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The initial boundary value problem for inviscid barotropic fluid motion

By Rentaro AGEMI

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0. Introduction.

In his paper [2], Ebin showed the local in time existence of solutions to the initial boundary value problem for inviscid barotropic fluid motion in a bounded domain provided that the initial velocity is subsonic and the initial density is nearly constant. The purpose of the present article is to prove without the above assumptions the existence and continuous dependence of solutions for the data.

The inviscid barotropic fluid motion in a bounded domain $\Omega\subset \boldsymbol{R}^3$ with smooth boundary $\partial\Omega$ is governed by the standard equations of fluid mechanics;

$$\text{(0.1)}\qquad \begin{aligned} &\frac{dv}{dt}+\frac{p'(\rho)}{\rho}\nabla\rho=K \\ &\frac{d\rho}{dt}+\rho\,\mathrm{div}\,v=0 \end{aligned}\qquad \text{in } (0, T)\times\Omega .$$

Here d/dt denotes the material derivative,

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v\cdot\nabla,\quad v\cdot\nabla=\sum_{j=1}^{3}v_j\frac{\partial}{\partial x_j}$$

and $v(t, x)=(v_1, v_2, v_3)$, $K(t, x)=(K_1, K_2, K_3)$, $\rho(t, x)$, $p(\rho(t, x))$ denote the velocity, the external force, the density, the pressure of fluid motion at time t and position $x\in\Omega$, respectively. For physical reasons we assume that ρ and the derivative $p'(\rho)$ in ρ are both positive. In addition to (0.1) we prescribe the initial data at $t=0$ and the boundary condition on $(0, T)\times\partial\Omega$;

$$\text{(0.2)}\qquad v(0)=v_0,\quad \rho(0)=\rho_0 \qquad \text{on } \Omega ,$$

$$\text{(0.3)}\qquad \langle v, n\rangle=0 \qquad \text{on } (0, T)\times\partial\Omega$$

where $n=n(x)$ denotes the unit outerward normal to Ω at $x\in\partial\Omega$ and $\langle v, n\rangle$ or $v\cdot n$ denotes the standard inner product of v and n in $\boldsymbol{R}^3$.

The main result is

THEOREM. *Suppose that the pressure p belongs to $C^{s+1}(\boldsymbol{R})$, the data (v_0, ρ_0, K) belong to $H^s(\Omega, \boldsymbol{R}^4)\times X^s(T, \Omega, \boldsymbol{R}^3)$ ($s\geqq 3$; integer) and satisfy the compatibility conditions up to order s. Then there exists a positive constant $T_1\leqq T$ such that the initial boundary value problem* (0. 1)-(0. 3) *has a unique solution (v, ρ) in $X^s(T_1, \Omega, \boldsymbol{R}^4)$. Moreover, let $(v_0^{(n)}, \rho_0^{(n)}, K^{(n)})\in H^s(\Omega, \boldsymbol{R}^4)\times X^s(T, \Omega, \boldsymbol{R}^3)$ converge to $(v_0, \rho_0, K)\in H^s(\Omega, \boldsymbol{R}^4)\times X^s(T, \Omega, \boldsymbol{R}^3)$ in $H^{s-1}(\Omega, \boldsymbol{R}^4)\times X^{s-1}(T, \Omega, \boldsymbol{R}^3)$. Then there exist a positive constant $T_2\leqq T$ such that the solution $(v^{(n)}, \rho^{(n)})$ for $(v_0^{(n)}, \rho_0^{(n)}, K^{(n)})$ converge to the solution (v, ρ) for (v_0, ρ_0, K) in $X^{s-1}(T_2, \Omega, \boldsymbol{R}^4)$.*

For the notations see Section 2.

We can also prove the similar results for another dimensional flows.

For the uniqueness of solutions see Serrin [10] and for the initial value problem see Kato [5].

The proof of existence of solutions will be done in the direction of Ebin's paper. However, to remove his assumptions we need three new points of view. The first point is to use a system of integro-differential equations in stead of his system of differential equations which is equivalent to (0. 1)-(0. 3). The second point is to give a different interpretation of the second order hyperbolic equation in this system and to use the idea of Rauch and Massey [9]. The third point is to show the sharp estimates for gradient part of v (see Proposition 3. 4 and Lemma 5. 1).

In preparation for manuscripts of this article, T. Nishida imformed to me Veiga's article; Un theoreme d'existence dans la dynamique des fluides compressibles, C. R. Acad. Sc. Paris, Serie B. t. 289 (17 Decembre 1979). The equivalent system used in his article is different from our system (1. 13)-(1. 15) and he has announced the existence theorem in L^∞ or L^1-category with respect to time.

Our plan of this article is as follows.

1. Equivalent system of equations.
2. Function spaces and compatibility conditions.
3. Linearized equations and estimates for their solutions.
4. Invariant set under iterations.
5. Proof of Theorem.
6. Estimates for log ρ.
7. Estimates for v.

1. Equivalent system of equations

Following [2] we introduce new functions

$$g=\log\rho, \quad a(g)=p'(e^g).$$

Then we obtain a system of equations for (v, g);

$$(1.1)\qquad \begin{aligned}&(\alpha)\quad \frac{dv}{dt}+a(g)\nabla g=K\\ &(\beta)\quad \frac{dg}{dt}+\operatorname{div} v=0\end{aligned}\qquad \text{in } (0, T)\times\Omega,$$

$$(1.2)\qquad v(0)=v_0,\quad g(0)=g_0 \qquad \text{on } \Omega,$$

$$(1.3)\qquad \langle v, n\rangle=0 \qquad \text{on } (0, T)\times\partial\Omega.$$

We shall derive an equivalent system to (1.1)-(1.3). Applying d/dt to (1.1 β), we get

$$\frac{d^2g}{dt^2}+\operatorname{div}\frac{\partial v}{\partial t}+v\cdot\nabla(\operatorname{div} v)=0.$$

Using (1.1 α) and the identity

$$(1.4)\qquad \operatorname{div}\big((v\cdot\nabla)u\big)=v\cdot\nabla(\operatorname{div} u)+\operatorname{tr}\big((Dv)(Du)\big)$$

where $(Dv)(Du)$ means the product of matrices $Dv=(\partial v_j/\partial x_k)$ and $(\partial u_j/\partial x_k)$, we then obtain

$$(1.5)\qquad \frac{d^2g}{dt^2}-\operatorname{div}\big(a(g)\nabla g\big)=\operatorname{tr}\big((Dv)^2\big)-\operatorname{div} K \qquad \text{in } (0, T)\times\Omega.$$

Differentiating (1.3) with respect to t and using (1.1 α), we get

$$(1.6)\qquad \big\langle(v\cdot\nabla)v+a(g)\nabla g-K, n\big\rangle=0 \qquad \text{on } (0, T)\times\partial\Omega.$$

From (1.1 β) and (1.2) we have

$$(1.7)\qquad g(0)=g_0,\quad \frac{\partial g}{\partial t}(0)=-(v_0\cdot\nabla g_0+\operatorname{div} v_0) \qquad \text{on } \Omega.$$

To derive the rest of the system, we decompose a vector field v into its solenoidal and gradient parts,

$$v=w+\nabla f,$$

where the solenoidal part of v satisfies

$$(1.8)\qquad \operatorname{div} w=0 \quad \text{in } \Omega,\quad \langle w, n\rangle=0 \quad \text{on } \partial\Omega.$$

This decomposition is orthogonal in $L^2(\Omega, \boldsymbol{R}^3)$. As is well known (for instance see Ladyzhenskaya [8]), a function f can be taken as a solution of Neumann problem;

$$(1.9)\qquad \Delta f=\operatorname{div} v \qquad \text{in } \Omega,\quad \langle\nabla f, n\rangle=\langle v, n\rangle \quad \text{on } \partial\Omega,$$

and w is defined by $v-\nabla f$, We also define the projection operators P and Q,

$$Pv=w, \quad Qv=\nabla f.$$

From (1.1 β), (1.3) and (1.9) we have

$$\begin{aligned} &\Delta f=-\frac{dg}{dt} && \text{in } (0, T)\times\Omega, \\ &\langle \nabla f, n\rangle=0 && \text{on } (0, T)\times\partial\Omega. \end{aligned} \tag{1.10}$$

Applying P to (1.1 α), we get

$$\frac{\partial w}{\partial t}+P\Big((v\cdot\nabla)w+(w\cdot\nabla)\nabla f-K\Big)=-P\Big(a(g)\nabla g+(\nabla f\cdot\nabla)\nabla f\Big).$$

Since

$$\begin{aligned} &a(g)\nabla g=\nabla\int_0^g p'(e^y)\,dy, \\ &(\nabla f\cdot\nabla)\nabla f=\nabla\langle\nabla f, \nabla f\rangle/2 \end{aligned} \tag{1.11}$$

and P annihilates gradient parts, we obtain

$$\begin{aligned} &\frac{\partial w}{\partial t}+P\Big((v\cdot\nabla)w+(w\cdot\nabla)\nabla f\Big)=PK && \text{in } (0, T)\times\Omega, \\ &w(0)=Pv_0 && \text{on } \Omega. \end{aligned} \tag{1.12}$$

Conversely, let (w, f, g) be a solution of equations (1.5)-(1.7), (1.10) and (1.12). Then it was proved in [4] that (v, g), $v=w+\nabla f$, is a solution of (1.1)-(1.3).

In view of the solvability for elliptic boundary value problems, we use here the following system of integro-differential equations for (w, f, g) with $v=w+\nabla f$;

$$\begin{aligned} &\frac{d^2g}{dt^2}-\operatorname{div}\Big(a(g)\nabla g\Big)=\operatorname{tr}\Big((Dv)^2\Big)-\operatorname{div} K \\ &\Delta f=-\frac{dg}{dt}+\frac{1}{|\Omega|}\int_\Omega\frac{dg}{dt}\,dx && \text{in } (0, T)\times\Omega, \\ &\frac{\partial w}{\partial t}+P\Big((v\cdot\nabla)w+(w\cdot\nabla)\nabla f\Big)=PK \end{aligned} \tag{1.13}$$

$$\begin{aligned} &w(0)=Pv_0, \quad g(0)=g_0 \\ &\frac{\partial g}{\partial t}(0)=-(v_0\cdot\nabla g_0+\operatorname{div} v_0) && \text{on } \Omega, \end{aligned} \tag{1.14}$$

$$\text{(1.15)}\qquad \begin{aligned} &\big\langle a(g)\nabla g+(v\cdot\nabla)v,\, n\big\rangle=\langle K, n\rangle \\ &\langle \nabla f, n\rangle=0 \end{aligned} \qquad \text{on } (0, T)\times\partial\Omega .$$

Here $|\Omega|$ stands for the volume of Ω.

We shall show that this system is equivalent to (1.1)-(1.3). Let (v, g), $v=w+\nabla f$, be a solution of (1.13)-(1.15) and let

$$b(t)=\int_{\Omega}\frac{dg}{dt}dx .$$

Then, using (1.14) and the divergence theorem, we get

$$b(0)=-\int_{S}\langle v_0, n\rangle dS=0 \qquad (S=\partial\Omega),$$

where the last equality follows from the compatibility condition for v_0. Note that w is solenoidal and thus $\Delta f=\operatorname{div} v$. Then it follows from (1.4) and (1.13) that

$$\begin{aligned} \frac{\partial}{\partial t}\frac{dg}{dt}&=\frac{d^2g}{dt^2}-v\cdot\nabla\Big(\frac{dg}{dt}\Big) \\ &=\operatorname{div}\Big((v\cdot\nabla)v+a(g)\nabla g-K\Big). \end{aligned}$$

Using (1.15) and the divergence theorem, we get

$$\frac{\partial b}{\partial t}(t)=0 .$$

Thus we know that the system (1.13)-(1.15) is equivalent to (1.1)-(1.3).

2. Function spaces and compatibility conditions.

We first state the definitions of function spaces and their basic properties which are used frequently in this article (for instance, see Nierenberg [8], Sobolev [11]). We consider generally functions defined in a bounded domain $\Omega\subset\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$. Let $H^s(\Omega, \boldsymbol{R}^m)$ denote the Sobolev space consisting of all $\boldsymbol{R}^m$-valued functions which have L^2-derivatives up to order s. Then $H^s(\Omega, \boldsymbol{R}^m)$ is a Hilbert space equipped the standard inner product and norm;

$$\begin{aligned} &(f, g)_s=\sum_{|\alpha|\leq s}\int_{\Omega}\langle\partial^\alpha f, \partial^\alpha g\rangle dx, \\ &\|f\|_s=(f, f)_s^{1/2} \qquad \big(f, g\in H^s(\Omega, \boldsymbol{R}^m)\big), \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product of $\boldsymbol{R}^m$, $\partial^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$ $(\alpha=(\alpha_1, \cdots, \alpha_n))$. We can also define the Sobolev space H^s

$(\partial\Omega, \boldsymbol{R}^m)$ for $s\in\boldsymbol{R}$ whose inner product and norm will be denoted by $(\cdot,\cdot)_{s,\partial\Omega}$ and $\|\cdot\|_{s,\partial\Omega}$. Furtheremore, we use the Sobolev space $H^s((0, T)\times\Omega)$ ($s\geqq 0$; integer) whose norm will be denoted by $\|\cdot\|_{s,(0,T)\times\Omega}$. We also use $H^s((0, T)\times\partial\Omega)$ consisting of all distributions f in $(0, T)\times\partial\Omega$ such that $f=F$ in $(0, T)\times\partial\Omega$ for some $F\in H^s(\boldsymbol{R}\times\partial\Omega)$ (s, real). Its norm is defined by

$$\|f\|_{s,(0,T)\times\partial\Omega}=\inf_F \|F\|_{s,\boldsymbol{R}\times\partial\Omega},$$

where infimum is taken over all such F.

We shall arrange the basic properties of function spaces.

(*P I*) Let k, s be non-negative integers such that $k+n/2<s$. Then

$$\sum_{|\alpha|\leqq k} \sup_{x\in\Omega} |\partial^\alpha f(x)|\leqq c\|f\|_s$$

for any $f\in H^s(\Omega, \boldsymbol{R}^m)$, where c is a constant depending on k, s, n, Ω and $|\cdot|$ denotes the norm of $\boldsymbol{R}^m$.

(*P II*) Let r, s, t be non-negative integers such that $r\leqq\min\{s, t, s+t-n/2\}$. Then

$$\|fg\|_r\leqq c\|f\|_s\|g\|_t$$

for any $f\in H^s(\Omega, \boldsymbol{R})$, $g\in H^t(\Omega, \boldsymbol{R}^m)$, where c is a constant depending on r, s, t, n, Ω.

We introduce the space $\boldsymbol{C}^k([0, T], H^s(\Omega, \boldsymbol{R}^m))$ consisting of bounded k-times continuously differentiable $H^s(\Omega, \boldsymbol{R}^m)$-valued functions in $[0, T]$. Then we can define a Banach space $X_r^s(T, \Omega, \boldsymbol{R}^m)$ which is simply denoted by $X_r^s(T)$, $X_r^s(\Omega)$ or X_r^s,

$$X_r^s=\bigcup_{k=0}^{s-r} \boldsymbol{C}^k\left([0, T],\ H^{s-k}(\Omega, \boldsymbol{R}^m)\right).$$

Its norm is defined by

$$\|f\|_{X_r^s}=\sup_{0\leqq t\leqq T}\|f(t)\|_{X_r^s},\quad \|f(t)\|_{X_r^s}=\sum_{k=0}^{s-r}\|\frac{\partial^k f}{\partial t^k}(t)\|_{s-k}.$$

We also introduce a Banach space $Y_r^s(T, \Omega, \boldsymbol{R}^m)$;

$$Y_r^s=\Big\{f;\ \partial^k f/\partial t^k\ (k=0, 1, \cdots, s-r) \text{ are bounded measurable } H^{s-k}(\Omega, \boldsymbol{R}^m)\text{-valued functions in } [0, T]\Big\}.$$

Its norm is the same as in X_r^s. Similarly we can define such function spaces on $\partial\Omega$. Hereafter X_0^s and Y_0^s will be denoted by X^s and Y^s.

Next we describe the compatibility conditions for the initial boundary value problem (1. 1)-(1. 3). Let $(v, g)\in X^{s}$ be a solution of (1. 1)-(1. 3). Then we have

$$\frac{\partial v}{\partial t}(0)=-\Big((v_0\cdot\nabla)v_0+a(g_0)\nabla g_0-K(0)\Big)$$

$$\frac{\partial g}{\partial t}(0)=-(v_0\cdot\nabla g_0+\operatorname{div} v_0)\,.$$

Furthermore, differentiating (1. 1) with respect to t, we verify that the initial values of $\partial^k v/\partial t^k$ and $\partial^k g/\partial t^k$ $(k=1, \cdots, s)$ can be written successively by the data (v_0, g_0, K) and their derivatives. When the data and their derivatives satisfy these relations, we say that $\partial^k v/\partial^k t$ and $\partial^k g/\partial^k t$ satisfy the equation (1. 1) at $t=0$. On the other hand, differentiating (1. 3) with respect t, we get

$$(2.1)_k \qquad \Big\langle\frac{\partial^k v}{\partial t^k}(0), n\Big\rangle=0 \qquad \text{on } \partial\Omega\ (k=0, 1, \cdots, s-1)\,.$$

These are the relations among the data (v_0, g_0, K). Thus we call these the compatibility conditions up to order s for the data (v_0, g_0, K).

3. Linearized equations and estimates for their solutions.

In this section we shall consider the linearized equations of (1. 13)-(1. 15) and state the estimates for their solutions which are used in the proof of Theorem. The proofs will be found in Sections 6 and 7.

We assume that $(v, g)\in X^{s}$ $(s\geqq 3)$ are given so that $\langle v, n\rangle=0$ on $\partial\Omega$, $a(g)-\langle v, n\rangle^2\geqq d>0$ in a neighborhood of $\partial\Omega$ and $a(g)>d$.

First we solve the equations for $\hat{g}$;

$$(3.1)\qquad \begin{aligned} &(\alpha)\quad \frac{d^2\hat{g}}{dt^2}-\operatorname{div}\big(a(g)\nabla\hat{g}\big)=F && \text{in } (0, T)\times\Omega\,,\\ &(\beta)\quad \hat{g}(0)=g_0,\ \ \frac{\partial\hat{g}}{\partial t}(0)=g_1 && \text{on } \Omega\,,\\ &(\gamma)\quad \langle\nabla\hat{g}, n\rangle=h && \text{on } (0, T)\times\partial\Omega\,.\end{aligned}$$

Here

$$F=\operatorname{tr}\big((Dv)^2\big)-\operatorname{div} K,\quad g_1=-(v_0\cdot\nabla g_0+\operatorname{div} v_0)$$

$$h=\big\langle K-(v\cdot\nabla)v, n\big\rangle/a(g)\,.$$

The equation (3. 1 α) is strictly hyperbolic and the boundary is non-characteristic for (3. 1 α) because $a(g)\geqq d$ and $\langle v, n\rangle=0$ on the boundary.

Assume that $(v(0), g(0))=(v_0, g_0)$ and $\partial^k v/\partial^k t$, $\partial^k g/\partial^k t$ $(k=1, \cdots, s-1)$ satisfy the equation (1.1) at $t=0$. Then the compatibility conditions $(2.1)_k$ lead to the ones for the data (g_0, g_1, h, F).

PROPOSITION 3.1. *Suppose that the data (g_0, g_1, h, F) satisfy the compatibility conditions up to order s. Then the initial boundary value problem* (3.1) *has a unique solution $\hat{g}\in X^s(T)$ such that*

$$(3.2)_k \quad e^{-p_1 t}\|\hat{g}(t)\|_{X^k}\leqq p_2\Big(\|\hat{g}(0)\|^2_{X^k}+\|F(0)\|^2_{X^{k-2}} + \int_0^T \|F(t)\|^2_{X^{k-1}}\,dt+\|h\|^2_{k-1/2,(0,T)\times\partial\Omega}\Big)$$

$$(1\leqq k\leqq s-1,\ \|F(0)\|_{X^{-1}}=0)\,,$$

$$(3.3) \quad e^{-p_2 t}\|\hat{g}(t)\|^2_{X^s}\leqq q_1(t)\Big(\|\hat{g}(0)\|^2_{X^s}+\|F(0)\|^2_{X^{s-2}} + p_4\Big(\int_0^T \|F(t)\|^2_{X^{s-1}}\,dt+\|h\|^2_{s-1/2,(0,T)\times\partial\Omega}\Big)$$

where p's are polynomials of $\|v\|_{X^s}$, $\|a(g)\|_{X^s}$ and $q_1(t)$ is a polynomial of $\|v(t)\|_{X^{s-1}}$, $\|a(g(t)\|_{X^{s-1}}$.

REMARK. Since $\langle v, n\rangle=0$ on $\partial\Omega$, $v\cdot\nabla$ is a tangential differential operator on $\partial\Omega$. Therefore $\langle(v\cdot\nabla)v, n\rangle$ can be regarded as a function $-\langle v, (v_\tau\cdot\nabla)\tilde{n}\rangle$ which belongs to X^s. Here $v_\tau=v-\langle v, \tilde{n}\rangle\tilde{n}$ and the unit vector $\tilde{n}$ is the extension of n to a neighbourhood of $\partial\Omega$.

Next we solve the requations for $\hat{f}$;

$$(3.4) \quad \begin{aligned} &(\alpha)\ \ \Delta\hat{f}=G && \text{in } (0, T)\times\Omega\,, \\ &(\beta)\ \ \langle\nabla\hat{f}, n\rangle=0 && \text{on } (0, T)\times\partial\Omega\,, \end{aligned}$$

where

$$G=-\frac{d\hat{g}}{dt}+\frac{1}{|\Omega|}\int_\Omega \frac{d\hat{g}}{dt}\,dx$$

and $\hat{g}$ is the solution of (3.1).

REMARK. The linearized equation used in [4] is

$$-\Delta\hat{f}=\frac{\partial\hat{g}}{\partial t}+w\cdot\nabla\hat{g}+\nabla\hat{f}\cdot\nabla\hat{g}$$

and then it has been necessary that $\|\hat{g}(0)\|_{X^s}$ is small.

Since $\int_\Omega G\,dx=0$, we can obtain

PROPOSITION 3.2. *Then boundary value problem* (3.4) *has a unique solution* $\hat{f}\in X_1^s$ *modulo constants such that*

$$(3.5)_k \qquad \|\nabla\hat{f}(t)\|_{X^k}\leqq c_1\|G(t)\|_{X^{k-1}} \qquad (1\leqq k\leqq s-1),$$

$$(3.6) \qquad \|\nabla\hat{f}(t)\|_{X_1^s}\leqq q_2(t)\|\hat{g}(t)\|_{X^s}$$

where $q_2(t)$ *is a polynomial of* $\|v(t)\|_{X^{s-1}}$ *and* c_1 *is a constant depending on* Ω *and* k.

Finally we solve the equations for $\hat{w}$;

$$(3.7) \quad \begin{cases} (\alpha) \quad \dfrac{\partial\hat{w}}{\partial t}+P\Big((v\cdot\nabla)\hat{w}+(\hat{w}\cdot\nabla)\nabla\hat{f}\Big)=PK & \text{in } (0,T)\times\Omega, \\ (\beta) \quad \hat{w}(0)=Pv_0 & \text{on } \Omega, \end{cases}$$

where $\hat{f}$ is the solution of (3.4).

PROPOSITION 3.3. *The initial value problem* (3.7) *has a unique solution* $\hat{w}\in X^s$ *such that*

$$(3.8)_k \qquad \|\hat{w}(t)\|_{X^k}\leqq e^{p_5 t}\|\hat{w}(0)\|_{X^k}+\int_0^t e^{p_5(t-\tau)}\|PK(\tau)\|_{X^k}\,d\tau \quad (1\leqq k\leqq s-1),$$

$$(3.9) \qquad \|\hat{w}(t)\|_{X_1^s}\leqq e^{p_5 t}\|\hat{w}(0)\|_{X_1^s}+\int_0^t e^{p_5(t-\tau)}\|PK(\tau)\|_{X_1^s}\,d\tau,$$

$$(3.10) \qquad \Big\|\frac{\partial^s\hat{w}}{\partial t^s}(t)\Big\|_0\leqq c_2\Big(\|v(t)\|_{X^{s-1}}+\|\nabla\hat{f}\|_{X_1^s}\Big)\|\hat{w}(t)\|_{X_1^s},$$

where $p_5=c_3(\|v\|_{X_1^s}+\|\nabla\hat{f}\|_{X_1^s})$ *and c's are constants depending on k, s and* Ω.

Using the equations (3.1), (3.4) and (3.7) we can obtain

PROPOSITION 3.4. $\nabla\hat{f}$ *belongs to* X^s *and satisfies*

$$(3.11) \qquad \Big\|\frac{\partial^s}{\partial t^s}\nabla\hat{f}(t)_0\Big\|_0\leqq q_3(t)\,(\|\hat{g}\|_{X_1^s}+\|\hat{v}\|_{X_1^s})$$
$$+c_4\Big(\|v(t)\|_{X^{s-1}}\|v\|_{X_1^s}+\|K\|_{X_1^{s-1}}\Big),$$

where $\hat{v}=\hat{w}+\nabla\hat{f}$, $q_3(t)$ *is a polynomial of* $\|v(t)\|_{X^{s-1}}$, $\|a(g(t))\|_{X^{s-1}}$ *and* c_4 *is a constant depending on s and* Ω.

4. Invariant set under iterations.

Let $(v_0, g_0, K)\in H^s(\Omega, \boldsymbol{R}^4)\times X^s(T, \Omega, \boldsymbol{R}^3)$ $(s\geqq 3)$ be satisfies the compatibility conditions up to orders s and let $(v, g)\in X^s$ be given so that $\langle v, n\rangle=0$ on the boundary, $(v(0), g(0))=(v_0, g_0)$ and $\partial^k v/\partial t^k$, $\partial^k g/\partial t^k$ $(k=1, \cdots, s-1)$ satisfy the equation (1.1) at $t=0$. Then it follows from the results in Section 3 that the system of problems (3.1), (3.4) and (3.7) has a unique solution

$(\hat{v}, \hat{g}) \in X^s$ with $\hat{v} = \hat{w} + \nabla \hat{f}$. Therefore, we have a mapping Φ defined by

$$\Phi(v, g) = (\hat{v}, \hat{g}).$$

Let $B_1 \subset H^s(\Omega, \boldsymbol{R}^4)$, $B_2 \subset X^s(T, \Omega, \boldsymbol{R}^3)$ be closed balls with center 0 and radius R_1, R_2, respectively. Then we define a positive constant d;

$$4d = \text{infimum of } a(g) \text{ in } |g| \leqq cR_1$$

where c is the constant in Sobolev lemma (**PI**). Furtheremore, we choose a fixed neighborhood U of $\partial\Omega$ such that

$$a(g_0) - \langle v_0, n \rangle^2 \geqq 2d \text{ for all } (v_0, g_0) \in B_1.$$

To find the fixed point of Φ we first consider a subset $E(C, T)$ of X^s;

$$\begin{aligned} E(C, T) = \Big\{ & (v, g) \in X^s(T)\,;\ \big(v(0), g(0)\big) = (v_0, g_0) \in B_1, \\ & \partial^k v/\partial t^k \text{ and } \partial^k g/\partial t^k\ (k=1, \cdots, s-1) \text{ satisfy} \\ & \text{the equation (1. 1), } \langle v, n \rangle = 0 \text{ on } (0, T) \times \partial\Omega, \\ & a(g) - \langle v, n \rangle^2 \geqq d \text{ on } U,\ a(g) \geqq 2d \text{ and} \\ & \|v\|_{X^s(T)} + \|g\|_{X^s(T)} \leqq C \Big\}. \end{aligned}$$

Now we shall show the following

PROPOSITION 4. 1. *Suppose that* $K \in B_2$. *Then there exist positive constants* C, T *and* δ *depending only on* Ω *and* R_j $(j=1, 2)$ *such that* $E(C, T)$ *is invariant under* Φ *provided that* $\|v(0)\|_{X^{s-1}} \leqq \delta$.

To prove Proposition 4. 1 we first need

LEMMA 4. 1. *Let* $(\hat{v}, \hat{g}) = \Phi(v, g)$ *for* $(v, g) \in E(C, T)$. *Then* $\partial^k \hat{v}/\partial t^k$ *and* $\partial^k \hat{g}/\partial t^k$ $(k=1, \cdots, s-1)$ *satisfy the equation* (1. 1) *at* $t=0$.

PROOF. From the initial conditions for $\hat{g}$ and $\hat{w}$ we have

$$\hat{g}(0) = g_0, \quad \hat{w}(0) = w_0$$
$$\frac{\partial \hat{g}}{\partial t}(0) = -(v_0 \cdot \nabla g_0 + \operatorname{div} v_0), \tag{4. 1}$$

where $v_0 = w_0 + \nabla f_0$ is the orthogonal decomposition of v_0. To show

$$\hat{v}(0) = v_0, \tag{4. 2}$$

it is enough to prove that $\nabla \hat{f}(0) = \nabla f_0$, that is, $\nabla(\hat{f}(0) - f_0)$ is solenoidal. In fact, it follows from $(2. 1)_0$, (3. 4) and (4. 1) that

$$\operatorname{div}\big(\nabla \hat{f}(0) - \nabla f_0\big) = -\frac{1}{|\Omega|} \int_{\partial\Omega} \langle v_0, n \rangle dS = 0 \qquad \text{in } \Omega,$$

$$\left\langle \nabla \hat{f}(0) - \nabla f_0, n \right\rangle = \langle v_0, n \rangle = 0 \qquad \text{on} \quad \partial\Omega .$$

Next we shall show

$$\frac{\partial \hat{v}}{\partial t}(0) = -\left((v_0 \cdot \nabla) v_0 + a(g_0) \nabla g_0 - K(0)\right). \tag{4.3}$$

Note that $\hat{w}$ is solenoidal and thus $\Delta \hat{f} = \operatorname{div} \hat{v}$. Then it follows from (1.4), (3.1 α) and (3.4 α) that

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial \hat{g}}{dt} &= \frac{d^2 \hat{g}}{dt^2} - v \cdot \nabla \left(\frac{d\hat{g}}{dt} \right) \\ &= \operatorname{div}\left((v \cdot \nabla) \hat{v} + a(g) \nabla \hat{g} - K\right) + \operatorname{tr}\left((Dv)^2 - (Dv)(D\hat{v})\right). \end{aligned} \tag{4.4}$$

Differentiating (3.4) with respect to t and evaluating (4.4) at $t=0$, we can verify from (4.1) and (4.2) that

$$\begin{aligned} &\operatorname{div}\left(\left(\frac{\partial}{\partial t} \nabla \hat{f}\right)(0) + Q\left((v_0 \cdot \nabla) v_0 + a(g_0) \nabla g_0 - K(0)\right)\right) \\ &\quad = \frac{1}{|\Omega|} \int_{\partial\Omega} \left\langle (v_0 \cdot \nabla) v_0 + a(g_0) \nabla g_0 - K(0), n \right\rangle dS, \end{aligned}$$

where Q denotes the projection to gradient parts of vector fields. Using the compatibility condition $(2.1)_1$, that is, $\langle (v_0 \cdot \nabla) v_0 + a(g_0) \nabla g_0 - K(0), n \rangle = 0$, we obtain

$$\left(\frac{\partial}{\partial t} \nabla \hat{f}\right)(0) = -Q\left((v_0 \cdot \nabla) v_0 + a(g_0) \nabla g_0 - K(0)\right). \tag{4.5}$$

On the other hand, using the relation (1.11) we can rewrite (3.7 α) as

$$\frac{\partial \hat{w}}{\partial t} + P\left((v \cdot \nabla) \hat{w} + (\hat{w} \cdot \nabla) \nabla \hat{f} + (\nabla f \cdot \nabla) \nabla f + a(g) \nabla g\right) = PK. \tag{4.6}$$

Then we have

$$\frac{\partial \hat{w}}{\partial t}(0) = -P\left((v_0 \cdot \nabla) v_0 + a(g_0) \nabla g_0 - K(0)\right). \tag{4.7}$$

Therefore (4.3) follows from (4.5) and (4.7).

That $\partial^2 \hat{g}/\partial t^2$ satisfies the equation (1.1) at $t=0$ follows from (4.3) and (4.4). Differentiating (4.4) and (4.6) with respect to t, we can prove successively the assertion of Lemma 4.1.

PROOF OF PROPOSITION 4.1. We shall show that there exist positive constants C and T depending R_j $(j=1, 2)$ such that

$$\|\hat{v}\|_{X^s(T)} + \|\hat{g}\|_{X^s(T)} \leqq C \tag{4.8}$$

provided that

$$(4.9)\qquad \|v\|_{X^s(T)}+\|g\|_{X^s(T)}\leqq C.$$

To prove this assertion we first note that

$$\|v(t)\|_{X^{s-1}}+\|g(t)\|_{X^{s-1}}\leqq\|v(0)\|_{X^{s-1}}+\|g(0)\|_{X^{s-1}}+2Ct.$$

From $(P\text{II})$ we have

$$\|a(g)\|_{X^s}\leqq M(C)(1+\|g\|_{X^s})^s$$

where $M(C)$ depends on the supremum of a and its derivatives. Then the polynomials $q_j(t)$ in (3.3), (3.6) and (3.11) are estimated by

$$p_1(R_1)+p_2(C)t.$$

Here and hereafter we denote various polynomials by p_j and consider the supremum of a and its derivatives as their coefficients. Using properties $(P\text{II})$ and the remark after Proposition 3.1, we can verify

$$\int_0^T\|F(t)\|_{X^{s-1}}dt,\ \|h\|_{s-1/2,(0,T)\times\partial\Omega}\leqq p_3(R_1)+P_3\left(C,R_2\right)T,$$
$$\|g(0)\|_{X^s},\ \|F(0)\|_{X^{s-2}}\leqq p_4(R_1,R_2),$$

Therefore if follows from (3.3) that

$$\|\hat{g}(t)\|^2_{X^s}\leqq e^{p_5(C)t}\left(p_6(R_1,R_2)+p_7(R_1,R_2,C)T\right).$$

Choose $T_1>0$ such that

$$p_5(C)T_1\leqq 1,\ p_7(R_1,R_2,C)T_1\leqq p_6(R_1,R_2),$$

we then obtain

$$(4.10)\qquad \|\hat{g}(t)\|^2_{X^s}\leqq 4\sqrt{p_6(R_1,R_2)}\qquad\text{for } 0\leqq t\leqq T_1.$$

Similarly, using (3.6), (3.9), (3.10) and (4.10), we can verify that there is a small constant $T_2>0$ depending on R_1, R_2, C and Ω such that

$$(4.11)\qquad \|\hat{g}(t)\|_{X^s}+\|\hat{w}(t)\|_{X^s}+\|\nabla\hat{f}(t)\|_{X_1^s}\leqq p_8(R_1,R_2)$$

for $0\leqq t\leqq\min(T_1,T_2)$. From (3.11) and (4.11) we have

$$\left\|\frac{\partial^s}{\partial t^s}\nabla\hat{f}(t)\right\|_0\leqq\Big(p_1(R_1)+p_2(C)p_8(R_1,R_2)$$
$$+c_4C\|v(0)\|_{X^{s-1}}+Ct+R_2\Big)$$

Choose $T_3>0$ and $\|v(0)\|_{X^{s-1}}$ such that

$$\left(p_2(C)\,p_8(R_1, R_2)+c_4C^2\right)T_3 \leqq p_1(R_1)p_8(R_1, R_2)+c_4R_2,$$

$$c_4C\|v(0)\|_{X^{s-1}} \leqq p_1(R_1)\,p_8(R_1, R_2)+c_4R_2\,,$$

we then obtain

$$(4.12)\qquad \left\|\frac{\partial^s}{\partial t^s}\nabla \hat{f}(t)\right\|_0 \leqq 3p_1(R_1)\,p_8(R_1, R_2)+3C_4R_2$$

for $0 \leqq t \leqq \min(T_1, T_2, T_3)$. Therefore, taking C and T as

$$C \geqq 4\sqrt{p_6(R_1, R_2)}+\left(1+3p_1(R_1)\right)p_8(R_1, R_2)+3c_4R_2\,,$$

$$T \leqq \min(T_1, T_2, T_3)\,,$$

we see from (4.10)-(4.12) that the assertion is valid for such C and T.

By the definitions of d and U we see that $a(g(t))-\langle v(t), n\rangle^2 \geqq d$ in U and $a(g) \geqq 2d$ for small t. Hence it is enough to show the existence of an element $(v, g) \in E(C, T)$ for some C and small T. To this end we shall consider the following problem instead of (3.1);

$$(4.13)\qquad \begin{aligned} &\frac{d^2\hat{g}}{dt^2}-\operatorname{div}\left(a(g)\nabla\hat{g}\right)=F \qquad \text{in } (0, T)\times\Omega\,,\\ &\hat{g}(0)=g_0,\quad \frac{\partial\hat{g}}{\partial t}(0)=g_1 \qquad \text{on } \Omega\,. \end{aligned}$$

Extend (v_0, g_0) and (v, g) to the space $\boldsymbol{R}^3$ such that their norms are estimated by constant times of norms in Ω and consider (4.13) as the initial value problem in $(0, T)\times\boldsymbol{R}^3$. Then we can verify that (4.13) has a solution $(\hat{v}, \hat{g}) \in X^s$ which satisfies (3.3) without boundary norms. Thus we have a mapping Φ' defined by $(\hat{v}, \hat{g})=\Phi'(v, g)$, where $(\hat{v}, \hat{g})$ is a solution of the system (3.4), (3.7) and (4.13).

We now prove that $(v, g)=(\Phi')^{s-1}(v_0, g_0)$ belongs to $E(C, T)$ for some C and small T. In the same way as the proof of Lemma 4.1 we can verify that $\partial^k v/\partial^k t$ and $\partial^k g/\partial^k t$ $(k=1, \cdots, s-1)$ satisfy the equation (1.1) at $t=0$, because the boundary condition (3.1 γ) is not used there. The X^s-norm of $\Phi'(v_0, g_0)$ is estimated by $p_9(R_1, R_2)+p_{10}(R_1, R_2)\,T$. Let $C=2p_9(R_1, R_2)$. Then, using the foregoing method, we can verify that for small T the X^s-norm of (v, g) is smaller than C. Thus the proposition is proved.

5. Proof of theorem.

We shall first show the existence of solutions of (1.13)-(1.15). To this end we introduce a metric in $E(C, T)$ which is defined by the $X^1(T)$-norm (compare with the metric in [4]). We denote this metric by d_T;

$$d_T\Big((v_1, g_1), (v_2, g_2)\Big) = \|v_1 - v_2\|_{X^1(T)} + \|g_1 - g_2\|_{X^1(T)}$$

for $(v_j, g_j) \in E(C, T)$ $(j=1, 2)$.

The key step is to prove.

PROPOSITION 5.1. *For every ε with $0<\varepsilon<1$ there exist positive constants T and δ depending on ε, Ω, and R_j $(j=1, 2)$ such that*

$$d_T\Big(\Phi(v_1, g_1), \Phi(v_2, g_2)\Big) \leqq \varepsilon d_T\Big((v_1, g_1), (v_2, g_2)\Big)$$

provided that $\|v_0\|_2 \leqq \delta$, where R_j are defined in Section 4.

PROOF. Set $(\hat{v}_j, \hat{g}_j) = \Phi(v_j, g_j)$; $\hat{v}_j = \hat{w}_j + \nabla \hat{f}_j$, for $(v_j, g_j) \in E(C, T)$ $(j=1, 2)$ and set $\tilde{g} = \hat{g}_1 - \hat{g}_2$, $\tilde{w} = \hat{w}_1 - \hat{w}_2$, $\nabla \tilde{f} = \nabla(\hat{f}_1 - \hat{f}_2)$. Then, by the definition of Φ, $\tilde{g}$, $\nabla \tilde{f}$ and $\tilde{w}$ satisfy the following systems of equations;

$$(5.1)\qquad \begin{cases} \dfrac{d^2 \tilde{g}}{dt^2} - \operatorname{div}\Big(a(g_1)\nabla \tilde{g}\Big) = \tilde{F} & \text{in } (0, T)\times\Omega, \\ \tilde{g}(0) = \dfrac{\partial \tilde{g}}{\partial t}(0) = 0 & \text{on } \Omega, \\ \langle \nabla \tilde{g}, n\rangle = \tilde{h} & \text{on } (0, T)\times\partial\Omega, \end{cases}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_1\cdot\nabla,$$

$$\tilde{h} = \Big\langle K - (v_1\cdot\nabla)v_1, n\Big\rangle / a(g_1) - \Big\langle K - (v_2\cdot\nabla)v_2, n\Big\rangle / a(g_2),$$

$$\tilde{F} = \operatorname{tr}\Big((Dv_1)^2 - (Dv_2)^2\Big) + \operatorname{div}\Big(\Big(a(g_1) - a(g_2)\Big)\nabla \hat{g}_2\Big) - 2(v_1 - v_2)\cdot\nabla\Big(\frac{\partial \hat{g}_2}{\partial t}\Big)$$
$$- \frac{\partial}{\partial t}(v_1 - v_2)\cdot\nabla \hat{g}_2 + (v_2\cdot\nabla)(v_2\cdot\nabla \hat{g}_2) - (v_1\cdot\nabla)(v_2\cdot\nabla \hat{g}_2).$$

$$(5.2)\qquad \begin{cases} \Delta \tilde{f} = \tilde{G} & \text{in } (0, T)\times\Omega, \\ \langle \nabla \tilde{f}, n\rangle = 0 & \text{on } (0, T)\times\partial\Omega, \end{cases}$$

where

$$\tilde{G} = -\Big(\frac{d\tilde{g}}{dt} + (v_1 - v_2)\cdot\nabla \hat{g}_2\Big) + \frac{1}{|\Omega|}\int_\Omega \Big(\frac{d\tilde{g}}{dt} + (v_1 - v_2)\cdot\nabla \hat{g}_2\Big)dx.$$

$$(5.3)\qquad \begin{cases} \dfrac{\partial \tilde{w}}{\partial t} + P\Big((v_1\cdot\nabla)\tilde{w} + (\tilde{w}\cdot\nabla)\nabla f_1\Big) = P\tilde{K} & \text{in } (0, T)\times\Omega, \\ \tilde{w}(0) = 0 & \text{on } \Omega, \end{cases}$$

where

$$-\tilde{K}=(v_1-v_2)\cdot\nabla\hat{w}_2+(\hat{w}_2\cdot\nabla)(\nabla f_1-\nabla f_2).$$

Applying the estimate $(3.2)_1$ to a solution $\tilde{g}$ of (5.1) and using properties (P II) and Remark after Proposition 3.1, we can verify

$$\|\hat{g}_1-\hat{g}_2\|_{X^1(T)}\leqq p_1(C, R_2)\,Td_T\Big((v_1, g_1), (v_2, g_2)\Big), \tag{5.4}$$

where we consider the supremum of a and its derivative which denpends on C as a coefficient of a polynomial p_1. From the estimate $(3.5)_1$ and (5.4) we have

$$\|\nabla\hat{f}_1-\nabla\hat{f}_2\|_{X^0(T)}\leqq p_2(C, R_2)\,Td_T\Big((v_1, g_1), (v_2, g_2)\Big). \tag{5.5}$$

Similarly we find from $(3.8)_1$ that

$$\|\hat{w}_1-\hat{w}_2\|_{X^1(T)}\leqq p_3(C)\,Td_T\Big((v_1, g_1), (v_2, g_2)\Big). \tag{5.6}$$

To estimate $\partial(\nabla\hat{f})/\partial t$ we need

LEMMA 5.1.

$$\begin{aligned}\Big\|\frac{\partial}{\partial t}\nabla(\hat{f}_1-\hat{f}_2)\Big\|_{X^0(T)}&\leqq c\|v_0\|_2\|v_1-v_2\|_{X^1_1(T)}\\&\quad+p_4(C, R_2)\,Td_T\Big((v_1, g_1), (v_2, g_2)\Big),\end{aligned} \tag{5.7}$$

where c is a constant depending on Ω.

Proposition 5.1. follows from estimates (5.4)-(5.7).

PROOF OF LEMMA 5.1. Differentiating the equations for $\hat{f}_j$ with respect to t and using the relation (4.4), we get

$$\begin{aligned}\Delta\frac{\partial\hat{f}_j}{\partial t}&=\operatorname{div}u_j-\operatorname{tr}\Big((Dv_j)^2-(Dv_j)(D\hat{v}_j)\Big)\\&\quad-\frac{1}{|\Omega|}\int_\Omega\Big(\operatorname{div}u_j-\operatorname{tr}\Big((Dv_j)^2-(Dv_j)(D\hat{v}_j)\Big)\Big)dx,\end{aligned}$$

where

$$u_j=K-a(g_j)\nabla\hat{g}_j-(v_j\cdot\nabla)\hat{v}_j.$$

Furthermore, setting

$$\begin{aligned}G_j&=\operatorname{div}u_j-\operatorname{tr}\Big((Dv_j)^2-(Dv_j)(D\hat{v}_j)\Big),\\\nabla b_j&=Q\,u_j\end{aligned}$$

we obtain

$$\Delta\left(\frac{\partial}{\partial t}(\hat{f}_1-\hat{f}_2)-(b_1-b_2)\right)$$

(5. 8)
$$= -\operatorname{tr}\left((Dv_1)^2-(Dv_2)^2+(Dv_2)(D\hat{v}_2)-(Dv_1)(D\hat{v}_1)\right) + \frac{1}{|\Omega|}\int_\Omega (G_2-G_1)\,dx\,.$$

From boundary conditions for $\hat{f}_j$ and $\hat{g}_j$ we obtain

(5. 9)
$$\left\langle \nabla\frac{\partial}{\partial t}(\hat{f}_1-\hat{f}_2)-\nabla(b_1-b_2),\, n\right\rangle = \left\langle (v_1\cdot\nabla)(\hat{v}_1-v_1)-(v_2\cdot\nabla)(\hat{v}_2-v_2),\, n\right\rangle .$$

Choosing b_j suitably, we can assume

$$\int_\Omega\left(\frac{\partial}{\partial t}(\hat{f}_1-\hat{f}_2)-(b_1-b_2)\right)dx=0\,.$$

Set $f=\partial(\hat{f}_1-\hat{f}_2)/\partial t-(b_1-b_2)$. Then Green formula for Laplacian gives

(5. 10)
$$\|\nabla f\|_0^2 \leqq \int_{\partial\Omega} |<(v_1\cdot\nabla)(\hat{v}_1-v_1)-(v_2\cdot\nabla)(\hat{v}_2-v_2),\, n>f|\,dS + \int_\Omega \left|\operatorname{tr}\left((Dv_1)^2-(Dv_2)^2+(Dv_2)(D\hat{v}_2)-(Dv_1)(D\hat{v}_1)\right)f\right|dx\,.$$

The integrand of the volume integral in (5. 10) is a sum of the following terms;

$$\left|f\,\partial v_j \text{ or } f\,\partial\hat{v}_j\right| \times \left|\partial({}_1v-v_2) \text{ or } \partial(\hat{v}^1-\hat{v}^2)\right|$$

where ∂ denotes one of $\partial/\partial x_k$ $(k=1, 2, 3)$. The integral of these terms are estimated by

(5. 11)
$$\varepsilon\|f\|_1^2+\varepsilon^{-1}(c_\Omega\|v_0\|_2\|v_1-v_2\|_1 + p_5(C, R_2)\,Td_T\left((v_1, g_1), (v_2, g_2)\right) \qquad (\varepsilon>0)\,.$$

For instance, we have

$$\begin{aligned}\int_\Omega &\left|f\,\partial v_1\partial(\hat{v}_1-\hat{v}_2)\right|dx \\ &\leqq \|f\,\partial v_1\|_0\|\partial(\hat{v}_1-\hat{v}_2)\|_0 \\ &\leqq c_\Omega\|f\|_1\|v_1\|_2\|\hat{v}_1-\hat{v}_2\|_1 \\ &\leqq \varepsilon\|f\|_1^2+\varepsilon^{-1}c_\Omega^2(\|v_0\|_2+tC)^2\|\hat{v}_1-\hat{v}_2\|_1^2\,.\end{aligned}$$

From this (5. 5) and (5. 6) we get (5. 11). Similarly, we can also verify using $(P\,\mathrm{II})$ and the remark after Proposition 3. 1 that the boundary integral in

(5. 10) is estimated by (5. 11). Using Poincaré lemma (for instance, see Courant and Hilbert [1]);

$$||f||_0 \leqq c_\Omega\Big(||\nabla f||_0 + |\int_\Omega f(x)dx|\Big),$$

and taking $\varepsilon>0$ small, we obtain from (5. 10) and (5. 11)

$$\text{(5. 12)}\qquad \begin{aligned} ||\nabla f||_0 \leqq & c_\Omega ||v_0||_2 ||v_1-v_2||_1 \\ & + p_6(C, R_2)\, T d_T\Big((v_1, g_1), (v_2, g_2)\Big). \end{aligned}$$

Therefore, by definitions of f, b_j and estimates (5. 4), (5. 5), (5. 6) and (5. 12), we see that (5. 7) is valid.

PROOF OF THEOREM. Let T and $||v(0)||_{X^{s-1}}$ be small so that Propositions 4. 1 and 5. 1 are valid. Then we know that Φ has a unique fixed point in the closure of $E(T, C)$ with respect to X^1-norm. This is a limit point (v, g) of a sequence $(v_j, g_j) \in E(C, T)$ which is recursively defined by $(v_j, g_j) = \Phi(v_{j-1}, g_{j-1})$, $v_j = w_j + \nabla f_j$. Since

$$(v_j, g_j) \longrightarrow (v, g) \text{ in } X^1 \text{ and } ||v_j||_{X^s} + ||g_j||_{X^s} \leqq C,$$

we can verify that

$$\text{(5. 13)}\qquad (v, g) \in X^{s-1} \text{ and } ||v(t)||_{X^s} + ||g(t)||_{X^s} \leqq C$$

for any $0 \leqq t \leqq T$.

We shall show that (v, g), $v = w + \nabla f$, is a solution of the system (1. 13)-(1. 15). Clearly w and f satisfy the corresponding equations. To see the equations for g we use a weak form of the initial boundary value problem (3. 1). Using the divergence theorem and the equations for g_j, we obtain

$$\begin{aligned} -\int_0^T dt \int_\Omega \Big(\frac{\partial g_j}{\partial t} + v_{j-1}\cdot\nabla g_j\Big)\Big(\frac{\partial \varphi}{\partial t} + v_{j-1}\cdot\nabla\varphi + \varphi \operatorname{div} v_{j-1}\Big) dx \\ + \int_0^T dt \int_\Omega < a(g_{j-1})\nabla g_j, \nabla\varphi > dx \\ = \int_0^T dt \int_\Omega \varphi\Big(\operatorname{tr}\big((Dv_{j-1})^2 - \operatorname{div} K\Big) dx + \int_\Omega \varphi(0) \operatorname{div} v_0 dx \\ + \int_0^T dt \int_{\partial\Omega} < (v_{j-1}\cdot\nabla) v_{j-1} - K, n > \varphi\, dS, \end{aligned}$$

for any $\varphi \in C_0^\infty([0, T) \times \bar{\Omega})$. Taking $n\to\infty$ and again using the divergence theorem, we can verify that $g \in X^{s-1}$ is a solution of the initial boundary value problem. Furthermore, we also verify from (5. 13) that

$$[0, T] \ni t \longrightarrow (\partial^k v/\partial t^k, \partial^k g/\partial t^k) \in H^{s-k} \qquad (k = 0, 1, \cdots, s)$$

is weakly continuous (hence strongly measurable). Then we have $(v, g)\in Y^{s}$. We shall consider (v, g) as a solution of the linear system with coefficients in Y^{s}. From the proof of estimates in Section 3 we find that the estimates replacing $(\hat{v}, \hat{g})$ by (v, g) hold. Assume $g\in X^{s}$ which will be proved in Section 6. Then $\nabla f\in X_{1}^{s}$ follows from (3. 6). Since the problem (3. 7) is reversible with respect to time, we find from (3. 9) with initial data at $t=t_{1}$ that the continuity of $\|w(t)\|_{X_{1}^{s}}$ in t. Thus $\nabla f\in X^{s}$ follows from (3. 11) and $w\in X^{s}$ follows from (3. 7).

Next we consider the continuous dependence of solutions on data. Let (v, g), $(v^{(n)}, g^{(n)})$ be solutions in X^{s} for data (v_{0}, g_{0}, K), $(v_{0}^{(n)}, g_{0}^{(n)}, K^{(n)})\in H^{s}\times X^{s}$ and $(v_{0}^{(n)}, g_{0}^{(n)}, K^{(n)})$ converge to (v_{0}, g_{0}, K) in $H^{s-1}\times X^{s-1}$-norm. Note that T and δ in Proposition 5. 1 depend only on R_{1}, R_{2} for a fixed $0<\varepsilon<1$. Then T is determined uniformly for (v, g), $(v^{(n)}, g^{(n)})$ if $\|v(0)\|_{X^{s-1}}\leqq\delta\leqq R_{1}$, $\|g_{0}^{(n)}\|_{s}\leqq R_{1}$ and $\|K^{(n)}\|_{X^{s}}\leqq R_{2}$. Furthermore, we have

$$\|v\|_{X^{s}(T)}+\|g\|_{X^{s}(T)}\leqq C,$$

where C is in $E(C, T)$.

Set $\tilde{g}=g-g^{(n)}$, $\nabla\tilde{f}=Q(v-v^{(n)})$ and $\tilde{w}=P(v-v^{(n)})$. Then we can verify that $\tilde{g}$, $\tilde{f}$ and $\tilde{w}$ satisfy the equations corresponding to (5. 1), (5. 2) and (5. 3), respectively. Applying estimates $(3.1)_{s-1}$, $(3.4)_{s-1}$ and $(3.7)_{s-1}$ to $\tilde{g}$, $\tilde{f}$ and $\tilde{w}$, we can verify that

$$\|(g-g^{(n)})(t)\|_{X^{s-1}}+\|(v-v^{(n)})(t)\|_{X^{s-1}}$$

is estimated by

$$p_{7}(C)\Big(\|g_{0}-g_{0}^{(n)}\|_{s-1}+\|v_{0}-v_{0}^{(n)}\|_{s-1}$$
$$+\int_{0}^{t}\Big(\|K-K^{(n)}\|_{X^{s-1}}+\|g-g^{(n)}\|_{X^{s-1}}+\|v-v^{(n)}\|_{X^{s-1}}\Big)dt\,,$$

Using Gronwall inequality we have

$$\|(g-g^{(n)})(t)\|_{X^{s-1}}+\|(v-v^{(n)})(t)\|_{X^{s-1}}$$
$$\leqq e^{p_{7}(C)t}\Big(\|g_{0}-g_{0}^{(n)}\|_{s-1}+\|v_{0}-v_{0}^{(n)}\|_{s-1}$$
$$+p_{7}(C)\int_{0}^{t}\|K-K^{(n)}\|_{X^{s-1}}\,dt\Big)\,.$$

Thus we obtain the continuous dependence of solutions on data.

Finally we remove the restriction on the initial data. Following [2] we scale variables setting

$$g_\lambda=g,\ v_\lambda=\lambda v,\ a_\lambda=\lambda^2 a,\ K_\lambda=\lambda^2 K,\ t_\lambda=t/\lambda \qquad (\lambda>0).$$

Then we find that $(v_\lambda(t_\lambda, x), g_\lambda(t_\lambda, x), a_\lambda, K_\lambda)$ satisfy the equation (1. 1) if and only if (v, g, a, K) do. Since

$$\frac{\partial^k v_\lambda}{\partial t_\lambda^k}=\lambda^{k+1}\frac{\partial^k v}{\partial t^k},$$

we see taking λ as small that the restriction is removed.

6. Estimates for $\log \rho$.

In this section we shall prove Proposition 3. 1.

First of all, to simplify the notations in (3. 1), we write $a(t, x)=a(g(t, x))$ and $g(t, x)=\hat{g}(t, x)$. Then we obtain the initial boundary value problem for g instead of (3. 1);

$$(6.1)\quad \begin{aligned} &(\alpha)\quad \frac{d^2 g}{dt^2}-\operatorname{div}(a\nabla g)=\mathrm{F} && \text{in } (0, T)\times\Omega, \\ &(\beta)\quad g(0)=g_0,\ \frac{\partial g}{\partial t}(0)=g_1 && \text{on } \Omega, \\ &(\gamma)\quad \langle \nabla g, n\rangle=h && \text{on } (0, T)\times\partial\Omega. \end{aligned}$$

Here

$$F=\operatorname{tr}\left((Dv)^2\right)-\operatorname{div} K,\quad g_1=-(v_0\cdot\nabla g_0+\operatorname{div} v_0),$$

$$h=\left\langle K-(v\cdot\nabla)\, v, n\right\rangle/a.$$

In this section we assume that $(v, a)\in X^s$ are given so that $\langle v, n\rangle=0$ on the boundary, $a\geqq 2d$, and $a-\langle v, n\rangle^2\geqq d$ in U. We remark that $-a+\langle v, n\rangle^2$ is the coefficient of second order normal derivative in (6. 1 α).

Following the idea of [9] we reduce the problem (6. 1) to the one with homogeneous boundary conditions. Choose $f_1\in H^{k+1}((0, T)\times\Omega)$ such that

$$(6.2)\quad \begin{aligned} &\langle \nabla f_1, n\rangle=h \qquad \text{on } (0, T)\times\partial\Omega, \\ &\|f_1\|_{k+1,(0,T)\times\Omega}\leqq c_\Omega\|h\|_{k-1/2,(0,T)\times\partial\Omega} \qquad (k\geqq 1). \end{aligned}$$

We then consider the initial boundary problem for f_2;

$$(6.3)\quad \begin{aligned} &Lf_2=F-Lf_1 && \text{in } (0, T)\times\Omega, \\ &f_2(0)=g_0-f_1(0) && \text{on } \Omega, \\ &\frac{\partial f_2}{\partial t}(0)=g_1-\frac{\partial f_1}{\partial t}(0) && \\ &\langle \nabla f_2, n\rangle=0 && \text{on } (0, T)\times\partial\Omega, \end{aligned}$$

where

$$L=\frac{d^2}{dt^2}-\operatorname{div}(a\nabla\cdot)\,.$$

Since the data (g_0, g_1, h, F) satisfy the compatibility conditions up to order s for (6. 1), the data in (6. 3) also satisfy the ones for (6. 3). Thus, if there exists a solution $f_2\in X^s$ satisfying the estimates corresponding to $(3.2)_k$ and (3. 3) without boundary norms, then we can verify from (6. 2) that $g=f_1+f_2$ is a solution of (6. 1) and satisfies the estimates $(3.2)_k$ and (3. 3).

Hereafter we shall consider the problem (6. 1) for any data $(g_0, g_1, F)\in H^s\times H^{s-1}\times X^{s-1}$ with $h=0$ satisfying compatibility conditions up to order s.

To derive a priori estimates for g, we introduce the energy integral

$$E(g\,;\,t)=\int_\Omega\left(\frac{dg}{dt}(t)^2+\langle a\nabla g, \nabla g\rangle(t)\right)dx\,.$$

Then we have

LEMMA 6. 1.

$$\left(d/q(t)\right)\left(\|\frac{\partial g}{\partial t}(t)\|_0^2+\|\nabla g(t)\|_0^2\right)\leqq E(g\,;\,t)$$
$$\leqq q(t)\left(\|\frac{\partial g}{\partial t}(t)\|_0^2+\|\nabla g(t)\|_0^2\right),$$

where

$$q(t)=\sup_{x\in\Omega}\left(1+a(t, x)+|v(t, x)|^2\right)\,.$$

PROOF. We consider $E(g\,;\,t)$ as a symmetric bilinear form in $\partial g/\partial t$, $\partial g/\partial x_j$. Let $A(t, x)$ be the matrix associated to this bilinear form. Then we can verify that the eigenvalues λ of A satisfy

$$(\lambda-a)^2\left(\lambda^2-(1+a+|v|^2)\,\lambda+a\right)=0\,.$$

Thus the lemma is proved.

LEMMA 6. 2. *It holds for a solution* $g\in X^s$ *of* (6. 1) *with* $\langle\nabla g, n\rangle=0$ *on the boundary that*

$$E(g\,;\,t)=E(g\,;\,0)-2\int_0^t\left(\frac{dg}{dt}, F\right)_0 dt+R_g\,, \tag{6.4}$$

where

$$|R_g|\leqq p_1\left|\int_0^t E(g\,;\,t)\,dt\right|,$$

and p_1 is a polynomial of $||v||_{X^3}$, $||a||_{X^3}$.

REMARK. The assertion of Lemma 6.2 is also valid for $g\in Y^2$ and $(v, a)\in Y^3$.

PROOF. Since $\langle v, n\rangle=0$ on the boundary, it follows from the divergence theorem that

$$(f, v\cdot\nabla g)_0=-(v\cdot\nabla f, g)_0-(f, g\,\mathrm{div}\, v)_0\,.$$

Using this formula and integration by parts, we get

$$2\int_0^t\left(\frac{d^2g}{dt^2}, \frac{dg}{dt}\right)_0 dt=||\frac{dg}{dt}||_0^2\Big|_0^t-\int_0^t\left(\frac{dg}{dt}, \frac{dg}{dt}\,\mathrm{div}\, v\right)_0 dt\,.$$

Again using the divergence theorem, we obtain

$$2\int_0^t\left(\mathrm{div}\,(a\nabla g), \frac{dg}{dt}\right)_0 dt=-\int_\Omega\langle a\nabla g, \nabla g\rangle\, dx\Big|_0^t+\int_0^t dt\int_\Omega\left\langle\nabla g, \frac{da}{dt}\nabla g\right\rangle dx$$
$$-\int_0^t dt\int_\Omega\left\langle a\nabla g, (\mathrm{div}\, v)\,\nabla g+2\left[\nabla, \frac{d}{dt}\right]g\right\rangle dx\,.$$

Therefore we obtain Green formula (6.4). The estimate of R_g follows from Sobolev lemma (PI).

Using Lemma 6.1, 6.2 and Gronwall inequality we obtain

PROPOSITION 6.1. *It holds for a solution $g\in X^2$ of* (6.1) *with* $\langle\nabla g, n\rangle=$ 0 *on the boundary that*

$$e^{-p_2t}||g(t)||_{X^1}^2\leqq q_1(t)||g(0)||_{X^1}^2+p_3\int_0^t||F||\,dt\,, \tag{6.5}$$

where $q_1(t)$ is a polynomial of $||v(t)||_{X^2}$, $||a(t)||_{X^2}$ *and p's are polynomials of* $||v||_{X^3}$, $||a||_{X^3}$.

A routine deriving higher order estimates from (6.5) is usually as follows. Using a partition of unity of Ω;

$$\sum\varphi_j(x)=1\,,$$

we first reduce the problem (6.1) to the one in the quadrant. Here the support of φ_j is contained in U if it intersects to $\partial\Omega$. Next, Applying $\partial/\partial t$ and tangential differential operators to these equations, we use (6.5). The terms involving normal derivatives are estimated finally by using the equations. Thus we can obtain the estimates $(3.2)_k$ for $g\in X^s$, (3.3) for $g\in X^{s+1}$ and

(6. 6) $$e^{-p_4 t}\|g(t)\|^2_{X^s} \leqq p_5\Big(\|g(0)\|^2_{X^s}+\|F(0)\|^2_{X^{s-2}}+\int_0^T \|F\|_{X^{s-1}}\,dt\Big)$$

where p's are polynomials of $\|v\|_{X^s}$, $\|a\|_{X^s}$.

Now we show the existence of a solution $g\in X^s$ of (6. 1) with $h=0$. Let $g^{(1)}$ be a solutionof the initial value problem for $(g_0, g_1, F)\in H^{s+1}\times H^s\times X^s$;

$$Lg^{(1)}=F,\quad g^{(1)}(0)=g_0,\quad \frac{\partial g^{(1)}}{\partial t}(0)=g_1$$

where $L=d^2/dt^2-\mathrm{div}\,(a\nabla\cdot)$. Let $g^{(2)}$ be a solution of the problem;

$$Lg^{(2)}=0,\ \langle \nabla g^{(2)}, n\rangle=-\langle \nabla g^{(1)}, n\rangle, g^{(2)}(0)=\frac{\partial g^{(2)}}{\partial t}(0)=0\,.$$

Then we see that $g=g^{(1)}+g^{(2)}$ is a solution of the problem (6. 1). Using the compatibility conditions for (g_0, g_1, F), we can verify that $-\langle \nabla g^{(1)}, n\rangle$ has an extension in the space $H_{s-1/2,\gamma}(\boldsymbol{R}\times\partial\Omega)=\{f;\ e^{-\gamma t}f\in H_{s-1/2}(\boldsymbol{R}\times\partial\Omega)\}$ $(\gamma>0)$ setting zero in $t<0$. By the aid of the existence theorem in $H_{s,\gamma}$ $(\boldsymbol{R}\times\Omega)$ we find that (6. 1) has a solution g satisfying $(3. 2)_{s-1}$ such that $\partial^k g/\partial t^k\in L^1([0, T], H^{s-k})$ $(k=0, \cdots, s)$. Let v and a belong to X^{s+1} and the data

$$(g_0, g_1, F)\in H^s\times H^{s-1}\times X^{s-1}$$

satisfy the compatibility condition up to order s. Then we can verify that this data is approximated in the norm of the right hand side in (6. 6) by the data $\in H^{s+1}\times H^s\times X^s$ satisfying the compatibility conditions up to order $s+1$ (see Ikawa [6]). Therefore we conclude that (6. 1) has a unique solution $g\in X^s$ satisfying the estimates (3. 3) and (6. 6). From Proposition 6. 1 we see that solutions of (6. 1) is unique in X^2. Thus Proposition (3. 1) is proved.

By the above arguments we can verify that if $g\in Y^s$ is a solution of (6. 1) for $(v, a)\in Y^s$ then $g\in X^s$.

7. Estimates for v.

In this section we shall prove Propositions 3. 2, 3. 3 and 3. 4.

We first consider the boundary value problem for $\hat{f}$;

(7. 1) $$\begin{aligned} \Delta\hat{f}&=G && \text{in } \Omega\,,\\ \langle \nabla\hat{f}, n\rangle&=h && \text{on } \partial\Omega\,. \end{aligned}$$

If a solution of (7. 1) exists, the data (G, h) satisfy the compatibility condition;

(7.2) $$\int_{\Omega} G\,dx = \int_{\partial\Omega} h\,ds\,.$$

As is well known (for instance, see Lions and Magenes [7]), we obtain

PROPOSITION 7.1. *Suppose that the data* $(G, h) \in H^s(\Omega) \times H^{s+1/2}(\partial\Omega)$ *satisfy the compatibility condition. Then the boundary value problem* (7.1) *has a unique solution* $\hat{f} \in H^{s+1}(\Omega)$ *modulo constants such that*

(7.3) $$\|\nabla \hat{f}\|_{s+1} \leqq c\|G\|_s + \|h\|_{s+1/2,\partial\Omega})\,,$$

where a constant c depends on Ω *and s.*

Proposition 3.2 follows directly from this proposition since the data in (3.4) satisfy the compatibility condition.

We next consider the initial value problem for $\hat{w}$;

(7.4) $$\begin{aligned} &\frac{\partial \hat{w}}{\partial t} + P\Big((v\cdot\nabla)\hat{w} + (\hat{w}\cdot\nabla)\nabla\hat{f}\Big) = PK \qquad \text{in } (0, T)\times\Omega\,,\\ &\hat{w}(0) = Pv_0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Omega\,, \end{aligned}$$

where $v \in X^s$ is given so that $\langle v, n\rangle = 0$ on the boundary and $\hat{f}$ is the solution of (3.4) such that $\nabla\hat{f} \in X_1^s$. Clearly a solution $\hat{w}$ of (7.4) is solenoidal.

Proposition 3.3 was stated in [4] without proof in details. Our proof is somewhat different from the method anounced in [4].

PROPOSITION 7.2. *It holds for a solution* $\hat{w} \in X^s$ *that*

(7.5) $$\|\hat{w}(t)\|_{X^1} \leqq e^{p_1 t}\|\hat{w}(0)_X{}^1\| + \int_0^t e^{p_1(t-\tau)}\|PK(\tau)\|_{X^1}\,dt\,.$$

where $p_1 = c(\|v\|_{X_1^3} + \|\nabla\hat{f}\|_{X_1^3})$ *and c is a constant depending on* Ω.

REMARK. Proposition 7.2 is also valid for $w \in Y^2$, $v \in Y^3$ and $\nabla\hat{f} \in Y_1^3$. The estimate (7.5) is nothing but $(3.8)_1$.

PROOF OF PROPOSITION 7.1. We first show

(7.6) $$\frac{\partial}{\partial t}\|\hat{w}(t)\|_1 \leqq p_2\|\hat{w}(t)\|_1 + \|PK(t)\|_1\,.$$

From (7.4) we have

$$\frac{\partial}{\partial t}\|\hat{w}\|_1^2 = -2\Big(\hat{w}, P\Big((v\cdot\nabla)\hat{w} + (\hat{w}\cdot\nabla)\nabla\hat{f} - K\Big)\Big)_1\,.$$

To estimate $(\hat{w}, P((v\cdot\nabla)\hat{w}))_1$ we decompose $(v\cdot\nabla)\hat{w}$ into

$$(v\cdot\nabla)\,\hat{w} = P\Big((v\cdot\nabla)\,\hat{w}\Big) + \nabla f_1\,.$$

Here

$$\Delta f_1 = \operatorname{div}\left((v\cdot\nabla)\hat{w}\right) = \operatorname{tr}\left((Dv)(D\hat{w})\right) \qquad \text{in } \Omega\,,$$

$$\langle \nabla f_1, n\rangle = \left\langle (v\cdot\nabla)\hat{w}, n\right\rangle \qquad \text{on } \partial\Omega\,,$$

where we use $\operatorname{div}\hat{w}=0$. Since the operator $v\cdot\nabla$ is tangential on $\partial\Omega$, it follows from Proposition 7.1 that

$$\|\nabla f_1\|_1 \leqq c_1\|v\|_3\|\hat{w}\|_1\,,$$

where a constant c_1 depends on Ω. Using the divergence theorem we can verify

$$\left|\left(\hat{w}, (v\cdot\nabla)\hat{w}\right)_1\right| \leqq c_2\|v\|_3\|\hat{w}\|_1^2\,.$$

Therefore we obtain

$$\left|\left(\hat{w}, P\left((v\cdot\nabla)\hat{w}\right)\right)_1\right| \leqq c_3\|v\|_3\|\hat{w}\|_1^2\,.$$

Since

$$\left|\left(\hat{w}, P\left((\hat{w}\cdot\nabla)\nabla\hat{f}\right)_1\right| \leqq c_4\|\nabla\hat{f}\|_3\|\hat{w}\|_1^2\,,$$

we thus obtain (7.6).

We next show

$$\text{(7.7)} \qquad \frac{\partial}{\partial t}\left\|\frac{\partial\hat{w}}{\partial t}(t)\right\|_0 \leqq p_3\left(\left\|\frac{\partial\hat{w}}{\partial t}(t)\right\| + \|\hat{w}(t)\|_1\right) + \left\|\frac{\partial PK}{\partial t}(t)\right\|_0\,.$$

Differentiating (7.1) with respect to t, we have

$$\frac{\partial}{\partial t}\left\|\frac{\partial\hat{w}}{\partial t}(t)\right\|_0^2 = -2\left(\frac{\partial\hat{w}}{\partial t}, \frac{\partial}{\partial t}\left((v\cdot\nabla)\hat{w}+(\hat{w}\cdot\nabla)\nabla\hat{f}-K\right)\right)_0\,,$$

where we use that $\partial\hat{w}/\partial t$ is solenoidal. Using the divergence theorem we can obtain (7.7). Since the estimate (7.5) follows from (7.6) and (7.7), the porposition is proved.

By a similar way as the proof of Proposition 7.2 we can verify the estimates $(3.8)_k$ $(k=2, \cdots, s-1)$. To obtain the estimate (3.9) for $\hat{w}\in X_1^{s+1}$, the proof of the following lemma is not routine, since $(\hat{w}\cdot\nabla)\nabla(\partial^k\hat{f}/\partial t^k)\notin H^{s-k}(\Omega)$.

LEMMA 7.1. *It holds for any vector fields v, $w\in X_1^s$ that*

$$\text{(7.8)} \qquad \|P(v\cdot\nabla)Qw\|_{X_1^s} \leqq c_5\|Qw\|_{X_1^s}\|v\|_{X_1^s}\,.$$

PROOF. As is well known (for instance, see Friedrichs [3]), we have

$$(7.9)\qquad \begin{aligned} &\|\nabla u\|_0 \leqq c_\Omega\Big(\|\operatorname{div} u\|_0+\|\operatorname{curl} u\|_0+\|u\|_0\Big),\\ &\langle u, n\rangle=0 \qquad \text{on } \partial\Omega . \end{aligned}$$

Letting $u=P(\partial^{s-1}((v\cdot\nabla)Qw)/\partial t^{s-1})$ and using the identity $\operatorname{curl}\cdot\nabla=0$, we get

$$\|\nabla P\Big(\frac{\partial^{s-1}}{\partial t^{s-1}}\big((v\cdot\nabla)Qw\big)\Big)\|_0 \leqq c_6\|\operatorname{curl}\Big(\frac{\partial^{s-1}}{\partial t^{s-1}}\big((v\cdot\nabla)Qw\big)\Big)\|_0 .$$

Again using the above identity we can verify that $\|\partial^{s-1}P((v\cdot\nabla)Qw)/\partial t^{s-1}\|_1$ is estimated by the right hand side in (7.8). Similarly we obtain the estimate (7.8).

Now we prove the existence of a solution $\hat{w}\in X^s$ of the initial value problem (7.4). To this end we consider the equations for $\hat{u}$;

$$(7.10)\qquad \begin{aligned} &\Big(\frac{\partial}{\partial t}+v\cdot\nabla\Big)\hat{u}=P\big((v\cdot\nabla)Q\hat{u}\big)+H \qquad &&\text{in } (0, T)\times\Omega ,\\ &\hat{u}(0)=u_0 &&\text{on } \Omega . \end{aligned}$$

Applying P to (7.10) and setting $\hat{w}=P\hat{u}$, we see that $\hat{w}$ is a solution of the problem

$$\begin{aligned} &\frac{\partial\hat{w}}{\partial t}+P\big((v\cdot\nabla)\hat{w}\big)=PH \qquad &&\text{in } (0, T)\times\Omega ,\\ &\hat{w}(0)=Pu_0 &&\text{on } \Omega . \end{aligned}$$

This procedure is owed by T. Shirota.

Extending v, u_0 to $\tilde{v}$, $\tilde{u}_0$ in $\boldsymbol{R}^3$ so that their norms estimated by ones of v, u_0 in Ω, we consider the initial value problem in stead of (7.10),

$$(7.11)\qquad \begin{aligned} &\Big(\frac{\partial}{\partial t}+\tilde{v}\cdot\nabla\Big)u=G \qquad &&\text{in } (0, T)\times\boldsymbol{R}^3 ,\\ &u(0)=\tilde{u}_0 &&\text{on } \boldsymbol{R}^3 . \end{aligned}$$

Since (7.11) is symmetric system of the first order, the initial value problem (7.11) has a unique solution $u\in X_1^s$ such that

$$(7.12)\qquad \|u(t)\|_{X_1^s}\leqq e^{p_4 t}\|u_0\|_s+\int_0^t e^{p_4(t-\tau)}\|G(\tau)\|_{X_1^s}\,d\tau .$$

For the stable C_0-semi group method see [5].

Consider the operator

$$u\longrightarrow \widetilde{P}\big((v\cdot\nabla)Q(u|_\Omega)\big)-\widetilde{P}\big((u|_\Omega\cdot\nabla)\nabla f\big)$$

where $\sim$ and $|_\Omega$ denote the extension to $\boldsymbol{R}^3$ and the restriction on Ω. Then we known from Lemma 7.1 that this operator is bounded from $X_1^s(\boldsymbol{R}^3)$ into $X_1^s(\boldsymbol{R}^3)$. Using the estimate (7.12) and the method of iterations, we also known the existence of a solution $\tilde{w}\in X_1^s$ of (7.4). Using the equation (7.4) we see that $\hat{w}\in X^s$ and satisfies the estimate (3.10).

Finally we prove Proposition 3.4. The method is the same as in Lemma 5.1. In a similar way as deriving the boundary value problem (5.8), (5.9), we can obtain

$$\Delta\Big(\frac{\partial \hat{f}}{\partial t}-b\Big)=-\operatorname{tr}\Big((Dv)^2-(Dv)(D\hat{v})\Big)$$

$$-\frac{1}{|\Omega|}\int_\Omega\Big(\operatorname{div} u-\operatorname{tr}\Big((Dv)^2-(Dv)(D\hat{v})\Big)dx\,,$$

$$\Big\langle\nabla\Big(\frac{\partial \hat{f}}{\partial t}-b\Big),n\Big\rangle=\Big\langle(v\cdot\nabla)(\hat{v}-v),n\Big\rangle\,,$$

where

$$u=K-a(g)\nabla\hat{g}-(v\cdot\nabla)\hat{v},\quad \nabla b=Qu\,.$$

Applying $\partial^{s-1}/\partial t^{s-1}$ to the above equations and using Green formula for Laplacian, we can verify that $\nabla\hat{f}\in X^s$ and satisfies the estimate (3.11).

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Department of Mathematics
Hokkaido University