i} u(s))\right\rangle_{\frac{1}{2},\gamma}^2\right) ds\right\}.
$$

Observe that

$$
P(D_t^i \Lambda_\gamma^{p-i} u) = D_t^i \Lambda_\gamma^{p-i} f - [D_t^i \Lambda_\gamma^{p-i}, C]\, u
$$
$$
- \sum_{j=1}^{n=1} [D_t^i \Lambda_\gamma^{p-i}, A_j]\, D_j u - [D_t^i \Lambda_\gamma^{p-i}, A_n]^t \left({}^t(D_n u_{\mathrm{I}}), 0\right),
$$
$$
B(D_t^i \Lambda_\gamma^{p-i} u) = D_t^i \Lambda_\gamma^{p-i} g - [D_t^i \Lambda_\gamma^{p-i}, B_{\mathrm{I}}]\, u_{\mathrm{I}}.
$$

Then we find from (1. 5) and (1. 6) that the first and second terms in the brackets ( ) on the right in (3. 8) are dominated respectively by constant times

$$
\left\|f(s)\right\|_{p,\gamma}^2 + \left\|u(s)\right\|_{p,\gamma}^2 + \left\|D_n u_{\mathrm{I}}(s)\right\|_{p-1,\gamma}^2,
$$
$$
\left\langle\left\langle g(s)\right\rangle\right\rangle_{p+\frac{1}{2},\gamma}^2 + \left\langle\left\langle u_{\mathrm{I}}(s)\right\rangle\right\rangle_{p-\frac{1}{2},\gamma}^2.
$$

Therefore (3. 8) leads to

$$
e^{-\gamma t}\left\|u(t)\right\|_{p,\gamma}^2 + \gamma \int_{t_1}^{t} e^{-\gamma s}\left(\left\|u(s)\right\|_{p,\gamma}^2 + \left\langle\left\langle u_{\mathrm{I}}(s)\right\rangle\right\rangle_{p-\frac{1}{2},\gamma}^2\right) ds
$$
$$
\leq C\left\{e^{-\gamma t_1}\left\|u(t_1)\right\|_{p,\gamma}^2 + \gamma^{-1}\int_{t_1}^{t} e^{-\gamma s}\left(\left\|f(s)\right\|_{p,\gamma}^2 + \left\langle\left\langle g(s)\right\rangle\right\rangle_{p+\frac{1}{2},\gamma}^2\right) ds\right.
$$

$$+\gamma^{-1}\int_{t_1}^{t} e^{-\gamma s}\left(|\!|\!| u(s) |\!|\!|_{p,\gamma}^2+\langle\!\langle u_{\mathrm{I}}(s)\rangle\!\rangle_{p-\frac{1}{2},\gamma}^2+|\!|\!| D_n u_{\mathrm{I}}(s)|\!|\!|_{p-1,\gamma}^2\right) ds\Big\}.$$

Applying (3.4) to the last term we obtain (3.7).

LEMMA 3.5. *Under the conclusion of Theorem 1, there exist constants C_p, $\gamma_p>0$ such that for every $\gamma\geq\gamma_p$ and $u\in H^{p+2}([t_1, t_2]\times G)$*

$$\tag{3.9} \begin{aligned}\gamma\int_{t_1}^{t} e^{-\gamma s}\langle\!\langle \Lambda_\gamma^{-\frac{1}{2}} A_n u(s) \rangle\!\rangle_{p,\gamma}^2 ds &\leq C_p\Big\{e^{-\gamma t_1}|\!|\!| u(t_1)|\!|\!|_{p,\gamma}^2+\gamma\int_{t_1}^{t} e^{-\gamma s}|\!|\!| u(s)|\!|\!|_{p,\gamma}^2 ds\\ &\quad+\gamma^{-1}\int_{t_1}^{t} e^{-\gamma s}\left(|\!|\!| f(s)|\!|\!|_{p,\gamma}^2+\langle\!\langle g(s)\rangle\!\rangle_{p+\frac{1}{2},\gamma}^2\right) ds\Big\}.\end{aligned}$$

PROOF. According to (1.7) we have $\langle\!\langle\!\langle \Lambda_\gamma^{-\frac{1}{2}} A_n u(s)\rangle\!\rangle\!\rangle_{p,\gamma}\leq C\langle\!\langle\!\langle \Lambda_\gamma^{-\frac{1}{2}} u_{\mathrm{I}}(s)\rangle\!\rangle\!\rangle_{p,\gamma}$ and hence

$$\tag{3.10} \begin{aligned}\gamma\int_{t_1}^{t} e^{-\gamma s}\langle\!\langle\!\langle \Lambda_\gamma^{-\frac{1}{2}} A_n u(s)\rangle\!\rangle\!\rangle_{p,\gamma}^2 ds &\leq C\gamma\int_{t_1}^{t} e^{-\gamma s}\Big(\langle\!\langle\!\langle \Lambda_\gamma^{-\frac{1}{2}} u_{\mathrm{I}}(s)\rangle\!\rangle\!\rangle_{p,\gamma}^2+\sum_{\substack{j+k\leq p\\ k\geq 1}}\left\langle \Lambda_\gamma^{p-\frac{1}{2}-j-k} D_t^j D_n^k u_{\mathrm{I}}(s)\right\rangle_0^2\Big) ds.\end{aligned}$$

By (3.7) and the relation $||\cdot||_{p,\gamma}\leq|\!|\!|\cdot|\!|\!|_{p,\gamma}$, the first term on the right in (3.10) is dominated by the right of (3.9). Therefore it is enough to estimate the second term.

Let $k\geq 1$ and $j+k\leq p$ and apply $\Lambda_\gamma^{p-\frac{1}{2}-j-k} D_t^j D_n^{k-1}$ to (0.8). Then we have

$$\begin{aligned}\Lambda_\gamma^{p-\frac{1}{2}-j-k} D_t^j D_n^k u_{\mathrm{I}} &= -\Lambda_\gamma^{p-\frac{1}{2}-j-k} D_t^j D_n^{k-1} A^{-1} D_t u_{\mathrm{I}}\\ &\quad-\Lambda_\gamma^{-\frac{1}{2}}\cdot\Lambda_\gamma^{p-j-k} D_t^j D_n^{k-1} A^{-1}\left(A_{\mathrm{I\,I}} u_{\mathrm{I}}+A_{\mathrm{I\,II}} u_{\mathrm{II}}+(Cu)_{\mathrm{I}}-f_{\mathrm{I}}\right).\end{aligned}$$

Here we remark that $\Lambda_\gamma^{-\frac{1}{2}}=\Lambda_\gamma^{\frac{1}{2}}\Lambda_\gamma^{-1}$ and

$$\langle \Lambda_\gamma^{\frac{1}{2}} u\rangle_0\leq C|u|_{1,\gamma} \qquad u\in H^1(G).$$

Then we see from (1.4) that

$$\begin{aligned}\left\langle \Lambda_\gamma^{p-\frac{1}{2}-j-k} D_t^j D_n^k u_{\mathrm{I}}(t)\right\rangle_0^2 &\leq C\Big(\left\langle \Lambda_\gamma^{p-\frac{1}{2}-j-k} D_t^{j+1} D_n^{k-1} u_{\mathrm{I}}(t)\right\rangle_0^2\\ &\quad+\sum_{i,l}\left\langle \Lambda_\gamma^{p-\frac{1}{2}-j-k} D_t^i D_n^l u_{\mathrm{I}}(t)\right\rangle_0^2+|\!|\!| u(t)|\!|\!|_{p,\gamma}^2+|\!|\!| \Lambda_\gamma^{-1} f_{\mathrm{I}}(t)|\!|\!|_{p,\gamma}^2\Big),\end{aligned}$$

where the sum is taken over $i+l\leq j+k-1$, $i\leq j+1$ and $l\leq k-1$. Observe that the first and second terms on the right are, for $k=1$ and $j=0,\cdots,p-1$, dominated by $C\langle\!\langle \Lambda_\gamma^{-\frac{1}{2}} u_{\mathrm{I}}(t)\rangle\!\rangle_{p,\gamma}^2$ according to (1.5). Employing the above inductively for $k=2,\cdots,p$ with $j=0,\cdots,p-k$ we obtain, for any $k\geq 1$ and $j+k\leq p$,

$$\left\langle \Lambda_\gamma^{p-\frac{1}{2}-j-k} D_t^j D_n^k u_{\mathrm{I}}(t)\right\rangle_0^2$$
$$\leq C\left(\left\langle \Lambda_\gamma^{-\frac{1}{2}} u_{\mathrm{I}}(t)\right\rangle_{p,\gamma}^2 + \left|\!\left|\!\left| u(t)\right|\!\right|\!\right|_{p,\gamma}^2 + \left|\!\left|\!\left| \Lambda_\gamma^{-1} f_{\mathrm{I}}(t)\right|\!\right|\!\right|_{p,\gamma}^2\right).$$

Combining this with (3. 7) we conclude that the second term of (3. 10) is dominated by the right of (3. 9). Thus the lemma is proved.

PROPOSITION 3. 6. *Under the conclusion of Theorem* 1, *that of Proposition* 2. 3 *is valid.*

PROOF. It suffices to prove the conclusions of Lemma 2. 5 under our assumption. (2. 14) for $p=0$ and $u\in C_0^\infty(\boldsymbol{R}^1\times G)$ follows from (0. 2) with $p=0$ by replacing γ by 2γ, since $\langle u_{\mathrm{I}}(s)\rangle_{-\frac{1}{2},\gamma}\leq C\langle A_n u(s)\rangle_{-\frac{1}{2},\gamma}$ and t_1, t_2 are arbitrary. To prove (2. 15) for $p=0$ we claim that for every $f\in C_0^\infty(\boldsymbol{R}^1\times G)$ and $g\equiv 0$ $(P, B)_0$ has a solution u satisfying (2. 10) with $p=0$. In fact, choose t_1 so that supp $f\subset\{t>t_1\}$, and let u be an extension to $\{t<t_1\}$ of the solution to the mixed problem (P, B) with $g=h=0$ such that $u=0$ in $\{t<t_1\}$. Then since t_2 is arbitrary and the solution is unique in $[t_1, t_2]\times G$, u is well defined for all $t\in\boldsymbol{R}^1$ and is a desired solution to $(P, B)_0$ according to (0. 2) with $p=0$. Therefore (2. 15) for $p=0$ and $v\in C_0^\infty(\boldsymbol{R}^1\times G)$ follows from Green's formula (cf. [2], Lemma 4. 3 and (A. 4) below) and Proposition 2. 1 applied to the problem (2. 17). (2. 14) and (2. 15) with $p\geq 1$ follow from those with $p=0$.

§ 4. Normal estimates

We first show the following lemma concerning the equation

$$\left\{D_t + A_{\mathrm{II\,II}}(t, x; D_{x'}) + C_{\mathrm{II\,II}}(t, x)\right\} u_{\mathrm{II}} = f_{\mathrm{II}} \quad \text{in} \quad \boldsymbol{R}^1\times G:$$

LEMMA 4. 1. (i) *For any integer* $p\geq 0$ *there exist constants* C_p, $\gamma_p>0$ *such that for every* $\gamma\geq\gamma_p$ *and* $u_{\mathrm{II}}(t, x)\in H^{p+1}([t_1, t_2]\times G)$

$$e^{-\gamma t}\left|\!\left|\!\left| u_{\mathrm{II}}(t)\right|\!\right|\!\right|_{p,\gamma}^2 \leq C_p\left(e^{-\gamma t_1}\left|\!\left|\!\left| u_{\mathrm{II}}(t_1)\right|\!\right|\!\right|_{p,\gamma}^2 + \gamma^{-1}\int_{t_1}^t e^{-\gamma s}\left|\!\left|\!\left| f_{\mathrm{II}}(s)\right|\!\right|\!\right|_{p,\gamma}^2 ds\right). \tag{4. 1}$$

(ii) *For any integer* p *there exist constants* C_p, $\gamma_p>0$ *such that for every* $\gamma\geq\gamma_p$ *and* $f_{\mathrm{II}}\in H_{0,p;\gamma}(\boldsymbol{R}^1\times G)$ *there exists a unique solution* $u_{\mathrm{II}}\in H_{0,p;\gamma}$ $(\boldsymbol{R}^1\times G)$ *satisfying*

$$||u_{\mathrm{II}}||_{p,\gamma}^2 \leq C_p\gamma^{-2}||f_{\mathrm{II}}||_{p,\gamma}^2. \tag{4. 2}$$

PROOF. Since $A_{\mathrm{II\,II}}$ is hermitian and does not contain D_n, as we derived (2. 13) we have

$$\frac{d}{dt}\left(e^{-\gamma t}\left|u_{\mathrm{II}}(t)\right|_0^2\right) \leq C\gamma^{-1}e^{-\gamma t}\left|f_{\mathrm{II}}(t)\right|_0^2. \tag{4. 3}$$

Integrating (4. 3) over $[t_1, t]$ yields (4. 1) with $p=0$; applying it to $(\Lambda_\gamma^{p-i} D_t^i u_{\mathrm{II}})$ with $u_{\mathrm{II}} \in H^{p+1}([t_1, t_2] \times G)$ and $i \leq p$ we get (4. 1) with $p \geq 1$ as in the proof of Lemma 3. 4. (ii) follows from (4. 3) by the same fashon as in the proofs of Lemma 2. 5 and Proposition 2. 3.

In what follows, we suppose $p \geq 1$ and the condition (A). We shall first derive a priori estimates (0. 2) and then the differentiability in the normal direction of the solution to $(P, B)_0$.

Now, in order to prove (0. 2) we use

LEMMA 4. 2. *Let $1 \leq k \leq p$. Then there exist constants C_p, $\gamma_p > 0$ such that for every $\gamma \geq \gamma_p$ and $u \in H^{p+1}([t_1, t_2] \times G)$*

$$
\begin{aligned}
(4.4) \qquad & e^{-\gamma t} \left\| D_n^k u_{\mathrm{II}}(t) \right\|_{p-k,\gamma}^2 + \gamma \int_{t_1}^t e^{-\gamma s} \left\| D_n^k u_{\mathrm{II}}(s) \right\|_{p-k,\gamma}^2 ds \\
& \leq C_p \Big\{ \Big(e^{-\gamma t_1} \left\| u(t_1) \right\|_{p,\gamma}^2 + \gamma^{-1} \int_{t_1}^t e^{-\gamma s} \left\| f(s) \right\|_{p,\gamma}^2 ds \Big) \\
& \quad + \Big(e^{-\gamma t} \left\| u(t) \right\|_{p,\gamma}^2 + \gamma \int_{t_1}^t e^{-\gamma s} \left\| u(s) \right\|_{p,\gamma}^2 ds \Big) \\
& \quad + \sum_{j=1}^{k-1} \Big(e^{-\gamma t} \left\| D_n^j u_{\mathrm{I}}(t) \right\|_{p-j,\gamma}^2 + \gamma \int_{t_1}^t e^{-\gamma s} \left\| D_n^j u(s) \right\|_{p-j,\gamma}^2 ds \Big) \\
& \quad + \gamma^{-1} \int_{t_1}^t e^{-\gamma s} \left\| D_n^k u_{\mathrm{I}}(s) \right\|_{p-k,\gamma}^2 ds \Big\} .
\end{aligned}
$$

PROOF. The definition of f yields (0.9) and (A) implies that there are no second order derivatives on the right side of (0. 9). Hence, applying D_n^{k-1} with $1 \leq k \leq p$ to (0. 9) we have

$$
(4.5) \qquad (D_t + A_{\mathrm{II\,II}} + C_{\mathrm{II\,II}})(D_n^k u_{\mathrm{II}} - D_n^{k-1} A_{\mathrm{II\,I}} A^{-1} u_{\mathrm{I}}) = F \qquad \text{for} \quad x_n \geq 0 ,
$$

where

$$
\begin{aligned}
(4.6) \qquad F = {} & D_n^{k-1} \{ D_n f_{\mathrm{II}} - A_{\mathrm{II\,I}} A^{-1} f_{\mathrm{I}} - C_{\mathrm{II\,I}} D_n u_{\mathrm{I}} + K_5 u \} \\
& + [A_{\mathrm{II\,II}} + C_{\mathrm{II\,II}}, D_n^{k-1}] (D_n u_{\mathrm{II}} - A_{\mathrm{II\,I}} A^{-1} u_{\mathrm{I}}) .
\end{aligned}
$$

Since $u \in H^{p+1}([t_1, t_2] \times G)$, (4. 5) together with Lemma 4. 1 (i) with p replaced by $p-k$ implies that

$$
\begin{aligned}
& e^{-\gamma t} \left\| D_n^k u_{\mathrm{II}}(t) - D_n^{k-1} A_{\mathrm{II\,I}} A^{-1} u_{\mathrm{I}}(t) \right\|_{p-k,\gamma}^2 \\
& \leq C_p \Big\{ e^{-\gamma t_1} \left\| D_n^k u_{\mathrm{II}}(t_1) - D_n^{k-1} A_{\mathrm{II\,I}} A^{-1} u_{\mathrm{I}}(t_1) \right\|_{p-k,\gamma}^2 \\
& \quad + \gamma^{-1} \int_{t_1}^t e^{-\gamma s} \left\| F(s) \right\|_{p-k,\gamma}^2 ds \Big\} .
\end{aligned}
$$

Hence

$$e^{-rt}\left\|D_n^k u_{\mathrm{II}}(t)\right\|_{p-k,r}^2 \leq C\left\{e^{-rt_1}\left\|\!\left| u(t_1)\right|\!\right\|_{p,r}^2\right.$$
$$\left.+\sum_{j=0}^{k-1} e^{-rt}\left\|D_n^j u_{\mathrm{I}}(t)\right\|_{p-j,r}^2+\gamma^{-1}\int_{t_1}^t e^{-rs}\left\|F(s)\right\|_{p-k,r}^2 ds\right\}.$$

Note that K_5 in (4. 6) is a first order differential operators not containing D_n whose coefficients depend only on those of P. Then from (4. 6) we see that for some $C>0$

$$\left\|F(t)\right\|_{p-k,r}^2 \leq C\left(\left\|\!\left| f(t)\right|\!\right\|_{p,r}^2+\sum_{j=0}^{k-1}\left\|D_n^j u(t)\right\|_{p-j,r}^2+\left\|D_n^k u_{\mathrm{I}}(t)\right\|_{p-k,r}^2\right).$$

Therefore we get for $1\leq k\leq p$

$$e^{-rt}\left\|D_n^k u_{\mathrm{II}}(t)\right\|_{p-k,r}^2$$
$$\leq C\left\{\left(e^{-rt_1}\left\|\!\left| u(t_1)\right|\!\right\|_{p,r}^2+\gamma^{-1}\int_{t_1}^t e^{-rs}\left\|\!\left| f(s)\right|\!\right\|_{p,r}^2 ds\right)\right.$$
$$+\sum_{j=0}^{k-1}\left(e^{-rt}\left\|D_n^j u_{\mathrm{I}}(t)\right\|_{p-j,r}^2+\gamma^{-1}\int_{t_1}^t e^{-rs}\left\|D_n^j u(s)\right\|_{p-j,r}^2 ds\right)$$
$$\left.+\gamma^{-1}\int_{t_1}^t e^{-rs}\left\|D_n^k u_{\mathrm{I}}(s)\right\|_{p-k,r}^2 ds\right\}.$$

This means that the first term on the left in (4. 4) is estimated by the right. The estimate of the second term is obtained from the above inequality as we derived (2. 20) from (2. 19).

From Lemmas 3. 2 and 4. 2 we have

LEMMA 4. 3. *There exist constants C_p, $\gamma_p>0$ such that for every $\gamma\geq\gamma_p$ and $u\in H^{p+1}([t_1, t_2]\times G)$*

$$e^{-rt}\left\|\!\left| u(t)\right|\!\right\|_{p,r}^2+\gamma\int_{t_1}^t e^{-rs}\left\|\!\left| u(s)\right|\!\right\|_{p,r}^2 ds \tag{4. 7}$$
$$\leq C_p\left\{\left(e^{-rt_1}\left\|\!\left| u(t_1)\right|\!\right\|_{p,r}^2+\gamma^{-1}\int_{t_1}^t e^{-rs}\left\|\!\left| f(s)\right|\!\right\|_{p,r}^2 ds\right)\right.$$
$$\left.+\left(e^{-rt}\left\| u(t)\right\|_{p,r}^2+\gamma\int_{t_1}^t e^{-rs}\left\| u(s)\right\|_{p,r}^2 ds\right)\right\}.$$

PROOF. From (3. 3) and (4. 4) we get for $1\leq k\leq p$

$$e^{-rt}\left\|D_n^k u(t)\right\|_{p-k,r}^2+\gamma\int_{t_1}^t e^{-rs}\left\|D_n^k u(s)\right\|_{p-k,r}^2 ds$$
$$\leq C\left\{K+\sum_{j=1}^{k-1}\left(e^{-rt}\left\|D_n^j u(t)\right\|_{p-j,r}^2+\gamma\int_{t_1}^t e^{-rs}\left\|D_n^j u(s)\right\|_{p-j,r}^2 ds\right)\right\},$$

where K is the right side of (4. 7). Using this inductively for $k=1, \cdots, p$ yields (4. 7).

From Lemmas 4. 3, 3. 4 and 3. 5 we obtain (0. 2), that is,

PROPOSITION 4. 4. *Under the assumptions of Theorem 2, there exist constants* C_p, $\gamma_p>0$ *such that for every* $\gamma\geq\gamma_p$ *and* $u\in H^{p+2}([t_1,t_2]\times G)$

$$e^{-\gamma t}|||u(t)|||_{p,\gamma}^2+\gamma\int_{t_1}^{t}e^{-\gamma s}\Big(|||u(s)|||_{p,\gamma}^2+\langle\langle\langle\Lambda_\gamma^{-\frac{1}{2}}A_n u(s)\rangle\rangle\rangle_{p,\gamma}^2\Big)ds$$
$$\leq C_p\Big\{e^{-\gamma t_1}|||u(t_1)|||_{p,\gamma}^2+\gamma^{-1}\int_{t_1}^{t}e^{-\gamma s}\Big(|||f(s)|||_{p,\gamma}^2+\langle\langle g(s)\rangle\rangle_{p+\frac{1}{2},\gamma}^2\Big)ds\Big\}.$$

The rest of this section is devoted to the proof of

PROPOSITION 4. 5. *Under the assumptions of Theorem 2, there exist constants* C_p, $\gamma_p>0$ *such that for every* $\gamma\geq\gamma_p$, $f\in H_{p,0;\gamma}(\boldsymbol{R}^1\times G)$ *and* g *with* $\Lambda_\gamma^{\frac{1}{2}}g\in H_{p,\gamma}(\boldsymbol{R}^1\times\partial G)$ *there exists a unique solution* $u\in H_{p,0;\gamma}(\boldsymbol{R}^1\times G)$ *to* $(P,B)_0$ *satisfying the inequality*

$$|||u|||_{p,\gamma}^2\leq C_p\gamma^{-2}\Big(|||f|||_{p,\gamma}^2+\langle\langle\Lambda_\gamma^{\frac{1}{2}}g\rangle\rangle_{p,\gamma}^2\Big).$$

First, from (3. 5) and (3. 6) we deduce the regularity result corresponding to Lemma 3. 2:

LEMMA 4. 6. *Suppose that* $u\in H_{0,p;\gamma}(\boldsymbol{R}^1\times G)$ *and* $f\in H_{p,0;\gamma}(\boldsymbol{R}^1\times G)$. *Let* $1\leq k\leq p$. *If* $||D_n^j u||_{p-j,\gamma}$ *are finite for all* $j=1,\cdots,k-1$, *then* $||D_n^k u_{\mathrm{I}}||_{p-k,\gamma}^2$ *is finite and the inequality*

$$||D_n^k u_{\mathrm{I}}||_{p-k,\gamma}^2\leq C\Big(|||f|||_{p-1,\gamma}^2+||u||_{p,\gamma}^2+\sum_{j=1}^{k-1}||D_n^j u||_{p-j,\gamma}^2\Big)$$

holds.

Next we have the regularity result corresponding to Lemma 4. 2:

LEMMA 4. 7. *Suppose that* $u\in H_{0,p;\gamma}(\boldsymbol{R}^1\times G)$ *and* $f\in H_{p,0;\gamma}(\boldsymbol{R}^1\times G)$. *Let* $1\leq k\leq p$. *If* $||D_n^j u||_{p-j,\gamma}$ *are finite for all* $j=1,\cdots,k-1$ *and so is* $||D_n^k u_{\mathrm{I}}||_{p-k,\gamma}$, *then* $||D_n^k u_{\mathrm{II}}||_{p-k,\gamma}$ *is finite and the inequality*

$$||D_n^k u_{\mathrm{II}}||_{p-k,\gamma}^2\leq C\Big(\gamma^{-2}|||f|||_{p,\gamma}^2+||u||_{p,\gamma}^2$$
$$+\sum_{j=1}^{k-1}||D_n^j u||_{p-j,\gamma}^2+\gamma^{-2}||D_n^k u_{\mathrm{I}}||_{p-k,\gamma}^2\Big)$$

holds.

PROOF. Since (A) implies (4. 5) with $F\equiv F_k$ defined by (4. 6), it follows from the assumptions and $k\leq p$ that F_k belongs to $H_{0,p-k;\gamma}(\boldsymbol{R}^1\times G)$ and that

$$||F_k||_{p-k,\gamma}^2\leq C\Big(|||f|||_{p,\gamma}^2+\sum_{j=0}^{k-1}||D_n^j u||_{p-j,\gamma}^2+||D_n^k u_{\mathrm{I}}||_{p-k,\gamma}^2\Big). \tag{4. 8}$$

So applying Lemma 4. 1 (ii) with p replaced by $p-k$ to (4. 5), we get

$$||D_n^k u_{\mathrm{II}} - D_n^{k-1} A_{\mathrm{II}\,\mathrm{I}} A^{-1} u_{\mathrm{I}}||_{p-k,\gamma}^2 \leq C\gamma^{-2} ||F_k||_{p-k,\gamma}^2 ,$$

hence

$$||D_n^k u_{\mathrm{II}}||_{p-k,\gamma}^2 \leq C\Big(\gamma^{-2} ||F_k||_{p-k,\gamma}^2 + \sum_{j=0}^{k-1} ||D_n^j u_{\mathrm{I}}||_{p-j,\gamma}^2\Big).$$

This together with (4. 8) implies the lemma.

PROOF OF PROPOSITION 4. 5. Since $f \in H_{p,0;\gamma}(\boldsymbol{R}^1 \times G)$ implies $f \in H_{0,p;\gamma}(\boldsymbol{R}^1 \times G)$, we have the unique solution $u \in H_{0,p;\gamma}(\boldsymbol{R}^1 \times G)$ described in Proposition 3. 6. For such f and u apply Lemmas 4. 6 and 4. 7 alternately. Then we see that $u \in H_{p,0;\gamma}(\boldsymbol{R}^1 \times G)$ and

$$|||u|||_{p,\gamma}^2 \leq C\Big(||u||_{p,\gamma}^2 + \gamma^{-2} |||f|||_{p,\gamma}^2\Big).$$

This together with (2. 10) implies the proposition.

§ 5. Proofs of Theorems 1 and 2

To define the compatibility conditions for (P, B), let the solution $u(t, x) \in \bigcap_{i=0}^{k} C^i([t_1, t_2] ; H^{p-i}(G))$. Then from $Bu|_{\partial G} = g$ the relations

$$(5.1) \qquad (D_t^i g)(t_1, x) = (D_t^i Bu)(t_1, x) = \sum_{j=0}^{i} \binom{i}{j} (D_t^{i-j} B)(t_1, x)(D_t^j u)(t_1, x), \quad \text{on } \partial G, \ (0 \leq i \leq k)$$

must hold. Representing $(D_t^j u)(t_1, x)$ by f and h and inserting them into the above, we arrive at the following

DEFINITION 5. 1. Let $p \geq 1$, $f \in H^p([t_1, t_2] \times G)$, $\Lambda_\gamma^{\frac{1}{2}} g \in H^p([t_1, t_2] \times \partial G)$ and $h \in H^p(G)$. Let these data satisfy

$$(5.2) \qquad \sum_{j=0}^{i} \binom{i}{j} (D_t^{i-j} B)(t_1, x)\, h^{(j)}(x) = (D_t^i g)(t_1, x) \quad \text{on } \partial G \ \text{ for } \ i = 0, \cdots, k,$$

where $k \leq p-1$, $h^{(0)}(x) = h(x)$,

$$(5.3) \qquad h^{(i)}(x) = (D_t^{i-1} f)(t_1, x) - \sum_{j=0}^{i-1} \binom{i-1}{j} L_{i-1-j}(t_1)\, h^{(j)}(x), \quad (1 \leq i \leq p),$$

$$(5.4) \qquad L_i(t_1) = \Big(D_t^i (P - D_t)\Big)(t_1) = \sum_{j=1}^{n} (D_t^i A_j)(t_1, x)\, D_j + (D_t^i C)(t_1, x).$$

Then we say that the compatibility conditions of order k or, for convenience, $\{f, g, h\, ;\ p, k\}$ are fulfilled.

Now in order to prove our theorems we need

LEMMA 5.2. *Suppose that* $p\geq 1$ *and* $\{f, g, h\,;\, p, p-1\}$ *are fulfilled and let* $q\geq p+1$ *be an integer. Then there exist sequences* $\{f_n\}\subset H^q([t_1, t_2]\times G)$, $\{\Lambda^{\frac{1}{2}}_T g_n\}\subset H^q([t_1, t_2]\times\partial G)$ *and* $\{h_n\}\subset H^p(G)$ *such that* $\{f_n, g_n, h_n\,;\, p+1, p\}$ *are fulfilled and that* $f_n\to f$ *in* $H^p([t_1, t_2]\times G)$, $\Lambda^{\frac{1}{2}}_T g_n\to\Lambda^{\frac{1}{2}}_T g$ *in* $H^p([t_1, t_2]\times\partial G)$ *and* $h_n\to h$ *in* $H^p(G)$ *as* $n\to\infty$.

PROOF. Since this lemma can be proved by a minor modification of the proof of Rauch and Massey III [9], Lemma 3.3, we only describe different points preserving the same notations as in [9], p 309 as far as possible. First let $B(t, x)$ be independent of t. Then the data fulfills $Bh^{(i)}=(D^i_t g)(t_1)$ on ∂G for $0\leq i\leq p-1$, and we must approximate f, g, h by sequences $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ satisfying $Bh^{(i)}_n=(D^i_t g_n)(t_1)$ on ∂G for $0\leq i\leq p$.

Let $q\geq p+1$ and first take sequences $\{f_n\}\subset H^{p+q}([t_1, t_2]\times G)$, $\{\Lambda^{\frac{1}{2}}_T g_n\}\subset H^{p+q}([t_1, t_2]\times\partial G)$, $\{\tilde h_n\}\subset H^{p+q}(G)$ with $f_n\to f$ in $H^p([t_1, t_2]\times G)$, $\Lambda^{\frac{1}{2}}_T g_n\to\Lambda^{\frac{1}{2}}_T g$ in $H^p([t_1, t_2]\times\partial G)$, $\tilde h_n\to h$ in $H^p(G)$ and write the desired sequence $\{h_n\}$ as $h_n=\tilde h_n-h'_n$. Here $h'_n\in H^q(G)$ must be chosen so that $h'_n\to 0$ in $H^p(G)$ and

$$BB_i h'_n = B(B_i\tilde h_n+E_i f_n)-(D^i_t g_n)(t_1) \quad \text{on } \partial G \text{ for } 0\leq i\leq p\,,$$

where B_i and E_i are such operators as in p 309, *i.e.*, (5.3) is rewritten as $h^{(i)}=B_i h+E_i f\in H^{p-i}(G)$. Let $\tilde T={}^t({}^t(B^*_{\rm I}(B_{\rm I}B^*_{\rm I})^{-1}, 0)$ be an $m\times l$ matrix, where $B_{\rm I}$ is the $l\times d$ matrix in (0.4). Then $\tilde TB=I_l$, so it suffices to solve the equation $(B_i h'_n)_{\rm I}=(a_{i,n})_{\rm I}$ on ∂G for $0\leq i\leq p$, where $a_{i,n}=\tilde T\{B(B_i\tilde h_n+E_i f_n)-(D^i_t g_n)(t_1)\}\equiv{}^t({}^t(a_{i,n})_{\rm I}, {}^t(a_{i,n})_{\rm II})$.

According to (5.3) and (5.4), the operator B_i has the form

$$B_i h = \begin{pmatrix} -A & 0\\ 0 & 0\end{pmatrix}^i D^i_n h+\sum_{j=0}^{i-1} C_{i,i-j}D^j_n h\,,$$

where $C_{i,i-j}$ are $m\times m$ matrix valued operators of order $i-j$ which only invlove differentiations tangential to ∂G. We now choose h'_n so that $(h'_n)_{\rm II}=0$. Then the equation $(B_i h'_n)_{\rm I}=(a_{i,n})_{\rm I}$ can be written as $D^i_n(h'_n)_{\rm I}=b_{i,n}$ where

$$b_{i,n}=(-A)^{-i}\Big((a_{i,n})_{\rm I}-\sum_{j=0}^{i-1}(C_{i,i-j})_{\rm I\,I}\,b_{j,n}\Big)$$

and $(C_{i,i-j})_{\rm I\,I}$ is the upper left $d\times d$ matrix of $C_{i,i-j}$. Now the assumption implies that $(a_{i,n})_{\rm I}$ belongs to $H^{p+q-i-\frac{1}{2}}(\partial G)$ for $0\leq i\leq p$ and tends to zero in $H^{p-i-\frac{1}{2}}(\partial G)$ for $0\leq i\leq p-1$, hence so do $b_{i,n}$. Therefore we can construct $(h'_n)_{\rm I}\in H^q(G)$ so that $(h'_n)_{\rm I}\to 0$ in $H^p(G)$ and $D^i_n(h'_n)_{\rm I}=b_{i,n}$ on ∂G for $0\leq i\leq p$ by the same fashion as h_n is done in [9], p 310 (decompose $(h'_n)_{\rm I}=v_n+w_n$ so that, on ∂G, $D^i_n v_n=b_{i,n}$ $(0\leq i\leq p-1)$, $D^i_n v_n=0$ $(p\leq i\leq p+q-1)$ and $D^i_n w_n=0$ $(0\leq i\leq p-1)$, $D^p_n w_n=b_{p,n}-D^p_n v_n)$.

Next, when B depends on t, we can reduce our arguments to the above case, by using such a transformation r of dependent variables as in the proof of [9], Lemma 3.1 (in our case let $r(t,x)=\begin{bmatrix} H(t_1,x)\,H^{-1}(t,x) & 0 \\ 0 & I_{m-d}\end{bmatrix}$ where $H=(B_{\mathrm{I}}^{*},(B_{\mathrm{I}}')^{*})$ and B_{I}' is the matrix in (2.16)).

COROLLARY 5.3. *Suppose that* $p\geq 0$ *and* $f\in H^{p}([t_1,t_2]\times G)$, $\Lambda_{\gamma}^{\frac{1}{2}}g\in H^{p}([t_1,t_2]\times\partial G)$ *and* $h\in H^{p}(G)$. *When* $p\geq 1$ *suppose further that* $\{f,g,h\,;\,p,p-1\}$ *are fulfilled. Then, for* $q\geq p+3$, *there exist sequences* $\{f_n\}\subset H^{q}([t_1,t_2]\times G)$, $\{\Lambda_{\gamma}^{\frac{1}{2}}g_n\}\subset H^{q}([t_1,t_2]\times\partial G)$ *and* $\{h_n\}\subset H^{q}(G)$ *such that* $\{f_n,g_n,h_n\,;\,p+3,p+2\}$ *are fulfilled and that* $f_n\to f$ *in* $H^{p}([t_1,t_2]\times G)$, $\Lambda_{\gamma}^{\frac{1}{2}}g_n\to\Lambda_{\gamma}^{\frac{1}{2}}g$ *in* $H^{p}([t_1,t_2]\times\partial G)$ *and* $h_n\to h$ *in* $H^{p}(G)$ *as* $n\to\infty$.

PROOF. First let $p=0$. Take sequences $\{f_n\}\subset H^{1}([t_1,t_2]\times G)$, $\{\Lambda_{\gamma}^{\frac{1}{2}}g_n\}\subset H^{1}([t_1,t_2]\times\partial G)$ and $\{h_n\}\subset H^{1}(G)$ such that $f_n\to f$ in $L^2([t_1,t_2]\times G)$, $\Lambda_{\gamma}^{\frac{1}{2}}g_n\to\Lambda_{\gamma}^{\frac{1}{2}}g$ in $L^2([t_1,t_2]\times\partial G)$ and $h_n\to h$ in $L^2(G)$, $h_n|_{\partial G}=\tilde{T}g_n(t_1)\in H^{\frac{1}{2}}(\partial G)$ respectively, where $\tilde{T}$ is the $m\times l$ matrix defined in the proof of Lemma 5.2. Then $\{f_n,g_n,h_n\,;\,1,0\}$ are fulfilled. Applying Lemma 5.2 with $p=1$ we approximate, for every fixed n, the sequences $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ above taken. Then we obtain new ones $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ such that $\{f_n,g_n,h_n\,;\,2,1\}$ are fulfilled. Applying once more Lemma 5.2 with $p=2$ to these sequences we obtain the desired ones. Next, when $p\geq 1$, the desired sequences are obtained by the analogous uses of Lemma 5.2.

PROOF OF THEOREM 2. First let $p\geq -1$ for later convenience and suppose that $f\in H^{p+3}([t_1,t_2]\times G)$, $\Lambda_{\gamma}^{\frac{1}{2}}g\in H^{p+2}([t_1,t_2]\times\partial G)$, $h\in H^{p+3}(G)$ and $\{f,g,h\,;\,p+2,p+1\}$ are fulfilled. Let $u_0\in H^{p+3}([t_1,t_2]\times G)$ be a solution to the Cauchy problem

$$\begin{cases} Pu_0=f & \text{in} \quad [t_1,t_2]\times G\,, \\ u_0(t_1,x)=h & \text{for} \quad x\in G\,; \end{cases}$$

we shall consider the boundary value problem

$$\begin{cases} Pu_1=0 & \text{in} \quad \boldsymbol{R}^1\times G\,, \\ Bu_1=\tilde{g} & \text{on} \quad \boldsymbol{R}^1\times\partial G\,, \end{cases} \tag{5.5}$$

where

$$\tilde{g}=\begin{cases} 0 & \text{for } t<t_1 \text{ and for large } t\,, \\ g-Bu_0|_{\partial G} & \text{for } t_1<t<t_2\,. \end{cases}$$

Since $\{f,g,h\,;\,p+2,p+1\}$ are fulfilled, it follows from (5.1), (5.2) and (5.3) that

$$(D_t^i Bu_0)(t_1)=(D_t^i g)(t_1) \qquad \text{on } \partial G \qquad (0\leq i\leq p+1)\,.$$

So, we can take $\tilde{g}$ to be $\Lambda_\gamma^{\frac{1}{2}}\tilde{g}\in H_{p+2,\gamma}(\boldsymbol{R}^1\times\partial G)$. Now let $p\geq 1$. Applying Proposition 4.5 we obtain the solution $u_1\in H_{p+2,\gamma}(\boldsymbol{R}^1\times G)$ to (5.5), which satisfies $u_1(t_1, x)=0$ by Proposition 3.6. Thus $u=u_0+u_1$ is a solution to (P, B) belonging to $H^{p+2}([t_1, t_2]\times G)$ $\left(\text{hence to } \bigcap_{i=0}^{p+1} C^i([t_1, t_2]\,;\, H^{p+1-i}(G))\right)$ and satisfies (0.7) and (0.2) according to Lemma 3.1 and Proposition 4.4 respectively.

Next suppose f, g, h are general data satisfying the hypotheses of the theorem and let $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ be such approximating sequences as described in Corollary 5.3. If u_n is such a solution to (P, B) as above corresponding to the data f_n, g_n, h_n, then inequality (0.2) with (0.7) applied to u_n-u_m shows that the sequence $\{u_n\}$ converges in $C^i([t_1, t_2]\,;\, H^{p-i}(G))$ for all $i\leq p$. Since u_n tends to a function u in $L^2([t_1, t_2]\times G)$, u belongs to $\bigcap_{i=0}^{p} C^i([t_1, t_2]\,;\, H^{p-i}(G))$ and is a (strong) solution to (P, B) with data f, g, h satisfying (0.2) and (0.7). Thus the conclusions of the theorem follow from the uniquness in $C^0([t_1, t_2]\,;\, L^2(G))$ of the solution to (P, B).

Proof of Theorem 1. First let $f\in H^2([t_1, t_2]\times G)$, $\Lambda_\gamma^{\frac{1}{2}} g\in H^1([t_1, t_2]\times\partial G)$, $h\in H^2(G)$ and $\{f, g, h\,;\, 1, 0\}$ be fulfilled. We apply the arguments in the first part of the proof of Theorem 2 to $p=-1$. Then we see from Proposition 2.3 and (0.8) that there exists a solution $u_1\in H_{0,1;\gamma}(\boldsymbol{R}^1\times G)$ to (5.5) such that $D_n(u_1)_{\mathrm{I}}$ belongs to $H_{0,0;\gamma}(\boldsymbol{R}^1\times G)$ and $u_1(t_1, x)=0$. Therefore $u\equiv u_0+u_1$ is a solution to (P, B) which belongs to $\mathscr{H}^1$ so that to $C^0([t_1, t_2]\,;\, L^2(G))$; by this and Proposition 2.7 u satisfies (2.18) (hence (0.2) with $p=0$). Now the existence in $C^0([t_1, t_2]\,;\, L^2(G))$ of such a solution for f, g, h as in Theorem 1 follows from the same arguments as in the proof of Theorem 2 by using Corollary 5.3 with $p=0$ and inequality (0.2) with $p=0$. The uniquness of solutions in $C^0([t_1, t_2]\,;\, L^2(G))$ follows from the uniqueness of solutions to $(P, B)_0$ owing to Proposition 2.3 with $p=0$.

§ 6. Examples

We first present three examples of hermitian hyperbolic operators which (i) are of constant multiplicity, (ii) are uniformly characteristic and (iii) fulfill the condition (A).

Example 6.1. (*the* curl *operator*)

$$P(D_t, D_x)=I_3 D_t\pm\mathrm{curl}=I_3 D_t+\sum_{j=1}^{n} A_j D_j\,, \quad m=n=3\,:$$

$$P(\tau, \sigma, \lambda) = I_3\tau \pm i\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\sigma_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\sigma_2 + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\lambda\right),$$

$$\det P(\tau, \sigma, \lambda) = \tau(\tau^2 - |\sigma|^2 - \lambda^2) \qquad (\text{(i)}),$$
$$\operatorname{rank} A_n = 2 \qquad (\text{(ii)}).$$

We only show (iii) for the operator $I_3D_t + \text{curl}$ (similarly for $I_3D_t - \text{curl}$). Let T_n be the following 3×3 matrix

$$T_n = \begin{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}, \text{ so } T_n^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}.$$

Then P is transformed so that (0. 3) is valid, obtaining

$$T_n^{-1}A_nT_n = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}, \quad T_n^{-1}A_1T_n = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{bmatrix},$$

$$T_n^{-1}A_2T_n = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & 1 & 0 \end{bmatrix}.$$

Hence, $T_n^{-1}PT_n$ fulfills (A) since $A_{\mathrm{I}\,\mathrm{I}} = A_{\mathrm{II}\,\mathrm{II}} = 0$ and $A_{\mathrm{II}\,\mathrm{I}}A^{-1}A_{\mathrm{I}\,\mathrm{II}} = 0$.

EXAMPLE 6. 2. (*Maxwell system*)

$$P(D_t, D_x) = I_6D_t + \frac{1}{i}\begin{bmatrix} 0 & -\text{curl} \\ \text{curl} & 0 \end{bmatrix} = I_6D_t + \sum_{j=1}^{n} A_jD_j, \quad m=6,\ n=3:$$

$$\det P(\tau, \sigma, \lambda) = \tau^2(\tau^2 - |\sigma|^2 - \lambda^2)^2 \qquad (\text{(i)}).$$

Since the curl operator is invariant under rotations of the coordinates, make the change of variables: $x_1 \to x_n$, $x_2 \to x_1$, $x_3 \to x_2$ in [3], pp 153–154. Then we see (designate there T_1, A_1, A_2, A_3 by T_n, A_n, A_1, A_2 respectively) that, by the orthogonal matrix

$$T_n = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

P is transformed so that (0. 3) is valid, obtaining

$$T_n^{-1}A_nT_n=\begin{bmatrix}1&&&&&\\&1&&&&\\&&-1&&&\\&&&-1&&\\&&&&0&\\&&&&&0\end{bmatrix},\quad T_n^{-1}A_1T_n=\frac{1}{\sqrt{2}}\left[\begin{array}{c|c}\mathbf{0}&\begin{matrix}0&-1\\1&0\\1&0\\0&-1\end{matrix}\\ \hline \begin{matrix}0&1&1&0\\-1&0&0&-1\end{matrix}&\mathbf{0}\end{array}\right],$$

$$T_n^{-1}A_2T_n=\frac{1}{\sqrt{2}}\left[\begin{array}{c|c}\mathbf{0}&\begin{matrix}-1&0\\0&-1\\0&1\\1&0\end{matrix}\\ \hline \begin{matrix}-1&0&0&1\\0&-1&1&0\end{matrix}&\mathbf{0}\end{array}\right].$$

So, rank $T_n^{-1}A_nT_n=4$, *i. e.*, (ii) is fulfilled, furthermore $T_n^{-1}PT_n$ fulfills (A) since $A_{\mathrm{I\,I}}=A_{\mathrm{II\,II}}=0$ and $A_{\mathrm{II\,I}}A^{-1}A_{\mathrm{I\,II}}=0$.

EXAMPLE 6. 3. (*The linearized shallow water equations with uniformly characteristic boundary* (*c.f.* [6]))

$$P(D_t,D_x)=I_3D_t-\begin{bmatrix}a&0&c\\0&a&0\\c&0&a\end{bmatrix}D_1-\begin{bmatrix}0&0&0\\0&0&c\\0&c&0\end{bmatrix}D_2,$$

$$c>|a|>0, \qquad m=3,\ n=2:$$

$$\det P(\tau,\sigma,\lambda)$$
$$=(\tau-a\sigma)(\tau-a\sigma-c\sqrt{\sigma^2+\lambda^2})(\tau-a\sigma+c\sqrt{\sigma^2+\lambda^2}) \quad (\text{(i)}),$$
$$\text{rank } A_n=2 \quad (\text{(ii)}).$$

$P(\tau,\sigma,\lambda)$ is transformed by the 3×3 matrices

$$\begin{bmatrix}0&0&1\\ \frac{1}{\sqrt{2}}\begin{bmatrix}-1&1\\1&1\end{bmatrix}&&\begin{matrix}0\\0\end{matrix}\end{bmatrix}\ \text{and its inverse}\ \begin{bmatrix}\begin{matrix}0\\0\end{matrix}&\frac{1}{\sqrt{2}}\begin{bmatrix}-1&1\\1&1\end{bmatrix}\\1&0\quad 0\end{bmatrix},$$

to

$$I_3\tau-\begin{bmatrix}a&0&(\sqrt{2})^{-1}c\\0&a&(\sqrt{2})^{-1}c\\(\sqrt{2})^{-1}c&(\sqrt{2})^{-1}c&a\end{bmatrix}\sigma+\begin{bmatrix}c&&\\&-c&\\&&0\end{bmatrix}\lambda.$$

This fulfills (A) since $A_{\mathrm{II\,I}}A^{-1}A_{\mathrm{I\,I}}=(\sqrt{2})^{-1}a(1,1)=A_{\mathrm{II\,II}}A_{\mathrm{II\,I}}A^{-1}$ and $A_{\mathrm{II\,I}}A^{-1}A_{\mathrm{I\,II}}=0$.

We remark that, for the operators in Examples 6.1 and 6.3, all L^2-well posed boundary conditions are maximally non-positive after a change of dependent variables if necessary (see [1]). Furthermore for Maxwell system P, so is (P, B) if either B is real and (P, B) satisfies Hersh's condition or else (P, B) satisfies Kreiss' condition (see [3], [4]).

For each operator above, all roots λ of det $P(\tau, \sigma, \lambda)=0$ are bounded whenever (τ, σ) is bounded (cf. case (iv) of Theorem 3 in [6], where Kreiss' condition is assumed for (P, B)). There are further such operators fulfilling (A):

EXAMPLE 6.4. *Let*

$$P_1(\tau, \sigma, \lambda)=I_2\tau+\sum_{j=1}^{n-1}\begin{bmatrix} a_j & \bar{c}_j \\ c_j & b_j \end{bmatrix}\sigma_j+\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\lambda, \qquad a\neq 0,$$

$$P_2(\tau, \sigma, \lambda)=I_3\tau+\begin{bmatrix} a_1 & \bar{b} & -(\sqrt{2})^{-1} \\ b & a_2 & -(\sqrt{2})^{-1}i \\ -(\sqrt{2})^{-1} & (\sqrt{2})^{-1}i & a_3 \end{bmatrix}\sigma+\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}\lambda.$$

Then, if P_i fulfills (A), *all roots λ of the equation* det $P_i(\tau, \sigma, \lambda)=0$ *are bounded whenever (τ, σ) is bounded* $(i=1, 2)$.

PROOF. Since

$$\begin{aligned} \det P_1(\tau, \sigma, \lambda) &= \left(\tau+\sum_{j=1}^{n-1} b_j\sigma_j\right)a\lambda \\ &\quad +\left(\tau+\sum_{j=1}^{n-1} b_j\sigma_j\right)\left(\tau+\sum_{j=1}^{n-1} a_j\sigma_j\right)-\left|\sum_{j=1}^{n-1} c_j\sigma_j\right|^2, \end{aligned}$$

all roots λ of det $P_1(\tau, \sigma, \lambda)=0$ are bounded whenever (τ, σ) is, if and only if $\sum_{j=1}^{n-1} c_j\sigma_j=0$ for all σ, that is, $c_1=\cdots=c_{n-1}=0$, which is equivalent to (A) for P_1. Since

$$\begin{aligned} \det P_2(\tau, \sigma, \lambda) &= -(\tau+a_3\sigma)\lambda^2+(\tau+a_3\sigma)(a_2-a_1)\sigma\lambda \\ &\quad +(\tau+a_3\sigma)\left\{(\tau+a_1\sigma)(\tau+a_2\sigma)-|b|^2\sigma^2\right\} \\ &\quad -\sigma^2\left\{\tau+\left(2^{-1}(a_1+a_2)-\operatorname{Im} b\right)\sigma\right\}, \end{aligned}$$

we see that all roots λ of det $P_2(\tau, \sigma, \lambda)=0$ are bounded whenever (τ, σ) is, if and only if

$$2^{-1}(a_1+a_2)-\operatorname{Im} b=a_3.$$

This holds if

$$a_2 = a_1 + 2i(\mathrm{Re}\, b) \quad \text{and} \quad a_3 = a_1 + ib\,,$$

which is equivalent to (A−1). Since (A−2) is always fulfilled for P_2, the assertion for P_2 is proved.

We finally give a remark on (A). Let P_1 be the operator in Example 6.4 with $a>0$ and $n=2$ and $\ker B$ be maximally non-positive for P_1. If estimates (0.2) with $p \geq 1$ hold for the solution to (P_1, B) then the condition (A) must be fulfilled, that is, $c_1=0$ in Example 6.4 (see [12], Theorem 3).

Appendix

The following proposition is used for the proof of Proposition 2.7:

PROPOSITION A.1. *Suppose that (P, B) is maximally non-positive and $P_0(t, x;\, \tau, \sigma, \lambda)$ is of constant multiplicity in τ. Then there exist constants C, $\gamma_0 \geq 0$ such that for every $\gamma \geq \gamma_0$ and $u \in \mathscr{H}^1$*

$$\text{(A.1)} \qquad \gamma \int_{t_1}^{t} e^{-\gamma s} \langle \Lambda_\gamma^{-\frac{1}{2}} u_{\mathrm{I}}(s) \rangle_0^2 ds \leq C \Big\{ e^{-\gamma t_1} |h|_0^2 + \gamma^{-1} \int_{t_1}^{t} e^{-\gamma s} \Big(|f(s)|_0^2 + \langle \Lambda_\gamma^{\frac{1}{2}} g(s) \rangle_0^2 \Big) ds \Big\}.$$

To prove this proposition we use the methods as in Sakamoto [10]. We start with

LEMMA A.2. *Assume the same conditions as in Proposition A.1. Then for every fixed $t_0 (>t_1)$ and every $f' \in C_0^\infty((t_1, \infty) \times G)$ and $g' \in C_0^\infty((t_1, \infty) \times \partial G)$ with $f' = g' = 0$ $(t > t_0)$ the dual boundary value problem for $(P, B)_0$:*

$$(P^*, B')_0 \qquad \begin{cases} P^* v = f' & \text{in } \boldsymbol{R}^1 \times G\,, \\ B' v = g' & \text{on } \boldsymbol{R}^1 \times \partial G\,, \end{cases}$$

has a unique solution v with $v=0$ $(t>t_0)$ such that for large γ $v \in H_{0,1;-\gamma}(\boldsymbol{R}^1 \times G)$, $D_n v_{\mathrm{I}} \in H_{0,0;-\gamma}(\boldsymbol{R}^1 \times G)$ and

$$\text{(A.2)} \qquad e^{\gamma t_1} |v(t_1)|_0^2 + \gamma \int_{t_1}^{t_0} e^{\gamma s} \Big(|v(s)|_0^2 + \langle \Lambda_\gamma^{-\frac{1}{2}} v_{\mathrm{I}}(s) \rangle_0^2 \Big) ds \leq C \gamma^{-1} \int_{t_1}^{t_0} e^{\gamma s} \Big(|f'(s)|_0^2 + \langle \Lambda_\gamma^{\frac{1}{2}} g'(s) \rangle_0^2 \Big) ds\,,$$

where $C>0$ is independent of γ, t_0, t_1, f' and g'.

PROOF. The assumption implies that (2.17) is maximally non-positive and that $P_0^*(-t, x;\, -\tau, \sigma, \lambda)$ is of constant multiplicity in τ. So, let us apply Proposition 2.3 to (2.17) noting that any changes of f and g in $t > -t_1$

have no influence on the solution u to $(P, B)_0$ in $t<-t_1$. Then we obtain such a solution v as described in the lemma which satisfies, by (1.9), for large γ

$$\text{(A. 3)} \qquad \gamma\int_{t_1}^{t_0} e^{\gamma s}\Big(\big|v(s)\big|_0^2+\big\langle \Lambda_\gamma^{-\frac{1}{2}} v_{\mathrm{I}}(s)\big\rangle_0^2\Big)\, ds \leq C\gamma^{-1}\int_{t_1}^{t_0} e^{\gamma s}\Big(\big|f'(s)\big|_0^2+\big\langle \Lambda_\gamma^{\frac{1}{2}} g'(s)\big\rangle_0^2\Big)\, ds\,.$$

Since v belongs also to $\mathscr{H}^1$, the maximal non-positiveness of (2.17) yields an analogue of (2.13); (integrating it over $(-t_0, -t_1)$) we can derive the inequality

$$e^{\gamma t_1}\big|v(t_1)\big|_0^2\leq C\int_{t_1}^{t_0} e^{\gamma s}\Big\{\gamma\big\langle \Lambda_\gamma^{-\frac{1}{2}} v_{\mathrm{I}}(s)\big\rangle_0^2+\gamma^{-1}\Big(\big|f'(s)\big|_0^2+\big\langle \Lambda_\gamma^{\frac{1}{2}} g'(s)\big\rangle_0^2\Big)\Big\}\, ds\,,$$

since $v(t_0)=0$. This and (A. 3) imply (A. 2).

LEMMA A. 3. *Let B'_{I} be the matrix in* (2.16) *and set*

$$C_{\mathrm{I}}=(B_{\mathrm{I}}B_{\mathrm{I}}^*)^{-1}B_{\mathrm{I}}A^*\,, \qquad C'_{\mathrm{I}}=(0, I_{d-l})\Big(B_{\mathrm{I}}^*, (B'_{\mathrm{I}})^*\Big)^{-1}\,,$$

where C_{I}, C'_{I} are $l\times d$, $(d-l)\times d$ matrices respectively. Then every u_{I} and $v_{\mathrm{I}}\in \boldsymbol{C}^d$ satisfy the following:

$$\text{(A. 4)} \qquad Au_{\mathrm{I}}\cdot v_{\mathrm{I}}=B_{\mathrm{I}}u_{\mathrm{I}}\cdot C_{\mathrm{I}}v_{\mathrm{I}}+C'_{\mathrm{I}}u_{\mathrm{I}}\cdot(B'_{\mathrm{I}}A^*)\, v_{\mathrm{I}}\,,$$

$$\text{(A. 5)} \qquad |u_{\mathrm{I}}|^2\leq C\Big(|B_{\mathrm{I}}u_{\mathrm{I}}|^2+|C'_{\mathrm{I}}u_{\mathrm{I}}|^2\Big)\,,$$

where $C>0$ is a constant independent of u_{I}.

PROOF. Set $H=(B_{\mathrm{I}}^*, (B'_{\mathrm{I}})^*)$, then according to (2.16) H is a $d\times d$ nonsingular matrix; clearly, $(I_l, 0)\,H^*=B_{\mathrm{I}}$, $(0, I_{d-l})\,H^*=B'_{\mathrm{I}}$ and $(I_l, 0)H^{-1}=(B_{\mathrm{I}}B_{\mathrm{I}}^*)^{-1}B_{\mathrm{I}}$. Applying these relations to the identity

$$\begin{aligned} Au_{\mathrm{I}}\cdot v_{\mathrm{I}}&=H^{-1}u_{\mathrm{I}}\cdot H^*A^*v_{\mathrm{I}}\\ &=\Big((I_l, 0)\,H^{-1}\Big)\,u_{\mathrm{I}}\cdot\Big((I_l, 0)\,H^*\Big)\,A^*v_{\mathrm{I}}\\ &\quad+(0, I_{d-l})\,H^{-1}u_{\mathrm{I}}\cdot\Big((0, I_{d-l})\,H^*\Big)\,A^*v_{\mathrm{I}}\,, \qquad u_{\mathrm{I}},\ v_{\mathrm{I}}\in\boldsymbol{C}^d\,, \end{aligned}$$

we see (A. 4). (A. 5) follows from (A. 4) and $C|u_{\mathrm{I}}|^2\leq|Au_{\mathrm{I}}\cdot u_{\mathrm{I}}|$.

PROOF OF PROPOSITION A. 1. Let $u\in\mathscr{H}^1$ and apply (A. 5) to $\Lambda_\gamma^{-\frac{1}{2}}u_{\mathrm{I}}$. Then, noting $B_{\mathrm{I}}\Lambda_\gamma^{-\frac{1}{2}}u_{\mathrm{I}}=\Lambda_\gamma^{-\frac{1}{2}}g+[B_{\mathrm{I}}, \Lambda_\gamma^{-\frac{1}{2}}]\,u_{\mathrm{I}}$ on ∂G we get

$$\gamma\int_{t_1}^{t} e^{-\gamma s}\big\langle \Lambda_\gamma^{-\frac{1}{2}} u_{\mathrm{I}}(s)\big\rangle_0^2 ds \leq C\gamma\Big\{\int_{t_1}^{t} e^{-\gamma s}\big\langle \Lambda_\gamma^{-\frac{1}{2}} g(s)\big\rangle_0^2 ds+\int_{t_1}^{t} e^{-\gamma s}\big\langle \Lambda_\gamma^{-\frac{1}{2}} C'_{\mathrm{I}}u_{\mathrm{I}}(s)\big\rangle_0^2 ds\Big\}\,.$$

So it suffices to estimate the last integral.

Let v be the function satisfying the conditions of Lemma A. 2 with $f'\equiv 0$. Then (A. 4) implies that

$$\langle C_I' u_I, g'\rangle(s) = \langle Au_I, v_I\rangle(s) + R(u_I, v_I)(s),$$

where

$$\left|R(u_I, v_I)(s)\right|^2 \leq C\left\langle \Lambda_\gamma^{\frac{1}{2}} g(s)\right\rangle_0^2 \left\langle \Lambda_\gamma^{-\frac{1}{2}} v_I(s)\right\rangle_0^2.$$

Now integration by parts yields

$$\int_{t_1}^{t_0} \left\{(Pu, v)(s) - (u, P^* v)(s)\right\} ds$$
$$= i\int_{t_1}^{t_0} \left\langle Au_I, v_I\right\rangle(s)\, ds - i(u, v)(s)\Big|_{s=t_1}^{s=t_0}.$$

Hence noting that $P^* v\equiv 0$ and $v(t_0)=0$ and using (A. 2), we obtain

$$\text{(A. 6)}\qquad \left|\int_{t_1}^{t_0} \left\langle C_I' u_I, g'\right\rangle(s)\, ds\right| \leq C\gamma^{-\frac{1}{2}} F(t_0) \left(\int_{t_1}^{t_0} e^{\gamma s}\left\langle \Lambda_\gamma^{\frac{1}{2}} g'(s)\right\rangle_0^2 ds\right)^{\frac{1}{2}},$$

where

$$F^2(t) = e^{-\gamma t_1}\left|u(t_1)\right|_0^2$$
$$+\gamma^{-1}\int_{t_1}^{t} e^{-\gamma s}\left(\left|f(s)\right|_0^2 + \left\langle \Lambda_\gamma^{\frac{1}{2}} g(s)\right\rangle_0^2\right) ds.$$

Observing that (A. 6) holds for every $g'\in C_0^\infty((t_1, t_0)\times\partial G)$ and $t_0(>t_1)$ is arbitaray we get for $t>t_1$

$$\gamma\int_{t_1}^{t} e^{-\gamma s}\left\langle \Lambda_\gamma^{-\frac{1}{2}} C_I' u_I(s)\right\rangle_0^2 ds \leq CF^2(t).$$

Therefore we obtain (A. 1) and complete the proof.

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