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On the stability of planar step shock fronts in multi-dimensional spaces

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1. Introduction.

In this note we study the equations of 2-dimensional isentropic compressible flow

$$(1.1)\qquad \frac{\partial}{\partial t}\begin{pmatrix}\rho w_1\\ \rho w_2\\ \rho\end{pmatrix}+\frac{\partial}{\partial x_1}\begin{pmatrix}\rho w_1^2+P\\ \rho w_1 w_2\\ \rho w_1\end{pmatrix}+\frac{\partial}{\partial x_2}\begin{pmatrix}\rho w_1 w_2\\ \rho w_2^2+P\\ \rho w_2\end{pmatrix}=0,$$

where ${}^t(w_1, w_2)$ is the velocity, ρ is the density and $P=P(\rho)$ is a well defined function of $\rho(>0)$ with $P_\rho(\rho)>0$.

Since Lax had developed mathematical theory of system of hyperbolic conservation laws ([6]), 1-dimensional shock fronts were investigated by many mathematicians, but multi-dimensional cases were treated by a few authors ([2], [3], [9] also [13], [14]). Recently Majda has investigated the stability of multi-dimensional shock fronts and obtained the local in time shock front solutions under the uniform stability condition with respect to the linearized problem. ([7], [8]).

The problem considered in this note is what phenomena may occur if the condition mentioned above is replaced by the weak stability. Because of difficulty to obtain exact answer with mathematical rigor, here we shall treat our problem as a perturbation one in the simplest form.

Let us the nonconservative variables $U={}^t(w_1, w_2, P)$. Then equations (1.1) become

$$(1.1')\qquad \begin{pmatrix}\rho & & \\ & \rho & \\ & & \frac{1}{\rho c^2}\end{pmatrix}\frac{\partial U}{\partial t}+\begin{pmatrix}\rho w_1, & 0, & 1\\ 0, & \rho w_1, & 0\\ 1, & 0, & \frac{w_1}{\rho c^2}\end{pmatrix}\frac{\partial U}{\partial x_1}+\begin{pmatrix}\rho w_2, & 0, & 0\\ 0, & \rho w_2, & 1\\ 0, & 1, & \frac{w_2}{\rho c^2}\end{pmatrix}\frac{\partial U}{\partial x_2}=0,$$

where $c^2=P_\rho(\rho)$.

Let $U^{0\pm}$ be constant step states:

$${}^t(w_1^{0+}, w_2^{0+}, P^{0+})\qquad \text{for}\quad x_1\geqq\sigma t,$$

$$ {}^t(w_1^{0-}, w_2^{0-}, P^{0-}) \qquad \text{for} \quad x_1 \leqq \sigma t\,, $$

where *the Rankine-Hugoniot conditions*,

$$ \begin{aligned} & w_2^- = w_2^+ = w_2\,, \\ (1.2) \qquad & -\sigma[\rho] + [\rho w_1] = 0\,, \\ & -\sigma[\rho w_1] + \left[\rho w_1^2 + P(\rho)\right] = 0 \end{aligned} $$

are assumed to be satisfied with the basic shock speed σ. We also assume that *Lax's* 1-*shock inequalities*,

$$ (1.3) \qquad \begin{aligned} & w_1^+ - c^+ < \sigma < w_1^- - c^-\,, \\ & \sigma < w_1^+ \end{aligned} $$

are satisfied for $U^{0\pm}$.

We let $U^{\pm} = U^{0\pm} + U'^{\pm}$ and $\beta = \sigma t + \phi$ denote the perturbed shock front solution of (1, 1') such that for some T, $U'^{\pm} \in H^s(G^{\pm})$, $\phi \in H^{s+1}((0, T) \times R^1)$,

$$ (1.4) \qquad ||U'^{\pm}||_{s,G^{\pm}} \leqq k_0\,, \qquad \langle\!\langle \phi \rangle\!\rangle_{s+1,(0,T)\times R^1} \leqq k_0\,, $$

$\mathrm{Supp}_x(U'^{\pm})$ and $\mathrm{Supp}_{x_2}(\phi)$ are contained in $\{|x| \leqq 1\}$ and its intersection with $\{x_1 = 0\}$ respectively. Here k_0 is a sufficiently small constant depending only on $\{U^{0\pm}, \sigma\}$, $s = 2\left[\frac{n}{2}\right] + 7$ (here $n = 2$),

$$ G^{\pm} = \left\{(t, x) \middle| \beta(t, x_2) \gtrless x_1 \quad \text{for} \quad t \in (0, T)\right\}, $$

$||U||_{s,\Omega}$ is the Sobolev norm of order s of U relative to Ω and $\langle\!\langle U \rangle\!\rangle_{s,(0,T)\times R^1}$ is the 2-dimensional sobolev norm.

For initial data $\frac{\partial^i U'^{\pm}}{\partial t^i}(0, x)$, $\frac{\partial^{i+1}\phi}{\partial t^{i+1}}(0, x_2)$ $(0 \leqq i \leqq s + l)$ we impose the following conditions:

l is some positive integer, $\phi(0, x_2) = 0$, the supports of these functions $\subset \frac{1}{2}$-disks with 0 as the center, H^{s+l-i}-norms in the definition domains of these functions are not greater than some constant k_0' for $i \leqq s + l$, all of functions with $i \leqq s - 1$ vanish on $\{x_1 = 0\}$, $\frac{\partial^i U'^{\pm}}{\partial t^i}(0, x)$ with $i \leqq s - 1$ satisfy the compatibility conditions up to order $s - 1$ on $\{x_1 \gtrless 0\}$, respectively.

Hereafter these functions also are denoted by $\{U'^{\pm}, \phi\}$.

To state our theorem, we introduce the following

DEFINITION. *We say the basic states* $\{U^{0\pm}, \sigma t\}$ *to be stable in* L^2-*sense if*

(i) *for any non-vanishing initial data mentioned above with sufficiently* *large l and small k_0' there is a unique shock front solution satisfying the conditions* (1. 4) *for some T, such that*

(ii) $$\int |U'|^2(t, x)\,dx \leqq c\int |U'|^2(0, x)\,dx\,, \qquad \textit{for} \quad t\in[0, T]\,.$$

Here $\int |U'|^2(0, x)\,dx = \iint_{x_1>0} |U'^+|^2(0, x)\,dx_1 dx_2 + \iint_{x_1<0} |U'^-|^2(0, x)\,dx_1 dx_2$.
Furthermore c depends only on k_0', l, T.

THEOREM. *The basic states $\{U^{0\pm}, \sigma t\}$ are stable in L^2-sense if and only if the linearized problem with respect to $\{U^{0+}, \sigma\}$ is uniformly stable.*

The proof of Theorem is based on our investigation of the linear hyperbolic mixed problem. In fact we shall reduce our problem to examine the vanishing order of reflection coefficients of the linearized problem. ([1], [10]). Therefore in order to obtain an analogous conclusion of Theorem, we may weaken conditions in Definition of the stability in L^2-sense. But in (ii) only relevant physical quantities are appeared in L^2-sense and it seems to me that the definition is also significant for the nonlinear problems.

After preliminaries, in section 3 and 4 we give the proof of Theorem. Finally, in section 5 we discuss about its significance and an example.

2. Preliminaries.

We let (1. 1) and (1. 1′) rewrite as follows:

$$\frac{\partial}{\partial t} F_0(U) + \frac{\partial}{\partial x_1} F_1(U) + \frac{\partial}{\partial x_2} F_2(U) = 0\,,$$

$$A_0(U)\frac{\partial U}{\partial t} + A_1(U)\frac{\partial U}{\partial x_1} + A_2(U)\frac{\partial U}{\partial x_2} = 0$$

respectively. Then our problem is to find a set of functions $\{U^+, U^-, \beta\}$ called a shock front solution such that

$$A_0(U^\pm)\frac{\partial U^\pm}{\partial t} + A_1(U^\pm)\frac{\partial U^\pm}{\partial x_1} + A_2(U^\pm)\frac{\partial U^\pm}{\partial x_2} = 0$$

$$\text{in} \quad \{x_1 \gtrless \beta(t, x_2)\} \quad \text{respectively,}$$

$$n_t\big(F_0(U^+) - F_0(U^-)\big) + n_1\big(F_1(U^+) - F_1(U^-)\big) + n_2\big(F_2(U^+) - F_2(U^-)\big) = 0$$

$$\text{on} \quad \{x_1 = \beta(t, x_2)\}\,,$$

where (n_t, n_1, n_2) is the time space normal of the surface $\{x_1 = \beta(t, x_2)\}$. Let us denote by $U^\pm = U^{0\pm} + U'^\pm$, $\beta = \sigma t + \phi$ a perturbed solution for $t \geqq 0$. Then via the coordinate transformation:

$$\tilde{x}_1 = x_1 - \beta(t, x_2),$$
$$\tilde{x}_2 = x_2,$$
$$\tilde{t} = t,$$

the equations satisfied by $\{U'^{\pm}, \phi\}$ may be written as follows,

$$(2.1)^{\pm} \quad \begin{aligned} &A_0(U^{0\pm}+U'^{\pm})\frac{\partial U'^{\pm}}{\partial t} + \left(A_1(U^{0\pm}+U'^{\pm}) - \sigma A_0(U^{0\pm}+U'^{\pm})\right)\frac{\partial U'^{\pm}}{\partial x_1} \\ &+ A_2(U^{0\pm}+U'^{\pm})\frac{\partial U'^{\pm}}{\partial x_2} - \left(\phi_t A_0(U^0+U'^{\pm})\right. \\ &\left.+\phi_{x_2} A_2(U^0+U'^{\pm})\right)\frac{\partial U'^{\pm}}{\partial x_1} = 0 \end{aligned}$$

in $\{x_1>0\}$ and $\{x_1<0\}$ respectively,

$$(2.2) \quad \begin{aligned} &\sigma\left(F_0(U^{0+}+U'^{+}) - F_0(U^{0-}+U'^{-})\right) - \left(F_1(U^{0+}+U'^{+}) - F_1(U^{0-}+U^{-})\right) \\ &+\phi_t\left(F_0(U^{0+}+U'^{+}) - F_0(U^{0-}+U'^{-})\right) \\ &+\phi_{x_2}\left(F_2(U^{0+}+U'^{+}) - F_2(U^{0-}+U'^{-})\right) \\ &=0 \quad \text{on} \quad \{x_1=0\}. \end{aligned}$$

Since the basic states $U^{0\pm}$ are constant, the linearized equations with respect to $\{U^{0\pm}, \sigma\}$ are of the form:

$$(2.1')^{\pm} \quad \begin{aligned} &A_0(U^{0\pm})\frac{\partial U'^{\pm}}{\partial t} + \left(A_1(U^{0\pm}) - \sigma A_0(U^{0\pm})\right)\frac{\partial U'^{\pm}}{\partial x_1} + A_2(U^{0\pm})\frac{\partial U'^{\pm}}{\partial x_2} = 0 \\ &\qquad \text{in} \quad \{x_1>0\} \text{ and } \{x_1<0\} \quad \text{respectively,} \end{aligned}$$

$$(2.2') \quad \begin{aligned} &\phi_t\left(F_0(U^{0+}) - F_0(U^{0-})\right) + \phi_{x_2}\left(F_2(U^{0+}) - F_2(U^{0-})\right) \\ &+\left(\sigma F_0'(U^{0+}) - F_1'(U^{0+})\right) U'^{+} - \left(\sigma F_0'(U^{0-}) - F_1'(U^{0-})\right) U'^{-} = 0 \\ &\qquad \text{on} \quad \{x_1=0\}. \end{aligned}$$

Here $F_0'(U^0)\,U'$ is the Fréchet derivative at U^0 of $F_0(U^0+U')$ denoted usually by $F_0'(U^0;\ U')$ and $\{U'^{\pm}, \phi\}$ are unknown functions.

Following Majda ([7]) we now describe the Lopaninskii determinant of (2.1′) and (2.2′). First we note that U'^{-} is determined by the equation $(2.1')^{-}$ and the initial Cauchy data, for (1.3) is valid. Thus we may take $\{U'^{+}, \phi\}$ as unknown functions. The eigenvalues and corresponding eigenvectors of $A_0(U^{0+})^{-1}(A_1(U^{0+}) - \sigma A_0(U^{0+}))$ are given as follows: we set $w_i^0 = w_i^{0+}$, $\rho^0 = \rho^{0+}$ and $c^0 = P_\rho(\rho^0)^{\frac{1}{2}}$, then

$$\lambda_1^0 = w_1^0 - \sigma - c^0 < 0\,, \qquad \lambda_2^0 = w_1^0 - \sigma > 0\,,$$
$$\lambda_3^0 = w_1^0 - \sigma + c^0 > \lambda_2^0\,,$$
$$\gamma_1 = \frac{1}{\sqrt{2}}\,{}^t(-1, 0, c^0\rho^0),\ \gamma_2 = {}^t(0, 1, 0) \text{ and } \gamma_3 = \frac{1}{\sqrt{2}}\,{}^t(1, 0, c^0\rho^0)\,.$$

We let N_0 be the matrix $(\gamma_2, \gamma_3, \gamma_1)$. Then direct calculations yield:

$$N_0^{-1} = \begin{pmatrix} 0, & 1, & 0 \\ \frac{1}{\sqrt{2}}, & 0, & \frac{1}{\sqrt{2}\,c^0\rho^0} \\ -\frac{1}{\sqrt{2}}, & 0, & \frac{1}{\sqrt{2}\,c^0\rho^0} \end{pmatrix},$$

$$N_0^{-1}\begin{pmatrix} w_1' \\ w_2' \\ P' \end{pmatrix} = \begin{pmatrix} w_2' \\ \frac{1}{\sqrt{2}}\left(\frac{P'}{c^0\rho^0} + w_1'\right) \\ \frac{1}{\sqrt{2}}\left(\frac{P}{c^0\rho^0} - w_1'\right) \end{pmatrix} \equiv \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

where $(w_1', w_2', P') = (w_1'^{+}, w_2'^{+}, P'^{+})$.

The equations for $V = {}^t(v_1, v_2, v_3)$ become as follows:

$$D_{x_1} V - M(D_t, D_{x_2})\, V = 0\,, \tag{2.3}$$

$$M(D_t, D_{x_2}) = \begin{pmatrix} \frac{-1}{w_1^0 - \sigma}\tilde{D}_t & , \frac{-c^0}{\sqrt{2}\,(w_1^0-\sigma)} D_{x_2}, & \frac{-c^0}{\sqrt{2}\,(w_1^0-\sigma)} D_{x_2} \\ \frac{-c^0}{\sqrt{2}\,(w_1^0-\sigma+c^0)} D_{x_2}, & \frac{-1}{w_1^0-\sigma+c^0}\tilde{D}_t\ , & 0 \\ \frac{-c^0}{\sqrt{2}\,(w_1^0-\sigma-c^0)} D_{x_2}, & 0 & , \frac{-1}{w_1^0-\sigma-c^0}\tilde{D}_t \end{pmatrix},$$

where $D_{x_1} = \frac{1}{i}\frac{\partial}{\partial x_1}$ and $\tilde{D}_t = D_t + w_2^0 D_{x_2}$.

The boundary conditions for V on $\{x_1 = 0,\ t \geqq 0\}$ are:

$$B = \begin{pmatrix} \tilde{D}_t, & 0, & -\alpha D_{x_2} \\ 0\,, & 1, & -\beta \end{pmatrix},\ BV = 0 \text{ on } x_1 = 0\,,$$

$$\alpha = \frac{(P(\rho^{0+}) - P(\rho^{0-}))\,(w_1^0 - \sigma - c^0)}{(\rho^{0+} - \rho^{0-})\,(w_1^0 - \sigma)\sqrt{2}\,(w_1^0 - \sigma + c^0)} < 0\,, \tag{2.4}$$

$$\beta = -\left(\frac{c^0 - (w_1^0 - \sigma)}{c^0 + (w_1^0 - \sigma)}\right)^2 \quad \text{and} \quad 0 > \beta > -1\,.$$

The boundary conditions for ϕ on $\{x_1=0,\ t\geqq 0\}$ are:

$$D_{x_2}\phi=(-i)\frac{\rho^0(w_1^0-\sigma)}{P(\rho^{0+})-P(\rho^{0-})}v_1,$$

$$\text{(2.5)}\qquad D_t\phi=i\left\{\left(\frac{(w_1^0-\sigma)\,c^0\rho^0}{\sqrt{2}\,(\rho^{0+}-\rho^{0-})\,c^0}+\frac{\rho^0}{\sqrt{2}\,(\rho^{0+}-\rho^{0-})}\right)v_2\right.$$
$$\left.+\left(\frac{(w_1-\sigma)\,c^0\rho^0}{\sqrt{2}\,(\rho^{0+}-\rho^{0-})\,c^0}-\frac{\rho^0}{\sqrt{2}\,(\rho^{0+}-\rho^{0-})}\right)v_3\right\}.$$

Since $\phi(0, x_2)=0$, ϕ is determined from V.

Let the Fourier transformation of $f(t, x_2)$ be the following form:

$$\hat{f}(\tau, \omega)=\int e^{-i(\tau t+\omega x_2)}f(t, x_2)\,dtdx_2.$$

Then eigenvalues and the corresponding eigenvectors of $M(\tilde{\tau}, \omega)$ are: for $\tau=\eta-i\gamma$ and $\tilde{\tau}=\tau+w_2^0\omega$ $((\eta, \omega)\in R^2, \gamma>0)$

$$\lambda_1^+=\frac{-\tilde{\tau}}{w_1^0-\sigma},\quad \lambda_2^\pm=\frac{(w_1^0-\sigma)\,\tilde{\tau}\mp c^0\sqrt{\tilde{\tau}^2-\omega^2d^2}}{d^2},$$

$$e_1^+={}^t\!\left(\sqrt{2}\,\tilde{\tau},\ \omega(w_1^0-\sigma),\ -\omega(w_1^0-\sigma)\right),$$

$$e_2^\pm={}^t\!\left(-\omega\sqrt{2}\,c^0,\ \tilde{\tau}+\lambda_2^\pm(w_1^0-\sigma-c^0),\ \tilde{\tau}+\lambda_2^\pm(w_1^0-\sigma+c^0)\right),$$

where $d^2=(c^0)^2-(w_1^0-\sigma)^2$, $\sqrt{1}=1$ and $\operatorname{Im}\lambda^+>0$ for $\gamma>0$. The determinant of the matrix:

$$\begin{pmatrix}\tilde{\tau}, & 0, & -\omega d\\ 0, & 1, & -\beta\end{pmatrix}(e_1^+, e_2^+)$$

is $\sqrt{2}\,(\tilde{\tau}+\lambda_2^+(w_1^0-\sigma))\,((1-\beta)\,\tilde{\tau}^2+\sqrt{2}\,\alpha(w_1^0-\sigma)\,\omega^2+(1+\beta)\,\tilde{\tau}\sqrt{\tilde{\tau}^2-d^2\omega^2})$. The vectors e_1^+ and e_2^+ are linearly independent if and only if $\tilde{\tau}+\lambda_2^+(w_1^0-\sigma)\neq 0$. From the above two fact, it follows that for $(\tau, \omega)\neq(0, 0)$ the Lopatinskii determinant is always

$$\text{(2.6)}\qquad \begin{aligned}&L=a_1\tilde{\tau}^2+a_2\,\omega^2-\tilde{\tau}\sqrt{\tilde{\tau}^2-d^2\,\omega^2},\\ &a_1=-\frac{1-\beta}{1+\beta}<0,\quad a_2=-\frac{\sqrt{2}\,\alpha(w_1^0-\sigma)}{1+\beta}>0.\end{aligned}$$

In [7], Majda shows that the conditions (1.2) and (1.3) for $\{U^{0\pm}, \sigma\}$ imply the weak stability of the linearized problem (2.3) and (2.4) with respect to $\{U^{0+}, \sigma\}$, i.e., $L(\tau, \omega)\neq 0$ for $\gamma>0$. Furthermore he prove that the above problem is uniformly stable, i.e., $L(\tau, \omega)\neq 0$ for $\gamma\geqq 0$, if and only if

$$\text{(2.7)}\qquad a_1+a_2\frac{1}{d^2}<0.$$

Under the above uniform stability, form his theorem it follows that for initial data described in section 1 for $l=1$ and sufficiently small k_0' there exists the shock front solution with the smoothness (1.4) for some T.

3. Proof of Theorem.

To show the sufficiency of the assertion of Theorem, we may only to prove the inequality (ii) of section 1 for the smooth shock front solution $\{U^{\pm}, \beta\}$ with (1.4). Furthermore we may assume that $\{U'^{\pm}, \phi\}$ satisfy (2.1) and (2.2). Then we need only to estimate them. Note that the linearized problem with respect to $\{U^{0\pm}, \sigma\}$ is assumed to be uniformly stable.

To obtain linear problem for $\{U'^{\pm}, \phi\}$, using (1.2) we rewrite (2.2) as follows:

$$\begin{aligned}
&\phi_t\Big(F_0(U^{0+}+U'^{+})-F_0(U^{0-}+U'^{-})\Big)+\phi_{x_2}\Big(F_2(U^{0+}+U'^{+})-F_2(U^{0-}+U'^{-})\Big)\\
&\quad+\left\{\sigma\int_0^1 F_0'(U^{0+}+\theta U'^{+})\,d\theta-\int_0^1 F_1'(U^{0+}+\theta U'^{+})\,d\theta\right\}\cdot U'^{+}\\
&\quad-\left\{\sigma\int_0^1 F_0'(U^{0-}+\theta U'^{-})\,d\theta-\int_0^1 F_1'(U^{0-}+\theta U'^{-})\,d\theta\right\}\cdot U'^{-}=0\,.
\end{aligned}\tag{3.1}$$

Then regarding $A_i(U^{0\pm}+U'^{\pm})$, $F_i(U^{0\pm}+U'^{\pm})$, $\phi_{x_i}A_i$ and $\int_0^1 F_i'\cdot d\theta$ in (2.1) and (3.1) with fixed $\{U'^{\pm}, \phi\}$ as coefficients, and reflecting U'^{-} with respect to $x_1=0$, we obtain a mixed problem for $U=\{U'^{+}, U'^{-}\}$ and ϕ which is denoted by

$$\begin{aligned}
&L^{+}U'^{+}=0,\ L^{-}U'^{-}=0 &&\text{in}\ \ [0, T]\times\{x_1>0\}\,,\\
&B(U'^{+}, U^{-}, \phi)=0 &&\text{on}\ \ [0, T]\times\{x_1=0\}\,,\\
&U'^{+}(0, x)=h^{+}(x)\ \ \text{and}\ \ U'^{-}(0, x)=h^{-}(x)\,.
\end{aligned}\tag{3.2}$$

We may choose T and k_0 in (1.4) sufficiently small numbers such that $\{L^{\pm}\}$ in (3.2) is strongly hyperbolic and the linear mixed problem (3.2) is also uniformly stable in Kreiss-Majda's sense. We let $U_1=\{U_1^{+}, U_1^{-}\}$ and ϕ_1 be the solution of (3.2) with zero initial data and with $B(U_1^{+}, U_1^{-}, \phi_1)=g$ for some $g\in L^2$ $(t>0, x_1=0)$. We obtain the estimate for U_1 and ϕ_1:

$$\begin{aligned}
&\|U_1\|^2_{0,(0,T)\times\{x_1>0\}}+\langle\!\langle U_1\rangle\!\rangle^2_{0,(0,T)\times\{x_1=0\}}+\langle\!\langle \phi_1\rangle\!\rangle^2_{0,(0,T)\times\{x_1=0\}}\\
&\quad\leqq c_1\langle\!\langle g\rangle\!\rangle^2_{0,(0,\ T)\times\{x_1=0\}}\,.
\end{aligned}$$

We let also $U_2=(U_2^{+}, U_2^{-})$ be the solution of the problem:

$$\begin{aligned}
&L^{+}U_2^{+}=0,\ L^{-}U_2^{-}=0 &&\text{in}\ \ [0, T]\times\{x_1>0\}\,,\\
&N^{+}U_2^{+}=0,\ N^{-}U_2^{-}=0 &&\text{in}\ \ [0, T]\times\{x_1=0\}\,,\\
&U_2^{+}(0, x)=h^{+}(x)\ \ \text{and}\ \ U_2^{-}(0, x)=h^{-}(x)\,,
\end{aligned}$$

where $N^+(x)$ is the projection of the subspace in C^3 spaned by positive eigenvectors of the coefficient of $\frac{\partial}{\partial x_1}$ in L^+. Then since $L^\pm$ is symmetric, by Rauch's method ([11]) we obtain the desired estimate corresponding to (ii). That is, by the Energy method it follows that

$$\begin{aligned} &||U_2(t, x)||^2_{0,\{x_1>0\}} + \langle\!\langle U_2 \rangle\!\rangle^2_{0,(0,T)\times\{x_1=0\}} \\ &\leqq c_2||h||^2_{0,\{x_1>0\}} \qquad \text{for} \quad t\in[0, T]\,. \end{aligned}$$

Thus setting $g=-B(U_2^+, U_2^-, 0)$ and $\tilde{U}'=U_1+U_2$, by the uniqueness theorem of the linear problem we obtain that $\tilde{U}'=U'$, $\phi_1=\phi$ and the desired estimate for $\{U', \phi\}$. For, the symmetricity of $L^\pm$ yields the estimate: for $t\in[0, T]$.

$$||U_1(t, x)||^2_{0,\{x_1>0\}} \leqq c_3\langle\!\langle U_1 \rangle\!\rangle^2_{0,(0,T)\times\{x_1=0\}}\,.$$

Here we remark also that the uniqueness of solutions with (1.4) is derived as above by using a linear equations satisfied by the difference of solutions.

Next, to prove the necessity of the assertion of Theorem, we assume that (i) and (ii) are valid. Now we shall show that the linearized problem (2.3) and (2.4) is L^2-well posed in our sense ([1]). We let $\{U'^\pm, \phi\}$ be functions such that their derivatives satisfy the conditions with respect to initial data in section 1, $\{U'^\pm, \phi\}\not\equiv 0$, $\mathrm{Supp}\,(U'^+)\cap\{x_1\leqq\delta\}=\emptyset$ for a fixed $\delta\ll 1$ and $U'^-\equiv 0$. We let $\{U^{0\pm}+U_\varepsilon'^\pm, \sigma t+\phi_\varepsilon\}$ be the solution with the initial data $\{U^{0+}+\varepsilon U'^+, U^{0-}, \sigma t+\varepsilon\phi\}$ with $\varepsilon>0$, whose existence is assumed in (i). Then $U_\varepsilon'^-=0$ and $(U_\varepsilon'^+, \phi_\varepsilon\}$ is a solution to the problem $(2.1)^+$ and (3.1), for which (1.4) and (ii) are valid. Let $h(t)$ be a cut off function such that $h(t)\in C^\infty(R)$, $h(t)=1$ for $t\leqq T-\delta$ and $h(t)=0$ for $t\geqq T$. Here we note that ϕ and U'^+ vanish on $\{(t, x)|t=0,\ 0<x_1<\delta\}$.

We let $\{V_\varepsilon, \psi_\varepsilon\}$ be

$$V_\varepsilon = h\cdot\frac{1}{\varepsilon}U_\varepsilon^{+\prime}\,,$$

$$\psi_\varepsilon = h\cdot\frac{1}{\varepsilon}\phi_\varepsilon \qquad \text{for} \quad t\geqq 0\,.$$

Then from (ii) it follows that for some $c_1>0$

$$||V_\varepsilon||_{0,\{t>0,x_1>0\}} \leq cT||V_\varepsilon||_{0,\{t=0,x_1>0\}} \leqq c_1\,.$$

We have from $(2.1)^+$ and (3.1) that

$$\begin{aligned} &L_\varepsilon(V_\varepsilon) \\ &\equiv A_0(U^{0+}+\varepsilon V_\varepsilon)\frac{\partial V_\varepsilon}{\partial t} + \big(A_1(U^{0+}+\varepsilon V_\varepsilon) - \sigma A_0(U^{0+}+\varepsilon V_\varepsilon)\big)\frac{\partial V_\varepsilon}{\partial x_1} \end{aligned}$$

$$
\begin{aligned}
(3.3) \qquad & +A_2(U^{0+}+\varepsilon V_\varepsilon)\frac{\partial V_\varepsilon}{\partial x_2}-\Big\{\varepsilon\psi_{\varepsilon,t}A_0(U^{0+}+\varepsilon V_\varepsilon) \\
& +\varepsilon\psi_{\varepsilon,x_2}A_2(U^{0+}+\varepsilon V_\varepsilon)\Big\}\frac{\partial V_\varepsilon}{\partial x_1} \\
& =\frac{1}{\varepsilon}f_\varepsilon \qquad \text{in } \Big\{(t,x)\Big|t\geqq 0,\ x_1\geqq 0\Big\},
\end{aligned}
$$

$$
\begin{aligned}
& \psi_{\varepsilon,t}\Big(F_0(U^{0+}+\varepsilon V_\varepsilon)-F_0(U^{0-})\Big)+\psi_{\varepsilon,x_2}\Big(F_2(U^{0+}+\varepsilon V_\varepsilon)-F_2(U^{0-})\Big) \\
(3.4) \qquad & +\Big\{\sigma\int_0^1 F_0'(U^{0+}+\theta\varepsilon V_\varepsilon)\,d\theta-\int_0^1 F_1'(U^{0+}+\theta\varepsilon V_\varepsilon)\,d\theta\Big\}\,V_\varepsilon \\
& =0 \qquad \text{on } \Big\{(t,x)\Big|0\leqq t\leqq T-\delta,\ x_1=0\Big\},
\end{aligned}
$$

where $\operatorname{Supp}(f_\varepsilon)\subset\{T-\delta<t<T,\ x_1\geqq 0\}$. By the usual calculations with respect to Sobolev norms we see that $f_\varepsilon|\{t\geq 0\}$ is a sum of terms containg $U_\varepsilon'^{+}$ with bounded factors, and hence that $\|\frac{1}{\varepsilon}f_\varepsilon\|_{0,\{t\geq 0\}}\leqq c_2$ for some constant c_2. Now, we use the estimate

$$
\begin{aligned}
\langle\!\langle U\rangle\!\rangle_{-\frac{1}{2},\{x_1=0\}} &\leqq c_3\|U\|_{1,-1,\{x_1>0\}} \\
&\leqq c_4\Big(\|L_\varepsilon(U)\|_{0,\{x_1>0\}}+\|U\|_{0,\{x_1>0\}}\Big),
\end{aligned}
$$

where $U\in H^1(\{x_1>0\})$. ([4]). We set χ a cut off function $\in C_0^\infty(R')$ such that $\chi(x_1)=1$ for $x_1<\delta/2$ and $\chi(x_1)=0$ for $x_1>\delta$. Then from the above estimate we have:

$$
\begin{aligned}
& \langle\!\langle V_\varepsilon\rangle\!\rangle_{-\frac{1}{2},\{x_1=0\}} \\
& \leqq c_4\Big(\|L_\varepsilon\chi V_\varepsilon\|_{0,\{x_1>0\}}+\|\chi V_\varepsilon\|_{0,\{x_1>0\}}\Big) \\
& \leqq c_5\Big(\|L_\varepsilon V_\varepsilon\|_{0,\{t>0,x_1>0\}}+\|V_\varepsilon\|_{0,\{t>0,x_1>0\}}\Big) \\
& \leqq c_6,
\end{aligned}
$$

where c_6 also dependent on χ' and hence only on δ, $Tc\|V_\varepsilon\|_{0,\{t=0,x_1\geqq 0\}}$ and k_0, but not on ε. Furthermore for sufficiently small k_0, (1. 2) and (1. 3) imply that $F_0(U^{0+}+\varepsilon V_\varepsilon)-F_0(U^{0-})$ and $F_2(U^{0+}+\varepsilon V_\varepsilon)-F_2(U^{0-})$ are linearly independent and hence (3. 4) yields that $\psi_{\varepsilon,t}$ and ψ_{ε,x_2} are represented as linear combinations of elements of V_ε with coefficients in $H^{s-\frac{1}{2}}(\{t<T-\delta,\ x_1=0\})$ Using the boundedness of V_ε and the above fact, there exist a weak limit $\{V,\phi\}$ such that

$$
V_{\varepsilon'}\xrightarrow[w]{}V \qquad \text{in } H^0\Big(\{0<t<T,\ x_1>0\}\Big),
$$

$$
(3.5)\qquad
\begin{aligned}
&V_{\varepsilon'} \xrightarrow[w]{} V && \text{in } H^{-\frac{1}{2}}\big(\{x_1=0\}\big),\\
&\varepsilon' V_{\varepsilon'} \longrightarrow 0 && \text{in } H^{s-1}\big(\{0<t<T,\ x_1>0\}\big),\\
&V_{\varepsilon'} = U'^{+}(0,x) && \text{in } H^{s}\big(\{t=0,\ x_1>0\}\big) \quad \text{and}\\
&\phi_{\varepsilon'} \xrightarrow[w]{} \phi && \text{in } H^{\frac{1}{2}}\big(\{t<T-\delta,\ x_1=0\}\big)
\end{aligned}
$$

as $\varepsilon'\to 0$. Here we note that $\mathrm{Supp}_{x_2}(\phi_{\varepsilon'})\subset\{|x_2|\leqq 1\}$.
Accordingly for any $W\in C_0^\infty(\{0<\mathrm{t}<\mathrm{T}-\delta,\ 0<x_1<\infty\})$

$$\int\Big\langle\Big(A_0(U^{0+}+\varepsilon' V_{\varepsilon'}^{+})-A_0(U^{0+})\Big)\frac{\partial V_{\varepsilon'}^{+}}{\partial t},\ W\Big\rangle\, dtdx\to 0$$

as $\varepsilon'\to 0$. Because then

$$A_0(U^{0+}+\varepsilon' V_{\varepsilon'}^{+})-A_0(U^{0+})\longrightarrow 0$$

in $H^{s-1}(\{0<t<T,\ x_1>0\})$.
Therefore we have that V is a weak solution of the linearized problem (2.1')$^+$ and (2.2') with initial data $V=U'^{+}(0,x)\in H_0^{s+1}(\{x_1>\delta\})$ and (3.5) such that

$$\|V(t,x)\|_{0,\{0,<t<T-\delta,x_1>0\}}\leqq cT\|U'^{+}(0,x)\|_{0,\{x_1>0\}},$$

$$\int\Big\langle L_0^{+}(V_{\varepsilon'}),\ W\Big\rangle\, dtdx=-\int\langle V_{\varepsilon'},\ L_0^{+}W\rangle\, dtdx\to 0,$$

$$\int\Big\langle \mathrm{B}_0(V_{\varepsilon'},\ \phi_{\varepsilon'}),\ W_1\Big\rangle\, dtdx_2\to 0$$

as $\varepsilon'\to 0$.

Here L_0^{+}, B_0 are the linearized operators in (2.1')$^+$ and (2.2') respectively and $W_1\in C_0^\infty(\{0\leqq t<T-\delta,\ x_1=0\})$.

On the other hand, the weak stability of the linear problem (2.3) and (2.4) implies that for initial data $N_0^{-1}U'^{+}(0,x)$, as it is shown below in section 3, there exist a unique solution $N_0^{-1}V_1\in H^{s-1}(\{0<t<T-\delta,\ x_1>0\})$ and hence from (2.5) there is a unique solution $\{V_1,\phi_1\}$ of the problem (2.1')$^+$, (2.2'). Therefore we have that $\{V_{\varepsilon'}-V_1,\ \phi_{\varepsilon},\ -\phi_1\}$ satisfies (3.5) replaced V, U'^{+} $(0,x)$ and ϕ by $V-V_1$, 0 and $\phi-\phi_1$ respectively. Furthermore we have, as $\varepsilon'\to 0$

$$\int\langle V_{\varepsilon'}-V_1,\ L_0^{+}\widehat{W}\rangle\, dtdx\to 0,$$

$$\int\Big\langle B_0(V_{\varepsilon'}-V_1,\ \phi_{\varepsilon'}-\phi_1),\ W_1\Big\rangle\, dtdx_2\to 0,$$

where $\widetilde{W}\in C_0^\infty(\{0\leqq t<T-\delta, 0<x_1<\infty\})$. Hence considering the convolution by a function $q_n(t, x_2)=n^2g(nt, nx_2)\in C_0^\infty(R^2)$, we see that

$$L_0^+\Big(q_n*(V-V_1)\Big)=0 \qquad \text{for} \quad t\leqq T-2\delta,\ x_1>0\,,$$

$$B_0\Big(q_n*(V-V_1),\ q_n*(\phi-\phi_1)\Big)=0 \qquad \text{for} \quad t\leqq T-2\delta\,,$$

$$q_n*(V-V_1)=q_n*(\phi-\phi_1)=0 \qquad \text{for} \quad t\leqq -\frac{1}{n}\,.$$

These equalities imply that $N_0^{-1}q_n*(V-V_1)$ is a solution of (2. 3) and (2. 4) for $t\leqq T-2\delta$ with zero initial data at $t=-\dfrac{1}{n}$.

By the uniqueness of the smooth solution of the above problem, we have $N_0^{-1}V\in H^{s-1}(\{0<t<T-2\delta, x_1>0\})$. Thus we see that there exists the solution of (2. 3) and (2. 4) $N_0^{-1}V\in H^{s-1}(\{0<t<T-2\delta, x_1>0\})$ such that $||N_0^{-1}V(t, x)||_{0,\{0<t<T-2\delta,x_1>0\}}\leqq cT||N_0^{-1}U'^+(0, x)||_{0,\{x_1\geqq 0\}}$.

Obviously for the above inequality the restrictions with respect to norms of initial data are removed. Regarding the pure Cauchy problem and the partition of unity we can also remove the condition that support of data $\subset\left\{|x|\leqq\dfrac{1}{2}\right\}$.

Extending V_ε and $\varepsilon V_\varepsilon$ suitably to $\{t<0\}$, by the same way as above we can also remove the condition that (support of data) $\cap\{x_1\leqq\delta\}=\emptyset$. Thus by Duhamel's principle we obtain that for all $F(t, x)\in C^1([0, T-2\delta], H_0^{s+l}(\{x_1>0\}))$ the solution $U(t, x)$ of (2. 3) with the right hand side F and (2. 4) with zero initial data satisfies: for some $c_0>0$

$$||U(t, x)||_{0,\{0<t<T-2\delta,x_1>0\}}\leqq c_0||F(t, x)||_{0,\{0<t<T-2\delta,x_1>0\}}\,,$$

which is just the L^2-well posedness of the linearized problem (2. 3) and (2. 4). For, by limit process, we can weaken the condition of F above to that $F\in H_0^1(\{0<t<T-2\delta, 0<x_1\})$. The proof of the assertion of Theorem is complete, provided the above L^2-well posedness yields the uniform stability of the same problem. Its proof is given in the next section.

4. Reflection coefficients.

In this section we study the linearized problem (2. 3) and (2. 4) with respect to the step shock front satisfying (1. 2) and (1. 3).

We let N be the matrix:

$$N=(e_1^+, e_2^+, e_2^-)\,,$$

whose components are functions of $(\tilde{\tau}, \omega)$ for fixed $\{U^{0\pm}, \sigma\}$. Hereafter we consider only statements with respect to (τ, ω) which is normalized and has small imaginary part γ of τ. Then as stated in section 2 e_1^+ and e_2^+ are linearly independent and the three vectors are also linearly independent whenever $\gamma \neq 0$. Assuming $\gamma > 0$ we let $\hat{U} = {}^t(U_1, U_2, U_3)$ be the vector such that for a solution $V(t, x)$ of (2. 3) vanishing for $t<0$

$$\hat{U} = N^{-1} \hat{V}.$$

Then $\hat{U}$ satisfies the following relations:

$$D_x \hat{U} = \begin{pmatrix} \lambda_1^+ & & \\ & \lambda_2^+ & \\ & & \lambda_2^- \end{pmatrix} \hat{U} \quad \text{in} \quad \{x_1 \geqq 0\}, \tag{4. 1}$$
$$\hat{B}(e_1^+, e_2^+) {}^t(U_1, U_2) + \hat{B}(e_2^-) U_3 = \hat{B} \hat{V} \quad \text{on} \quad \{x_1 = 0\}.$$

By assumption of the weak stability of the linearized problem we have:

$$\left|\hat{B}(e_1^+, e_2^+)\right| \neq 0 \qquad \text{for} \qquad \gamma > 0,$$
$${}^t(U_1, U_2) + \hat{B}(e_1^+, e_2^+)^{-1} \cdot \hat{B}(e_2^-) U_3 = \hat{B}(e_1^+, e_2^+)^{-1} \hat{B} \hat{V},$$

where

$$\left(\hat{B}(e_1^+, e_2^+)\right)^{-1} \cdot \hat{B}(e_2^-) \equiv {}^t(\widetilde{b_{12}}, \widetilde{b_{22}})$$

is called by *reflection coefficients* with respect to our problem.

We already have that for our problem is L^2-well posed if and only if for some $c \geq 0$

$$\begin{aligned} |\widetilde{b_{12}}| &\leqq c\gamma^{-1} |\mathrm{Im}\, \lambda_1^+|^{\frac{1}{2}} |\mathrm{Im}\, \lambda_2^-|^{\frac{1}{2}} |\lambda_2^- - \lambda_2^+|, \\ |\widetilde{b_{22}}| &\leqq c\gamma^{-1} |\mathrm{Im}\, \lambda_2^+|^{\frac{1}{2}} |\mathrm{Im}\, \lambda_2^-|^{\frac{1}{2}} |\lambda_2^- - \lambda_2^+|, \end{aligned} \tag{4. 2}$$

whenever Lopatinskii determinant $|\hat{B}(e_1^+, e_2^+)|(\tau^0, \omega^0) = 0$. Here τ^0 is real and (τ, ω) are in a neighborhood of (τ^0, ω^0) with $\gamma > 0$. ([10], [12]).

By direct calculations yield:

$$\begin{aligned} \widetilde{b_{12}} \times \left|\hat{B}(e_1^+, e_2^+)\right| &= \left|\hat{B}(e_2^-), \hat{B}(e_2^+)\right| \\ &= \tilde{\tau}(\lambda_2^+ - \lambda_2^-)\, \omega \sqrt{2}\, c^0 \left\{\sqrt{2}\, \alpha + (\beta+1)\, c^0 + (\beta-1)\, (w_1^0 - \sigma)\right\}, \\ \widetilde{b_{22}} \times \left|\hat{B}(e_1^+, e_2^+)\right| &= \left|\hat{B}(e_1^+), \hat{B}(e_2^-)\right| \\ &= \left\{\tilde{\tau} + \lambda_2^- (w_1^0 - \sigma)\right\} \left\{(1-\beta)\tilde{\tau}^2 + \sqrt{2}\, \alpha (w_1^0 - \sigma)\, \omega^2 - (1+\beta) \sqrt{\tilde{\tau}^2 - d^2 \omega^2}\, \tilde{\tau}\right\}, \end{aligned} \tag{4. 3}$$

where $\mathrm{Im}\,(-\sqrt{\ }) > 0$ for $\gamma > 0$.

Here we remark that if the stability function $a_1+a_2d^{-2}>0$, the zeros of Lopatinskii determinant occur at (η, ω) contained in the projection to the hyperplane $\{\lambda=0\}$ of the interior of the normal cone. In this case the problem is not L^2-well posed. ([12]). Therefore we need only to treat the case where $a_1+a_2d^{-2}=0$. But for the case where $a_1+a_2d^{-2}\geqq 0$ we examine the vanishing order of reflection coefficients in order to clarify the degnerate order of well posedness of our problem.

To do so, we first show that the following two functions are equivalent in a neighborhood of $(\tilde{\tau}_0, \omega_0)$ i. e.,

$$\widetilde{b_{12}}\left|\hat{B}(e_1^+, e_2^+)\right| \sim |\lambda_2^+ - \lambda_2^-| \,. \tag{4.4}$$

we let $(\tilde{\tau}, \omega)$ belong to a neighborhood of $(\tilde{\tau}_0, \omega_0)$ for which $L(\tilde{\tau}_0, \omega_0)=0$. If $\tilde{\tau}_0=0$, (2.6) implies $L(\tilde{\tau}_0, \omega_0)=-a_2\omega_0^2$ and hence $\omega_0=0$, which is contrary to our assumption $(\tau_0, \omega_0)\neq 0$. By the same way we see that $\omega_0\neq 0$ and we need only to show:

$$\sqrt{2}\,\alpha+(\beta+1)\,c^0+(\beta-1)\,(w_1^0-\sigma)\neq 0\,.$$

We divide this by $\beta+1>0$, then from (2.6) it becomes

$$-a_2\,(w_1^0-\sigma)^{-1}+c^0+a_1(w_1^0-\sigma)\,.$$

Since we can rewrite a_1 and a_2 as follows:

$$a_1=-\frac{1}{2}\left\{(c^0)^2+(w_1^0-\sigma)^2\right\}(c^0)^{-1}(w_1^0-\sigma)^{-1}\,,$$

$$a_2=\frac{1}{2}[P]\left([\rho]\,c^0\right)^{-1}d^2(w_1^0-\sigma)^{-1}\,,$$

where $[\rho]=\rho^{0+}-\rho^{0-}$, we need only to show that

$$(c^0)\neq\frac{1}{2}\left\{\frac{(c^0)^2+(w_1^0-\sigma)^2}{c^0}+\frac{[P]}{[\rho]\,c^0}\,\frac{(c^0)^2-(w_1^0-\sigma)^2}{(w_1^0-\sigma)^2}\right\}.$$

Using the fact that

$$\frac{[P]}{[\rho]}=\frac{\rho^{+0}}{\rho^{-0}}(w_1^0-\sigma)^2\,,$$

we reduce our problem to show:

$$(c^0)^2\neq\frac{1}{2}\left(1+\frac{\rho^{0+}}{\rho^{0-}}\right)(c^0)^2+\frac{1}{2}\left(1-\frac{\rho^{0+}}{\rho^{0-}}\right)(w_1^0-\sigma)^2\,.$$

From (1.2) and (1.3) it implies that $\rho^{0+}\neq\rho^{0-}$ and by (1.3) we have that $(c^0)^2>(w_1^0-\sigma)^2$. Hence the above inequality is valid.

Next we remark that

$$\tilde{\tau}+\lambda_2^-(w_1^0-\sigma)\neq 0\,.$$

For, if $\tilde{\tau}+\lambda_2^-(w_1^0-\sigma)=0$, then from the relation $(c^0)^2=d^2+(w_1^0-\sigma)^2$ it follows that $\tilde{\tau}=-i|\omega|(w_1^0-\sigma)$. But $\mathrm{Im}\,\lambda_2^-<0$ and hence $\tilde{\tau}+\lambda_2^-(w_1^0-\sigma)\neq 0$, which is a contradiction. Thus we have that

$$(4.5)\qquad \widetilde{b_{22}}\cdot\left|\tilde{B}(e_1^+, e_2^+)\right|\sim a_1\tilde{\tau}^2+a_2\omega^2+\tilde{\tau}\sqrt{\tilde{\tau}^2-d^2\omega^2}\,.$$

Let $a_1+a_2d^{-2}=0$, then for the point $(\tilde{\tau}_0, \omega_0)$ such that $\tilde{\tau}_0^2=d^2\omega_0^2$ Lopatinskii determinant $L(\tau_0, \omega_0)=0$. We let $\tilde{\tau}\pm d\omega_0=\zeta$ for $\tilde{\tau}_0=\mp d\omega_0$ respectively. Then from (4.4) and (4.5) it follows that

$$|\widetilde{b_{12}}|\sim\frac{k_0\sqrt{\zeta}}{k_1\sqrt{\zeta}+k_2\zeta}\,,$$

$$|\widetilde{b_{22}}|\sim\frac{k_1\sqrt{\zeta}+k_2'\zeta}{k_1\sqrt{\zeta}+k_2\zeta}\,,$$

where k_0 and $k_1\neq 0$. Therefore we have that both of $|\widetilde{b_{12}}|$ and $|\widetilde{b_{22}}|$ are bounded and do not vanish. Since $|\mathrm{Im}\,\lambda_1^+|\sim\gamma$, $\gamma\sim(\mathrm{Re}\sqrt{\zeta})\cdot(\mathrm{Im}\sqrt{\zeta})$ and $\lambda_2^- - \lambda_2^+\sim\sqrt{\zeta}$, we see that the first term of the right hand side in (4.2) $\sim\sqrt{\zeta}(\mathrm{Re}\sqrt{\zeta})^{-\frac{1}{2}}$, which tends to zero when $\zeta=-i\gamma$. Furthermore the second term $\sim\sqrt{\zeta}(\mathrm{Re}\sqrt{\zeta})^{-1}$, which is bounded from below by positive constant. Thus we have that (4.2) are valid for $\widetilde{b_{22}}$, but not for $\widetilde{b_{12}}$.

Next, we let $a_1+a_2d^{-2}>0$. Then also from (4.4), (4.5) and the simple multiplicity of the zeros of L it follows that both of $\widetilde{b_{12}}$ and $\widetilde{b_{22}}$ are $0(\gamma^{-1})$. But in this case terms of the right hand side of (4.2) are bounded, since both of λ_1^+ and λ_2^+ are simple. Hence we see that (4.2) are not valid and the problem is L^2-well posed if and only if it is uniformly stable.

Finally we remark that for some $c>0$

$$|\widetilde{b_{12}}|\leqq c\gamma^{-2}|\mathrm{Im}\,\lambda_1^+|^{\frac{1}{2}}\,|\mathrm{Im}\,\lambda_2^+|^{\frac{1}{2}}\,|\lambda_2^- - \lambda_2^+|\,,$$

$$|\widetilde{b_{22}}|\leqq c\gamma^{-2}|\mathrm{Im}\,\lambda_2^+|^{\frac{1}{2}}\,|\mathrm{Im}\,\lambda_2^-|^{\frac{1}{2}}\,|\lambda_2^- - \lambda_2^+|$$

even if $a_1+a_2d^{-2}\geqq 0$. Therefore the linearized problem is always L^2-well posed with decreasing order 1. Note that the plane $\{x_1=0\}$ is non-characteristic for operators (2.3) and (2.4). Hence for a given vector $F(t, x)\in H_0^{s+1}(\{t>0, x_1>0\})$, there exists a unique solution U to problem (2.3) with the right hand side F and (2.4) with homogeneous boundary conditions such that $\mathrm{supp}\,(U)\cap\{t<0\}=\emptyset$, and for some c_T

$$||U||_{s,\{0<t<T,x_1>0\}}\leqq c_T||F||_{s+1,\{0<t<T,x_1>0\}}\,.$$

([1], [10]). Therefore we have: for a given initial data $U(0, x) \in H_0^{s+1}(\{x_1>0\})$ there is a unique solution $U(t, x) \in H^{s-1}(\{t>0, x_1>0\})$ to the problem (2. 3) and (2. 4) such that

$$||U||_{s-1,\{0<t<T,x_1>0\}} \leqq c_T ||U(0, x)||_{s+1,\{x_1>0\}} .$$

5. Conclusion and an example.

1) From Theorem we have that if the linearized problem with respect to $\{U^{0\pm}, \sigma\}$ is only weakly stable, the stability condition (ii) is not satisfied for any short time, even if (i) is valid until some time. Though here we deal mainly with piecewise smooth shock front solutions, this fact will correspond to the instability in Fluid mechanics, ([5], [14]). However, it is not known even whether there exists an other such shock front solution with the same initial data as $\{U^{0\pm}, \sigma t\}$, when the linearized problem with respect to $\{U^{0\pm}, \sigma\}$ is only weakly stable.

The same treatment as above is applicable to 3-dimensional problems.

2) Example.

The stability of the linearized problem with respect to $\{U^{0\pm}, \sigma\}$ depends only on $\{\rho^{0\pm}\}$ under the conditions (1. 2) and (1. 3). That is, for 1-shock front it is uniformly stable if and only if

$$P(\rho^{0+}) - P(\rho^{0-}) < P_\rho(\rho^{0+})\, \rho^{0+} .$$

([7]).

We let

$$P(\rho) = 1 + \left(1 - \frac{1}{\rho}\right)\left(1 + \varepsilon(\rho - 1)\right)^2 \qquad \text{for} \quad \rho > 1 \text{ and}$$

for *some positive* $\varepsilon \ll 1$. ([9]).

Since $P_\rho = (1+\varepsilon(\rho-1))\, \rho^{-2}\{1+\varepsilon(\rho-1)(1+2\rho)\}$ and

$$P_{\rho\rho} = 2\rho^{-3}\left(\rho^3\varepsilon^2 - (1-\varepsilon)^2\right),$$

setting $\rho_0 = ((1-\varepsilon)\, \varepsilon^{-1})^{\frac{2}{3}}$, we have that

$$P_\rho > 0 \qquad \text{for } \rho > 1,\ P_{\rho\rho} < 0 \qquad \text{for } \rho \in (1, \rho_0),$$

$$P_{\rho\rho}(\rho_0) = 0 \text{ and } P_{\rho\rho} > 0 \qquad \text{for } \rho > \rho_0 .$$

If $\rho^+ > \rho^-$, for any w_1^+ and $w_2^+ = w_2^-$ we can find w_1^- and σ such that $U^+ = {}^t(w_1^+, w_2^+, P(\rho^+))$, $U^- = {}^t(w_1^-, w_2^-, P(\rho^-))$ and σ satisfy (1. 2) and (1. 3). In fact the 1-shock is genuinely nonlinear. Now the maximum of $P(\rho) - P_\rho(\rho) \cdot \rho$ occus at ρ_0. We see that for sufficiently small ε

$$P(\rho_0)-P_\rho(\rho_0)\cdot\rho_0>1\,.$$

We let $\rho^{0+}=\rho^0$. Then there exists ρ_1 such that $1<\rho_1<\rho_0$, the linearized problem is uniformly stable for $\{\rho^{0+}, \rho^{0-}\}$ provided: $\rho^{0-}\in(\rho_1, \rho_0)$, and is only weakly stable for $\{\rho^{0+}, \rho^{0-}\}$ provided: $1<\rho^{0-}\leqq\rho_1$.

References

[1] R. AGEMI-T. SHIROTA: On necessary and sufficient conditions for L^2-well-posedness of mixed problems for hyperbolic equations I, II, J. Fac. Sci. Hokkaido Univ., Vol. 21, No. 2, 133–151 (1970) and Vol. 22, No. 3, 4, 137–149 (1972).

[2] J. J. ERPENBECK: Stability of steady-state equilibrium detonations, The physics of Fluids, Vol. 5, No. 5, 604–614 (1962).

[3] J. J. ERPENBECK: Stability of step shocks, The Physics of Fluids, Vol. 5, No. 10, 1181–1187 (1962).

[4] L. HÖRMANDER: Pseudo-differential operators and non-elliptic boundary problems, Ann. Math., Vol. 83, 130–209 (1966).

[5] L. D. LANDAU-E. M. LIFSHITZ: Fluid Mechanics, Course of Theoretical Physics, Vol. 6, Pergamon Press. 1966.

[6] P. D. LAX: Hyperbolic systems of conservation laws, II, Comm. pure Appl. Math., Vol. 10, 537–556 (1957).

[7] A. MAJDA: The stability of multi-dimensional shock fronts —a new problem for linear hyperbolic eguations, (to appear).

[8] A. MAJDA: The existence of multi-dimensional shock fronts, (to appear).

[9] V. P. MASLOV: Propagation of shock waves in an isentropic nonviscous gas, Sovrem. Probl. Math., Vol. 8, 199–271 (1977). (J. Soviet Mate., Vol. 13, No. 1, 119–162 (1980).)

[10] T. OKUBO-T. SHIROTA: On structures of L^2-well posed mixed problems for hyperbolic systems of first order, Hokkaido Math. J., Vol. 4, 82–158 (1975).

[11] J. RAUCH: Energy inequalities for hyperbolic initial boundary value problems, Thesis, New York Univ., February, (1981).

[12] T. SHIROTA: On the propagation speed of hyperbolic operator with mixed boundary conditions, J. Fac. Sci. Hokkaido Univ., Ser. I, Vol. 22, 25–31 (1972).

[13] K. S. ECKHOFF-L. STORESLETTEN: A note on the stability of steady inviscide helical gas flows, J. Fluid Mech., Vol. 89, part 3, 401–411 (1978).

[14] K. S. ECHKHOFF: On stability of symmetric hyperbsolic sytems, 1, J. Diff. Eq., Vol. 40, 94–115 (1981).

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