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Geometrization of Jet bundles

By Keizo YAMAGUCHI

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Introduction

Let (M, N, p) be a fibred manifold of fibre dimension m. Let $J^{k}(M, N, p)$ be the bundle of k-jets of local sections of (M, N, p). It is classically known (cf. [2], [1]) that the pseudo-group of local diffeomorphisms which preserve the canonical differential system C^{k} on $J^{k}(M, N, p)$ is isomorphic with the pseudo-group of local contact transformations of $J^{1}(M, N, p)$ if $m=1$ and with the pseudo-group of local diffeomorphisms of M if $m\geqq 2$. Thus there is a marked difference in C^{k} between these cases.

In the present note we characterize the canonical system C^{k} on $J^{k}(M, N, p)$ for $m\geqq 2$ (the characterization for $m=1$ was given in [8]) and explain the distinction between the case of $m=1$ and that of $m\geqq 2$. For this purpose, we introduce the notion of contact mainfolds of order k of bidegree (n, m) (Definition 2.2): *Let D be a differential system on a manifold K. Then (K, D) is called a contact manifold of order k of bidegree (n, m) if and only if the following conditions are satisfied*;

(1) *There exists a family $\{D^{1}, \cdots, D^{k}\}$ of differential systems on K such that $D^{k}=D$ and $D^{r}=\partial D^{r+1}$ for $r=1, \cdots, k-1$, where ∂D^{r+1} is the derived system of D^{r+1}.*

(2) *D^{1} is a differential system of codimension m.*

(3) *There exists a completely integrable subbundle F of D^{1} of codimension n such that $F\supset Ch(D^{1})$.*

(4) *The Cauchy-Cartan characteristic system $Ch(D^{r})$ of D^{r} is a subbundle of D^{r+1} of codimension n for $r=1, \cdots, k-1$.*

(5) $Ch(D^{k})(x)=\{0\}$ *at each* $x\in K$.

(6) $Ch(D^{k-1})=D^{k}\cap F \quad (k\geqq 2)$.

(7) $\dim K=m\times\left(\sum_{r=1}^{k}{}_{n}H_{r}\right)+m+n$, *where* ${}_{n}H_{r}=\binom{n+r-1}{r}$.

Then our main result is stated as follows;

THEOREM 2.4′. *Let D be a differential system on a manifold K. Then (K, D) is a contact manifold of order k of bidegree (n, m) if and only if it is locally isomorphic with $(J^{k}(M, N, p), C^{k})$, where* $\dim N=n$ *and* $\dim M=m+n$.

One should observe that, without (7), other conditions of Definition 2.2 characterize the canonical differential system $D=C^k|_K$ restricted to a submanifold K of $J^k(M, N, p)$ of some class (cf. the proof of Theorem 2.4 and §§ 4, 5 [8]). In this case, in general, the completely integrable system F of condition (3) is not uniquely determined by D. However, in the presence of condition (7), F is uniquely determined by D if $m \geqq 2$ (Remark 2.5 (1)). This is the marked difference from the case $m=1$.

In § 1, we will characterize C^1 on $J^1(M, N, p)$ as a regular differential system of some type. Recently we learned that Dr. Bryant, in his thesis [3], also gave a characterization of C^1 on $J^1(M, N, p)$, which essentially includes our result. However we will give it, since our formulation is somewhat different from his, and also for the sake of completeness. § 2 is concerned with the notion of higher order contact manifolds of bidegree (n, m). Finally in § 3, we will give a remark on global contact diffeomorphisms of Jet bundles. Throughout the present note, we always assume the differentiability of class C^∞ and use the terminology in [8].

§ 1. Canonical systems on Grassmann bundles

1.1. *Canonical systems on Grassmann bundles.* Let M be a manifold of dimension $m+n$. We consider the Grassmann bundle $J(M, n)$ over M consisting of n-dimensional contact elements to M, i.e.,

$$J(M, n)=\bigcup_{x \in M} Gr\big(T_x(M), n\big),$$

where $Gr(T_x(M), n)$ denotes the Grassmann manifold of n-dimensional subspaces of $T_x(M)$ (cf. [5, p. 44]). Let π be the bundle projection of $J(M, n)$ onto M. Let $z \in J(M, n)$. Then π_*; $T_z(J(M, n)) \rightarrow T_x(M)$ is onto and z is an n-dimensional subspace of $T_x(M)$, where $x=\pi(z)$.

DEFINITION 1.1. *The canonical (differential) system C on $J(M, n)$ is the differential system of codimension m defined by*

$$C(z)=\big\{X \in T_z\big(J(M, n)\big) \,\big|\, \pi_*(X) \in z\big\}\big(=\pi_*^{-1}(z)\big) \quad \textit{for } z \in J(M, n).$$

Let V and W be vector spaces over $\boldsymbol{R}$ of dimension n and m respectively. Let $\mathfrak{C}^1(n, m)=\mathfrak{C}^1(V, W)$ be the contact algebra of first order of bidegree (n, m) (Definition 3.5 [8]). Recall that $\mathfrak{C}^1(V, W)$ is defined as follows;

$$\mathfrak{C}^1(V, W)=\mathfrak{C}_{-2}^1(V, W) \oplus \mathfrak{C}_{-1}^1(V, W) \quad \text{(direct sum)},$$

where $\mathfrak{C}_{-2}^1(V, W)=W$, $\mathfrak{F}^1(V, W)=W \otimes V^*$ and $\mathfrak{C}_{-1}^1(V, W)=V \oplus \mathfrak{F}^1(V, W)$ (direct sum). The bracket operation of $\mathfrak{C}^1(V, W)$ is given by

$$[w, w'] = 0\,, \qquad [w, v] = 0\,, \qquad [v, v'] = 0\,,$$
$$[w \otimes \xi, v] = \langle \xi, v \rangle\, w\,,$$

for $v,\ v' \in V,\ w,\ w' \in W$ and $\xi \in V^*$. Then, by using canonical coordinates (§ 1. 5 [8]), we easily have (cf. Proposition 3. 6 [8])

PROPOSITION 1. 2 *Let M be a manifold of dimension $m+n$. Then $(J(M, n), C)$ is a regular differential system of type $\mathfrak{C}^1(n, m)$. Furthermore $(J(M, n), C)$ is locally isomorphic with the standard differential system of type $\mathfrak{C}^1(n, m)$.*

One should note that, if $m=1$, $(J(M, n), C)$ is a contact manifold of dimension $2n+1$.

1. 2. *Contact transformations.* First, let $A^1(V, W)$ be the group of graded Lie algebra automorphisms of $\mathfrak{C}^1(V, W)$. Then, for the contact algebra $\mathfrak{C}^1(V, W)$, we have

PROPOSITION 1. 3 *If* $\dim W \geqq 2$, *then*

$$\mathfrak{F}^1(V, W) = \left\langle \left\{ X \in \mathfrak{C}^1_{-1}(V, W) \,\middle|\, \mathit{rank}\ \mathit{ad}(X) \leqq 1 \right\} \right\rangle.$$

Hence, if $\dim W \geqq 2$, *each $\varphi \in A^1(V, W)$ leaves $\mathfrak{F}^1(V, W)$ invariant, i. e., $\varphi(\mathfrak{F}^1(V, W)) = \mathfrak{F}^1(V, W)$.*

PROOF. Let $X = v_X + f_X$ be any element of $\mathfrak{C}^1_{-1}(V, W)$, where $v_X \in V$ and $f_X \in \mathfrak{F}^1(V, W) = \mathrm{Hom}\,(V, W)$. Then we have

$$\mathrm{ad}\,(X)\,(v) = [X, v] = f_X(v) \qquad \text{for} \quad v \in V\,,$$
$$\mathrm{ad}\,(X)\,(f) = [X, f] = -f(v_X) \qquad \text{for} \quad f \in \mathfrak{F}^1(V, W)\,.$$

Since $\mathrm{ad}\,(X)\,(\mathfrak{C}^1_{-2}(V, W)) = \{0\}$, we see that $\mathrm{rank}\,\mathrm{ad}\,(X) = \dim W$ if $v_X \neq 0$ and $\mathrm{rank}\,\mathrm{ad}\,(X) = \mathrm{rank} f_X$ if $v_X = 0$. On the other hand it is clear that $\mathfrak{F}^1(V, W) = \mathrm{Hom}\,(V, W)$ is generated by elements of rank 1. Set $E = \langle \{ X \in \mathfrak{C}^1_{-1}(V, W) |\ \mathrm{rank}\,\mathrm{ad}\,(X) \leqq 1 \} \rangle$. Then E is an $A^1(V, W)$-invariant subspace of $\mathfrak{C}^1(V, W)$ and it follows that $E = \mathfrak{C}^1_{-1}(V, W)$ if $\dim W = 1$ and $E = \mathfrak{F}^1(V, W)$ otherwise.

q. e. d.

Now we have

THEOREM 1. 4 *Let M and $\hat{M}$ be manifolds of dimension $m+n$. Then a diffeomorphism φ of M onto $\hat{M}$ induces a unique isomorphism $p\varphi$ of $(J(M, n), C)$ onto $(J(\hat{M}, n), \hat{C})$ defined by $p\varphi(z) = \varphi_*(z)$ for $z \in J(M, n)$. Conversely, if $m \geqq 2$, an isomorphism ψ of $(J(M, n), C)$ onto $(J(\hat{M}, n), \hat{C})$ induces a unique diffeomorphism φ of M onto $\hat{M}$ such that $\psi = p\varphi$.*

PROOF. The first assertion is clear by Definition 1. 1. In order to prove the converse, let z be any point of $J(M, n)$. Let $\mathfrak{c}(z)$ (resp. $\hat{\mathfrak{c}}(\phi(z))$) be the graded algebra of $(J(M, n), C)$ (resp. $(J(\hat{M}, n), \hat{C})$) at z (resp. $\phi(z)$). Then there exist graded Lie algebra isomorphisms ν; $\mathfrak{C}(V, W)\to\mathfrak{c}(z)$ and $\hat{\nu}$; $\mathfrak{C}(V, W)\to\hat{\mathfrak{c}}(\phi(z))$ such that $\nu(\mathfrak{F}^1(V, W))=\mathfrak{f}(z)$ and $\hat{\nu}(\mathfrak{F}^1(V, W))=\hat{\mathfrak{f}}(\phi(z))$, where $\mathfrak{f}(z)=\operatorname{Ker}(\pi_*)_z$ and $\hat{\mathfrak{f}}(\phi(z))=\operatorname{Ker}(\hat{\pi}_*)_{\phi(z)}$. Since ϕ is an isomorphism of $(J(M, n), C)$ onto $(J(\hat{M}, n), \hat{C})$, ϕ induces an isomorphism $\tilde{\phi}_*$ of $\mathfrak{c}(z)$ onto $\hat{\mathfrak{c}}(\phi(z))$. Then, by Proposition 1. 3, we get $\phi_*(\operatorname{Ker}\pi_*)=\operatorname{Ker}\hat{\pi}_*$. Since each fibre of $J(M, n)$ and $J(\hat{M}, n)$ is connected, we see that ϕ is fibre-preserving. Hence ϕ induces a unique diffeomorphism φ of M onto $\hat{M}$ such that $\hat{\pi}\cdot\phi=\varphi\cdot\pi$. $\phi=p\varphi$ easily follows from $\phi_*(C)=\hat{C}$ and Definition 1. 1. (cf. the proof of Proposition 3. 1 [8]). q. e. d.

Theorem 1. 4 is, in its local form, due to A. V. Bäcklund [2] (cf. [1]). In case $m=1$, $(J(M, n), C)$ is a contact manifold of dimension $2n+1$. Hence it is well known that the last assertion is false in this case.

1. 3. *Characterization by graded algebra.* In this paragraph we will consider the converse of Proposition 1. 2. Let K be a manifold of dimension $m+n+mn$ and let D be a regular differential system on K of type $\mathfrak{C}^1(n, m)$ (cf. [6]). In case $m=1$. (K, D) is a contact manifold of dimension $2n+1$ (cf. Example (1) [6, p. 10]). Hence, by the Darboux's theorem, (K, D) is locally isomorphic with $(J(M, n), C)$, where $\dim M=n+1$. Assume that $m\geqq 2$. Let $\mathfrak{d}(x)=\mathfrak{d}_{-2}(x)+\mathfrak{d}_{-1}(x)$ be the graded algebra of (K, D) at $x\in K$ (cf. [6]). Since (K, D) is a regular differential system of type $\mathfrak{C}^1(n, m)$, there exists a graded Lie algebra isomorphism $\nu(x)$ of $\mathfrak{C}^1(V, W)$ onto $\mathfrak{d}(x)$. We define the subspace $\mathfrak{f}(x)$ of $\mathfrak{d}_{-1}(x)=D(x)$ by setting

$$\mathfrak{f}(x)=\nu(x)\left(\mathfrak{F}^1(V, W)\right).$$

By Proposition 1. 3, $\mathfrak{f}(x)$ is well defined, i. e., the above definition is independent of the choice of $\nu(x)$. Hence the assignment $x\mapsto\mathfrak{f}(x)$ defines a subbundle F of D of codimension n. Obviously F is a covariant system of (K, D) (cf. Remark 1. 4 [8]) and is called the *symbol system* of (K, D). First we have

PROPOSITION 1. 5 *Let K be a manifold of dimension $m+n+mn$ and let D be a regular differential system on K of type $\mathfrak{C}^1(n, m)$. Let M be a manifold of dimension $m+n$. If $m\geqq 2$, then (K, D) is locally isomorphic with $(J(M, n), C)$ if and only if the symbol system F of (K, D) is completely integrable.*

PROOF. The only if part is clear by Proposition 1. 3 (cf. the proof of Theorem 1. 4). In order to prove the if part, assume that F is completely integrable. Let x be any point of K. Since F is a completely integrable subbundle of $T(K)$ of codimension $m+n$, we can take first integrals z^{α} and x_i ($\alpha=1, \cdots, m$, $i=1, \cdots, n$) defined on a neighborhood U of x such that $\mathrm{d}z^{\alpha}$ and $\mathrm{d}x_i$ ($\alpha=1, \cdots, m$, $i=1, \cdots, n$) are linearly independent at each $y \in U$. Furthermore, since F is a subbundle of D of codimension n, we may assume that $\mathrm{d}x_1, \cdots, \mathrm{d}x_n$ are linearly independent on $D(y)$ at each $y \in U$, i.e., $\mathrm{d}x_1, \cdots, \mathrm{d}x_n$ are linearly independent $(\mathrm{mod}\, D^{\perp}(y))$ at each $y \in U$. Then there exist unique functions p_i^{α} ($\alpha=1, \cdots, m$, $i=1, \cdots, n$) on U such that D is defined on U by the following 1-forms,

$$\varpi^{\alpha} = \mathrm{d}z^{\alpha} - \sum_{i=1}^{n} p_i^{\alpha} \mathrm{d}x_i \qquad (\alpha=1, \cdots, m).$$

On the other hand, for $\mathfrak{C}^1(V, W)$, it is easy to see that, if $X \in \mathfrak{C}^1_{-1}(V, W)$ and $[X, \mathfrak{C}^1_{-1}(V, W)] = 0$, then $X=0$. Hence $Ch(D)(x) = \{0\}$ at each $x \in K$. Since $\dim K = m+n+mn$, this implies that ϖ^{α}, $\mathrm{d}x_i$ and $\mathrm{d}p_i^{\alpha}$ ($\alpha=1, \cdots, m$, $i=1, \cdots, n$) are linearly independent at each $y \in U$. Therefore the system of functions z^{α}, x_i and p_i^{α} ($\alpha=1, \cdots, m$, $i=1, \cdots, n$) is a coordinate system of K around x. q. e. d.

Furthermore, if $m \geqq 3$, the symbol system F is necessarily completely integrable (cf. p. 81 [4]). In fact we have

THEOREM 1.6 (cf. [3]). *Let K be a manifold of dimension $m+n+mn$ and let D be a regular differential system on K of type $\mathfrak{C}^1(n, m)$. If $m \neq 2$, then, at each point $x \in K$, there exists a coordinate system $(z^{\alpha}, x_i, p_i^{\alpha})$ ($\alpha=1, \cdots, m$, $i=1, \cdots, n$) defined on a neighborhood U of x such that D is defined on U by the following 1-forms,*

$$\varpi^{\alpha} = \mathrm{d}z^{\alpha} - \sum_{i=1}^{n} p_i^{\alpha} \mathrm{d}x_i \qquad (\alpha=1, \cdots, m).$$

PROOF. If $m=1$, the assertion is precisely the Darboux's theorem for contact manifolds. If $m \geqq 3$, in view of Proposition 1.5, it suffices to show that the symbol system F is completely integrable. First one should note that (K, D) is a regular differential system of type $\mathfrak{C}^1(n, m)$ if and only if, at each $x \in K$, there exist 1-forms ϖ^{α}, ω_i and ϖ_i^{α} ($\alpha=1, \cdots, m$, $i=1, \cdots, n$) defined on a neighborhood U of x, which form a basis of 1-forms at each $y \in U$, such that D is defined on U by $\varpi^1, \cdots, \varpi^m$ and that the following equalities hold (cf. the proof of Theorem 6.5 [8]),

$$(1.1) \qquad \mathrm{d}\varpi^{\alpha} \equiv \sum_{i=1}^{n} \omega_i \wedge \varpi_i^{\alpha} \quad (\mathrm{mod}\, \varpi^1, \cdots, \varpi^m) \text{ for } \alpha=1, \cdots, m.$$

Then it is easy to see that F is defined on U by ϖ^{α} and ω_i ($\alpha=1, \cdots, m$,

$i=1, \cdots, n)$. Since $d\varpi^\alpha \equiv 0 \pmod{\varpi^1, \cdots, \varpi^m, \omega_1, \cdots, \omega_n}$, it suffices to show that $d\omega_i \equiv 0 \pmod{\varpi^1, \cdots, \varpi^m, \omega_1, \cdots, \omega_n}$ for $i=1, \cdots, n$. First, from (1.1), we get

$$(1.2) \qquad \sum_{i=1}^{n} d\omega_i \wedge \varpi_i^\alpha \equiv 0 \quad (\mathrm{mod}\, F^\perp) \text{ for } \alpha=1, \cdots, m\,.$$

On the other hand, there exist functions A_α^{ijk} and $B_{\alpha\beta}^{ijk}$ on U such that

$$(1.3) \qquad d\omega_i \equiv \sum_{\beta=1}^{m} \sum_{j<k} A_\beta^{ijk} \varpi_j^\beta \wedge \varpi_k^\beta + \sum_{j,k=1}^{n} \sum_{\beta<\gamma} B_{\beta\gamma}^{ijk} \varpi_j^\beta \wedge \varpi_k^\gamma$$
$$(\mathrm{mod}\, F^\perp) \text{ for } i=1, \cdots, n\,.$$

Substituting (1.3) into (1.2), we get

$$(1.4) \qquad \sum_{i=1}^{n} \sum_{\beta=1}^{m} \sum_{j<k} A_\beta^{ijk} \varpi_i^\alpha \wedge \varpi_j^\beta \wedge \varpi_k^\beta + \sum_{i,j,k=1}^{n} \sum_{\beta<\gamma} B_{\beta\gamma}^{ijk} \varpi_i^\alpha \wedge \varpi_j^\beta \wedge \varpi_k^\gamma$$
$$\equiv 0 \qquad (\mathrm{mod}\, F^\perp) \text{ for } \alpha=1, \cdots, m\,.$$

Take any β and (j, k) such that $j<k$. Since $m \geqq 2$, there exists α such that $\alpha \neq \beta$. Then, for any i, we see that the coefficient of $\varpi_i^\alpha \wedge \varpi_j^\beta \wedge \varpi_k^\beta$ of the left hand side of (1.4) equals to A_β^{ijk}. Hence we get $A_\beta^{ijk}=0$. Similarly take any (β, γ), j and k such that $\beta<\gamma$. Since $m \geqq 3$, there exists α such that $\alpha \neq \beta$ and $\alpha \neq \gamma$. Then, for any i, we see that the coefficient of $\varpi_i^\alpha \wedge \varpi_j^\beta \wedge \varpi_k^\gamma$ of the left hand side of (1.4) equals to $B_{\beta\gamma}^{ijk}$. Hence we get $B_{\beta\gamma}^{ijk}=0$. Thus we obtain

$$d\omega_i \equiv 0 \qquad (\mathrm{mod}\, F^\perp) \text{ for } i=1, \cdots, n\,. \qquad \text{q. e. d.}$$

For $m=2$, the assertion of Theorem 1.6 does not hold as shown by the following example.

EXAMPLE. Let $(x_i, z^1, z^2, p_i^1, p_i^2)$ $(i=1, \cdots, n)$ be the natural coordinate of $\boldsymbol{R}^N$, where $N=3n+2$. Let D be the differential system on $K=\boldsymbol{R}^N$ of codimension 2 defined by the following 1-forms,

$$\varpi^1 = dz^1 - \sum_{i=1}^{n} p_i^1 dx_i - p_1^1 p_1^2 dp_1^1\,,$$

$$\varpi^2 = dz^2 - \sum_{i=1}^{n} p_i^2 dx_i\,.$$

Then we have

$$d\varpi^\alpha = \sum_{i=1}^{n} \omega_i \wedge dp_i^\alpha \qquad (\alpha=1, 2)\,,$$

where $\omega_1=dx_1-p_1^1 dp_1^2$ and $\omega_i=dx_i$ $(i=2, \cdots, n)$. Hence (K, D) is a regular differential system of type $\mathfrak{C}^1(n, 2)$. Furthermore the symbol system F is defined by ϖ^α and ω_i ($\alpha=1, \cdots, m,\ i=1, \cdots, n$). It is easy to see that F is not completely integrable.

REMARK 1.7. (1) Let R be a submanifold of $J(M, n)$ such that $\pi: R\to M$ is a submersion. Set $D=C|_R$ and $F=\mathrm{Ker}\,\pi_*$. One should note, for a submanifold R of $J(M, n)$, $(R; D, F)$ is characterized by Lemma 1.5 [8] (see also [7]). In particular, $(J(M, n), C)$ is characterized as follows: *Let D be a differential system on a manifold K. Assume that D satisfies the following.*

(1) *D is a differential system of codimension m such that $Ch(D)(x)=\{0\}$ at each $x\in K$.*

(2) *There exists a completely integrable subbundle F of D of codimension n.*

Then $\dim K\leqq m+n+mn$ (In fact, by Lemma 1.5 [8], $(K; D, F)$ is locally realized as a submanifold of $J(M, n)$, where $M=K/F$). The equality holds if and only if (K, D) is locally isomorphic with $(J(M, n), C)$. This characterization is due to N. Tanaka [7]. Furthermore, if the equality holds ($m\geqq 2$), F is a covariant system of D by Proposition 1.3 (cf. Remark 2.5 (1)).

(2) We can apply Theorem 1.6 to the (contact) equivalence and the integration problems of the following type of involutive systems of second order (cf. p. 80 [4]): Let $\mathfrak{C}^2(V, W)$ be the contact algebra of second order of degree n (Definition 3.5 [8]). Let E be an r-dimensional subspace of V. We define the involutive subalgebra $\mathfrak{s}(E)$ of $\mathfrak{C}^2(V, W)$ by setting

$$\mathfrak{s}(E)=\mathfrak{s}_{-3}(E)\oplus\mathfrak{s}_{-2}(E)\oplus\mathfrak{s}_{-1}(E)\,,$$
$$\mathfrak{s}_{-3}(E)=W\,,\qquad \mathfrak{s}_{-2}(E)=W\otimes V^*\,,$$
$$\mathfrak{s}_{-1}(E)=V\oplus W\otimes A(E)\,,\qquad A(E)=E^\perp\otimes_S V^*\subset S^2(V^*)\,.$$

Let $(R; D^1, D^2)$ be a regularly involutive PD manifold of second order of type $\mathfrak{s}(E)$ (cf. § 5 [8]). Then, if $r\geqq 2$, $(R; D^1, D^2)$ is locally isomorphic with the involutive system R_r of second order defined by

$$R_r=\left\{\frac{\partial^2 z}{\partial x_i\partial x_j}=0\,\middle|\,1\leqq i, j\leqq r\right\}.$$

Furthermore every solution of (R, D^2) can be obtained locally by solving ordinary differential equations.

We will treat this application in a forthcoming paper.

§ 2. Geometrization of Jet bundles

2.1. *Higher prolongation of $(J(M, n), C)$.* If $m\geqq 2$, $Q^1=\mathrm{Ker}\,\pi_*$ is a covariant system of C and is a subbundle of C of codimension n. Hence $(J(M,n), C)$ is a differential system with (geometrically defined) n independent variables (cf. § 2 [8]). Now, for $k\geqq 2$, we define the k-th order prolongation $(J^k(M,n), C^k)$ of $(J(M,n), C)$ $(m\geqq 2)$ inductively as follows. For $k=1$, we set

$$J^1(M,n)=J(M,n), \qquad C^1=C, \qquad \rho_0^1=\pi \quad \text{and} \quad Q^1=\operatorname{Ker}\pi_* .$$

(1) *The bundle $J^k(M,n)$ of k-th order*: For each $u\in J^{k-1}(M,n)$, let J_u^k be the set of all n-dimensional integral elements v of $(J^{k-1}(M,n), C^{k-1})$ at u such that $v\cap Q^{k-1}(u)=\{0\}$, where $Q^{k-1}=\operatorname{Ker}(\rho_{k-2}^{k-1})_*$. Then $J^k(M,n)$ is defined by

$$J^k(M,n)=\bigcup_{u\in J^{k-1}(M,n)} J_u^k \qquad (k\geqq 2).$$

$J^k(M,n)$ is a regular submanifold of $J(J^{k-1}(M,n),n)$ and is a fiber bundle over $J^{k-1}(M,n)$ with standard fibre $\boldsymbol{R}^{N(k)}$, where $N(k)=m\times\binom{n+k-1}{k}$. Let ρ_{k-1}^k be the bundle projection of $J^k(M,n)$ onto $J^{k-1}(M,n)$. We set

$$\rho_r^k=\rho_{k-1}^k\cdot\cdots\cdot\rho_r^{r+1} \quad \text{for} \quad k>r \quad \text{and} \quad \rho_k^k=id_{J^k(M,n)} .$$

(2) *The canonical (differential) system C^k on $J^k(M,n)$*: For $v^k\in J^k(M,n)$, (ρ_{k-1}^k); $T_{v^k}(J^k(M,n))\to T_{v^{k-1}}(J^{k-1}(M,n))$ is onto and v^k is an n-dimensional subspace of $T_{v^{k-1}}(J^{k-1}(M,n))$, where $v^{k-1}=\rho_{k-1}^k(v^k)$. Then C^k is defined by

$$C^k(v^k)=(\rho_{k-1}^k)_*^{-1}(v^k) \quad \text{for} \quad v^k\in J^k(M,n).$$

In other words C^k is the restriction to $J^k(M,n)$ of the canonical system on $J(J^{k-1}(M,n),n)$.

Thus we have completed our inductive definition of $(J^k(M,n), C^k)$.

Let M_k be the set of all k-tuples of integers $1,\cdots,n$. We denote by S_r the set of all $I=(i_1,\cdots,i_r)\in M_r$ such that $1\leqq i_1\leqq\cdots\leqq i_r\leqq n$ and set $\Sigma_k=\bigcup_{r=1}^k S_r$. Let v^k be any point of $J^k(M,n)$. By using a canonical coordinate $(x_1,\cdots,x_n,z^1,\cdots,z^m)$ of $J^1(M,n)$ at $v^1=\rho_1^k(v^k)$, we get a coordinate system $(x_i, z^\alpha, p_I^\alpha)$ $(1\leqq i\leqq n,\ 1\leqq\alpha\leqq m,\ I\in\Sigma_k)$ such that C^k is defined by the following 1-forms (cf. Lemma 2.3 [8]);

$$\varpi^\alpha=dz^\alpha-\sum_{i=1}^n p_i^\alpha dx_i \qquad (\alpha=1,\cdots,m),$$

$$\varpi_I^\alpha=dp_I^\alpha-\sum_{j=1}^n p_{I,j}^\alpha dx_j \qquad (\alpha=1,\cdots,m,\ I\in\Sigma_{k-1}).$$

Let $\mathfrak{C}^k(n,m)=\mathfrak{C}^k(V,W)$ be the contact algebra of k-th order of bidegree

(n, m) (Definition 3.5 [8]). Let C_r^k and Q_r^k be differential systems on $J^k(M, n)$ defined by

$$C_r^k=(\rho_r^k)_*^{-1}(C^r) \quad (r=1, \cdots, k), \qquad Q_r^k=\operatorname{Ker}(\rho_r^k)_* \quad (r=0, \cdots, k-1).$$

Then we have easily

PROPOSITION 2.1 (cf. Proposition 2.5 [8]). (1) *C^k is a regular differential system of type $\mathfrak{C}^k(n, m)$. Furthermore C_r^k coincides with the first derived system ∂C_{r+1}^k of C_{r+1}^k, i.e., $C_r^k=\partial^{k-r}C^r$ $(r=1, \cdots, k-1)$, where $\partial^\alpha C^k$ is the α-th derived system of C^k.*

(2) *Q_r^k is a subbundle of C_{r+1}^k of codimension n and coincides with the Cauchy-Cartan characteristic system $Ch(C_r^k)$ of C_r^k $(r=1, \cdots, k-1)$. Furthermore Q_0^k is a completely integrable subbundle of C_1^k of codimension n.*

(3) *$Ch(C^k)(v^k)=\{0\}$ at each $v^k\in J^k(M, n)$.*

(4) *$Q^k=C^k\cap Q_0^k$ $(k\geqq 2)$.*

2.2. *Contact manifold (K, D) of order k of bidegree (n, m).* In view of Proposition 2.1, we first give the following definition (cf. Definition 6.1 [8]).

DEFINITION 2.2 *Let D be a differential system on a manifold K. Then (K, D) is called a contact manifold of order k of bidegree (n, m) if and only if the following conditions are satisfied:*

(1) *There exists a family $\{D^1, \cdots, D^k\}$ of differential systems on K such that $D^k=D$ and $D^r=\partial D^{r+1}$ for $r=1, \cdots, k-1$, i.e., the r-th derived sheaf $\partial^r\mathscr{D}$ of D defines a differential system $\partial^r D=D^{k-r}$ for $r=1, \cdots, k-1$.*

(2) *D^1 is a differential system of codimension m.*

(3) *There exists a completely integrable subbundle F of D^1 of codimension n such that $F\supset Ch(D^1)$.*

(4) *The Cauchy-Cartan characteristic system $Ch(D^r)$ of D^r is a subbundle of D^{r+1} of codimension n for $r=1, \cdots, k-1$.*

(5) *$Ch(D^k)(x)=\{0\}$ at each $x\in K$.*

(6) *$Ch(D^{k-1})=D^k\cap F$ $(k\geqq 2)$.*

(7) *$\dim K=m\times\left(\sum_{r=1}^k {}_nH_r\right)+m+n$, where ${}_nH_r=\binom{n+r-1}{r}$.*

Obviously, by Proposition 2.1, $(J^k(M, n), C^k)$ is a contact manifold of order k of bidegree (n, m).

First we have easily

LEMMA 2.3. (cf. Lemma 5.2 [8]). (i) *$F\supset Ch(D^1)\supset\cdots\supset Ch(D^k)$.*
(ii) *$Ch(D^1)=F\cap D^2$ and $Ch(D^r)=Ch(D^{r-1})\cap D^{r+1}$ for $r=2, \cdots, k-1$.*

Now we have (cf. Theorem 6.2 [8])

THEOREM 2.4 *Let D be a differential system on a manifold K. Then (K, D) is a contact manifold of k-th order of bidegree (n, m) if and only if, at each $x \in K$, there exist functions x_i, z^α and p_I^α $(i=1, \cdots, n,\ \alpha=1, \cdots, m,\ I \in \bigcup_{r=1}^{k} M_r)$ defined on a neighborhood U of x such that*

(1) *p_I is symmetric with respect to I.*

(2) *The system of functions x_i, z^α and p_I^α $(i=1, \cdots, n,\ \alpha=1, \cdots, m,\ I \in \Sigma_k)$ is a coordinate system on U.*

(3) *D is defined on U by the following 1-forms;*

$$\varpi^\alpha = dz^\alpha - \sum_{i=1}^{n} p_i^\alpha dx_i \qquad (\alpha=1, \cdots, m),$$

$$\varpi_I^\alpha = dp_I^\alpha - \sum_{j=1}^{n} p_{I,j}^\alpha dx_j \qquad (\alpha=1, \cdots, m,\ I \in \Sigma_{k-1}).$$

PROOF. By Proposition 2.1, it suffices to show the only if part. Let (K, D) be a contact manifold of order k of bidegree (n, m). The following proof is quite similar to that of Theorem 5.3 [8]. Matters being of local nature, we may assume that K is regular with respect to F, i.e., the set $M = K/F$ of all leaves of the foliation defined by F has a differentiable structure such that the natural projection $\bar{\rho}_0$; $K \to M$ is a submersion (cf. §5.1 [8]).

Now we will show that there exists an immerison ι of K into $J^k(M, n)$ such that $\rho_0^k \cdot \iota = \bar{\rho}_0$ and $D = \iota_*^{-1}(C^k)$. For this purpose, by induction on r, we construct a map $\bar{\rho}_r$; $K \to J^r(M, n)$ satisfying $\rho_{r-1}^r \bar{\rho}_r = \bar{\rho}_{r-1}$, $\mathrm{Ker}\,(\bar{\rho}_r)_* = Ch(D^r)$ and $D^r = (\bar{\rho}_r)_*^{-1}(C^r)$ as follows. For $r=1$, since $F = \mathrm{Ker}\,(\bar{\rho}_0)_*$ is a subbundle of D^1 of codimension n, we can define $\bar{\rho}_1$ by

$$\bar{\rho}_1(x) = (\bar{\rho}_0)_*\big(D^1(x)\big) \in J^1(M, n) \qquad \text{for} \quad x \in K.$$

Then it is easy to see that $\rho_0^1 \cdot \bar{\rho}_1 = \bar{\rho}_0$ and $D^1 = (\bar{\rho}_1)_*^{-1}(C^1)$. Furthermore, since $F \cap Ch(D^1) = Ch(D^1)$, $\mathrm{Ker}\,(\bar{\rho}_1)_* = Ch(D^1)$ follows from Lemma 1.5 [8].

Now suppose that we have constructed a map $\bar{\rho}_r$; $K \to J^r(M, n)$ satisfying $\rho_{r-1}^r \cdot \bar{\rho}_r = \bar{\rho}_{r-1}$, $\mathrm{Ker}\,(\bar{\rho}_r)_* = Ch(D^r)$ and $D^r = (\bar{\rho}_r)_*^{-1}(C^r)$. Since $\mathrm{Ker}\,(\bar{\rho}_r)_* = Ch(D^r)$ is a subbundle of D^{r+1} of codimension n, we can define $\tilde{\rho}_{r+1}$; $K \to J(J^r(M, n), n)$ by setting

$$\tilde{\rho}_{r+1}(x) = (\bar{\rho}_r)_*\big(D^{r+1}(x)\big) \in J\big(J^r(M, n), n\big) \qquad \text{for} \quad x \in K.$$

Then, by the definition of the canonical system C_{r+1}^* on $J(J^r(M, n), n)$, we have $D^{r+1} = (\rho_r)_*^{-1}(C_{r+1}^*)$ and $\pi^{r+1} \cdot \tilde{\rho}_{r+1} = \bar{\rho}_r$, where π^{r+1} is the projection of $J(J^r(M, n), n)$ onto $J^r(M, n)$. Furthermore $\mathrm{Ker}\,(\tilde{\rho}_{r+1})_* = Ch(D^{r+1})$ follows from $\mathrm{Ker}\,(\bar{\rho}_r)_* \cap Ch(D^{r+1}) = Ch(D^{r+1})$ and Lemma 1.5 [8]. Hence it remains to show that $\tilde{\rho}_{r+1}(K)\subset J^{r+1}(M, n)$. First, from $D^r=\partial D^{r+1}$ and $D^r=(\bar{\rho}_r)_*^{-1}(C^r)$, we see that $\tilde{\rho}_{r+1}(x)$ is an integral element of $(J^r(M, n), C^r)$. On the other hand, by Lemma 2. 3 (ii), we have $Ch(D^r)=Ch(D^{r-1})\cap D^{r+1}$, i. e., $\mathrm{Ker}\,(\bar{\rho}_r)_*=Ch(D^{r-1})\cap D^{r+1}$. Furthermore, from $\rho_{r-1}^r\cdot\bar{\rho}_r=\bar{\rho}_{r-1}$ and $Q^r=\mathrm{Ker}\,(\rho_{r-1}^r)_*$, we get $Ch(D^{r-1})=(\bar{\rho}_r)_*^{-1}(Q^r)$. Hence we have $(\bar{\rho}_r)_*(D^{r+1}(x))\cap Q^r(\bar{\rho}_r(x))=\{0\}$. Therefore we obtain $\tilde{\rho}_{r+1}(x)\in J^{r+1}(M, n)$. Thus we construct a map $\bar{\rho}_{r+1}$; $K\to J^{r+1}(M,n)$ satisfying $\rho_r^{r+1}\cdot\bar{\rho}_{r+1}=\bar{\rho}_r$, $\mathrm{Ker}(\bar{\rho}_{r+1})_*=Ch(D^{r+1})$ and $D^{r+1}=(\bar{\rho}_{r+1})_*^{-1}(C^{r+1})$.

Accordingly, for $r=k$, we get a map $\iota=\bar{\rho}_k$; $K\to J^k(M, n)$ satisfying $\rho_{k-1}^k\cdot\iota=\bar{\rho}_{k-1}$, $\mathrm{Ker}\,\iota_*=Ch(D^k)$ and $D^k=\iota_*^{-1}(C^k)$. Since $Ch(D^k)=\{0\}$, ι is an immersion of K into $J^k(M, n)$. Furthermore, by (7) of Definition 2. 2, ι is a local isomorphism of (K, D) into $(J^k(M, n), C^k)$. q. e. d.

REMARK 2. 5. (1) Let (K, D) be a contact manifold of order k of bidgree (n, m). The differential system D^r of Definition 2. 2 (1) is a $(k-r)$-th derived system of D. Hence the family $\{D^1, \cdots, D^k\}$ is uniquely determined by D. Furthermore, if $m\geqq 2$, the completely integrable system F of Definition 2. 2 (3) is also uniquely determined by D. In fact we can show the uniqueness of F as follows: Matters being of local nature, we may assume that K is regular with respect to $Ch(D^1)$. Then there exists a differential system $\bar{D}$ on $\bar{K}=K/Ch(D^1)$ such that $D^1=\bar{\rho}_*^{-1}(\bar{D})$, where $\bar{\rho}$ is the projection of K onto $\bar{K}$. On the other hand, by Theorem 2. 4, there exist (independent) first integrals x_i, z^α and p_i^α $(i=1, \cdots, n,\ \alpha=1, \cdots, m)$ of $Ch(D^1)$ such that D^1 is defined by $\varpi^\alpha=dz^\alpha-\sum_{i=1}^n p_i^\alpha dx_i$ $(\alpha=1, \cdots, m)$. Hence $\bar{K}$ is a manifold of dimension $m+n+mn$ and $\bar{D}$ is a differential system on $\bar{K}$ of codimension m. Furthermore, $(\bar{K}, \bar{D})$ is locally isomorphic with $(J(M, n), C)$. Hence $\bar{D}$ is a regular differential system of type $\mathfrak{C}^1(n, m)$. Let $\bar{F}$ be the symbol system of $(\bar{K}, \bar{D})$. Then we have $F=\bar{\rho}_*^{-1}(\bar{F})$. This shows the uniqueness of F.

(2) Let (K, D) be a contact manifold of order k of bidegree (n, m). Then (K, D) is a regular differential system of type $\mathfrak{C}^k(n, m)$. Concerning with this fact, we have the following (cf. Remark 6. 7 [8]): Let K be a manifold of dimension $m\times\left(\sum_{r=1}^k {}_nH_r\right)+m+n$ and let D be a differential system on K $(m\geqq 3)$. Let x be any point of K. Then we can find a coordinate system $(x_i, z^\alpha, p_I^\alpha)$ $(1\leqq i\leqq n,\ 1\leqq\alpha\leqq m,\ I\in\Sigma_k)$ of K around x such that D is defined by the following 1-forms;

$$\varpi^\alpha=dz^\alpha-\sum_{i=1}^n p_i^\alpha dx_i \qquad (\alpha=1, \cdots, m),$$

$$\varpi_I^\alpha=dp_I^\alpha-\sum_{j=1}^n p_{I,j}^\alpha dx_j \qquad (\alpha=1, \cdots, m,\ I\in\Sigma_{k-1}),$$

if and only if there exist 1-forms $\omega_1, \cdots, \omega_n$, $\varpi^1, \cdots, \varpi^m$ and ϖ^α_I ($1 \leqq \alpha \leqq m$, $I \in \Sigma_k$) defined around x, which form a basis of 1-forms around x, such that D is defined by ϖ^α and ϖ^α_I ($1 \leqq \alpha \leqq m$, $I \in \Sigma_{k-1}$) and that the following equalities hold;

$$d\varpi^\alpha \equiv \sum_{i=1}^n \omega_i \wedge \varpi^\alpha_i \qquad (\bmod\ \varpi^1, \cdots, \varpi^m),$$

$$d\varpi^\alpha_I \equiv \sum_{j=1}^n \omega_j \wedge \varpi^\alpha_{I,j} \qquad \left(\bmod\ \varpi^\beta, \varpi^\beta_J\ (1 \leqq \beta \leqq m,\ J \in \Sigma_r)\right),$$

for $I \in S_r$ ($r=1, \cdots, k-1$). For $m=2$, we must further impose the condition that $F=\{\varpi^1=\cdots=\varpi^m=\omega_1=\cdots=\omega_n=0\}$ is completely integrable.

Let (K, D) be a contact manifold of order k of bidegree (n, m). (K, D) is a regular differential system of type $\mathfrak{C}^k(n, m)$. Now we will mention about the prolongations of (K, D). Let x be any point of K and let $\mathfrak{d}(x)=\sum_{p=-1}^{-(k+1)} \mathfrak{d}_p(x)$ be the graded algebra of D at x (cf. [6]). Let $K^{(1)}$ be the prolongation of (K, D), i. e.,

$$K^{(1)} = \bigcup_{x \in K} K^{(1)}(x),$$

where $K^{(1)}(x)$ is the set of n-dimensional integral elements w of (K, D) at x such that $D(x)=w \oplus Ch(\partial D)(x)$ (direct sum) (cf. § 6 [8]). Set $V(x)= D(x)/Ch(\partial D)(x)$. Then, by Proposition 3. 7 [8] (cf. Proposition 5. 10 [8]), it follows that $K^{(1)}(x)$ is an affine space modeled on $\mathfrak{d}_{-(k+1)}(x) \otimes S^{k+1}(V^*(x))$. Let $S(K)$ be the vector bundle over K defined by

$$S(K) = \mathfrak{d}_{-(k+1)} \otimes S^{k+1}(V^*),$$

where $\mathfrak{d}_{-(k+1)}=T(K)/\partial^{k-1}D$ and $V=D/Ch(\partial D)$. Then $K^{(1)}$ is an affine bundle over K modeled on $S(K)$. Let $\rho^{(1)}$ be the projection of $K^{(1)}$ onto K and let $D^{(1)}$ be the restriction to $K^{(1)}$ of the canonical system on $J(K, n)$. Then $(K^{(1)}, D^{(1)})$ is a contact manifold of order $k+1$ of bidegree (n, m) and Ker $\rho^{(1)}_*=Ch(\partial D^{(1)})$. Thus we obtain

PROPOSITION 2. 6. (1) *Let (K, D) be a contact manifold of order k of bidegree (n, m) $(m \geqq 2)$. Then the prolongation $(K^{(1)}, D^{(1)})$ of (K, D) is a contact manifold of order $k+1$ of bidegree (n, m) and $K^{(1)}$ is an affine bundle over K modeled on $S(K)$.*

(2) *Let (K, D) and $(\hat{K}, \hat{D})$ be contact manifolds of order k of bidegree (n, m) and let $(K^{(1)}, D^{(1)})$ (resp. $(\hat{K}^{(1)}, Y^{(1)})$) be the prolongation of (K, D) (resp. (K, D)). Let φ be an isomorphism of (K, D) onto $(\hat{K}, \hat{D})$. Then φ induces a unique isomorphism $p\varphi$ of $(K^{(1)}, D^{(1)})$ onto $(\hat{K}^{(1)}, \hat{D}^{(1)})$ defined by $p\varphi(w)=$*

$\varphi_*(w)$ *for* $w\in K^{(1)}$. *Conversely an isomorphism* ϕ *of* $(K^{(1)}, D^{(1)})$ *onto* $(\hat{K}^{(1)}, \hat{D}^{(1)})$ *induces a unique isomorphism* φ *of* (K, D) *onto* $(\hat{K}, \hat{D})$ *such that* $\phi=p\varphi$.

§ 3. A remark on contact diffeomorphisms of Jet bundles

Let (M, N, p) be a fibred manifold of fibre dimension m, i.e., p is a submersion of M onto N such that $\dim N=n$ and $\dim M=m+n$. Let $J^k(M, N, p)$ be the bundle of k-jets of local sections of (M, N, p). Then $J^k(M, N, p)$ has a canonical differential system C^k ([5, p. 85]). Obviously $(J^k(M, N, p), C^k)$ is a contact manifold of order k of bidegree (n, m).

For $k=1$, there exists a canonical open imbedding ι of $J^1(M, N, p)$ into $J(M, n)$ defined by $\iota(z)=f_*(T_x(N))$ for $x=p^1_{-1}(z)$ and $z=j^1_x f$. Furthermore, if $m=1$, $\mathfrak{C}=(J^1(M, N, p), C^1)$ is a contact manifold. In this case, there exists a canonical open imbedding ι^2 of $J^2(M, N, p)$ into $L(\mathfrak{C})$ defined by $\iota^2(w)=(j^1 f)_*(T_x(N))$ for $x=p^2_{-1}(w)$ and $w=j^2_x f$, where $(L, (\mathfrak{C}), E)$ is the Lagrange-Grassmann bundle over $\mathfrak{C}$ (§ 2 [8]). In both cases one should note that $C^1=\iota_*^{-1}(C)$ and $C^2=(\iota^2)_*^{-1}(E)$.

Let (M, N, p) and $(\hat{M}, \hat{N}, \hat{p})$ be fibred manifolds of fibre dimension m. Let φ be a fibre-preserving diffeomorphism of M onto $\hat{M}$. Obviously φ induces a unique isomorphism $p^k\varphi$ of $(J^k(M, N, p), C^k)$ onto $(J^k(\hat{M}, \hat{N}, \hat{p}), \hat{C}^k)$. For the converse problem we note

THEOREM 3.1. *Let* (M, N, p) *and* $(\hat{M}, \hat{N}, \hat{p})$ *be fibred manifolds of fibre dimension* m *such that each fibre is connected. Let* ϕ *be an isomorphism of* $(J^k(M, N, p), C^k)$ *onto* $(J^k(\hat{M}, \hat{N}, \hat{p}), \hat{C}^k)$. *Then, if* $(k, m)\neq(1, 1)$, *there exists a unique fibre-preserving diffeomorphism* φ *of* M *onto* $\hat{M}$ *such that* $\phi=p^k\varphi$.

PROOF. By Proposition 3.1 [8] and Proposition 2.6, it suffices to prove the assertion in the following two cases.

(i) $k=1$ and $m\geqq 2$. As in the proof of Theorem 1.4, ϕ induces a diffeomorphism φ of M onto $\hat{M}$ such that $\hat{p}^1_0\phi=\varphi\circ p^1_0$. Then, applying Lemma 1.5 [8] to $(J^1(M, N, p), C^1, \varphi\cdot p^1_0, \hat{M})$, we get $\hat{\iota}\cdot\phi=p\varphi\cdot\iota$, where ι (resp. $\hat{\iota}$) is the canonical open imbedding of $J^1(M, N, p)$ (resp. $J^1(\hat{M}, \hat{N}, \hat{p})$) into $J(M, n)$ (resp. $J(\hat{M}, n)$) and $p\varphi$ is the lift of φ (Theorem 1.4). On the other hand, for $J^1_y(M, N, p)=(p^1_0)^{-1}(y)$, $y\in M$, we have

$$\iota\left(J^1_y(M, N, p)\right)=B\left(F(y)\right),$$

where $F(y)=\operatorname{Ker}(p^0_{-1})_{*y}$ and

$$B\left(F(y)\right)=\left\{z\in Gr\left(T_y(M), n\right)\middle| z\cap F(y)=\{0\}\right\}.$$

Hence, from $\varphi_*(B(F(y))=B(\hat{F}(\varphi(y)))$, we get $\varphi_*(\mathrm{Ker}\,(p^0_{-1})_*)=\mathrm{Ker}\,(\hat{p}^0_{-1})_*$ Therefore φ is fibre-preserving.

(ii) $k=2$ and $m=1$. Since $(p^2_1)^{-1}_*(C^1)=\partial C^2$ and $\mathrm{Ker}\,(p^2_1)_*=Ch(\partial C^2)$, ϕ induces a unique isomorphism $\tilde{\varphi}$ of $(J^1(M,N,p),C^1)$ onto $(J^1(\hat{M},\hat{N},\hat{p}),\hat{C}^1)$ such that $\hat{p}^2_1\cdot\phi=\tilde{\varphi}\cdot p^2_1$. Then, by Corollary 5.4 [8], we have $\hat{\iota}^2\circ\phi=q\tilde{\varphi}\circ\iota^2$, where ι^2 (resp. $\hat{\iota}^2$) is the canonical open imbedding of $J^2(M,N,p)$ (resp. $J^2(\hat{M},\hat{N},\hat{p})$) into $L(\mathfrak{C})$ (resp. $L(\hat{\mathfrak{C}})$) and $q\tilde{\varphi}$ is the lift of $\tilde{\varphi}$ (Theorem 3, 2 [8]). On the other hand, $L_z=\pi^{-1}(z)$ being considered as the Lagrange-Grassmann manifold $\Lambda(n)$ (§ 2 [8[), $\iota^2(J^2_z(M,N,p))$ coincides with the open cell $\Lambda(F(z))\subset L_z$ consisting of those elements of L_z which are transversal to $F(z)=\mathrm{Ker}\,(p^1_0)_*\in L_z$. From $\hat{\iota}^2\circ\phi=q\tilde{\varphi}\circ\iota^2$, we get $q\tilde{\varphi}(\Lambda(F(z)))=\Lambda(\hat{F}(\tilde{\varphi}(z)))$. Furthermore, from $\bigcap_{w\in\Lambda(F(z))}\Lambda(w)=\{F(z)\}$, we obtain $\tilde{\varphi}_*(F(z))=\hat{F}(\tilde{\varphi}(z))$, i.e., $\tilde{\varphi}_*(\mathrm{Ker}\,(p^1_0)_*)=\mathrm{Ker}\,(\hat{p}^1_0)_*$. Hence $\tilde{\varphi}$ induces a diffeomorphism φ of M such that $\hat{p}^1_0\circ\tilde{\varphi}=\varphi\circ p^1_0$. Then, as in (i), it follows that φ is fibre-preserving.

q. e. d.

COROLLARY 3.2. *Let X be an infinitesimal isomorphism of $(J^k(M,N,p),C^k)$ which is complete, i.e., X generates a global 1-parameter group of isomorphisms of $(J^k(M,N,p),C^k)$. Then, if $(k,m)\neq(1,1)$, X is projectable to N, i.e., there exists a vector field $\hat{X}$ on N such that X is p^k_{-1}-related to $\hat{X}$.*

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Department of Mathematics
Hokkaido University
and
Mathematical Sciences
Research Institute, Berkeley