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On the nontriviality of cohomology of finite groups

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1. Introduction

In this note we are concerned with the cohomology groups of a finite group G with coefficients in a finitely generated $F[G]$-module, where F is a field of characteristic p dividing the order of G. By an $F[G]$-module we shall always mean a finitely generated $F[G]$-module. Note that if an indecomposable $F[G]$-module U does not belong to the principal block, then the cohomology groups of G with coefficients in U are trivial.

Recently Ogawa [8] has shown the following theorem of great interest. (See also Ogawa [7].)

THEOREM 1. (Ogawa [8], THEOREM 2) *Let G be a finite p-solvable group with a nontrivial abelian Sylow p-subgroup and V be an irreducible $\boldsymbol{F}_p[G]$-module in the principal block, where $\boldsymbol{F}_p$ is the prime field of characteristic p. Then there exists an i $(1 \leq i \leq |G: O_{p',p}(G)|)$ for which $H^{2i}(G, V) \neq 0$.*

Our purpose of this note is to generalize this theorem as follows.

THEOREM 2. *Let G be a finite group, F a field of characteristic p dividing the order of G, and U be an irreducible $F[G]$-module in the principal block. Suppose that U has an abelian vertex D $(\neq 1)$ and a trivial source. Then there exists an i $(1 \leq i \leq |N_G(D): C_G(D)|)$ for which $H^{2i}(G, U) \neq 0$.*

THEOREM 3. *Let G be a finite group with an abelian Sylow p-subgroup P $(\neq 1)$, F a field of characteristic p, and U be an indecomposable $F[G]$-module in the principal block. Suppose that U has P as a vertex and a trivial source. Then there exists an i $(1 \leq i \leq |N_G(P): C_G(P)|)$ for which $H^{2i}(G, U) \neq 0$.*

These theorems will be proven by combining the following Proposition which is a direct generalization of Theorem 1 and a transfer theorem for cohomology groups obtained in Sasaki [10].

PROPOSITION. *Let G be a finite group, F a field of characteristic p*

dividing the order of G, *and* U *be an irreducible* $F[G]$*-module in the principal block. Suppose that* U *has a normal abelian vertex* $(\neq 1)$. *Then there exists an* i $(1 \leq i \leq |G : O_{p',p}(G)|)$ *for which* $H^{2i}(G, U) \neq 0$.

Let G be a finite p-solvable group with $O_{p'}(G)=1$ and suppose that G has an abelian Sylow p-subgroup P. Then P is normal in G so that every irreducible $F[G]$-module has P as a vertex. Therefore THEOREM 1 follows from Proposition.

PROPOSITION will be proven by making use of Ogawa's idea of [8] and modular representation theory of finite groups, and this will be done in section 2. THEOREM 2 and THEOREM 3 will be proven in section 3.

Our notation and terminologies are standard. See Dornhoff [1], Feit [3], Gorenstein [5], and Weiss [11].

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2. Proof of Proposition

Throughout this section let K denote the prime field $\boldsymbol{F}_p$ of characteristic p. Let D be a vertex of U. Then by our assumption D is a normal abelian p-subgroup of G.

STEP 1. *The irreducible* $F[G]$*-module* U *has a trivial source. Also we have* $D=O_p(G)$.

PROOF. The vertex D acts on U trivially, since D is normal in G. Hence U has a trivial source. As $O_p(G)$ is included in the vertices of all irreducible $F[G]$-modules, we have $D=O_p(G)$.

STEP 2. *We may assume that* $O_{p'}(G)=1$.

PROOF. Since U is in the principal block, the subgroup $O_{p'}(G)$ acts on U trivially. As an $F[G/O_{p'}(G)]$-module, U is an irreducible module in the principal block and has $DO_{p'}(G)/O_{p'}(G)$ as a vertex and a trivial source. Furtheremore $H^i(G, U)$ and $H^i(G/O_{p'}(G), U)$ are isomorphic for all $i \geq 1$, since $H^i(O_{p'}(G), U)=0$. Thus we may assume that $O_{p'}(G)=1$.

STEP 3. *The cohomology group* $H^i(G, U)$ *is isomorphic with* $H^i(D, U)^G$ *for all* $i \geq 0$.

PROOF. Since U is D-projective, by Proposition 3. 2 of Dress [2], the restriction map

$$\mathrm{res}_{G,D}: H^i(G, U) \longrightarrow H^i(D, U)$$

is a splitting monomorphism and moreover $\mathrm{Im}\,\mathrm{res}_{G,D}$ consists of all G-stable elements of $H^i(D, U)$. Namely, as D is normal in G, we have

$$H^i(G, U) \simeq \mathrm{Im}\,\mathrm{res}_{G,D} = H^i(D, U)^G .$$

STEP 4. *Let $E=\Omega_1(D)$. Then $(E\otimes_K F)^*=\mathrm{Hom}_F(E\otimes_K F, F)$ is a faithful $F[G/D]$-module.*

PROOF. It will suffice to show that $E^*=\mathrm{Hom}_K(E, K)$ is a faithful $K[G/D]$-module, since $(E\otimes_K F)^*$ is isomorphic with $E^*\otimes_K F$. By Corollary 3.7 of Knörr [6] there exists a Sylow p-subgroup P of G for which $C_P(D) \leq D$. Since D is abelian, we have $C_P(D)=D$ so that D is a Sylow p-subgroup of $C_G(D)$. This implies that $C_G(D)=D$, as $O_{p'}(G)=1$. Let $N=\mathrm{Ker}\,E^*$. Then $N=C_G(E)\geq D$. As D is abelian and $E=\Omega_1(D)$, a p'-element of N/D centralizes D. Thus N/D is a normal p-subgroup of G/D. Hence by Step 1 we have $N=D$, as desired.

STEP 5. *There exists an i $(1\leq i\leq |G:O_p(G)|)$ for which $H^{2i}(G, U)\neq 0$.*

PROOF. By Step 4 and Theorem 1 of Ogawa [8] the irreducible $F[G/D]$-module $U^*=\mathrm{Hom}_F(U, F)$ is isomorphic with a submodule of $S^i(E^*\otimes_K F)$ for some i $(1\leq i\leq |G/D|)$. By Proposition 1 of Ogawa [8] the i-th term $S^i(E^*\otimes_K F)$ is isomorphic with an $F[G/D]$-sumbodule of $H^{2i}(D, F)$. Hence U^* is siomorphic with a submodule of $H^{2i}(D, F)$ as $F[G/D]$-modules. Thus $\mathrm{End}_F(U)$ is isomorphic with a submodule of $H^{2i}(D, U)$, since $\mathrm{End}_F(U)$ and $H^{2i}(D, U)$ are isomorphic with $U^*\otimes_F U$ and $H^{2i}(D, F)\otimes_F U$, respectively. Consequently $H^{2i}(G, U)\simeq H^{2i}(D, U)^G$ has a submodule isomorphic with $\mathrm{End}_{F[G]}(U)\neq 0$. The proof is complete.

REMARK. (1) In [8] Ogawa discussed only in the case of $F=\boldsymbol{F}_p$, the prime field. But the conclusion of Proposition 1 of [8] is also valid for an arbitrary field F. (2) Since $O_p(G/O_{p'}(G))=DO_{p'}(G)/O_{p'}(G)$ and $C_G(D)=DO_{p'}(G)$, we have $O_{p',p}(G)=C_G(D)$.

3. Proof of Theorem 2 and Theorem 3

Let G be a finite group, F a field of characteristic $p$dividing the order of G, and U be an indecomposable $F[G]$-module with a vertex D in the principal block. Then a Green correspondent V of U with respect to $(G, D, N_G(D))$ also lies in the principal block of $N_G(D)$. If the vertex D is abelian and U has a trivial source, then, by Theorem 4 of Sasaki [10], the cohomology groups $H^i(G, U)$ and $H^i(N_G(D), V)$ are isomorphic for all i.

If furthermore the Green correspondent V of U is an irreducible $F[N_G(D)]$-module, then by Proposition there exists an i $(1 \leq i \leq |N_G(D):C_G(D)|)$ for which $H^{2i}(N_G(D), V) \neq 0$. Thus to prove theorems it will suffice to show that V is irreducible.

If U is irreducible, then V is also irreducible by Lemma 2.2 of Okuyama [9]. Theorem 2 is proven.

If the vertex D is a Sylow p-subgroup P of G, then V is an indecomposable $F[N_G(P)/P]$-module so that V is irreducible. Theorem 3 is proven.

REMARK. In [9] Okuyama discussed in the case that the field F is a splitting field for G. But the conclusion of Lemma 2.2 of [9] is also valid for an arbitrary field.

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