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Ideals of hereditary noetherian prime rings

Dedicated to Professor Kentaro Murata
on his 60th birthday

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Introduction. Let R be a hereditary noetherian prime ring (abbr. HNP ring), and let I be an ideal of the form $M_1\cap\cdots\cap M_k$, where $M_1, \cdots, M_k$ are distinct idempotent maximal ideals of R with $O_r(M_1)=O_l(M_2), \cdots, O_r(M_{k-1})=O_l(M_k)$. In the study of ideals of HNP rings, it is important to consider such ideals (cf. [2]). When $M_1, \cdots, M_k$ form a cycle (i. e., moreover $O_r(M_k)=O_l(M_1)$ holds), I is an invertible ideal and the properties of such ideals were broadly studied in [2, 6]. In this paper, we present some properties of the ideal I when $M_1, \cdots, M_k$ form an open cycle (i. e., moreover $O_r(M_k)\neq O_l(M_1)$ holds (Theorem 1. 3). We also give the structure of an eventually idempotent ideal (Theorem 1. 4), and minimal idempotent ideals provided R has finitely many idempotent maximal ideals (Theorem 1. 5). We consider in section 2 an idealizer C of an HNP ring R and completely determine all maximal ideals of C and their relations stated by their maximal right (left) orders (Proposition 2. 2, Theorems 2. 4-6). By this we can give an example of an HNP ring which has finitely arbitrary many strictly open cycles of 'arbitrary size' (Corollary 2.8).

A part of Theorem 1. 3 has been independently obtained by S. Singh [8], however, inasmuch as our proof is not only different from his but also interesting itself, we shall present it in our context.

Throughout this paper, R is an HNP ring which is not artinian and Q is its maximal quotient ring. For submodules A, B of Q, we put $A^{\cdot}.B=\{q\in Q;\ Aq\subset B\}$, $B.{}^{\cdot}A=\{q\in Q;\ qA\subset B\}$, $O_r(A)=\{q\in Q;\ Aq\subset A\}$, and $O_l(A)=\{q\in Q;\ qA\subset A\}$. An ideal I of R is *invertible* (resp. *idempotent*) if $I(I^{\cdot}.R)=R=(R.{}^{\cdot}I)I$ (resp. $I=I^2$). As concerns the properties of HNP rings, the reader is referred to [2, 6, 7].

1. Idempotent ideals. A finite set of distinct idempotent maximal ideals $M_1, \cdots, M_k$ of an HNP ring R is called an *open cycle* (resp. *cycle*) if $O_r(M_k)\neq O_l(M_1)$ (resp. $O_r(M_k)=O_l(M_1)$) and $O_r(M_i)=O_l(M_{i+1})$ for $i=1, \cdots, k-1$. An open cycle $\{M_1, \cdots, M_k\}$ is *right* (resp. *left*) *strictly open* if $O_r(M_k)$

$\neq O_l(M)$ (resp. $O_r(M) \neq O_l(M_1)$) for any idempotent maximal ideal M. A right and left strictly open cycle is said to be a strictly open cycle. We begin with the following lemma in which one will see the importance of strictly open cycles.

LEMMA 1.1. *Each idempotent maximal ideal of R belongs to a unique cycle or else a unique strictly open cycle. Cycles and strictly open cycles either coincide or are disjoint each other.*

PROOF. It follows from [6, Theorem 11 and Corollary 21] that each idempotent maximal ideal belongs to a cycle or a strictly open cycle. Let M, N be idempotent maximal ideals of R. Then $M = R.\cdot(R.\cdot N)$ iff $O_r(M) = O_l(N)$ iff $(M\cdot.R)\cdot.R = N$. Therefore, the uniqueness and the latter statement hold.

LEMMA 1.2. (Robson [7]). *Let $M_1, \cdots, M_k$ be distinct idempotent maximal ideals of R. Then $O_r(M_i) \neq O_l(M_j)$ for any $i, j \in \{1, \cdots, k\}$ iff $M_1 \cap \cdots \cap M_k$ is idempotent iff $M_1 \cap \cdots \cap M_k = M_{\sigma(1)} \cdots M_{\sigma(k)}$ for any permutation σ of $\{1, \cdots, k\}$.*

PROOF. This follows from [7, Corollary 5.5].

In the following theorem we give some properties of an open cycle which are available to study idempotent ideals of HNP rings.

THEOREM 1.3. *Let $\{M_1, \cdots, M_k\}$ $(k \geq 1)$ be an open cycle of idempotent maximal ideals of R, and let $I = M_1 \cap \cdots \cap M_k$. Then*

(1) $I(I\cdot.R) = M_1$ *and* $(R.\cdot I)I = M_k$.

(2) $I = M_1 \cdots M_k$.

(3) $IM_i = M_{i+1}I$ *for* $i = 1, \cdots, k-1$.

(4) $I^i(I\cdot.R)^i = M_i \cdots M_1$ *and* $(R.\cdot I)^i I^i = M_k \cdots M_{k-i+1}$ *for* $i = 1, \cdots, k$. *In special,* $I^k = I^k(I\cdot.R)^k = (R.\cdot I)^k I^k = M_k \cdots M_1$ *is idempotent.*

(5) $I \supsetneqq I^2 \supsetneqq \cdots \supsetneqq I^k = I^{k+1} = \cdots$.

PROOF. (1) Since I is not invertible by [2, Proposition 2.4], $I(I\cdot.R) \neq R$ and $(R.\cdot I)I \neq R$ from the proof of [7, Lemma 5.1]. When $k=1$, the assertion follows from [2, Lemma 1.5]. Let $k \geq 2$. For $i = 2, \cdots, k$, let $S = O_l(M_i) = O_r(M_{i-1})$ and $A = M_1 \cdots M_{i-2} M_{i+1} \cdots M_k$. Since $M_i S = S$ and $M_{i-1} S = M_{i-1}$ by [2, Lemma 1.5], we have

$$I(I\cdot.R) \supset IS \supset AM_{i-1}M_iS = AM_{i-1}.$$

Therefore $I(I\cdot.R) \not\subset M_i$ for any $i = 2, \cdots, k$, so that $I(I\cdot.R) = M_1$. Similarly we have $(R.\cdot I)I = M_k$. (2) If $k=1$, the assertion clearly holds. Let $k \geq 2$ and $A = M_2 \cap \cdots \cap M_k$. Then by (1), $I = I(I\cdot.R)I = M_1 I \subset M_1 A \subset M_1 \cap A = I$.

Thus $I=M_1A$. Since $\{M_2, \cdots, M_k\}$ is an open cycle, we have $I=M_1\cdots M_k$ by induction on k. (3) For $k=2$, $I=M_1M_2\supset M_2M_1$ from (2). Hence $IM_1=M_1M_2M_1\supset M_2M_1\supset IM_1$ and $M_2I=M_2M_1M_2\supset M_2M_1\supset M_2I$. Thus $IM_1=M_1M_2M_1=M_2M_1=M_2M_1M_2=M_2I$. Let k be arbitrary. Then we have $M_jM_{j+1}M_j=M_{j+1}M_jM_{j+1}$ for $j=1, \cdots, k-1$. Since $O_r(M_s)\neq O_l(M_t)$ for $\{s, t\}=\{j, i+1\}$ $(1\leq j\leq i-1)$ or $\{s, t\}=\{j, i\}$ $(i+2\leq j\leq k)$, $M_{i+1}M_j=M_jM_{i+1}$ $(1\leq j\leq i-1)$ and $M_iM_j=M_jM_i (i+2\leq j\leq k)$ by Lemma 1.2. Thus we conclude that $M_{i+1}I=M_{i+1}M_1\cdots M_k=M_1\cdots M_{i-1}M_{i+1}M_iM_{i+1}\cdots M_k=M_1\cdots M_iM_{i+1}M_iM_{i+2}\cdots M_k=M_1\cdots M_kM_i=IM_i$. (4) We have already seen that $I(I^{\cdot}.R)=M_1$. Thus $I^2(I^{\cdot}.R)^2=IM_1(I^{\cdot}.R)=M_2I(I^{\cdot}.R)=M_2M_1$ by (3). By the similar way we have $I^i(I^{\cdot}.R)^i=M_i\cdots M_1$ and $(R.^{\cdot}I)^iI^i=M_k\cdots M_{k-i+1}$ for $i=1, \cdots, k$. Since I^k is idempotent from [2, Proposition 4.3], $I^k\subset I^k(I^{\cdot}.R)^k\subset I^k(I^{k\cdot}.R)=I^k$. Therefore, $I^k=I^k(I^{\cdot}.R)^k=(R.^{\cdot}I)^kI^k=M_k\cdots M_1$ is idempotent. (5) Since I^k is idempotent, $I^k=I^{k+1}=\cdots$. Assume that $I^i=I^{i+1}$ for some $1\leq i\leq k-1$. Then, by (4), $M_k\supset IM_i\cdots M_1=I^{i+1}(I^{\cdot}.R)^i=I^i(I^{\cdot}.R)^i=M_i\cdots M_1$ and so $M_k=M_j$ for some $1\leq j\leq i$, a contradiction.

Remark. The fact that $IM_i=M_{i+1}I$ is interesting in view of [6, Theorem 14].

Let $M_1, \cdots, M_k, N_1, \cdots, N_l$ be distinct idempotent maximal ideals of R. Then we call $M_1, \cdots, M_k$ and $N_1, \cdots, N_l$ to be separated if $O_r(M_i)\neq O_l(N_j)$ and $O_r(N_j)\neq O_l(M_i)$ for all $i=1, \cdots, k$ and $j=1, \cdots, l$.

It is shown in [2] that every ideal of R is the product XA with X an invertible ideal and A an eventually idempotent ideal, and that every eventually idempotent ideal A satisfies that $A^k=(M_1\cap\cdots\cap M_k)^k$ is idempotent where $M_1, \cdots, M_k$ are the maximal ideals containing A. Thus we state the structure of eventually idempotent ideals of the form $M_1\cap\cdots\cap M_k$.

Theorem 1.4. *Let $M_1, \cdots, M_k$ be distinct idempotent maximal ideals of R such that $I=M_1\cap\cdots\cap M_k$ is not contained in any invertible ideal. Then $I=I_1\cap\cdots\cap I_n=I_1\cdots I_n$ and $I_iI_j=I_jI_i$ for all $i, j\in\{1, \cdots, n\}$, where each $I_i=M_{i1}\cap\cdots\cap M_{i,m(i)}$ such that all $M_{ij}\in\{M_1, \cdots, M_k\}$ and that $\{M_{i1}, \cdots, M_{i,m(i)}\}$ $(i=1, \cdots, n)$ are open cycles which are disjoint and separated each other.*

Proof. If I is idempotent, then the assertion clearly holds by Lemma 1.2. Let I be not idempotent. Then $k\geq 2$. By the proof of Lemma 1.1 we have $I=I_1\cap\cdots\cap I_n$ such that each $I_i (i=1, \cdots, n)$ satisfies the condition of the theorem. The fact that $I_iI_j=I_jI_i$ follows from Lemma 1.2 and Theorem 1.3 (2). Thus we only prove $I=I_1\cdots I_n$. It holds by the same way as in the proof of Theorem 1.3 (1) that $I(I^{\cdot}.R)\not\subset M_{ih}$ for any $i=1$,

$\cdots, n$ and $h=2, \cdots, m(i)$. Thus the maximal ideals that can contain $I(I^{\cdot}.R)$ are $M_{11}, \cdots, M_{n1}$. Renumberring if necessary, we put $I(I^{\cdot}.R)=M_{11}\cap\cdots\cap M_{a1}$ for some $1\leq a\leq n$. We can conclude by the same way as in the proof of [2, Proposition 4. 3] that $I=M_{11}\cdots M_{a1}B$, where B is the intersection of all M_i's except $M_{11}, \cdots, M_{a1}$. By induction on k, we have $B=J_1\cdots J_a I_{a+1}\cdots I_n$, where $J_i=M_{i2}\cap\cdots\cap M_{i,m(i)}$ for $i=1, \cdots, a$. Hence $I=I_1\cdots I_n$ by Lemma 1. 2 and Theorem 1. 3 (2).

Next we state the structure of minimal idempotent ideals of an HNP ring with finitely many idempotent maximal ideals (cf. [3, Proposition 9]). It follows from Lemma 1. 1 that the idempotent maximal ideals of R are separated into cycles and strictly open cycles.

THEOREM 1. 5. *Let R have finitely many idempotent maximal ideals, say $\{M_{i1}, \cdots, M_{i,n(i)}\}$ $(1\leq i\leq k)$ are cycles and $\{P_{h1}, \cdots, P_{h,m(h)}\}$ $(1\leq h\leq l)$ are strictly open cycles. Then the minimal idempotent ideals are $I(j(1), \cdots, j(k))=I_{1,j(1)}\cdots I_{k,j(k)}J_1\cdots J_l$ where $I_{i,j(i)}=M_{i,j(i)-1}\cdots M_{i1}M_{1,n(i)}\cdots M_{i,j(i)+1}(1\leq i\leq k,$ $1\leq j(i)\leq n(i)$; replace $j(i)-1(j(i)+1)$ by $n(i)(1)$ when $j(i)=1(j(i)=n(i)))$ and $J_h=P_{h,m(h)}\cdots P_{h1}$ $(1\leq h\leq l)$. Therefore, the number of minimal idempotent ideals is $n(1)\times\cdots\times n(k)$.*

PROOF. Any $I_{i,j(i)}(1\leq i\leq k,\ 1\leq j(i)\leq n(i))$ and $J_h(1\leq h\leq l)$ are idempotent by Theorem 1. 3. Hence each $I(j(1), \cdots, j(k))$ is idempotent by Lemma 1. 2. The maximal ideals containing $I(j(1), \cdots, j(k))$ are all P_{hg}'s and M_{ij}'s other than $M_{1,j(1)}, \cdots, M_{k,j(k)}$. Assume that there is an idempotent ideal I which is strictly contained in $I(j(1), \cdots, j(k))$. Then, by [2, Corollary 4. 6], I has to be contained in $M_{i,j(i)}$ for some $1\leq i\leq k$. Hence I is contained in an invertible ideal $M_{i1}\cap\cdots\cap M_{i,n(i)}$. This contradicts to [2, Lemma 4. 1], and then each $I(j(1), \cdots, j(k))$ is a minimal idempotent ideal. The converse follows from Theorem 1. 4 and [2, Corollary 4. 6].

2. Idealizers and maximal ideals. Let R be an HNP ring, and let A be a semimaximal right ideal (i. e., an intersection of finitely many maximal right ideals) of R. Then the subring $I_R(A)=\{r\in R;\ rA\subset A\}$ (called the idealizer of A in R) is also an HNP ring, and which has very connected structure with R (cf. [7]). Idealizer subrings are useful to construct more complicated examples from a given one (cf. [5, 6, 7]). In this section, we shall decide the maximal ideals of $I_R(A)$ and the relations determined by the right (left) orders of them (Proposition 2. 2, Theorems 2. 4, 2. 5, and 2. 6). From this, one will see that the indicated example for [5, p. 113 (b)] is inadequate, however, a desired example will be given after Corollary 2. 7.

By [7, Proposition 1. 7] we can assume that $RA=R$. Let $K_1, \cdots, K_n$

be maximal right ideals of R such that $A=\bigcap_{i=1}^{n}K_i$. By [4, Lemma 4.18] we can assume that $R/A\cong\bigoplus_{i=1}^{n}R/K_i$ canonically. Furthermore, let $A_1=K_1\cap\cdots\cap K_{i_1},\cdots,A_k=K_{i_{k-1}+1}\cap\cdots\cap K_{i_k}$, where $1<i_1<\cdots<i_k=n$, such that R/A_j is isomorphic to a homogeneous component of R/A $(j=1,\cdots,k)$. Put $C=I_R(A)$, $N_j=A_j\cap C$ an ideal of C, and $U_j=R/K_{i_j}$ a simple right R-module $(j=1,\cdots,k)$. Then it follows from [7, Theorem 1.3] that U_j is a right C-module of length 2 and that $0\to S_j\to U_j\to T_j\to 0$ is a nonsplit exact sequence where $S_j\cong(C+K_{i_j})/K_{i_j}$ and $T_j\cong R/(C+K_{i_j})$. Note that $S_j\otimes_C R=U_j$ and $T_j\otimes_C R=0$, since ${}_CR$ is flat.

LEMMA 2.1. *$(T_j)_C$ is unfaithful iff $(U_j)_R$ is unfaithful.*

PROOF. If U_{jR} is unfaithful, $\operatorname{ann}_C U_j=\operatorname{ann}_R U_j\subset C\neq 0$ and $T_j(\operatorname{ann}_C U_j)=0$. Conversely, if T_{jC} is unfaithful, $V=R(\operatorname{ann}_C T_j)A$ is a nonzero ideal of R and $U_jB=0$.

Let $0\leq l\leq k$, and suppose that $U_1,\cdots,U_l$ are unfaithful and that $U_{l+1},\cdots,U_k$ are faithful. Note that if $l=0$ resp, $l=k$ then all U_j's are faithful resp. unfaithful. For $j=1,\cdots,l$, put $M_j=\operatorname{ann}_R U_j$ and $L_j=\operatorname{ann}_C T_j$. Set $\mathcal{M}=\{M;\ M$ is a maximal ideal of R such that $\operatorname{Hom}_R(R/A,R/M)=0\}$ and $\mathcal{M}\cap C=\{M\cap C;\ M\in\mathcal{M}\}$. Then

PROPOSITION 2.2. *The set of maximal ideals of C is equal to $\{N_1,\cdots,N_k,L_1,\cdots,L_l\}\cup(\mathcal{M}\cap C)$.*

PROOF. It follows from [7, Theorem 1.3 (a)] that each element of $\mathcal{M}\cap C$ is a maximal ideal of C. By [7, Proposition 1.1] $C/A\cong\operatorname{End}_R(R/A)$ is a semisimple artinian ring. For $j=1,\cdots,k$, since $S_j\cong C/K_{i_j}\cap C$, C/N_j is isomorphic to a direct sum of finitely many copies of S_j. Since $S_j=\operatorname{soc}U_j$ and $S_j\otimes_C R\cong U_j$, $S_j\not\cong S_h$ iff $U_j\not\cong U_k$, i.e., $j\neq h$. Thus C/N_j is isomorphic to a homogeneous component of C/A. Hence C/N_j is a simple artinian ring and so N_j is a maximal ideal of C. For $j=1,\cdots,l$, since T_j is an unfaithful simple right C-module by Lemma 2.1, L_j is a maximal ideal of C. Conversely, let L be a maximal ideal of C. Then it follows from [7, Corollary 2.4] that $L\in\mathcal{M}\cap C$ or $L=L_j$ or $L=N_j$. This completes the proof.

In the rest of this section, we shall fix the above notation except the following lemma.

LEMMA 2.3. *Let C be a ring, and let $0\to U\to W\to V\to 0$ be an exact sequence of right C-modules such that W_C is uniserial. Then*

(1) *For an epimorphism α: $U_C\to T_C$, there is an exact sequence $0\to T\to X\to V\to 0$ such that X_C is uniserial.*

(2) *For a monomorphism* $\gamma: S_C \to V_C$, *there is an exact sequence* $0 \to U \to Y \to S \to 0$ *such that* Y_C *is uniserial.*

PROOF. (1) Consider the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & W & \longrightarrow & V & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \| & & \\ 0 & \longrightarrow & T & \longrightarrow & X & \longrightarrow & V & \longrightarrow & 0\,. \end{array}$$

Since α is an epimorphism, so is β. Thus X_C is uniserial. Using a pullback diagram (2) can be shown dually.

A maximal ideal of an HNP ring is either invertible or idempotent, further, idempotent one belongs to either a cycle or a strictly open cycle (Lemma 1.1). In order to determine whether an idempotent maximal ideal M belongs to a cycle or a strictly open cycle we need to calculate $O_r(M)$ and $O_l(M)$. Since $\mathrm{Ext}_C{}^1(T_j, S_j) \neq 0$, it follows from [6, Theorem 8] that

$$(*) \qquad O_r(N_j) = O_l(L_j) \text{ for all } j=1, \cdots, l\,.$$

By the following the remaining relations among maximal ideals of $C=I_R(A)$ will be completely determined. Firstly, we study maximal ideals in $\mathcal{M} \cap C$.

THEOREM 2.4. *Let* $M \in \mathcal{M}$, *and* U *be a simple right* R*-module such that* $UM=0$. *Then*

(1) M *is invertible iff* $M \cap C$ *is invertible.*

(2) *If* M *is idempotent with* $O_r(M) = O_l(N)$ *for some* $N \in \mathcal{M}$, *then* $O_r(M \cap C) = O_l(N \cap C)$.

(3) *If* M *is idempotent with* $O_r(M) \neq O_l(N)$ *for all idempotent maximal ideals* N *of* R *and* $\mathrm{Ext}_R{}^1(U_j, U)=0$ *for all* $j=1, \cdots, k$, *then* $O_r(M \cap C) \neq O_l(L)$ *for all idempotent maximal ideals* L *of* C.

(4) *If* M *is idempotent with* $O_r(N) \neq O_l(M)$ *for all idempotent maximal ideals* N *of* R, *then* $O_r(L) \neq O_l(M \cap C)$ *for all idempotent maximal ideals* L *of* C.

PROOF. (1) Since $M=(M \cap C)R=R(M \cap C)$, if $M \cap C$ is invertible (or idempotent), so is M. Thus by [2, Proposition 2.2] M is invertible iff $M \cap C$ is invertible. (2) Let T be a simple right R-module such that $TN=0$. Then U and T are also simple right C-modules by [7, Theorem 1.3]. Since $O_r(M)=O_l(N)$ and $\mathrm{Ext}_R{}^1(T, U) \cong \mathrm{Ext}_C{}^1(T, U)$, $O_r(M \cap C)=O_l(N \cap C)$ by [6, Theorem 8]. (3) It follows from [6, Corollary 9(b)] that there is a faithful simple right R-module V such that $\mathrm{Ext}_R{}^1(V, U) \neq 0$. By the assumption, $V \neq U_j$ for all $j=1, \cdots, k$. Hence $\mathrm{Hom}_R(R/A, V)=0$. Thus V is a faithful simple right C-module and $\mathrm{Ext}_C{}^1(V, U_j) \cong \mathrm{Ext}_R{}^1(V, U_j) \neq 0$. Therefore, the assertion follows from [6, Corollary 9]. (4) Assume that $O_r(L)=O_l(M\cap C)$ for some idempotent maximal ideal L of C, and let S be a simple right C-module with $SL=0$, so that $\mathrm{Ext}_C{}^1(U, S)\neq 0$. If $S\cong S_j$ for some $1\leq j\leq k$, then by [6, Theorems 7, 8] $U_C\cong T_j$ and so $U\cong U\otimes_C R\cong T_j\otimes_C R=0$ which is a contradiction. Assume that $S\cong T_j$ for some $1\leq j\leq k$. Since C is hereditary, the homomorphism $\mathrm{Ext}_C{}^1(U, U_j)\rightarrow\mathrm{Ext}_C{}^1(U, T_j)$ induced by the epimorphism $U_j\rightarrow T_j$ is an epimorphism. So, since $\mathrm{Ext}_C{}^1(U, T_j)\neq 0$, $\mathrm{Ext}_R{}^1(U, U_j)\cong\mathrm{Ext}_C{}^1(U, U_j)\neq 0$. Since U_j is unfaithful from Lemma 2.1, $O_r(M_j)=O_l(M)$, a contradiction. Thus it follows from [7, Corollary 2.4] that $S_C\cong V_C$ for some simple right R-module V with $\mathrm{Hom}_R(R/A, V)=0$. Then V_R is unfaithful and $\mathrm{Ext}_R{}^1(U, V)\cong\mathrm{Ext}_C{}^1(U, V)\neq 0$. Hence $O_r(\mathrm{ann}_R V)=O_l(M)$, a contradiction.

The following two theorems show how the new maximal ideals N_j's and L_j's are connected. For L_j's, it is enough to study $O_r(L_j)$ according to (*).

THEOREM 2.5. *Let j be $1\leq j\leq l$, that is, U_j be unfaithful with $U_jM_j=0$. Then*

(a) *If M_j is invertible, then $\{N_j, L_j\}$ is a cycle of C.*

(b) *If M_j is idempotent with V a simple right R-module such that $\mathrm{Ext}_R{}^1(V, U_j)\neq 0$. Then*

(1) *Suppose that $\mathrm{Hom}_R(R/A, V)=0$. Then*

(i) *If V_R is unfaithful, $O_r(L_j)=O_l(M\cap C)$ for $M=\mathrm{ann}_R V$.*

(ii) *If V_R is faithful, $O_r(L_j)\neq O_l(L)$ for all idempotent maximal ideals L of C.*

(2) *Suppose that $\mathrm{Hom}_R(R/A, V)\neq 0$. Then $O_r(L_j)=O_l(N_h)$ for some $h(1\leq h\leq k)$ and $h\neq j$.*

PROOF. (a) It follows from [6, Theorem 11] that $\mathrm{Ext}_R{}^1(U_j, U_j)\neq 0$. Hence there is a nonsplit exact sequence $0\rightarrow U_j\rightarrow W\rightarrow U_j\rightarrow 0$. Then W_R is uniserial and by [7, Corollary 1.5] W_C is also uniserial. Using Lemma 2.2 we obtain nonsplit exact sequences of right C-modules $0\rightarrow T_j\rightarrow X\rightarrow U_j\rightarrow 0$ and $0\rightarrow T_j\rightarrow Y\rightarrow S_j\rightarrow 0$. Hence $\mathrm{Ext}_C{}^1(S_j, T_j)\neq 0$ and so $O_r(L_j)=O_l(N_j)$. Therefore, $\{N_j, L_j\}$ is a cycle. (b) At first, note that a simple right R-module V with $\mathrm{Ext}_R{}^1(V, U_j)\neq 0$ exists by [6, Theorem 7]. Thus there is a nonsplit exact sequence $0\rightarrow U_j\rightarrow W\rightarrow V\rightarrow 0$. Then W_C is uniserial. Using Lemma 2.3 (1) we obtain an exact sequence of right C-modules $0\rightarrow T_j\rightarrow X\rightarrow V\rightarrow 0$ with X_C uniserial. Hence $\mathrm{Ext}_C{}^1(V, T_j)\neq 0$. Now, for the case (1), since $\mathrm{Hom}_R(R/A, V)=0$, V_C is simple. (i) If V_R is unfaithful, so is V_C and hence $O_r(L_j)=O_l(M\cap C)$ by [6, Theorem 8]. (ii) If V_R is faithful, so is V_C and hence $O_r(L_j)\neq O_l(L)$ for all idempotent maximal ideals L of C by [6, Corollary 9].

For the case (2), since $\mathrm{Hom}_R(R/A, V) \neq 0$, $V \cong U_j$ for some $1 \leq h \leq k$. Since M_j is idempotent and $\mathrm{Ext}_R^1(U_h, U_j) \neq 0$, $U_h \not\cong U_j$, i.e., $h \neq j$. Using Lemma 2.3 (2) we have $\mathrm{Ext}_R^1(S_h, T_j) \neq 0$. Hence $\mathrm{O}_r(L_j) = \mathrm{O}_l(N_h)$ by [6, Theorem 8].

THEOREM 2.6. (1) *If U_j is faithful, that is, $l+1 \leq j \leq k$, then $\mathrm{O}_r(N_j) \neq \mathrm{O}_l(L)$ for all idempotent maximal ideals L of C.*

(2) *Suppose that* $\mathrm{Ext}_R^1(U_j, V) \neq 0$ *for some unfaithful simple right R-module V with $M = \mathrm{ann}_R V$ idempotent. Then*

(i) $\mathrm{O}_r(M \cap C) = \mathrm{O}_l(N_j)$, *whenever* $\mathrm{Hom}_R(R/A, V) = 0$.

(ii) $\mathrm{O}_r(L_h) = \mathrm{O}_l(N_j)$ *for some $1 \leq h \leq l$ and $h \neq j$, whenever* $\mathrm{Hom}_R(R/A, V) \neq 0$.

(3) *Suppose that* $\mathrm{Ext}_R^1(U_j, V) = 0$ *for any unfaithful simple right R-module V. Then $\mathrm{O}_r(L) \neq \mathrm{O}_l(N_j)$ for all idempotent maximal ideals L of C.*

PROOF. (1) Since T_j is faithful by Lemma 2.1 and $\mathrm{Ext}_C^1(T_j, S_j) \neq 0$, the assertion follows from [6, Corollary 9]. (2) Using Lemma 2.3 (1) we have $\mathrm{Ext}_C^1(S_j, V) \neq 0$ by assumption. Thus (i) is proved by the same way as Theorem 2.5 (b) (1) (i), while (ii) by the same way as Theorem 2.5 (b) (2). (3) is proved by the same way as Theorem 2.4 (4).

For an HNP ring R, let $D(R)$ denote the abelian group generated by the maximal invertible ideals of R (see [2, Theorem 2.9]). Then as an easy consequence of the theorems, we have the following

COROLLARY 2.7. $D(R) \cong D(C)$.

We shall give an example of an HNP ring which has exactly three idempotent maximal ideals M_1, M_2, M_3 such that $\{M_1, M_2\}$ is a cycle (see [5, p. 113 (b)]). The indicated one in [5] is an iterated idealizer from a simple HNP ring, and then it has no cycle by Corollary 2.7.

EXAMPLE. Let D be a primitive Dedekind prime ring with a nonzero maximal ideal M (for example see [1, p. 81 (ii) (a)]). Let K and L be maximal right ideals of D such that $K \supset M$ and $(D/L)_D$ is faithful. Let R be the full 2×2 matrix ring over D, and let $K_1 = \begin{pmatrix} K & K \\ D & D \end{pmatrix}$ and $K_2 = \begin{pmatrix} L & L \\ D & D \end{pmatrix}$ be maximal right ideals of R such that $R/K_1(R/K_2)$ is an unfaithful (faithful) simple right R-module. Then $C = I_R(K_1 \cap K_2)$ is an HNP ring, whose idempotent maximal ideals are $K_1 \cap C$, L_1, $K_2 \cap C$, where L_1 is the idempotent maximal ideal of C such that $\mathrm{O}_r(K_1 \cap C) = \mathrm{O}_l(L_1)$. Moreover, $\{K_1 \cap C, L_1\}$ is a cycle and $\{K_2 \cap C\}$ is a strictly open cycle.

COROLLARY 2.8. *There exists an HNP ring which has strictly open cycles $\{M_{i1}, \cdots, M_{i,m(i)}\}$ for arbitrary positive integers k and $m(i)$ $(1 \leq i \leq k)$.*

PROOF. Let R be a simple HNP ring which is not artinian and has infinitely many nonisomorphic faithful simple right R-modules (the existence of such a ring is well-known (cf. [6])). Let $A=K_1\cap\cdots\cap K_k$ be a semimaximal right ideal such that R/K_i's are mutually nonisomorphic faithful simple right R-modules and $C_1=I_R(A)$. Then by Theorem 2.6 (1) $\{K_1\cap C_1\},\cdots,\{K_k\cap C_1\}$ are strictly open cycles of C_1. Assume that $\{M_1,\cdots,M_t\}$ is a strictly open cycle of an HNP ring R' and S_t, T are simple right R'-modules such that $S_tM_t=0$ and $T=R'/K$ is faithful with $\mathrm{Ext}_{R'}{}^1(T,S_t)\neq 0$ (cf. [6, Corollary 9]). Since K is a maximal right ideal, $I_{R'}(K)=D$ is an HNP ring and $\{M_1\cap D,\cdots. M_t\cap D, K\}$ is a strictly open cycle of D by Theorems 2.4, 2.6 (1) and (2) (i), while if $\{N_1,\cdots,N_q\}$ is a strictly open cycle of R' which is different from $\{M_1,\cdots,M_t\}$ then $\{N_1\cap D,\cdots,N_q\cap D\}$ is also a strictly open cycle of D by Theorem 2.4. Hence we can construct the desired HNP ring C as an iterated idealizer, that is, there is a chain of subrings $C=C_s\subset C_{s-1}\subset\cdots\subset C_0=R$ such that C_i is an idealizer of a semimaximal right ideal in C_{i-1} for $i=1,\cdots,s$.

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